Polyhedral graph abstractions and an approach to the Linear Hirsch Conjecture

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Abstract

We introduce a new combinatorial abstraction for the graphs of polyhedra. The new abstraction is a flexible framework defined by combinatorial properties, with each collection of properties taken providing a variant for studying the diameters of polyhedral graphs. One particular variant has a diameter which satisfies the best known upper bound on the diameters of polyhedra. Another variant has superlinear asymptotic diameter, and together with some combinatorial operations, gives a concrete approach for disproving the Linear Hirsch Conjecture.

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1 Introduction

Studying the diameters of the graphs of polytopes and polyhedra has received a lot of attention (see [11]) due to Santos' recent counter-example (see [23]) to the Hirsch Conjecture. The Hirsch Conjecture asserts that the diameter of any d-dimensional polytope with n facets is never greater than $n - d$. Since this conjecture is now known to be false, the question of the Polynomial Hirsch Conjecture (which asserts that the diameter of any polytope with n facets is polynomial in n) is relevant. The first step in this line of investigation is to settle the Linear Hirsch Conjecture (asserting that the diameter is linear in n).

The original Hirsch Conjecture was stated for the graphs of polyhedra. Since Klee and Walkup (see [17]) showed that the Hirsch Conjecture is false for unbounded polyhedra, the Hirsch Conjecture (and the Linear Hirsch and Polynomial Hirsch Conjectures, which are both still open) is usually stated for bounded polytopes. However, we should note that any resolution to the Linear Hirsch or

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Polynomial Hirsch Conjectures would still be interesting for unbounded polyhedra.

These conjectures on the diameters of polytopes and polyhedra are interesting because of their relation to the efficiency of the simplex algorithm for linear programming. In particular, the diameter of the feasibility polyhedron is a lower bound on the number of pivot steps needed for the simplex algorithm. Thus, if the Polynomial Hirsch Conjecture is false, then no pivot rule for the simplex algorithm runs in polynomial time. For more information on the Hirsch Conjecture and its relationship to the behavior of the simplex method, see the survey [16] or the recent survey [14] written jointly with Santos. Our terminology on polytopes follows the language in [24].

Studying diameters of convex polyhedra via combinatorial abstractions of polyhedral graphs were considered by Adler, Dantzig, Murty, and Saigal (see [1], [2], [3], [4], and [19]), Kalai (see [12]), and Eisenbrand, Hähnle, Razborov, and Rothvoß (see [8] and [9]). Adler et al. introduced the first formally-defined combinatorial abstraction of the graphs of polytopes satisfying a collection of axioms. In [12], Kalai showed that a more general family of objects obtained by removing one of these axioms still satisfies the quasi-polynomial upper bound for the diameters of convex polyhedra proved in [13]. A further generalization of polyhedral graphs was studied in [8] by dropping an additional axiom. Eisenbrand et al. prove that even in this more general class of objects, the quasi-polynomial diameter upper bound of Kalai and Kleitman in [13] holds. On the other hand, they give a construction to prove that the diameters of this more general class of objects is superlinear. One can say that this superlinear lower bound in [8] is evidence against the Linear Hirsch Conjecture.

In this paper, we introduce *subset partition graphs* (defined in Section 3), a new family of combinatorial abstractions of the graphs of polyhedra. The new abstraction is a flexible framework inspired by the combinatorial properties found in previous abstractions, and provides many variants for abstractions of polyhedra. Our main theoretical result is the construction of a family of subset partition graphs whose diameter is superlinear even in fixed dimension (see Theorem 4.6), which can be considered evidence against the Linear Hirsch Conjecture. The remaining bounds on subset partition graphs proved in this paper provide evidence against the Linear Hirsch Conjecture. Moreover, we present a strategy to disprove the conjecture via combinatorial operations on subset partition graphs.

Outline of this paper: In Section 2, we formally define the previous abstractions and survey the known upper and lower bounds. Section 2.1 introduces the abstraction presented in [8] and Section 2.2 discusses combinatorial properties defining special cases. Motivated by this discussion, in Section 3 we define our new combinatorial abstraction, the subset partition graph. In Section 4 we prove upper and lower bounds on subset partition graphs that satisfy particular sets of properties. We give some final remarks in Section 5, and present a strategy for disproving the Linear Hirsch Conjecture.

2 Previous abstractions

Here we describe relevant previous combinatorial abstractions in the literature[1]. In all cases, the object of study is an abstract generalization of simple polyhedra. We say that a d-dimensional polyhedron P is *simple* if each of its vertices is contained in exactly d of the n facets of P. For the study of diameters of polyhedra, it is enough to consider simple polyhedra, since the largest diameter of d-polyhedra with n facets is found among the simple d-polyhedra with n facets (see, e.g., [14]).

Let $H(n, d)$ denote the maximum diameter of d-dimensional polytopes with n facets, and let $H_u(n, d)$ denote the maximum diameter of d-dimensional polyhedra with n facets. Since polytopes are polyhedra, we clearly have $H(n, d) \leq H_u(n, d)$. In [13], Kalai and Kleitman proved that $H_u(n, d) \leq n^{1+\log_2 d}$.

2.1 Base abstractions and connected layer families

We now introduce an abstraction of Eisenbrand et al. (see [8]). Fix a finite set S of cardinality n, called the *symbol set.* (Each $s \in S$ is called a *symbol.*) Let $\mathcal{A} \subseteq \binom{S}{d}$, where $\binom{S}{d}$ is the set of all d-element subsets of S. If the graph $G = (\mathcal{A}, E)$ with the vertex set $\mathcal{A}$ and edge set E satisfies

1. **finite diameter property:** the graph G is connected,
2. **face path property:** for each $A, A' \in \mathcal{A}$, there is a path from A to A' in the graph G using only vertices that contain $A \cap A'$,

then we say that G is a d-dimensional *base abstraction* of $\mathcal{A}$ on the symbol set S. The *diameter* of the base abstraction is the diameter of the graph G.

