Branch-and-Cut for Separable Piecewise Linear Optimization: New Inequalities and Intersection with Semi-Continuous Constraints

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Abstract

We give new facets and valid inequalities for the *separable piecewise linear optimization knapsack polytope*. We also extend the inequalities to the case in which some of the variables are semi-continuous. In a companion paper [12] we demonstrate the efficiency of the inequalities when used as cuts in a branch-and-cut scheme.

Keywords: piecewise linear optimization, mixed-integer programming, knapsack problem, special ordered set, semi-continuous variables, polyhedral method, branch-and-cut

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1 Introduction

Any nonlinear function can be approximated to an arbitrary degree of accuracy as a piecewise linear function. Piecewise linear approximation abounds in optimization. It is used in electronic circuit design [5, 19], portfolio selection [22, 27], and optimization of gas network [24]. In addition, piecewise linear optimization is of interest in its own. It arises, for example, in problems with economies of scale [2].

A particularly important special case is that of *separable* piecewise linear functions. Let n be a positive integer and $N = \{1, \ldots, n\}$. The separable piecewise linear optimization problem (SPLO) is:

$$\begin{aligned} maximize \quad & \textstyle\sum_{j \in N} g_j(x_j) \\ s.t. \quad & Ax \leq b \qquad (1) \\ & x_j \in [0, u_j], \quad j \in N, \qquad (2) \end{aligned}$$

where g_j is a piecewise linear function $\forall j \in N$.

When g_j is concave $\forall j \in N$, SPLO can be solved in polynomial time as a linear programming problem (LP), see for example [1]. So we assume that g_j is nonconcave for some $j \in N$. In this case, in general, SPLO is NP-hard [21]. When it is continuous and bounded, SPLO can be modeled as a mixed-integer programming (MIP) problem. Two such equivalent formulations [20] are the *convex combination* model of Dantzig [8] and the *incremental cost* model of Markowitz and Manne [23]. A different and interesting MIP formulation for SPLO was given recently by Vielma and Nemhauser [30], see also [28].

Beale and Tomlin [3] gave an approach alternative to MIP for continuous SPLO based on the concept of *special ordered set of type 2* (SOS2).

Definition 1 *A set of variables* $\{\lambda_0, \ldots, \lambda_T\}$ *is SOS2 when:*

at most two variables can be nonzero, and two nonzero variables must be adjacent. (3)

□

In the SOS2 approach, one keeps in the model only the variables of the special ordered sets and dispenses with the introduction of auxiliary binary variables to enforce (3). Rather, (3) is enforced by branching on the special ordered sets. For a discussion on the potential advantages and disadvantages of the MIP and SOS2 approaches see [14]. Recently, de Farias et al. [16] extended the SOS2 approach to solve discontinuous SPLO. Unlike MIP to discontinuous SPLO [6], which must assume that the g_j's are lower semi-continuous [25, 26, 29], the SOS2 approach requires no restrictive assumptions on the g_j's when some of them are discontinuous.

In this paper we give new valid inequalities for the convex hull of the feasible set of SPLO, regardless of whether it is approached as MIP or SOS2. More specifically, we consider the inequalities that are valid, and in some cases define facets, for the convex hull of a knapsack relaxation of SPLO. We also extend the inequalities for the case when some of the x_j variables in SPLO are semi-continuous [10, 11]. In a companion paper [12] we report on extensive computational experimentation that demonstrates the efficiency of the inequalities presented here. As part of our computational experimentation in [12], we also compare the MIP approaches of Dantzig and of Vielma and Nemhauser, and the SOS2 approach.

The remainder of the paper is organized as follows. In Section 2 we discuss previous work and present assumptions and notation. In Section 3 we present inequalities that are related to the *convexity constraints* of SPLO's formulation. In Section 4 we present *lifted cover inequalities*. In Section 5 we strengthen both convexity and cover cuts for the case where some of the x_j's are semi-continuous variables. In Section 6 we present directions for further research.

2 Previous Work, Assumptions, and Notation

To simplify notation, $\forall j \in N$, besides the points where $g_j(x_j)$ changes direction, we also call breakpoints the endpoints $x_j = 0$ and $x_j = u_j$; we assume that $g_j(x_j)$ has the same number of breakpoints $T + 1 \geq 2$; we assume that $0 < u_j < \infty$. The breakpoints of g_j are denoted as $d_j^0 = 0, \ldots, d_j^T = u_j$. Let $K = \{1, \ldots, T\}$. We write:

$$x_j = \sum_{k \in K} d_j^k \lambda_j^k, \tag{4}$$

$$\sum_{k \in K} \lambda_j^k \leq 1, \tag{5}$$

and

$$\lambda_j^k \geq 0 \ \forall k \in K. \tag{6}$$

Let

$$\sum_{j \in N} \alpha_j x_j \leq b \tag{7}$$

be one of the (knapsack) inequalities of (1). Because we focus our polyhedral analysis on a single knapsack, we assume that $\alpha_j \neq 0 \ \forall j \in N$. Plugging (4) into (7), we obtain:

$$\sum_{j \in N^+} \sum_{k \in K} a_j^k \lambda_j^k - \sum_{j \in N^-} \sum_{k \in K} a_j^k \lambda_j^k \leq b, \tag{8}$$

where $a_j^k = |\alpha_j| d_j^k \ \forall j \in N,\ k \in K,\ N^+ = \{j \in N : \alpha_j > 0\}$, and $N^- = \{j \in N : \alpha_j < 0\}$. We note that $N^+ \cup N^- = N$ and $a_j^T > \cdots > a_j^1 > 0 \ \forall j \in N$.

Now, consider the SOS2$'$ constraint defined in Keha et al. [21].

Definition 2 *The set* $\left\{\lambda_j^1, \ldots, \lambda_j^T\right\}$ *is SOS2$'$ when:*

$$\left\{\lambda_j^1, \ldots, \lambda_j^T\right\} \text{ is SOS2, and } \sum_{k \in K} \lambda_j^k = 1 \text{ whenever } \lambda_j^l > 0 \text{ for some } l > 1. \tag{9}$$

□

Clearly, any inequality valid for $S = \{\lambda \in \Re^{nT} : \lambda$ satisfies (5), (6), (8), (9) $\forall j \in N\}$ is valid for the feasible set of SPLO, regardless of whether additional binary variables and constraints are added to the model, as in [8, 30], or the SOS2 approach is used. For the remainder of the paper we will study the inequality description of the *PLO knapsack polytope* $P = conv(S)$.

To the best of our knowledge, P was first studied by Keha et al. [21]. They gave two families of inequalities valid for P, and computational results that demonstrated the efficiency of their inequalities when used as cuts in a branch-and-cut scheme to solve SPLO. Here, we give new families of inequalities valid for P, some of which generalize the inequalities in [21]. We are not aware of any other study of P.

We assume that:

$$b + \sum_{j \in N^-} a_j^T \geq 0, \tag{10}$$

$$a_j^T \leq b + \sum_{i \in N^-} a_i^T \ \forall j \in N^+, \tag{11}$$

and

$$-a_j^1 - \sum_{i \in N^- - \{j\}} a_i^T < b \ \forall j \in N^-. \tag{12}$$

If Assumption (10) does not hold, $S = \emptyset$. Assumption (11) holds without loss of generality, since otherwise we may decrease the value of d_j^T. As shown in [21], Assumption (12) is equivalent to assuming that P is full-dimensional.

We denote $LPS = \{\lambda \in \Re^{nT} : \lambda$ satisfies (5), (6), (8) $\forall j \in N\}$, i.e. the LP relaxation set of the knapsack SPLO. Given a polytope X, we denote its set of extreme points as $vert(X)$. Throughout the paper we use Lemma 1. Because its proof is simple, we omit it. Parts of Lemma 1 appear in [21].

Lemma 1 *Let* $\lambda \in vert(LPS)$*. Then* λ *has at most two fractional components, and in case it has a fractional component it must satisfy (8) at equality. Furthermore, if* $\lambda_{j_1}^{k_1}$, $\lambda_{j_2}^{k_2} \in (0,1)$, *then* $j_1 = j_2$ *and* $\lambda_{j_1}^{k_1} + \lambda_{j_2}^{k_2} = 1$*. A point* $\mu \in vert(P)$ *iff* $\mu \in vert(LPS)$*, and in case* $\mu_j^{k_1}, \mu_j^{k_2} \in (0,1)$ *for some* $j \in N$ *and* $k_1, k_2 \in K$*,* $|k_2 - k_1| = 1$. □

3 Lifted Convexity Inequalities

In this section we give two large families of inequalities valid for P. We also give sufficient conditions for the inequalities of the first family to define facets. Although we derive the inequalities in a much simpler way, they can be obtained by fixing a few of the λ_j's at 0 or 1 in (5), and then lifting the resulting inequality. Because (5) is usually referred to as *convexity constraint*, we call the inequalities of this section *lifted convexity inequalities*.

We now give our first family of lifted convexity inequalities.