Note that the graphs of simple d-dimensional polyhedra with n facets are base abstractions. Indeed, each of the n facets of P is associated with a symbol s in S. Since our polyhedron P is simple, each vertex of P is incident to exactly d facets, and so it is associated with the d-element subset of S consisting of the corresponding symbols. The graph G used in the base abstraction "is" the graph of the polyhedron. The first condition is satisfied since the graph of a polyhedron is connected. The second condition translates into the fact that for every pair of vertices y and z on a polyhedron P, there is a path from y to z on the smallest face of P containing both y and z.

Since the graphs of polyhedra are base abstractions, we clearly have $H_u(n, d) \leq B(n, d)$, where $B(n, d)$ denotes the maximum diameter among d-dimensional base abstractions on a symbol set of size n. In [8], Eisenbrand et al. prove that the Kalai-Kleitman bound of $n^{1+\log_2 d}$ is also an upper bound for $B(n, d)$. They

[1] Very recently (see [11]), new abstractions unrelated to Section 3 have formed the discussion of an web-based discussion: `http://gilkalai.wordpress.com/2010/09/29/polymath-3-polynomial-hirsch-conjecture/`, `http://gilkalai.wordpress.com/2010/10/03/polymath-3-the-polynomial-hirsch-conjecture-2/`, `http://gilkalai.wordpress.com/2010/10/10/polymath3-polynomial-hirsch-conjecture-3/`, `http://gilkalai.wordpress.com/2010/10/21/polymath3-polynomial-hirsch-conjecture-4/`, and `http://gilkalai.wordpress.com/2010/11/28/polynomial-hirsch-conjecture-5-abstractions-and-counterexamples/`.

also show $B(n, \frac{n}{4})$ is in $\Omega(n^2/\log n)$, i.e., the diameters of base abstractions obey a quadratic lower bound (up to a logarithmic factor).

Their upper and lower bounds for $B(n, d)$ were proved by analyzing the diameters of a related combinatorial object. A d-dimensional *connected layer family* of $\mathcal{A} \subseteq \binom{S}{d}$ on a set of $n = |S|$ symbols is a family $\mathcal{V} = \{\mathcal{V}_0, \ldots, \mathcal{V}_t\}$ of non-empty sets such that:

- **partition property:** $\mathcal{A} = \mathcal{V}_0 \cup \cdots \cup \mathcal{V}_t$,
- **disjointness property:** $\mathcal{V}_i \cap \mathcal{V}_j = \emptyset$ if $i \neq j$,
- **connectivity property:** for all $i < j < k$ and $A \in \mathcal{V}_i$, $A' \in \mathcal{V}_k$, there is an $A'' \in \mathcal{V}_j$ such that $A \cap A' \subseteq A''$.

Each individual $\mathcal{V}_i$ is called a *layer*. The *diameter* of the connected layer family $\mathcal{V} = \{\mathcal{V}_0, \ldots, \mathcal{V}_t\}$ is t. Recall that $B(n, d)$ denotes the maximal diameter of a d-dimensional base abstraction on a symbol set of size n. We use the notation $C(n, d)$ to denote the maximal diameter of a d-dimensional connected layer family on a symbol set of size n.

In [8], Eisenbrand et al. prove $B(n, d) = C(n, d)$. The proof of $B(n, d) \leq C(n, d)$ follows from the fact that a connected layer family is easily obtained from a base abstraction by the following layering process: let $G = (\mathcal{A}, E)$ be a d-dimensional base abstraction, and fix a particular d-subset $Z \in \mathcal{A}$. Then, let $\mathcal{V}_i := \{A \in \mathcal{A} : \mathrm{dist}_G(A, Z) = i\}$. Note $\mathcal{V} = \{\mathcal{V}_0, \ldots, \mathcal{V}_t\}$ is a d-dimensional connected layer family since the face path property immediately implies that the collection $\mathcal{V}$ satisfies the connectivity property (see Lemma 3.4.2 in [9] for a detailed proof). The bound is obtained by choosing a base abstraction $G = (\mathcal{A}, E)$ whose diameter is $B(n, d)$ and picking a pair (Z, Z') of d-sets in $\mathcal{A}$ at distance $B(n, d)$.

2.2 Previous abstractions satisfying additional properties

Our new abstraction defined in Section 3 is motivated by following the same layering process just described, but starting with special cases of base abstractions that were studied in [2] and [12] which satisfied additional combinatorial properties. Let $G = (\mathcal{A}, E)$ be a d-dimensional base abstraction. If the condition "(A, A') is an edge in E if and only if $|A \cap A'| = d - 1$" holds, then we say that G is an *ultraconnected set system*. This condition is called *ultraconnectedness*. Note that ultraconnectedness holds for the graphs of polyhedra: if two vertices y and z of a simple d-polyhedron P share all but one facet in common, then they are neighbors in the graph of P. Kalai (see [12]) proved that ultraconnected set systems satisfy the diameter bound in [13].

If the layering process described earlier was applied to a base abstraction satisfying the ultraconnectedness property, then the resulting collection $\mathcal{V} = \{\mathcal{V}_0, \ldots, \mathcal{V}_t\}$ of layers would satisfying the following adjacency property: if $A, A' \in \mathcal{A}$ and $|A \cap A'| = d - 1$, then A and A' are in the same or adjacent layers, i.e., if $A \in \mathcal{V}_i$ and $A' \in \mathcal{V}_j$ then $|i - j| \in \{0, 1\}$.

Adler, Dantzig, and Murty considered an abstraction where, in addition to ultraconnectedness, the following *polytopal endpoint-count condition* must hold: if $F \in \binom{S}{d-1}$, then $\{A \in \mathcal{A} : F \subset A\}$ has cardinality either 0 or 2. An ultraconnected set system satisfying this additional condition is called an *abstract polytope*. For polytopes, the polytopal endpoint-count condition translates into the fact that every 1-face of a polytope is incident to exactly two 0-faces. This condition fails for polyhedra because of 1-faces containing only one vertex, so we consider a *polyhedral endpoint-count condition*, where we allow $|\{A \in \mathcal{A} : F \subset A\}|$ to be 1 as well. In [3], Adler and Dantzig prove that d-dimensional abstract polytopes with n symbols satisfy the Hirsch bound if $n - d \leq 5$, the abstract analogue of [17].

If the layering process was applied to a base abstraction satisfying the polyhedral endpoint-count condition, then the resulting collection $\mathcal{V} = \{\mathcal{V}_0, \dots, \mathcal{V}_t\}$ would satisfy exactly the same property, namely: if $F \in \binom{S}{d-1}$, then $|\{A \in \mathcal{A} : F \subset A\}| \leq 2$.

3 Subset partition graphs

Now that we have seen how properties defining special classes of base abstractions imply certain structural properties on connected layer families obtained by the layering process, we are ready to define subset partition graphs. (As before, we have a set S called the symbol set, with each $s \in S$ being a symbol.)

Definition 3.1. *Fix a finite set S of cardinality n and a set $\mathcal{A} \subseteq \binom{S}{d}$ of subsets. Let $G = (\mathcal{V}, E)$ be a connected graph with vertex set $\mathcal{V} = \{\mathcal{V}_0, \dots, \mathcal{V}_t\}$. If $\mathcal{V}$ is a partition of $\mathcal{A}$ in the sense that:*

1. $\mathcal{A} = \mathcal{V}_0 \cup \cdots \cup \mathcal{V}_t$,
2. $\mathcal{V}_i \cap \mathcal{V}_j = \emptyset$ *if* $i \neq j$, *and*
3. $\mathcal{V}_i \neq \emptyset$ *for all* i,

then we say that G is a d-dimensional subset partition graph *of $\mathcal{A}$ on the symbol set S.*

Note that d-dimensional subset partition graphs on a symbol set S of size n are combinatorial abstractions of simple d-dimensional polyhedra with n facets: each of the n facets of a d-dimensional polyhedron P corresponds to a symbol $s \in S$, and a vertex of P corresponds to a d-set $A \in \mathcal{A}$ given by the incident facets.

As defined, the only condition on the edge set E is that the graph G is connected, and thus subset partition graphs do not yet give an interesting combinatorial abstraction of the graphs of polytopes and polyhedra. For this, one should require one or more of the combinatorial properties identified below, which are conditions on the set $\mathcal{A}$ of subsets or on the edge set E of the graph G. Before identifying the properties, we need to define the following operation on subset partition graphs:

Definition 3.2 (Restriction)**.** *Let $G = (\mathcal{V}, E)$ be a subset partition graph of $\mathcal{A}$ on the symbol set S, and let $F \subseteq S$ be a collection of symbols. We define a new subset partition graph $G_F = (\mathcal{V}_F, E_F)$ of $\mathcal{A}_F$ on the symbol set $S' := S$.*

We define $\mathcal{A}_F := \{A \in \mathcal{A} : F \subseteq A\}$. That is to say, $\mathcal{A}_F$ is obtained by deleting from $\mathcal{A}$ (and the containing $\mathcal{V}_i$) any d-set A which does not contain F. This deletion from the vertices in $\mathcal{V}$ may have made some of them empty. The vertex set $\mathcal{V}_F$ consists of those vertices in $\mathcal{V}$ which are still non-empty, and two vertices in $\mathcal{V}_F$ are connected by an edge in E_F exactly when the associated vertices were connected in E. The subset partition graph G_F is called the restriction *of G with respect to F.*

Based on the discussion in Sections 2.1 and 2.2, and together with the definition of restriction, we identify the main properties that should be considered for subset partition graphs:

- **dimension reduction:** if $F \subseteq S$ such that $|F| \leq d$, then (the underlying graph of) the restriction G_F is a connected graph.
- **adjacency:** if $A, A' \in \mathcal{A}$ and $|A \cap A'| = d - 1$, then A and A' are in the same or adjacent vertices of G.
- **endpoint-count:** if $F \in \binom{S}{d-1}$, then $|\{A \in \mathcal{A} : F \subset A\}| \leq 2$.

Example 3.3. *Figure 1 illustrates a 3-dimensional subset partition graph on a symbol set S with $n = |S| = 6$. The graph of G has six vertices and $|\mathcal{A}| = 2 \times 2 + 4 \times 1$.*

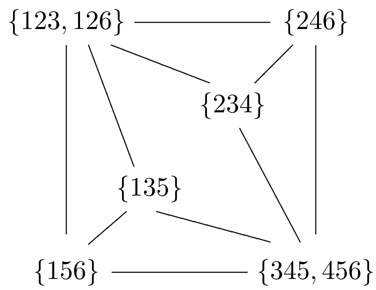

Figure 1: A subset partition graph G with $d = 3$ and $n = 6$.

We defined these (and the following) properties for subset partition graphs because we want a flexible framework where we consider certain collections of properties at a time. By considering certain properties "on" and other properties "off" we hope to understand which properties are crucial in proving diameter bounds.