Theorem 1 *Let* $j \in N^+$, $N_1^- \subseteq N^-$, *and* $b' = b + \sum_{i \in N_1^-} a_i^{m_i}$, *where* $m_i \in K$ $\forall i \in N_1^-$. *Let* $s \in K$ *be such that* $a_j^s < b'$, $I = \left\{ i \in N^+ - \{j\} : a_j^s + a_i^T > b' \right\}$, *and* $k_i = \min \left\{ k \in K : a_j^s + a_i^k > b' \right\}$ $\forall i \in I$. *Suppose that* $I \neq \emptyset$. *Then,*

$$\frac{1}{a_j^s} \sum_{k=1}^{s-1} a_j^k \lambda_j^k + \sum_{k=s}^{T} \lambda_j^k + \sum_{i \in I} \sum_{k=\max\{1, k_i - 1\}}^{T} \alpha_i^k \lambda_i^k - \sum_{i \in N_1^-} \sum_{k=m_i+1}^{T} \beta_i^k \lambda_i^k - \sum_{i \in N^- - N_1^-} \sum_{k \in K} \frac{a_i^k}{a_j^s} \lambda_i^k \leq 1 \quad (13)$$

is valid for P, *where*

$$\left(\alpha_i^{k_i-1}, \alpha_i^{k_i}\right) \in \left\{ (0,0), \left(\frac{a_j^s + a_i^{k_i-1} - b'}{a_j^s}, \frac{a_j^s + a_i^{k_i} - b'}{a_j^s} \right) \right\} \ \forall i \in I \text{ with } k_i > 1 \text{ and } a_j^s + a_i^{k_i-1} < b',$$

$$\left(\alpha_i^{k_i-1}, \alpha_i^{k_i}\right) = \left(0, \frac{a_j^s + a_i^{k_i} - b'}{a_j^s}\right) \ \forall i \in I \text{ with } k_i > 1 \text{ and } a_j^s + a_i^{k_i-1} = b',$$

$$\alpha_i^{k_i} = 0 \ \forall i \in I \text{ with } k_i = 1,$$

$$\alpha_i^k = \frac{a_j^s + a_i^k - b'}{a_j^s} \ \forall i \in I \text{ with } k > k_i,$$

and

$$\beta_i^k = \frac{a_i^k - a_i^{m_i}}{a_j^s}.$$

Inequality (13) is facet-defining when the following two conditions hold: $N^- = \emptyset$; *and in case* $s > 1$, $\forall k \in \{1, \ldots, s-1\}$ *there exist* $q \in I$ *and* $p \in \{k_q, \ldots, T\} - \{1\}$ *such that*

$$a_j^k + a_q^p > b > a_j^k + a_q^{p-1} \text{ and } \alpha_q^{p-1} = \frac{a_j^s + a_q^{p-1} - b}{a_j^s}. \quad (14)$$

□

The main result of Section 3 of [21] follows now as an easy corollary to Theorem 1:

Corollary 1 *When $N^- = \emptyset$ and $s = 1$, (13) defines a facet of P.* □

Before proving Theorem 1, we give two examples, one with $N^- \neq \emptyset$ and the other with $N^- = \emptyset$, which we will analyze further.

Example 1 Let $|N^+| = |N^-| = 2, T = 3$, and (8) be

$$2\lambda_1^1 + 6\lambda_1^2 + 8\lambda_1^3 + 5\lambda_2^1 + 9\lambda_2^2 + 20\lambda_2^3 - 4\lambda_3^1 - 6\lambda_3^2 - 8\lambda_3^3 - 2\lambda_4^1 - 5\lambda_4^2 - 8\lambda_4^3 \leq 10.$$

Let $j = 1$ and $s = 1$. The following inequalities define facets of P. For $N_1^- = \emptyset$:

$$2\lambda_1^1 + 2\lambda_1^2 + 2\lambda_1^3 - 3\lambda_2^1 + \lambda_2^2 + 12\lambda_2^3 - 4\lambda_3^1 - 6\lambda_3^2 - 8\lambda_3^3 - 2\lambda_4^1 - 5\lambda_4^2 - 8\lambda_4^3 \leq 2$$

(here $I = \{2\}$ and $k_2 = 2$). For $N_1^- = \{3\}$ and $m_3 \in \{1, 2\}$:

$$\begin{aligned} 2\lambda_1^1 + 2\lambda_1^2 + 2\lambda_1^3 - 3\lambda_2^2 + 8\lambda_2^3 - 2\lambda_3^2 - 4\lambda_3^3 - 2\lambda_4^1 - 5\lambda_4^2 - 8\lambda_4^3 &\leq 2, \\ 2\lambda_1^1 + 2\lambda_1^2 + 2\lambda_1^3 - 5\lambda_2^2 + 6\lambda_2^3 - 2\lambda_3^3 - 2\lambda_4^1 - 5\lambda_4^2 - 8\lambda_4^3 &\leq 2 \end{aligned}$$

(here $I = \{2\}$, $k_2 = 3$, and $b' \in \{14, 16\}$ respectively). For $N_1^- = \{4\}$ and $m_4 \in \{1, 2\}$:

$$\begin{aligned} 2\lambda_1^1 + 2\lambda_1^2 + 2\lambda_1^3 - \lambda_2^2 + 10\lambda_2^3 - 4\lambda_3^1 - 6\lambda_3^2 - 8\lambda_3^3 - 3\lambda_4^2 - 6\lambda_4^3 &\leq 2, \\ 2\lambda_1^1 + 2\lambda_1^2 + 2\lambda_1^3 - 4\lambda_2^2 + 7\lambda_2^3 - 4\lambda_3^1 - 6\lambda_3^2 - 8\lambda_3^3 - 3\lambda_4^3 &\leq 2 \end{aligned}$$

(here $I = \{2\}$, $k_2 = 3$, and $b' \in \{12, 15\}$ respectively). For $N_1^- = \{3, 4\}$ and $m_3 = m_4 = 1$:

$$2\lambda_1^1 + 2\lambda_1^2 + 2\lambda_1^3 - 5\lambda_2^2 + 6\lambda_2^3 - 2\lambda_3^2 - 4\lambda_3^3 - 3\lambda_4^2 - 6\lambda_4^3 \leq 2$$

(here $I = \{2\}$, $k_2 = 3$, and $b' = 16$). For $N_1^- = \{3, 4\}$, $m_3 = 1$, and $m_4 = 2$:

$$2\lambda_1^1 + 2\lambda_1^2 + 2\lambda_1^3 - 8\lambda_2^2 + 3\lambda_2^3 - 2\lambda_3^2 - 4\lambda_3^3 - 3\lambda_4^3 \leq 2$$

(here $I = \{2\}$, $k_2 = 3$, and $b' = 19$). For $N_1^- = \{3, 4\}$ and $m_3 = m_4 = 2$:

$$2\lambda_1^1 + 2\lambda_1^2 + 2\lambda_1^3 - 10\lambda_2^2 + \lambda_2^3 - 2\lambda_3^3 - 3\lambda_4^3 \leq 2$$

(here $I = \{2\}$, $k_2 = 3$, and $b' = 21$). For $N_1^- = \{3, 4\}$, $m_3 = 2$, and $m_4 = 1$:

$$2\lambda_1^1 + 2\lambda_1^2 + 2\lambda_1^3 - 7\lambda_2^2 + 4\lambda_2^3 - 2\lambda_3^3 - 3\lambda_4^2 - 6\lambda_4^3 \leq 2$$

(here $I = \{2\}$, $k_2 = 3$, and $b' = 18$). □

Example 2 Let $|N^+| = 4$, $N^- = \emptyset$, $T = 3$, and (8) be

$$2\lambda_1^1 + 6\lambda_1^2 + 8\lambda_1^3 + 3\lambda_2^1 + 7\lambda_2^2 + 10\lambda_2^3 + 4\lambda_3^1 + 8\lambda_3^2 + 10\lambda_3^3 + 5\lambda_4^1 + 7\lambda_4^2 + 9\lambda_4^3 \leq 10. \qquad (15)$$

Let $j = 2$. The following inequalities define facets of P. For $s = 1$:

$$\begin{aligned} -\lambda_1^2 + \lambda_1^3 + 3\lambda_2^1 + 3\lambda_2^2 + 3\lambda_2^3 - 3\lambda_3^1 + \lambda_3^2 + 3\lambda_3^3 + 2\lambda_4^3 &\leq 3, \\ -\lambda_1^2 + \lambda_1^3 + 3\lambda_2^1 + 3\lambda_2^2 + 3\lambda_2^3 + 3\lambda_3^3 + 2\lambda_4^3 &\leq 3, \\ 3\lambda_2^1 + 3\lambda_2^2 + 3\lambda_2^3 - 3\lambda_3^1 + \lambda_3^2 + 3\lambda_3^3 + 2\lambda_4^3 &\leq 3 \\ 3\lambda_2^1 + 3\lambda_2^2 + 3\lambda_2^3 + 3\lambda_3^3 + 2\lambda_4^3 &\leq 3. \end{aligned}$$

(here $I = \{1, 3, 4\}$, $k_1 = k_4 = 3$, $k_3 = 2$, we took $\alpha_3^1 = \alpha_3^2 = 0$ in the second inequality, $\alpha_1^2 = \alpha_1^3 = 0$ in the third inequality, and $\alpha_1^2 = \alpha_1^3 = \alpha_3^1 = \alpha_3^2 = 0$ in the fourth inequality). For $s = 2$:

$$\begin{aligned} -\lambda_1^1 + 3\lambda_1^2 + 5\lambda_1^3 + 3\lambda_2^1 + 7\lambda_2^2 + 7\lambda_2^3 + 5\lambda_3^2 + 7\lambda_3^3 + 4\lambda_4^2 + 6\lambda_4^3 &\leq 7, \\ 5\lambda_1^3 + 3\lambda_2^1 + 7\lambda_2^2 + 7\lambda_2^3 + 5\lambda_3^2 + 7\lambda_3^3 + 4\lambda_4^2 + 6\lambda_4^3 &\leq 7 \end{aligned}$$

(here $I = \{1, 3, 4\}$, $k_1 = 2$, $k_3 = k_4 = 1$, and we took $\alpha_1^1 = \alpha_1^2 = 0$ in the second inequality). □

We now prove Theorem 1.