Remark 3.4. *The class of subset partition graphs with the dimension reduction property together with the additional condition that the underlying graph* *of G is a path is exactly the class of connected layer families. Subset partition graphs interpolate between connected layer families and polyhedral graphs by the selection of properties.*

Remark 3.5. *All definitions can be given in the polar setting. A d-dimensional subset partition graph on n symbols can be equivalently defined as a graph $G = (\mathcal{V}, E)$ together with a pure $(d-1)$-dimensional (abstract) simplicial complex Δ on n vertices, and a map $\varphi : \mathrm{facets}(\Delta) \to \mathcal{V}$. The dimension reduction property becomes "the image of every star is connected". Adjacency turns into: if the facets F and F' have $d-1$ vertices of Δ in common, then $\mathrm{dist}_G(\varphi(F), \varphi(F')) \leq 1$.*

In addition to the three main properties described above, there are other combinatorial properties of polytopes which translate into natural properties to consider for subset partition graphs, for example:

- **d-connectedness:** the graph G is d-connected.
- **d-regularity:** the graph G is d-regular.
- **d-neighbors:** for every $A \in \mathcal{A}$, $|\{A' \in \mathcal{A} \setminus \{A\} : |A \cap A'| = d-1\}| = d$.
- **one-subset:** $|\mathcal{V}_i| = 1$ for each $i = 0, \ldots, t$.

The d-connectedness property for subset partition graphs is desireable due to Balinski's Theorem (see [5]), which says that the graph of a d-dimensional polytope is d-connected. The d-regularity and d-neighbors property hold for the graphs of simple d-polytopes: at each vertex v of a simple d-polytope P, there are d edges emanating from v, and in a bounded polytope, each of these edges leads to another vertex of P. These properties do not hold for unbounded polyhedra. The one-subset property, which says that each vertex should have a single d-subset, holds for the graphs of polytopes: each vertex in the subset partition graph should contain the d-set of incident facets.

There are two easy operations one can perform on a subset partition graph $G = (\mathcal{V}, E)$. Let $\mathcal{V}_i$ and $\mathcal{V}_j$ be two vertices in $\mathcal{V}$. Then we define:

1. **Contraction:** If $\mathcal{V}_i$ and $\mathcal{V}_j$ are connected by an edge in E, contraction on the edge produces a new subset partition graph with one less vertex: the two vertices $\mathcal{V}_i$ and $\mathcal{V}_j$ are replaced with a new vertex which contains all of the d-sets which were in $\mathcal{V}_i$ and $\mathcal{V}_j$.
2. **Edge addition:** If $\mathcal{V}_i$ and $\mathcal{V}_j$ were not connected by an edge in E, edge addition makes the two vertices adjacent. The resulting subset partition graph has one more edge than the originial subset partition graph G does.

Example 3.6. *The subset partition graph described in Example 3.3 was obtained from the natural subset partition graph for a 3-cube after two contractions.*

We remark that there is a clear analogue for contraction in the theory of connected layer families. We also note the following simple but potentially powerful effect of these operations, which will be important in Section 5:

Remark 3.7. *We wish to note what happens to the three main properties (dimension reduction, adjacency, and endpoint-count) for subset partition graphs under the operations of contraction and edge addition:*

1. *All three of these properties are preserved under both operations.*
2. *After a sufficient number of contractions and edge additions, the dimension reduction and adjacency conditions will hold.*

4 Upper and lower bounds

In this section, we prove upper and lower bounds on subset partition graphs satisfying various sets of the main properties. First, note the following easy bound:

Remark 4.1. *Recall (see Remark 3.4) that if we considered subset partition graphs with the dimension reduction property, together with the condition that the graph G is a path, then this is exactly the same as studying connected layer families.*

If we also added the one-subset property, then the Hirsch bound of $n-d$ holds: up to permutation of symbols, the unique longest path is $\{1,\ldots,d\},\{2,\ldots,d+1\},\ldots,\{n-d+1,\ldots,n\}$. Thus, in order to study path lengths in polytopes, it will be necessary to consider graphs G which are not paths.

4.1 Upper bound

Here we prove that the diameters of subset partition graphs satisfying the dimension reduction property obey the Kalai-Kleitman quasi-polynomial upper bound of $n^{1+\log_2 d}$. The proof is a hybrid of those found in [8] and [13].

Inspired by [8], we define the operation of *induction* on a symbol $s \in S$. (Performed on the graph of a polyhedron P, induction corresponds to examining the graph of the facet of P corresponding to s.) Induction of a subset partition graph $G = (\mathcal{V}, E)$ of $\mathcal{A}$ on a symbol $s \in S$ results in a new subset partition graph G' with symbol set S', as follows:

1. For all sets $A \in \mathcal{A}$ with $s \notin A$, remove A from the set $\mathcal{A}$ and from the vertex in $\mathcal{V}$ which contained A.
2. Remove empty vertices $\mathcal{V}_i \in \mathcal{V}$ and all incident edges in E.
3. Remove s from every remaining $A \in \mathcal{A}$.

Example 4.2. *Induction on $s = 3$ in the subset partition graph from Example 3.3 results in the subset partition graph shown in Figure 2.*

If G satisfies dimension reduction, the graph of G' is connected so:

Lemma 4.3. *Given a d-dimensional subset partition graph on n symbols which satisfies dimension reduction, induction on any symbol $s \in S$ produces a $(d-1)$-dimensional subset partition graph G' on $n-1$ symbols.*

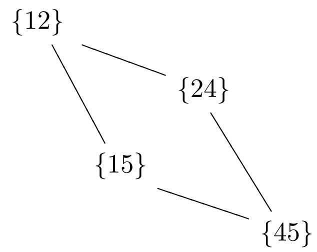

Figure 2: A subset partition graph obtained by induction.