Proof of Theorem 1 Let $\lambda \in P$. If $\sum_{i \in I} \sum_{k=\max\{1,k_i-1\}}^{T} \alpha_i^k \lambda_i^k \leq 0$, then (13) holds. So suppose that

$$\sum_{i \in I} \sum_{k=\max\{1,k_i-1\}}^{T} \alpha_i^k \lambda_i^k > 0. \qquad (16)$$

Let $I_1 = \left\{i \in I : k_i > 1, a_j^s + a_i^{k_i-1} < b', \text{ and } \alpha_i^{k_i-1} = 0\right\}$ and $I_2 = I - I_1$. Note that $\alpha_i^{k_i} = 0 \ \forall i \in I_1$ and $\alpha_i^{\max\{1,k_i-1\}} \leq 0 \ \forall i \in I_2$. As a result,

$$\sum_{i \in I_1} \sum_{k=k_i}^{T} \lambda_i^k + \sum_{i \in I_2} \sum_{k=\max\{1,k_i-1\}}^{T} \lambda_i^k \geq 1. \qquad (17)$$

Indeed, suppose that (17) does not hold. Because of SOS2′, $\lambda_i^k = 0 \ \forall i \in I_1, k > k_i$, and $\lambda_i^k = 0 \ \forall i \in I_2, k > \max\{1, k_i - 1\}$. So,

$$\sum_{i \in I} \sum_{k=\max\{1,k_i-1\}}^{T} \alpha_i^k \lambda_i^k = \sum_{i \in I_2} \alpha_i^{\max\{1,k_i-1\}} \lambda_i^{\max\{1,k_i-1\}} \leq 0. \qquad (18)$$

As (18) contradicts (16), (17) must hold.

Thus, we have

$$
\begin{aligned}
&\sum_{i\in I}\sum_{k=\max\{1,k_i-1\}}^{T}\alpha_i^k\lambda_i^k\\
=\ &\sum_{i\in I_1}\sum_{k=k_i}^{T}\alpha_i^k\lambda_i^k+\sum_{i\in I_2}\sum_{k=\max\{1,k_i-1\}}^{T}\alpha_i^k\lambda_i^k\\
\leq\ &\sum_{i\in I_1}\sum_{k=k_i}^{T}\frac{a_j^s+a_i^k-b'}{a_j^s}\lambda_i^k+\sum_{i\in I_2}\sum_{k=\max\{1,k_i-1\}}^{T}\frac{a_j^s+a_i^k-b'}{a_j^s}\lambda_i^k\\
\leq\ &\frac{a_j^s-b'}{a_j^s}\left(\sum_{i\in I_1}\sum_{k=k_i}^{T}\lambda_i^k+\sum_{i\in I_2}\sum_{k=\max\{1,k_i-1\}}^{T}\lambda_i^k\right)+\sum_{i\in N^+-\{j\}}\sum_{k\in K}\frac{a_i^k}{a_j^s}\lambda_i^k\\
\leq\ &\frac{a_j^s-b'}{a_j^s}+\sum_{i\in N^+-\{j\}}\sum_{k\in K}\frac{a_i^k}{a_j^s}\lambda_i^k.
\end{aligned}
$$

The last inequality holds because of (17) and $a_j^s < b'$.

Now,

$$
\begin{aligned}
&\sum_{i\in I}\sum_{k=\max\{1,k_i-1\}}^{T}\alpha_i^k\lambda_i^k-\sum_{i\in N_1^-}\sum_{k=m_i+1}^{T}\beta_i^k\lambda_i^k-\sum_{i\in N^--N_1^-}\sum_{k\in K}\frac{a_i^k}{a_j^s}\lambda_i^k\\
\leq\ &\frac{a_j^s-b'}{a_j^s}+\sum_{i\in N^+-\{j\}}\sum_{k\in K}\frac{a_i^k}{a_j^s}\lambda_i^k-\sum_{i\in N_1^-}\sum_{k=m_i+1}^{T}\frac{a_i^k-a_i^{m_i}}{a_j^s}\lambda_i^k-\sum_{i\in N^--N_1^-}\sum_{k\in K}\frac{a_i^k}{a_j^s}\lambda_i^k\\
\leq\ &\frac{a_j^s-b'}{a_j^s}+\frac{1}{a_j^s}\left(\sum_{i\in N^+-\{j\}}\sum_{k\in K}a_i^k\lambda_i^k-\sum_{i\in N^-}\sum_{k\in K}a_i^k\lambda_i^k\right)+\sum_{i\in N_1^-}\sum_{k\in K}\frac{a_i^{m_i}}{a_j^s}\lambda_i^k\\
\leq\ &\frac{a_j^s-b'}{a_j^s}+\frac{1}{a_j^s}\left(b-\sum_{k\in K}a_j^k\lambda_j^k\right)+\sum_{i\in N_1^-}\frac{a_i^{m_i}}{a_j^s}\\
=\ &1-\sum_{k\in K}\frac{a_j^k}{a_j^s}\lambda_j^k.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\frac{1}{a_j^s}\sum_{k=1}^{s-1} a_j^k\lambda_j^k + \sum_{k=s}^{T}\lambda_j^k + \sum_{i\in I}\sum_{k=\max\{1,k_i-1\}}^{T}\alpha_i^k\lambda_i^k - \sum_{i\in N_1^-}\sum_{k=m_i+1}^{T}\beta_i^k\lambda_i^k - \sum_{i\in N^- - N_1^-}\sum_{k\in K}\frac{a_i^k}{a_j^s}\lambda_i^k \\
\le\ &\frac{1}{a_j^s}\sum_{k=1}^{s-1} a_j^k\lambda_j^k + \sum_{k=s}^{T}\lambda_j^k + 1 - \sum_{k\in K}\frac{a_j^k}{a_j^s}\lambda_j^k \\
=\ &1 - \sum_{k=s}^{T}\frac{a_j^k - a_j^s}{a_j^s}\lambda_j^k \le 1.
\end{aligned}
$$

So (13) is valid for P.

We now assume that $N^- = \emptyset$ and (14) holds. We show that, in this case, (13) defines a facet of P. Specifically, we give nT linearly independent points v_i^k, $i \in N, k \in K$, of P that satisfy (13) at equality.

The points v_j^k, $k \ge s$, are given by $\lambda_j^k = 1$ and $\lambda_t^u = 0$ otherwise. The points v_i^k, $i \in N - (I \cup \{j\})$, $k \in K$, are given by $\lambda_i^k = \lambda_j^s = 1$ and $\lambda_t^u = 0$ otherwise. The points v_i^k, $i \in I$, $k < k_i - 1$, are given by $\lambda_i^k = \lambda_j^s = 1$ and $\lambda_t^u = 0$ otherwise.

We now consider the points v_i^k, $i \in I$, $k > k_i$. For $b - a_i^k \le a_j^1$, v_i^k is given by $\lambda_i^k = 1$,

$$
\lambda_j^1 = \frac{b - a_i^k}{a_j^1},
$$

and $\lambda_t^u = 0$ otherwise. Suppose that $b - a_i^k > a_j^1$. Then, $\exists r \in \{1, \ldots, s-1\}$ such that $a_j^r < b - a_i^k \le a_j^{r+1}$. In this case, v_i^k is given by $\lambda_i^k = 1$,

$$
\left(\lambda_j^r, \lambda_j^{r+1}\right) = \left(\frac{a_j^{r+1} + a_i^k - b}{a_j^{r+1} - a_j^r}, \frac{b - a_j^r - a_i^k}{a_j^{r+1} - a_j^r}\right),
$$

and $\lambda_t^u = 0$ otherwise.

Next we consider the points v_i^k, $i \in I$, $k \in \{k_i - 1, k_i\}$. If $k_i = 1$, $v_i^{k_i}$ is given by $\lambda_j^s = 1$,

$$
\lambda_i^{k_i} = \frac{b - a_j^s}{a_i^{k_i}},
$$

and $\lambda_t^u = 0$ otherwise. Suppose that $k_i > 1$. If $\alpha_i^{k_i} = 0$, $v_i^{k_i-1}$ is given by $\lambda_j^s = 1$, $\lambda_i^{k_i-1} = 1$, and $\lambda_t^u = 0$ otherwise; $v_i^{k_i}$ is given by $\lambda_j^s = 1$,

$$
\left(\lambda_i^{k_i-1}, \lambda_i^{k_i}\right) = \left(\frac{a_i^{k_i} + a_j^s - b}{a_i^{k_i} - a_i^{k_i-1}}, \frac{b - a_i^{k_i-1} - a_j^s}{a_i^{k_i} - a_i^{k_i-1}}\right),
$$

and $\lambda_t^u = 0$ otherwise. Suppose that $\alpha_i^{k_i} > 0$. The point $v_i^{k_i-1}$ is given by $\lambda_j^s = 1$,

$$
\left(\lambda_i^{k_i-1}, \lambda_i^{k_i}\right) = \left(\frac{a_i^{k_i} + a_j^s - b}{a_i^{k_i} - a_i^{k_i-1}}, \frac{b - a_i^{k_i-1} - a_j^s}{a_i^{k_i} - a_i^{k_i-1}}\right),
$$

and $\lambda_t^u = 0$ otherwise; for $b - a_i^{k_i} \leq a_j^1$, $v_i^{k_i}$ is given by $\lambda_i^{k_i} = 1$,

$$\lambda_j^1 = \frac{b - a_i^{k_i}}{a_j^1},$$

and $\lambda_t^u = 0$ otherwise. Suppose that $b - a_i^{k_i} > a_j^1$. Then, $\exists r \in \{1, \ldots, s-1\}$ such that $a_j^r < b - a_i^{k_i} \leq a_j^{r+1}$. In this case, $v_i^{k_i}$ is given by $\lambda_i^{k_i} = 1$,

$$\left(\lambda_j^r, \lambda_j^{r+1}\right) = \left(\frac{a_j^{r+1} + a_i^{k_i} - b}{a_j^{r+1} - a_j^r}, \frac{b - a_j^r - a_i^{k_i}}{a_j^{r+1} - a_j^r}\right),$$

and $\lambda_t^u = 0$ otherwise.