Theorem 4.4. *Let $J(d,n)$ denote the maximal diameter among d-dimensional subset partition graphs on n symbols which satisfies dimension reduction. Then, $J(d,n) \leq n^{1+\log_2 d}$.*

Proof. Let $G = (\mathcal{V}, E)$ be an arbitrary d-dimensional subset partition graph of $\mathcal{A}$ on n symbols which satisfies dimension reduction. Let A_1 and A_2 be two arbitrary subsets belonging to $\mathcal{A}$. Since the vertices of G partition $\mathcal{A}$, there are vertices $\mathcal{V}_1, \mathcal{V}_2 \in \mathcal{V}$ such that $A_1 \in \mathcal{V}_1$ and $A_2 \in \mathcal{V}_2$.

For $m = 1, 2$, let k_m be maximal such that at most $\lfloor \frac{n}{2} \rfloor$ symbols appear in $\{A \in \mathcal{W} \in \mathcal{V} : \mathrm{dist}(\mathcal{W}, \mathcal{V}_m) \leq k_m\}$. By maximality of k_m the sets $\{A \in \mathcal{W} \in \mathcal{V} : \mathrm{dist}(\mathcal{W}, \mathcal{V}_1) \leq k_1 + 1\}$ and $\{A \in \mathcal{W} \in \mathcal{V} : \mathrm{dist}(\mathcal{W}, \mathcal{V}_2) \leq k_2 + 1\}$ must share a common symbol, which we denote by s^*. Pick $A_1', A_2' \in \mathcal{A}$ and $\mathcal{V}_1', \mathcal{V}_2' \in \mathcal{V}$ such that

$$s^* \in A_1' \in \mathcal{V}_1' \in \{A \in \mathcal{W} \in \mathcal{V} : \mathrm{dist}(\mathcal{W}, \mathcal{V}_1) \leq k_1 + 1\}$$

and

$$s^* \in A_2' \in \mathcal{V}_2' \in \{A \in \mathcal{W} \in \mathcal{V} : \mathrm{dist}(\mathcal{W}, \mathcal{V}_2) \leq k_2 + 1\}.$$

Let G' be the subset partition graph obtained from G by induction on s^*. By Lemma 4.3,

$$\begin{aligned}
\mathrm{dist}_G(A_1, A_2) &= \mathrm{dist}_G(\mathcal{V}_1, \mathcal{V}_2) \\
&\leq k_1 + k_2 + \mathrm{dist}_{G'}(\mathcal{V}_1', \mathcal{V}_2') + 2 \\
&= k_1 + k_2 + \mathrm{dist}_{G'}(A_1', A_2') + 2 \\
&\leq 2 \cdot J\left(d, \left\lfloor \frac{n}{2} \right\rfloor\right) + J(d-1, n-1) + 2.
\end{aligned}$$

Since G was arbitrary, we obtain the recursion

$$J(n,d) \leq 2 \cdot J\left(d, \left\lfloor \frac{n}{2} \right\rfloor\right) + J(d-1, n-1) + 2,$$

and the result follows. $\square$

4.2 Lower bounds

In this section, we prove three lower bounds on the diameters of subset partition graphs satsifying the adjacency and endpoint-count conditions. First, we prove a general lower bound. Then we prove two lower bounds for special subclasses of subset partition graphs satisfying natural combinatorial properties coming from interesting classes of polytopes. The first special class is a combinatorial abstraction of spindles and the second bound is proved for an abstraction of transportation polytopes.

Remark 4.5. *The construction of Eisenbrand et al. in [8] together with Remark 3.4 proves that subset partition graphs with the dimension reduction property have superlinear diameter, which can be considered evidence against the Linear Hirsch Conjecture.*

The following construction due to Santos ([22], which improves the author's previously unpublished bound) gives a superlinear lower bound for subset partition graphs with the adjacency and endpoint-count conditions:

Theorem 4.6. *Let $K(n,d)$ denote the maximum diameter of d-dimensional subset partition graphs with n symbols satisfying the adjacency, endpoint-count, and one-subset properties. There is a universal constant κ such that*

$$\frac{K(n,d)}{n^{d/4}} \geq \kappa > 0$$

for infinitely many n and d.

Proof. Let $d \geq 8$ be a multiple of four. Let $n > d$ be even. We construct a d-dimensional subset partition graph $G = (\mathcal{V}, E)$ with n symbols. The symbol set is $S := \{1, \ldots, k\} \cup \{1', \ldots, k'\}$, where $k = \frac{n}{2}$.

Let P be a $\frac{d}{2}$-dimensional cyclic polytope with k vertices. The polar of P is a simple $\frac{d}{2}$-polytope with k facets. Let Q and Q' be two copies of the polar of P, with respective symbol sets $\Sigma = \{1, \ldots, k\}$ and $\Sigma' = \{1', \ldots, k'\}$ labelling the facets so that the involution $f : s \mapsto s'$ is a combinatorial bijection. Since Q has a Hamiltonian path Π (see [15]), we order the t vertices of Q as $Z_1, \ldots, Z_t$ using the path Π. Note

$$t = \frac{n}{n - \frac{d}{2}} \binom{\frac{n}{2} - \frac{d}{4}}{\frac{d}{4}}.$$

(Here, each Z_i is an $\frac{d}{2}$-subset of Σ, thus their images $(Z_i')_{i=1}^{t}$ with $Z_i' = f(Z_i)$ trace the same Hamiltonian path in Q'.)