Finally, we consider the points v_j^k, $k < s$. Let p and q be the indices of (14). Then, v_j^k is given by $\lambda_j^k = 1$,

$$\left(\lambda_q^{p-1}, \lambda_q^p\right) = \left(\frac{a_j^k + a_q^p - b}{a_q^p - a_q^{p-1}}, \frac{b - a_j^k - a_q^{p-1}}{a_q^p - a_q^{p-1}}\right),$$

and $\lambda_t^u = 0$ otherwise.

Clearly the points v_i^k, $i \in N - \{j\}$, $k \in K$, and v_j^k, $k \geq s$, are linearly independent. Also, the points v_j^k, $k \in K$, are linearly independent.

We now claim that v_j^1 cannot be written as a linear combination of the points v_i^k, $i \in N - \{j\}$, $k \in K$. The only such points that can enter the linear combination are v_q^{p-1} and v_q^p. Consider v_q^{p-1} (resp. v_q^p). We distinguish two cases. In the first case, the component λ_j^1 of v_q^{p-1} (resp. v_q^p) is equal to

$$\frac{b - a_q^{p-1}}{a_j^1} \left(\text{resp. } \frac{b - a_q^p}{a_j^1}\right).$$

Since the same component of v_j^1 is equal to 1, we must have $b - a_q^{p-1} = a_j^1$ (resp. $b - a_q^p = a_j^1$), which contradicts (14). In the second case, the component λ_j^2 of v_q^{p-1} (resp. v_q^p) is equal to

$$\frac{b - a_j^1 - a_q^{p-1}}{a_j^2 - a_j^1} \left(\text{resp. } \frac{b - a_j^1 - a_q^p}{a_j^2 - a_j^1}\right).$$

Since the same component of v_j^1 is equal to 0, we must have $b - a_j^1 - a_q^{p-1} = 0$ (resp. $b - a_j^1 - a_q^p = 0$), which again contradicts (14), thus establishing the claim.

By a similar argument, none of the points $v_j^2, \ldots, v_j^{s-1}$ can be written as a linear combination of the points v_i^k, $i \in N - \{j\}$, $k \in K$. Thus, the set $\left\{v_i^k : i \in N, k \in K\right\}$ is linearly independent. $\square$

We now give our second family of lifted convexity inequalities.

Theorem 2 *Let $j \in N^+$, $N_1^- \subseteq N^-$, and $b' = b + \sum_{i \in N_1^-} a_i^{m_i}$, where $m_i \in K$ $\forall i \in N_1^-$. Let $s \in K - \{1\}$ and $I = \{i \in N^+ - \{j\} : a_j^{s-1} + a_i^T \geq b'\}$. Suppose that $I \neq \emptyset$. Let $k_i = \min\{k \in K : a_j^{s-1} + a_i^k \geq b'\}$, $i \in I$, and $L = \left\{i \in I : a_j^s + a_i^{k_i-1} \geq b' \text{ and } k_i > 1\right\}$. Suppose that $L \neq \emptyset$, and let $a_L = \min\{a_i^{k_i-1} : i \in L\}$. Then,*

$$\sum_{k=1}^{s-1} \gamma_j^k \lambda_j^k + \sum_{k=s}^{T} \lambda_j^k + \sum_{i \in L} \sum_{k=k_i}^{T} \alpha_i^k \lambda_i^k - \sum_{i \in N_1^-} \sum_{k=m_i+1}^{T} \beta_i^k \lambda_i^k - \sum_{i \in N^- - N_1^-} \sum_{k \in K} \gamma_i^k \lambda_i^k \leq 1 \tag{19}$$

is valid for P, where

$$\alpha_i^k = \frac{a_i^k - a_L}{b' - a_L} \ \forall i \in L \text{ with } k \geq k_i,$$

$$\beta_i^k = \frac{a_i^k - a_i^{m_i}}{b' - a_L} \ \forall i \in N_1^- \text{ with } k \geq m_i + 1,$$

and

$$\gamma_i^k = \frac{a_i^k}{b' - a_L} \ \forall i \in N^- \cup \{j\} - N_1^- \text{ with } k \in K.$$

Before proving Theorem 2, we give an example.

Example 3 Let $|N| = |N^+| = 3$, $T = 3$, and (8) be

$$2\lambda_1^1 + 6\lambda_1^2 + 8\lambda_1^3 + 3\lambda_2^1 + 7\lambda_2^2 + 10\lambda_2^3 + 5\lambda_3^1 + 7\lambda_3^2 + 9\lambda_3^3 \leq 10. \tag{20}$$

Let $j = 1$ and $s = 2$. Then $I = \{2, 3\}$, $k_2 = k_3 = 3$, $L = \{2, 3\}$, $b' = 10$, and $a_L = 7$. The following inequality defines a facet of P:

$$\frac{2}{3}\lambda_1^1 + \lambda_1^2 + \lambda_1^3 + \lambda_2^3 + \frac{2}{3}\lambda_3^3 \leq 1.$$

$\square$

We now prove Theorem (2).

Proof of Theorem 2 Let λ be an extreme point of P. Since $b' - a_L > a_j^{s-1}$,

$$\sum_{k=1}^{s-1} \gamma_j^k \lambda_j^k + \sum_{k=s}^{T} \lambda_j^k \leq \sum_{k=1}^{T} \lambda_j^k \leq 1.$$

This means that (19) holds whenever

$$\sum_{i\in L}\sum_{k=k_i}^{T}\alpha_i^k\lambda_i^k = 0.$$

So suppose

$$\sum_{i\in L}\sum_{k=k_i}^{T}\alpha_i^k\lambda_i^k > 0, \tag{21}$$

and note that

$$\begin{aligned}
&\sum_{i\in L}\sum_{k=k_i}^{T}\alpha_i^k\lambda_i^k - \sum_{i\in N_1^-}\sum_{k=m_i+1}^{T}\beta_i^k\lambda_i^k - \sum_{i\in N^- - N_1^-}\sum_{k\in K}\gamma_i^k\lambda_i^k \\
=\;& \sum_{i\in L}\sum_{k=k_i}^{T}\frac{a_i^k - a_L}{b' - a_L}\lambda_i^k - \sum_{i\in N_1^-}\sum_{k=m_i+1}^{T}\frac{a_i^k - a_i^{m_i}}{b' - a_L}\lambda_i^k - \sum_{i\in N^- - N_1^-}\sum_{k\in K}\frac{a_i^k}{b' - a_L}\lambda_i^k \\
\le\;& \frac{1}{b' - a_L}\left(\sum_{i\in L}\sum_{k=k_i}^{T}a_i^k\lambda_i^k - \sum_{i\in N^-}\sum_{k\in K}a_i^k\lambda_i^k\right) - \frac{a_L}{b' - a_L}\sum_{i\in L}\sum_{k=k_i}^{T}\lambda_i^k + \sum_{i\in N_1^-}\sum_{k\in K}\frac{a_i^{m_i}}{b' - a_L}\lambda_i^k.
\end{aligned} \tag{22}$$

First assume that

$$\sum_{i\in L}\sum_{k=k_i}^{T}\lambda_i^k < 1. \tag{23}$$

Because of $SOS2'$, (23) implies that $\lambda_i^k = 0\ \forall i \in L, k > k_i$. Because of (21) and since λ is an extreme point of P, $\lambda_t^{k_t} \in (0, 1)$ for some $t \in L$, and it is the only nonzero value of $\lambda_i^{k_i}$ $\forall i \in L$.

In case $\lambda_j^k = 0\ \forall k \in K$, it follows from (22) that

$$\begin{aligned}
&\sum_{k=1}^{s-1}\gamma_j^k\lambda_j^k + \sum_{k=s}^{T}\lambda_j^k + \sum_{i\in L}\sum_{k=k_i}^{T}\alpha_i^k\lambda_i^k - \sum_{i\in N_1^-}\sum_{k=m_i+1}^{T}\beta_i^k\lambda_i^k - \sum_{i\in N^- - N_1^-}\sum_{k\in K}\gamma_i^k\lambda_i^k \\
\le\;& \frac{1}{b' - a_L}\left(b - a_t^{k_t-1}\lambda_t^{k_t-1}\right) - \frac{a_L}{b' - a_L}\lambda_t^{k_t} + \sum_{i\in N_1^-}\frac{a_i^{m_i}}{b' - a_L} \\
=\;& \frac{1}{b' - a_L}\left(b' - a_t^{k_t-1}\lambda_t^{k_t-1} - a_L\lambda_t^{k_t}\right) \\
\le\;& \frac{1}{b' - a_L}\left(b' - a_L\right) = 1.
\end{aligned}$$

In case $\lambda_j^v = 1$ for some $v \in K$,

$$\begin{aligned}
&\sum_{i\in L}\sum_{k=k_i}^{T}\alpha_i^k\lambda_i^k - \sum_{i\in N_1^-}\sum_{k=m_i+1}^{T}\beta_i^k\lambda_i^k - \sum_{i\in N^- - N_1^-}\sum_{k\in K}\gamma_i^k\lambda_i^k \\
\leq\ & \frac{1}{b'-a_L}\left(b - a_j^v - a_t^{k_t-1}\lambda_t^{k_t-1}\right) - \frac{a_L}{b'-a_L}\lambda_t^{k_t} + \sum_{i\in N_1^-}\frac{a_i^{m_i}}{b'-a_L} \\
=\ & \frac{1}{b'-a_L}\left(b' - a_j^v - a_t^{k_t-1}\lambda_t^{k_t-1} - a_L\lambda_t^{k_t}\right) \\
\leq\ & \frac{1}{b'-a_L}\left(b' - a_j^v - a_L\right).
\end{aligned}$$

If $v \leq s-1$,

$$\begin{aligned}
&\sum_{k=1}^{s-1}\gamma_j^k\lambda_j^k + \sum_{k=s}^{T}\lambda_j^k + \sum_{i\in L}\sum_{k=k_i}^{T}\alpha_i^k\lambda_i^k - \sum_{i\in N_1^-}\sum_{k=m_i+1}^{T}\beta_i^k\lambda_i^k - \sum_{i\in N^- - N_1^-}\sum_{k\in K}\gamma_i^k\lambda_i^k \\
\leq\ & \frac{a_j^v}{b'-a_L} + \frac{1}{b'-a_L}\left(b' - a_j^v - a_L\right) = 1;
\end{aligned}$$

if $v > s-1$,

$$\begin{aligned}
&\sum_{k=1}^{s-1}\gamma_j^k\lambda_j^k + \sum_{k=s}^{T}\lambda_j^k + \sum_{i\in L}\sum_{k=k_i}^{T}\alpha_i^k\lambda_i^k - \sum_{i\in N_1^-}\sum_{k=m_i+1}^{T}\beta_i^k\lambda_i^k - \sum_{i\in N^- - N_1^-}\sum_{k\in K}\gamma_i^k\lambda_i^k \\
\leq\ & 1 + \frac{1}{b'-a_L}\left(b' - a_j^v - a_L\right) \leq 1.
\end{aligned}$$

Next, assume

$$\sum_{i\in L}\sum_{k=k_i}^{T}\lambda_i^k \geq 1.$$

It follows from (22) that

$$\begin{aligned}
&\sum_{i\in L}\sum_{k=k_i}^{T}\alpha_i^k\lambda_i^k - \sum_{i\in N_1^-}\sum_{k=m_i+1}^{T}\beta_i^k\lambda_i^k - \sum_{i\in N^- - N_1^-}\sum_{k\in K}\gamma_i^k\lambda_i^k \\
\leq\ & \frac{1}{b'-a_L}\left(b - \sum_{k=1}^{T}a_j^k\lambda_j^k\right) - \frac{a_L}{b'-a_L} + \sum_{i\in N_1^-}\frac{a_i^{m_i}}{b'-a_L} \\
=\ & 1 - \sum_{k=1}^{T}\frac{a_j^k}{b'-a_L}\lambda_j^k.
\end{aligned}$$

So,

$$\sum_{k=1}^{s-1}\gamma_j^k\lambda_j^k+\sum_{k=s}^{T}\lambda_j^k+\sum_{i\in L}\sum_{k=k_i}^{T}\alpha_i^k\lambda_i^k-\sum_{i\in N_1^-}\sum_{k=m_i+1}^{T}\beta_i^k\lambda_i^k-\sum_{i\in N^--N_1^-}\sum_{k\in K}\gamma_i^k\lambda_i^k$$
$$\leq \sum_{k=1}^{s-1}\frac{a_j^k}{b'-a_L}\lambda_j^k+\sum_{k=s}^{T}\lambda_j^k+1-\sum_{k=1}^{T}\frac{a_j^k}{b'-a_L}\lambda_j^k\leq 1.$$

□

4 Lifted Cover Inequalities

In this section we give our second class of inequalities, *lifted cover inequalities*. We start with the definition of *cover* given in [21]. Then, we give *cover inequalities* that generalize the ones in [21]. We also stregthen our cover inequalities to obtain strong cutting planes valid for P. We now assume that $N = N^+$, except for the very end of this section, where it will be stated otherwise. Given $j \in N$, we henceforth denote

$$u_j^k = a_j^k - a_j^{k-1}\ \forall k \in K-\{1\} \text{ and } u_j^1 = a_j^1.$$

Definition 3 (Keha et. al. [21]) *Let $C \subseteq N$ and $l_j \in \{2,\dots,T\}\ \forall j \in C$ be such that*

$$\sum_{j\in C} a_j^{l_j} = b+\rho \tag{24}$$

with $\rho > 0$. The set C is a cover. It is a minimal cover if

$$\sum_{j\in C-\{i\}} a_j^{l_j} \leq b\ \forall i \in C.$$

□

For the remainder of the paper C will denote a cover. For the remainder of this section we denote $P' = conv\left\{\lambda \in S : \lambda_j^k = 0\ \forall k > l_j, k \in K, j \in C; \lambda_j^k = 0\ \forall j \in N-C, k\in K\ \right\}$.

Theorem 3 *Let C_1, C_2 be two disjoint subsets of C such that $C = C_1 \cup C_2$, and $l_j > 2$ $\forall j \in C_1$. The cover inequality*

$$\sum_{j\in C_1}(\alpha_j\lambda_j^{l_j-2}+\beta_j\lambda_j^{l_j-1}+\lambda_j^{l_j})+\sum_{j\in C_2}(\gamma_j\lambda_j^{l_j-1}+\lambda_j^{l_j})\leq |C|-1 \tag{25}$$

is valid for P, where

$$\alpha_j = min\left\{0, \frac{\rho - u_j^{l_j} - u_j^{l_j-1}}{\rho}\right\}, \ \beta_j = \frac{\rho - u_j^{l_j}}{\rho}, \ and \ \gamma_j = min\{0, \beta_j\}.$$

$\square$

We note that, unlike the cover inequalities in [21], we have, for the same cover C, several possible different cover inequalities, determined by how C is divided into C_1 and C_2. We now prove Theorem 3.

Proof of Theorem 3 Let $\lambda \in vert(P)$. It is clear that (25) is valid for P when $\lambda_j^k > 0$ for some $j \in C_1$, $k < l_j - 2$, or some $j \in C_2$, $k < l_j - 1$. So we assume that $\lambda_j^k = 0$ $\forall j \in C_1$, $k < l_j - 2$, and $j \in C_2$, $k < l_j - 1$.

Suppose that $\lambda_j^k \in \{0, 1\}$ $\forall j \in C, k \in K$. It suffuces to consider the case $\lambda_j^{l_j-1} = 1$ $\forall j \in T_1$, $\lambda_j^{l_j} = 1$ $\forall j \in T_2 \cup C_2$, where T_1 and T_2 are disjoint subsets of C_1 with $T_1 \cup T_2 = C_1$. Note that

$$\sum_{j \in C - T_1} a_j^{l_j} + \sum_{j \in T_1} a_j^{l_j-1} \leq b \text{ iff } \sum_{j \in T_1} \beta_j + |T_2| + |C_2| \leq |C| - 1.$$

Therefore (25) is valid for λ when $\lambda_j^k \in \{0, 1\}$ $\forall j \in C, k \in K$. We then now assume that $\lambda_i^t \in (0, 1)$ for some $i \in C, t \in K$.

Suppose that $i \in C_2$. It suffices to assume that $t = l_i$, and because of Lemma 1, $\lambda_j^{l_j} = 1$ $\forall j \in T_1 \cup C_2 - \{i\}$, and $\lambda_j^{l_j-1} = 1$ $\forall j \in T_2$, where T_1 and T_2 are again disjoint subsets of C_1 with $T_1 \cup T_2 = C_1$. Assume that $\lambda_i^{l_i-1}, \lambda_i^{l_i} \in (0, 1)$ (the proof for the case $\lambda_i^{l_i}, \lambda_i^{l_i+1} \in (0, 1)$ is similar).

We have that

$$\lambda_i^{l_i-1} = \frac{a_i^{l_i} - \delta}{u_i^{l_i}} \text{ and } \lambda_i^{l_i} = \frac{\delta - a_i^{l_i-1}}{u_i^{l_i}}, \tag{26}$$

where

$$\delta = b - \left(\sum_{j \in T_1 \cup C_2 - \{i\}} a_j^{l_j} + \sum_{j \in T_2} a_j^{l_j-1} \right). \tag{27}$$

This reduces (25) to

$$\sum_{j \in T_2} \beta_j + \gamma_i \cdot \frac{a_i^{l_i} - \delta}{u_i^{l_i}} + \frac{\delta - a_i^{l_i-1}}{u_i^{l_i}} \leq |T_2|,$$

or equivalently,

$$\gamma_i \cdot \frac{a_i^{l_i} - \delta}{u_i^{l_i}} + \frac{\delta - a_i^{l_i-1}}{u_i^{l_i}} \leq \frac{\sum_{j \in T_2} u_j^{l_j}}{\rho}. \tag{28}$$

Note that $\delta - a_i^{l_i-1} = \sum_{j\in T_2} u_j^{l_j} + u_i^{l_i} - \rho$ and $a_i^{l_i} - \delta = \rho - \sum_{j\in T_2} u_j^{l_j}$. If $\gamma_i = 0$, i.e. $\rho - u_i^{l_i} \geq 0$, (28) becomes

$$\frac{\sum_{j\in T_2} u_j^{l_j} + u_i^{l_i} - \rho}{u_i^{l_i}} \leq \frac{\sum_{j\in T_2} u_j^{l_j}}{\rho},$$

or

$$0 \geq \left(\sum_{j\in T_2} u_j^{l_j} - \rho\right)\left(\rho - u_i^{l_i}\right) = -\left(a_i^{l_i} - \delta\right)\left(\rho - u_i^{l_i}\right),$$

which holds since $a_i^{l_i} - \delta \geq 0$. If $\gamma_i \neq 0$, (28) becomes

$$\frac{\sum_{j\in T_2} u_j^{l_j} + u_i^{l_i} - \rho}{u_i^{l_i}} + \beta_i \cdot \frac{\rho - \sum_{j\in T_2} u_j^{l_j}}{u_i^{l_i}} \leq \frac{\sum_{j\in T_2} u_j^{l_j}}{\rho}. \tag{29}$$

Clearly, (29) holds (at equality).