The vertex set $\mathcal{V}$ of G is $\{\mathcal{V}_i : i = 1, \ldots, t\} \cup \{\mathcal{W}_{i,j} : i = 1, \ldots, t-1;\ j = 1, 2\}$ and each $\mathcal{W}_{i,j}$ has edges to $\mathcal{V}_i$ and $\mathcal{V}_{i+1}$. See Figure 3. Each vertex $\mathcal{V}_i$ and $\mathcal{W}_{i,j}$ contains exactly one d-set. Define $\mathcal{V}_i = \{A_i\}$, where $A_i = Z_i \cup Z_i'$. The sets A_i and A_{i+1} have $d-2$ elements in common, so each $D_i := A_i \bigtriangleup A_{i+1}$ has cardinality 4, where $\bigtriangleup$ denotes symmetric difference.

Consider the two d-sets $A_i \cup (A_{i+1} \cap D_i \cap \Sigma) \setminus (A_i \cap D_i \cap \Sigma)$ and $A_i \cup (A_{i+1} \cap D_i \cap \Sigma') \setminus (A_i \cap D_i \cap \Sigma')$. Let $\mathcal{W}_{i,1}$ contain one of these sets and let $\mathcal{W}_{i,2}$ contain

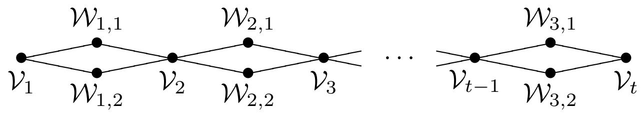

Figure 3: The underlying graph of $G = (\mathcal{V}, E)$

the other. It is easy to see that $|A_i \cap A_j| \leq d - 2$ if $i \neq j$. It follows that G satisfies the adjacency and endpoint-count properties. The diameter of G, which is the distance from $\mathcal{V}_1$ to $\mathcal{V}_t$ is $2(t-1)$, which is $\Omega(n^{d/4})$. □

This proves that subset partition graphs without the dimension reduction property do not satisfy the quasi-polynomial bound in [13]. Since the diameter of the subset partition graphs in Theorem 4.6 is larger than the best known bound for polytopes, this class may provide a good starting point for the strategy we present to disprove the Linear Hirsch Conjecture which we discuss in Section 5.

We now show that superlinear lower bounds hold for special cases of subset partition graphs, starting with a subclass inspired by spindles. Santos disproved the Hirsch Conjecture (see [23]) with a construction that comes from a polyhedral operation applied to a class of polytopes called spindles. A *spindle* is a polytope P with two special vertices A_1 and A_2 (called the *apices*) such that every facet of P contains exactly one of the apices. The *length* of a spindle P is the distance in the graph of P between A_1 and A_2. We say that a d-dimensional spindle is *long* if its length exceeds d. In [23], Santos proved that long 5-dimensional spindles exist, and moreover, the existence of a long spindle implies the existence of a long spindle with $n = 2d$ (i.e., a non-Hirsch *Dantzig figure*), thus the Hirsch Conjecture for polytopes is false.

The property that characterizes spindles is purely combinatorial, so we make an analogous definition for subset partition graphs. We say that a subset partition graph G of $\mathcal{A}$ on the symbol set S *satisfies the spindle property* if there are two distinguished subsets A_1 and A_2 (called the *apices*) belonging to $\mathcal{A}$, such that every symbol $s \in S$ belongs to exactly one of A_1 or A_2. The *length* of a subset partition graph G with the spindle property is the distance in G from one apex to the other.

Theorem 4.7. *Let $L(n)$ denote the maximum length of d-dimensional subset partition graphs on $n = 2d$ symbols satisfying the adjacency, endpoint-count, one-subset, and spindle properties. Let $\kappa = 2^{-3/2}$. For infinitely many n,*

$$\frac{L(n)}{n^{3/2}} \geq \kappa.$$

Proof. We define a subset partition graph $G_m = (\mathcal{V}, E)$ of $\mathcal{A}$ for each $m \in \mathbb{N}$. The symbol set is $S = [2m] \times [m]$, so $n = |S| = 2m^2$. Define the index set

$$I := \{(a, b, c) : 0 \leq a, b, c < m\} \cup \{(m, 0, 0)\}.$$

The triples $(a, b, c) \in I$ are totally ordered using the lexicographic order. For $(a, b, c) \in I$, define

$$\begin{aligned} A_{a,b,c} := & \{(i,j) : i = a+1, \dots, a+m-b-1 \text{ and } j = 1, \dots, m\} \\ & \cup \{(i,j) : i = a+m-b \text{ and } j = c+1, \dots, m\} \\ & \cup \{(i,j) : i = a+m-b+1 \text{ and } j = 1, \dots, c\} \\ & \cup \{(i,j) : i = a+m-b+2, \dots, a+m+1 \text{ and } j = 1, \dots, m\}. \end{aligned}$$

Let $\mathcal{A} := \{A_{a,b,c} : (a, b, c) \in I\}$. Figure 4 gives a schematic picture of a typical d-set $A_{a,b,c}$ in $\mathcal{A}$.

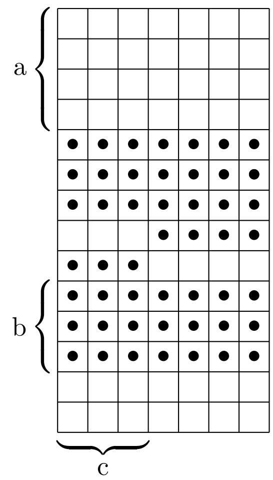

Figure 4: A schematic drawing of the d-set $A_{a,b,c} \in \mathcal{A}$, $m = 7$

The elements in $\mathcal{V}$ are the singleton sets $\mathcal{V}_{a,b,c} = \{A_{a,b,c}\}$ for $(a, b, c) \in I$, thus the one-subset property holds. Two vertices $\mathcal{V}_{a,b,c}$ and $\mathcal{V}_{a',b',c'}$ are connected by an edge if and only if the triples (a, b, c) and (a', b', c') are consecutive in the lexicographic order, which is a total order on I. Therefore the graph for G_m is a path, so the graph is connected. Hence, G_m is a subset partition graph.