Suppose that $i \in C_1$. It suffices to assume that $t = l_i - 1$ or $t = l_i$, $\lambda_j^{l_j} = 1\ \forall j \in T_1 \cup C_2$, and $\lambda_j^{l_j-1} = 1\ \forall j \in T_2$, where T_1 and T_2 are now disjoint subsets of $C_1 - \{i\}$ with $T_1 \cup T_2 = C_1 - \{i\}$. Assume that $\lambda_i^{l_i-1}, \lambda_i^{l_i} \in (0,1)$ (the proof for all other cases is similar).

We have that

$$\lambda_i^{l_i-1} = \frac{a_i^{l_i} - \delta}{u_i^{l_i}} \text{ and } \lambda_i^{l_i} = \frac{\delta - a_i^{l_i-1}}{u_i^{l_i}}, \tag{30}$$

where this time

$$\delta = b - \left(\sum_{j\in T_1\cup C_2} a_j^{l_j} + \sum_{j\in T_2} a_j^{l_j-1}\right).$$

This reduces (25) to

$$\sum_{j\in T_2} \beta_j + \beta_i \lambda_i^{l_i-1} + \lambda_i^{l_i} \leq |T_2|. \tag{31}$$

Note that

$$a_i^{l_i} - \delta = \rho - \sum_{j\in T_2} u_j^{l_j} \text{ and } \delta - a_i^{l_i-1} = \sum_{j\in T_2} u_j^{l_j} + u_i^{l_i} - \rho.$$

So (31) becomes

$$\beta_i \cdot \frac{\rho - \sum_{j\in T_2} u_j^{l_j}}{u_i^{l_i}} + \frac{\sum_{j\in T_2} u_j^{l_j} + u_i^{l_i} - \rho}{u_i^{l_i}} \leq \frac{\sum_{j\in T_2} u_j^{l_j}}{\rho}, \tag{32}$$

Clearly, (32) holds (again at equality). Therefore, we conclude that (25) is valid. □

We next give, in Theorems 4 and 5, sufficient conditions for (25) to be facet-defining for P'. The first (Theorem 4) refers to cover inequalities with $|C_1| \geq 2$, and the second (Theorem 5) refers to cover inequalities with $|C_1| = 1$.

Theorem 4 *Suppose that C is a minimal cover, $|C_1| \geq 2$,*

$$a_j^{l_j-3} \leq b - \sum_{i\in C-\{j\}} a_i^{l_i} \ \forall j \in C_1 \text{ with } l_j > 3, \tag{33}$$

and

$$a_j^{l_j-2} \leq b - \sum_{i\in C-\{j\}} a_i^{l_i} \ \forall j \in C_2 \text{ with } l_j > 2. \tag{34}$$

Finally, suppose that $\forall j \in C_1$ there are disjoint pairs of subsets $\{N_1, N_2\}$ and $\{N_1', N_2'\}$ of $C_1 - \{j\}$ such that $N_1 \cup N_2 = C_1 - \{j\}$, $N_1' \cup N_2' = C_1 - \{j\}$, and

$$b - a_j^{l_j-1} < \sum_{i\in C_2\cup N_1} a_i^{l_i} + \sum_{i\in N_2} a_i^{l_i-1} \leq b - a_j^{l_j-2} \tag{35}$$

and

$$b - a_j^{l_j} < \sum_{i\in C_2\cup N_1'} a_i^{l_i} + \sum_{i\in N_2'} a_i^{l_i-1} \leq \ b - a_j^{l_j-1}. \tag{36}$$

Then, (25) is facet-defining for P'. □

Note that (35) and (36) cannot be both satisfied when $|C_1| = 1$. Thus, the assumption that $|C_1| \geq 2$. Before proving Theorem 4, we give an example.

Example 4 Let $N = \{1, 2, 3\}$, $T = 3$, and (8) be

$$\lambda_1^1 + 3\lambda_1^2 + 4\lambda_1^3 + \lambda_2^1 + 3\lambda_2^2 + 4\lambda_2^3 + 2\lambda_3^1 + 4\lambda_3^2 + 9\lambda_3^3 \leq 10.$$

Let $C = \{1, 2, 3\}$, $l_1 = l_2 = 3$, $l_3 = 2$, $C_1 = \{1, 2\}$, and $C_2 = \{3\}$. We have that

$$-\frac{1}{2}\lambda_1^1 + \frac{1}{2}\lambda_1^2 + \lambda_1^3 - \frac{1}{2}\lambda_2^1 + \frac{1}{2}\lambda_2^2 + \lambda_2^3 + \lambda_3^2 \leq 2 \tag{37}$$

is valid for P. Note that C is a minimal cover, $|C_1| \geq 2$, conditions (33) and (34) are satisfied vacuously, and conditions (35) and (36) are satisfied for $j = 1$ (with $N_1 = N_2' = \{2\}$ and $N_1' = N_2 = \emptyset$) and $j = 2$ (with $N_1 = N_2' = \{1\}$ and $N_1' = N_2 = \emptyset$). So, (37) defines a facet of P'. □

We now prove Theorem 4.

Proof of Theorem 4 We show that when all assumptions hold, P' has $\sum_{j\in C} l_j$ linearly independent points that satisfy (25) at equality. (In what follows, we only give the nonzero components of the points). First we consider v_j^0 given by $\lambda_i^{l_i} = 1\ \forall i \in C - \{j\}$. Because C is a minimal cover, $v_j^0 \in P'$.

Next, we consider v_j^l, $j \in C_2$, $l \in \{1, \dots, l_j - 2\}$ given by $\lambda_i^{l_i} = 1\ \forall i \in C - \{j\}$ and $\lambda_j^l = 1$. Because of (34), $v_j^l \in P'$. The point $v_j^{l_j-1}$ is given by $\lambda_i^{l_i} = 1\ \forall i \in C - \{j\}$,

$$\lambda_j^{l_j-2} = \frac{\rho - u_j^{l_j}}{u_j^{l_j-1}} \text{ and } \lambda_j^{l_j-1} = 1 - \frac{\rho - u_j^{l_j}}{u_j^{l_j-1}}$$

when $b - \sum_{i\in C-\{j\}} a_i^{l_i} \le a_j^{l_j-1}$ (i.e. $\gamma_j = 0$), or

$$\lambda_j^{l_j-1} = \frac{\rho}{u_j^{l_j}} \text{ and } \lambda_j^{l_j} = 1 - \frac{\rho}{u_j^{l_j}}$$

otherwise.

The point v_j^l for $j \in C_1$, $l \in \{1, \dots, l_j - 3\}$ is given by $\lambda_i^{l_i} = 1\ \forall i \in C - \{j\}$ and $\lambda_j^l = 1$. Because of (33), $v_j^l \in P'$. Let N_1 and N_2 be as in (35). The point $v_j^{l_j-2}$ is given by $\lambda_i^{l_i} = 1$ $\forall i \in C_2 \cup N_1$, $\lambda_i^{l_i-1} = 1\ \forall i \in N_2$,

$$\lambda_j^{l_j-2} = \frac{a_j^{l_j-1} - \delta}{u_j^{l_j-1}} \text{ and } \lambda_j^{l_j-1} = \frac{\delta - a_j^{l_j-2}}{u_j^{l_j-2}},$$

where

$$\delta = b - \left(\sum_{i\in N_1\cup C_2} a_i^{l_i} + \sum_{i\in N_2} a_i^{l_i-1} \right).$$

Finally, let N_1' and N_2' be as in (36). The point $v_j^{l_j-1}$ is given by $\lambda_i^{l_i} = 1\ \forall i \in C_2 \cup N_1'$, $\lambda_i^{l_i-1} = 1\ \forall i \in N_2'$,

$$\lambda_i^{l_i-1} = \frac{a_i^{l_i} - \delta}{u_i^{l_i}} \text{ and } \lambda_i^{l_i} = \frac{\delta - a_i^{l_i-1}}{u_i^{l_i}},$$

and

$$\delta = b - \left(\sum_{j\in N_1'\cup C_2} a_j^{l_j} + \sum_{j\in N_2'} a_j^{l_j-1} \right).$$

Clearly, all these points belong to P', are linearly independent, and satisfy (25) at equality. Thus, (25), under the conditions given, is facet-defining for P'. □

Corollary 2 *Suppose that C is a minimal cover,*

$$\sum_{i\in C-\{j\}} a_i^{l_i} + a_j^{l_j-2} \le b \;\forall j \in C \text{ with } l_j > 2, \tag{38}$$

$|C_1| \ge 2$, $\beta_j > 0$ $\forall j \in C_1$, and for any $j \in C_1$ there exists $s \in C_1$ such that

$$\sum_{i\in C-\{j,s\}} a_i^{l_i} + a_s^{l_s-1} + a_j^{l_j-1} \le b. \tag{39}$$

Then, (25) is facet-defining for P'.

Proof Note that (38) implies (33) and (34). Because $\beta_j > 0$,

$$\sum_{i\in C-\{j\}} a_i^{l_i} + a_j^{l_j-1} > b. \tag{40}$$

We now take $N_2 = \emptyset$ and $N_2' = \{s\}$. The left-hand-sides of (35) and (36) follow from (40). The right-hand-side of (35) follows from (38) and the right-hand-side of (36) follows from (39). □

Theorem 5 *Suppose that C is a minimal cover,*

$$\sum_{j\in C-\{s\}} a_j^{l_j} + a_s^{l_s-2} \le b \;\forall s \in C \text{ with } l_s > 2, \tag{41}$$

$C_1 = \{t\}$, $\beta_t > 0$, and there exists $i \in C_2$ with

$$\gamma_i = \beta_i. \tag{42}$$

Then, (25) is facet-defining for P'. □

Before proving Theorem 5, we give an example.

Example 4 (Continued) Let, as before, $C = \{1, 2, 3\}$ and $l_1 = l_2 = 3$, $l_3 = 2$. But now we take $C_1 = \{1\}$ and $C_2 = \{2, 3\}$. We have that $(a_2^{l_2} + a_3^{l_3}) + a_1^{l_1-2} = 9 \le b$, $(a_1^{l_1} + a_3^{l_3}) + a_2^{l_2-2} = 9 \le b$, $\beta_1 = \frac{1}{2} > 0$, and $\gamma_3 = \beta_3$ $(= 0)$. Thus,

$$-\frac{1}{2}\lambda_1^1 + \frac{1}{2}\lambda_1^2 + \lambda_1^3 + \lambda_2^3 + \lambda_3^3 \le 2$$

is facet-defining for P'. □

We now prove Theorem 5.

Proof of Theorem 5 Note that (41) implies (33) and (34). Also, (41) and $\beta_t > 0$ imply (35). Therefore, all points in the proof of Theorem 3, not including $v_t^{l_t-1}$, belong to P'.

For $v_t^{l_t-1}$, we take $\lambda_j^{l_j} = 1\ \forall j \in C - \{t\}$, $\lambda_t^{l_t-1} = 1$, and $\lambda_i^{l_i-1}$ and $\lambda_i^{l_i}$ given by (26), with $T_2 = \{t\}$ in (27). Due to (42),

$$\sum_{j \in C-\{i,t\}} a_j^{l_j} + a_i^{l_i-1} + a_t^{l_t-1} < b,$$

and therefore $v_t^{l_t-1} \in P'$. Again, due to (42), and the fact that (29) is satisfied at equality, $v_t^{l_t-1}$ satisfies (25) at equality. This shows that (25) is facet-defining for P'. $\square$

We now make, in Theorem 6, a slight change in (25) for $|C_1| = 1$ that nevertheless will result, under certain conditions, in another inequality that is valid and facet-defining for P'. As a matter of fact, as we show in Theorem 7, the resulting inequality, when valid, will cut off any point of $vert(LPS)$ that is cut off by the inequality (25) that corresponds to it (i.e. with same C_1, C_2, and l_j's, $j \in C$), while in general it will cut off points of $vert(LPS)$ that (25) will not. Because Theorem 6 can be proved similarly to Theorem 3, we omit its proof.

Theorem 6 *Let $t \in N$. Suppose that $C_1 = \{t\}$, $\beta_t > 0$, and $\gamma_j = 0\ \forall j \in C_2$. The cover inequality*

$$\frac{\rho - u_t^{l_t} - u_t^{l_t-1}}{\omega}\lambda_t^{l_t-2} + \frac{\rho - u_t^{l_t}}{\omega}\lambda_t^{l_t-1} + \lambda_t^{l_t} + \sum_{j \in C_2} \lambda_j^{l_j} \le |C_2| \tag{43}$$

is valid for P, where

$$\omega = \ max\left\{u_j^{l_j} : j \in C_2\right\}.$$

Suppose that C is a minimal cover,

$$\sum_{i \in C-\{j\}} a_i^{l_i} + a_j^{l_j-2} \le b\ \forall j \in C,$$

and

$$\sum_{j \in C-\{t\}} a_j^{l_j} + a_t^{l_t-1} - \omega < b.$$

Then, (43) is facet-defining for P'. $\square$

Theorem 7 *Let $t \in N$. Suppose that $C_1 = \{t\}$, $\beta_t > 0$, and $\gamma_j = 0\ \forall j \in C_2$. Consider the inequalities (25) and (43) with same $C_1 = \{t\}$ (and consequently same C_2. Also same l_j's, $j \in C$). Let $\lambda \in vert(LPS)$ be such that it violates (25). Then λ also violates (43).* $\square$

Before proving Theorem 7, we give an example.

Example 5 Let $N = \{1, 2, 3\}$, $T = 3$, and (8) be

$$2\lambda_1^1 + 6\lambda_1^2 + 8\lambda_1^3 + 6\lambda_2^1 + 10\lambda_2^2 + 13\lambda_2^3 + 3\lambda_3^1 + 7\lambda_3^2 + 10\lambda_3^3 \leq 13.$$

For $C = \{1, 2\}$ with $C_1 = \{1\}$, $C_2 = \{2\}$, $l_1 = 3$, and $l_2 = 2$, (25) is

$$-\frac{1}{5}\lambda_1^1 + \frac{3}{5}\lambda_1^2 + \lambda_1^3 + \lambda_2^2 \leq 1, \tag{44}$$

while (43) is

$$-\frac{1}{4}\lambda_1^1 + \frac{3}{4}\lambda_1^2 + \lambda_1^3 + \lambda_2^2 \leq 1. \tag{45}$$

We note that all points of $vert(LPS)$ that (44) cuts off are also cut off by (45). But in addition, (45) cuts off $\lambda_1^2 = 1$, $\lambda_2^2 = \frac{2}{5}$, $\lambda_3^1 = 1$, $\lambda_j^k = 0$ otherwise, which is not cut off by (44). □

We now prove Theorem 7.

Proof of Theorem 7 First note that $\rho \geq \omega$. This is true because $\gamma_j = 0 \; \forall j \in C_2 \Rightarrow \rho \geq u_j^{l_j}$ $\forall j \in C_2$. So, if $\alpha_t = 0$, (43) implies (25). We then assume that $\alpha_t < 0$. Likewise, if $\lambda_t^{l_t - 2} = 0$, λ violates (43) in case it violates (25). So we assume that $\lambda_t^{l_t-2} > 0$. But if both $\lambda_t^{l_t-1}$ and $\lambda_t^{l_t}$ are 0, then λ cannot violate (25). So we also assume that at least one of $\lambda_t^{l_t-1}$ or $\lambda_t^{l_t}$ is positive.

From Lemma 1, it follows that $\lambda_j^k \in \{0, 1\} \; \forall k \in K, j \in N - \{t\}$. This means that, in order for λ to violate (25), we must have $\lambda_j^{l_j} = 1 \; \forall j \in C_2$, which we then assume. From Lemma 1 again, we have that at most one of $\lambda_t^{l_t-1}$ and $\lambda_t^{l_t}$ can be positive. Because (25) is feasible for P, λ can violate (25) only if $\lambda_t^{l_t}$ is the one that is positive, which we then assume.

Plugging into (25) what we already have assumed about λ, we have that it violates (25) only if

$$(\rho - u_t^{l_t} - u_t^{l_t-1})\lambda_t^{l_t-2} + \rho\lambda_t^{l_t} > 0. \tag{46}$$

But because Lemma 1 requires that λ satisfies (8) at equality, we have that

$$a_t^{l_t-2}\lambda_t^{l_t-2} + a_t\lambda_t^{l_t} + \sum_{j \in C_2} a_j^{l_j} = b. \tag{47}$$

Note, however, that (47) implies

$$(\rho - u_t^{l_t} - u_t^{l_t-1})\lambda_t^{l_t-2} + \rho\lambda_t^{l_t} = 0,$$

which contradicts (46). Thus, if λ does not violate (43), it must satisfy (25). □

Next, we give a necessary and sufficient condition for (25) with $C_1 = \emptyset$, together with the convexity and nonnegativity constraints, to give P'.

Theorem 8 *Suppose that C is a minimal cover. The polyhedron P' is given by (5), (6), and*

$$\sum_{j \in C} \left(\gamma_j \lambda_j^{l_j - 1} + \lambda_j^{l_j} \right) \leq |C| - 1 \tag{48}$$

iff

$$\gamma_j = \beta_j \; \forall j \in C. \tag{49}$$

Proof Let $P'' = \{\lambda \in \Re^{nT} : (5), (6), \text{ and } (48) \text{ hold}; \lambda_j^k = 0 \; \forall k > l_j, k \in K, j \in C; \lambda_j^k = 0 \; \forall j \in N - C, k \in K\}$. By taking $C_1 = \emptyset$, it follows from Theorem 3 that (48) is valid for P'. Therefore, $P' \subseteq P''$.

Condition (49) is necessary and sufficient for all vertices of P'' to be integral, with the possible exception of the points v_j, $j \in C$, given by $\lambda_i^{l_i} = 1 \; \forall i \in C - \{j\}$,

$$\lambda_j^{l_j - 1} = \frac{\rho}{u_j^{l_j}}, \lambda_j^{l_j} = \frac{u_j^{l_j} - \rho}{u_j^{l_j}},$$

and all other components equal to 0. Note that $v_j \in P' \; \forall j \in C$ (they satisfy (8) at equality).