For all $(a, b, c) \in I$, one has $|A_{a,b,c}| = m^2$, so the dimension of G_m is $d = m^2 = \frac{1}{2}n$. The d-sets $A_{0,0,0} = \{(i, j) : 1 \le i \le m, 1 \le j \le m\}$ and $A_{m,0,0} = \{(i, j) : m + 1 \le i \le 2m, 1 \le j \le m\}$ are a disjoint partition of S, thus $A_{0,0,0}$ and $A_{m,0,0}$ will be the apices of G_m, hence the spindle property holds. The remaining properties to verify are the adjacency and endpoint-count properties.

We prove the adjacency property by showing that if the d-sets $A_{a,b,c}$ and $A_{a',b',c'}$ have $d - 1$ symbols in common, then they are in adjacent vertices. We prove this statement by considering two d-sets $A_{a,b,c}$ and $A_{a',b',c'}$ that are not in adjacent vertices. Since $(a, b, c) \neq (a', b', c')$, without loss of generality we have $(a,b,c) < (a',b',c')$ in lexicographic ordering. If we define $A := A_{a,b,c}$ and $A' := A_{a',b',c'}$, we want to show that $|A \cap A'| < d-1$, or equivalently that $|A \triangle A'| \geq 2$. The proof is given in cases.

- Suppose $a' = a$.
  - If $b' = b$, then $c' - c > 1$ or otherwise the d-sets $A = A_{a,b,c}$ and $A' = A_{a',b',c'}$ are in adjacent layers. Then

    $$|\{(i,j) \in A : i = a + m - b\}| = m - c$$
    $$|\{(i,j) \in A' : i = a + m - b\}| = m - c',$$

    so the d-sets A and A' have less than $d-1$ elements in common.
  - If $b' \neq b$, then either $b' - b = 1$ or $b' - b > 1$.
    If $b' - b = 1$, then $(c,c') \neq (m-1,0)$ otherwise the d-sets A and A' would be adjacent. If $c < m-1$, then

    $$|\{(i,j) \in A' : i = a' + m - b' + 2\}| = m,$$
    $$|\{(i,j) \in A : i = a' + m - b' + 2\}| = c.$$

    If $c' > 0$, then

    $$|\{(i,j) \in A : i = a + m - b - 1\}| = m,$$
    $$|\{(i,j) \in A' : i = a + m - b - 1\}| = m - c',$$
    $$|\{(i,j) \in A : i = a' + m - b' + 2\}| = c,$$
    $$|\{(i,j) \in A' : i = a' + m - b' + 2\}| = m.$$

    Since $c < m$, $|A \triangle A'| \geq 2$.
    If $b' - b > 1$, then

    $$|\{(i,j) \in A : i = a + m - b, a + m - b + 1\}| = m$$
    $$|\{(i,j) \in A' : i = a + m - b, a + m - b + 1\}| = 2m.$$

- In the other case, $a' \neq a$. Here, we consider two subcases.
  - Suppose $a' = a + 1$. Then $(b,b') = (m-1,0)$ or $(b,b') \neq (m-1,0)$. If $(b,b') = (m-1,0)$, then it follows that $(c,c') \neq (m-1,0)$ otherwise the d-sets A and A' would be adjacent. Then,

    $$|\{(i,j) \in A : i = a + m - b\}| = m - c,$$
    $$|\{(i,j) \in A' : i = a + m - b\}| = 0,$$
    $$|\{(i,j) \in A : i = a' + m - b' + 1\}| = 0,$$
    $$|\{(i,j) \in A' : i = a' + m - b' + 1\}| = c'.$$

    So $|A \triangle A'| = (m-c) + c'$. Since either $c \leq m-2$, or $c' \geq 1$ and $m - c \geq 1$, we have $|A \triangle A'| \geq 2$.

If $(b, b') \neq (m-1, 0)$, then either $b \leq m-2$ or $b' \geq 1$. If $b \leq m-2$, then

$$|\{(i,j) \in A : i = a+1\}| = m$$
$$|\{(i,j) \in A' : i = a+1\}| = 0.$$

If $b' \geq 1$, then

$$|\{(i,j) \in A : i = a' + m + 1\}| = 0$$
$$|\{(i,j) \in A' : i = a' + m + 1\}| = m.$$

– Otherwise $a' - a > 1$. Then

$$|\{(i,j) \in A : i = a+1, a+2\}| \geq m,$$

but $\{(i,j) \in A' : i = a+1, a+2\}$ is empty.

In all cases, the non-adjacent d-sets A and A' have strictly less than $d-1$ symbols in common. Therefore, the adjacency property holds. Moreover, if A and A' are adjacent, then $|A \cap A'| = d-1$, so the endpoint-count property holds, since the degree of each vertex in G_m is at most two.

The path from $A_{0,0,0}$ to $A_{m,0,0}$ is a full m-ary counter from $(0,0,0)$ to $(m,0,0)$, so the number of edges in the path is $m^3 = \frac{1}{4}n^{3/2}$. Thus, the length of the family of subset partition graphs $\{G_m\}$ shows that $\limsup_{n\to\infty} \frac{L(n)}{\kappa \cdot n^{3/2}} \geq 1$, where $\kappa = 2^{-3/2}$. □

Since the length is a lower bound for the diameter:

Corollary 4.8. *Let $M(n)$ denote the maximum diameter of d-dimensional subset partition graphs on $n = 2d$ symbols satisfying the adjacency, endpoint-count, and spindle properties. Then,*

$$\limsup_{n\to\infty} \frac{M(n)}{n^{3/2}} \geq \kappa = \frac{1}{\sqrt{8}}.$$

4.3 Transportation polytopes

We briefly describe a lower bound for subset partition graphs which can be considered an abstraction of 3-way transportation polytopes. Transportation polytopes are a well-known family of polytopes. Our abstract analogue for transportation polytopes is obtained from the fact that the vertices of 3-way axial transportation polytopes satisfy a combinatorial property. If $v = (v_{i,j,k})_{i\in[p],j\in[q],k\in[r]}$ is a vertex of a 3-way axial transportation polytope P of size $p \times q \times r$, then the support $\mathrm{supp}(v) = \{(i,j,k) : v_{i,j,k} > 0\}$ of v is not a subset of the support of any vector in the kernel of $B_{p,q,r}$, where $B_{p,q,r}$ is the matrix of linear equations defining P.

We can consider a natural combinatorial property for subset partition graphs. Let $p, q, r \in \mathbb{N}_{>0}$. We say that a subset partition graph $G = (\mathcal{V}, E)$ defined on the symbol set $S = [p] \times [q] \times [r]$ satisfies the 3-way $p \times q \times r$ *transportation property* if for all $A \in \mathcal{A}$, the $(pqr - d)$-set $S \setminus A$ is not a subset of the support of any vector in $\ker(B_{p,q,r})$.

Remark 4.9. *Let $T(m)$ denote the maximal diameter of subset partition graph on the symbol set $S = [m] \times [m] \times [m]$ satisfying the adjacency, endpoint-count, and 3-way $m \times m \times m$ transportation properties. Then $T(m) \geq (m-1)^3$.*

The subset partition graph is obtained by considering the graph of the monotone staircase triangulation of $\Delta_{m-1} \times \Delta_{m-1} \times \Delta_{m-1}$, descibed in Remark 6.2.17 in the book [7] by De Loera, Rambau, and Santos. Adjacency follows from the defintion of a triangulation and the endpoint-count property is implied by the ICoP property in Corollary 4.5.19 of [7]. The diameter of this graph is exactly $(m-1)^3$.

In joint work with De Loera, Onn, and Santos (see [6]), we prove that the diameter of 3-way axial transportation polytopes of size $p \times q \times r$ have diameter at most $2(p+q+r-3)^2$. Thus, the 3-way axial $m \times m \times m$ transportation polytope obeys a quadratic diameter upper bound of $O(m^2)$, but the combinatorial abstraction for this family of polytopes satisfies a cubic $\Omega(m^3)$ lower bound.

5 Final remarks and open problems

We saw that the Kalai-Kleitman diameter upper bound (in [13]) holds for subset partition graphs satisfying dimension reduction. While we do have a lower bound for diameters of subset partition graphs with the adjacency and endpoint-count conditions, we ask:

Problem 5.1. *Prove a non-trivial upper bound on subset partition graphs with the adjacency and endpoint-count conditions.*

Subset partition graphs that satisfy the first main property, namely dimension reduction (see Remark 4.5), or the last two main properties, namely adjacency and endpoint-count (see Theorem 4.6), have superlinear diameter even in fixed dimension. Both of these results, combined with the fact that complementary sets of conditions are used, can be considered evidence against the Linear Hirsch Conjecture. (Theorem 4.7 and Remark 4.9, which present superlinear diameters for special subclasses provide even further evidence against the Linear Hirsch Conjecture.) In light of this, we ask:

Problem 5.2. *Construct a family of subset partition graphs with superlinear diameter satisfying all three of the main properties.*

In fact, subset partition graphs provide an approach for satisfying all three properties and, moreover, an approach for disproving the Linear Hirsch Conjecture:

1. Start with a family of subset partition graphs satisfying at least the endpoint-count property with superlinear diameter growth, such as the family resulting from Theorem 4.6.

2. Gain the other main properties that do not yet hold with the contraction and edge addition operations (see Remark 3.7).

3. If the resulting family of graphs still has superlinear diameter, realize the sequence of graphs as a sequence of polytopes.

We identify the principal difficulties with this strategy: in the second step the contraction and edge addition operations are liable to significantly reduce the diameter of the subset partition graphs, and in the third step the realization problem for polytopes and the study of polytopality of graphs is still the subject of ongoing research (see, e.g., [10], [18], [20], [21], [25]).

In light of these difficulties, for the purpose of the above approach, we note that *any* superlinear construction of subset partition graphs satisfying the endpoint-count property is useful, since the method attempts to construct polytopes using some construction of subset partition graphs as a starting point: while the bound in Theorem 4.6 is much better than the one in Theorem 4.7, it could be that steps 2 and 3 above are easier to perform from the construction in Theorem 4.7. Thus, any superlinear construction of subset partition graphs satisfying the endpoint-count is interesting.

By considering different combinations of properties, subset partition graphs provide a framework for describing which conditions are crucial in proving upper and lower bounds. It is natural to ask which combination of properties are most "useful" in combinatorial abstractions for superlinear lower bounds:

Question 5.3. *Are there superlinear lower bounds for subset partition graphs satisfying other non-trivial combinations of properties?*

Finally, is a certain combination of properties sufficient for proving the Polynomial Hirsch Conjecture for subset partition graphs, and thus, for polyhedra?

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