In the same way, condition (49) is necessary and sufficient for all maximal integral vertices of P'' to be w_j, $j \in C$, given by $\lambda_i^{t_i} = 1 \; \forall i \in C$, where $t_i \in \{1, \ldots, l_i\} \; \forall i \in C - \{j\}$ and $t_j \in \{1, \ldots, l_j - 1\}$, and all other components equal to 0. Note that $w_j \in P' \; \forall j \in C$, and so $P' = P''$. □

It is possible to strengthen our cover inequalities effortlessly, i.e. without any computational burden, by approximately lifting some of the variables as in Lemma 2 from [21].

Lemma 2 (Keha et al. [21]) *Let $V \subset N$, $j \in N - V$, and $l \in K - \{1\}$. Suppose that*

$$\sum_{v \in V} \sum_{k \in K} \alpha_v^k \lambda_v^k + \sum_{k=1}^{l-1} \alpha_j^k \lambda_j^k \leq \gamma$$

is valid for $P \cap \left\{ \lambda \in \Re^{nT} : \lambda_j^k = 0 \; \forall k \geq l \right\}$. Then,

$$\sum_{v \in V} \sum_{k \in K} \alpha_v^k \lambda_v^k + \sum_{k=1}^{l-1} \alpha_j^k \lambda_j^k + \alpha_j^{l-1} \sum_{k=l}^{T} \lambda_j^k \leq \gamma$$

is valid for P. □

From Theorem 3 and Lemma 2, it follows immediately that

Theorem 9 *Let C_1, C_2, l_j, α_j, β_j, and γ_j be as in Theorem 3. Then,*

$$\sum_{j\in C_1}(\alpha_j\lambda_j^{l_j-2}+\beta_j\lambda_j^{l_j-1}+\sum_{j=l_j}^{T}\lambda_j^{l_j})+\sum_{j\in C_2}(\gamma_j\lambda_j^{l_j-1}+\sum_{j=l_j}^{T}\lambda_j^{l_j})\leq |C|-1 \tag{50}$$

is valid for P. □

We now strengthen (25) for the case $C_1=\emptyset$.

Theorem 10 *Let $a_C = max\ \{a_i^{l_i} : \forall i \in C\}$, $\bar{C} = \{j \in N-C : a_j^T \geq a_C$ and $a_j^{T-1} \geq a_C-\rho\}$, and $t_j = min\ \{k : a_j^k \geq a_C$ and $a_j^{k-1} \geq a_C-\rho\}$ $\forall j \in \bar{C}$. Suppose that $\bar{C} \neq \emptyset$. The lifted cover inequality*

$$\sum_{j\in C}(\gamma_j\lambda_j^{l_j-1}+\sum_{k=l_j}^{T}\lambda_j^{k})+\sum_{j\in\bar{C}}\sum_{k=t_j}^{T}\lambda_j^{k}\leq |C|-1 \tag{51}$$

is valid for P. □

Before proving Theorem 10, we give an example.

Example 2 (Continued) We take $C=\{1,2\}$, $l_1=l_2=2$, and $C_1=\emptyset$. Theorem 3 gives:

$$-\frac{1}{3}\lambda_1^1+\lambda_1^2-\frac{1}{3}\lambda_2^1+\lambda_2^2\leq 1.$$

From Lemma 2 we have:

$$-\frac{1}{3}\lambda_1^1+\lambda_1^2+\lambda_1^3-\frac{1}{3}\lambda_2^1+\lambda_2^2+\lambda_2^3\leq 1.$$

Now, $\bar{C}=\forall j\in\{3,4\}$ and $t_3=t_4=2$. So the inequality

$$-\frac{1}{3}\lambda_1^1+\lambda_1^2+\lambda_1^3-\frac{1}{3}\lambda_2^1+\lambda_2^2+\lambda_2^3+\lambda_3^2+\lambda_3^3+\lambda_4^2+\lambda_4^3\leq 1 \tag{52}$$

is valid for P. Inequality (52) turns out to be facet-defining for P. □

We now prove Theorem 10.

Proof of Theorem 10 From Theorem 3 (with $C_1=\emptyset$) and Lemma 2,

$$\sum_{j\in C}(\gamma_j\lambda_j^{l_j-1}+\sum_{k=l_j}^{T}\lambda_j^{k})\leq |C|-1 \tag{53}$$

is valid for P. We first fix $\lambda_j^k=0\ \forall k\geq t_j, j\in\bar{C}$. Let $i\in\bar{C}$ and

$$P_i^{t_i} = \left\{\lambda \in P : \lambda_j^k = 0 \ \forall k \geq t_j, j \in \bar{C} - \{i\}\,; \lambda_i^k = 0 \ \forall k > t_i; \lambda_i^{t_i} > 0\right\}.$$

We now lift (53) with respect to $\lambda_i^{t_i}$. The lifting problem is:

$$\eta_i^{t_i} = \min \left\{ \frac{|C| - 1 - \sum_{j\in C}(\gamma_j \lambda_j^{l_j-1} + \sum_{k=l_j}^{T} \lambda_j^k)}{\lambda_i^{t_i}} : \lambda \in vert(P_i^{t_i}) \right\}, \tag{54}$$

where $\eta_i^{t_i}$ is the lifting coefficient. Because $a_j^k > a_j^{l_j} \ \forall k > l_j, j \in C$, (54) has an optimal solution with $\lambda_j^k = 0 \ \forall k > l_j, j \in C$. So, we may rewrite the lifting problem as:

$$\eta_i^{t_i} = \min \left\{ \frac{|C| - 1 - \sum_{j\in C}(\gamma_j \lambda_j^{l_j-1} + \lambda_j^{l_j})}{\lambda_i^{t_i}} : \lambda \in vert(P_i^{t_i}) \right\}. \tag{55}$$

We will now show that $\eta_i^{t_i} \geq 1$.

Because $a_i^{t_i} \geq a_C$, if (55) has an optimal solution with $\lambda_i^{t_i} = 1$, $\sum_{j\in C}(\gamma_j \lambda_j^{l_j-1} + \lambda_j^{l_j}) \leq |C| - 2$, and $\eta_i^{t_i} \geq 1$. So we assume that $\lambda_i^{t_i} \in (0,1)$. In this case, in order for $\eta_i^{t_i} < 1$, we must have $|C| - 1$ variables $\lambda_j^{l_j} = 1$, $j \in C$. Let $r \in C$, and suppose $\lambda_j^{l_j} = 1 \ \forall j \in C - \{r\}$. Suppose also that $\lambda_i^{t_i-1}, \lambda_i^{t_i} \in (0,1)$ (the case $\lambda_i^{t_i}, \lambda_i^{t_i+1} \in (0,1)$ can be treated similarly). We then have

$$\sum_{j\in C-\{r\}} a_j^{l_j} + a_i^{t_i-1}\lambda_i^{t_i-1} + a_i^{t_i}\lambda_i^{t_i} \leq b,$$

and so,

$$a_i^{t_i-1} < b - \sum_{j\in C-\{r\}} a_j^{l_j} \leq b - \sum_{j\in C} a_j^{l_j} + a_C = a_C - \rho.$$

However, by assumption, $a_i^{t_i-1} \geq a_C - \rho$. So, even when $\lambda_i^{t_i}$ in an optimal solution to (55), we still must have $\eta_i^{t_i} \geq 1$.

This means that

$$\sum_{j\in C}(\gamma_j \lambda_j^{l_j-1} + \sum_{k=l_j}^{T} \lambda_j^k) + \lambda_i^{t_i} \leq |C| - 1$$

is valid for P. Let $\hat{C} \subset \bar{C}$ and suppose that

$$\sum_{j\in C}(\gamma_j \lambda_j^{l_j-1} + \sum_{k=l_j}^{T} \lambda_j^k) + \sum_{j\in \hat{C}} \lambda_j^{t_j} \leq |C| - 1 \tag{56}$$

is valid for P. We can now, as we did before, lift (56) with respect to $\lambda_r^{t_r}$, $r \in \bar{C} - \hat{C}$. Because $a_j^{t_j} \geq a_C \ \forall j \in \bar{C}$, the lifting problem must have an optimal solution with $\lambda_j^{t_j} = 0 \ \forall j \in \hat{C}$, and therefore, just as it happened before with $\lambda_i^{t_i}$, the lifting coefficient of $\lambda_r^{t_r}$ is at least 1. In other words,

$$\sum_{j\in C}(\gamma_j\lambda_j^{l_j-1}+\sum_{k=l_j}^{T}\lambda_j^{k})+\sum_{j\in\bar{C}}\lambda_j^{t_j}\leq|C|-1 \tag{57}$$

is valid for P. Applying now Lemma 2, (51) follows. □

Finally, we drop the assumption that $N^-=\emptyset$. We adopt, for the case $N^-\neq\emptyset$, the definition of a *generalized cover* given in [21].

Definition 4 (Keha et. al. [21]) *Let* $C^+\subseteq N^+$, $C^-\subseteq N^-$, $2\leq l_j\leq T\ \forall j\in C^+$, $1\leq l_j\leq T-1\ \forall j\in C^-$, *and* $C=C^+\cup C^-$. *If*

$$\sum_{j\in C^+}a_j^{l_j}-\sum_{j\in C^-}a_j^{l_j}=b+\rho$$

with $\rho>0$, C *is a generalized cover.* □

As shown in Theorem 11, to each division of C^+ into C_1 and C_2, we obtain a valid inequality for P, which we call a *lifted generalized cover inequality.*

Theorem 11 *Let* C *be a generalized cover. The inequality*

$$\sum_{j\in C_1}\left(\alpha_j\lambda_j^{l_j-2}+\beta_j\lambda_j^{l_j-1}+\sum_{j=l_j}^{T}\lambda_j^{l_j}\right)+\sum_{j\in C_2}\left(\gamma_j\lambda_j^{l_j-1}+\sum_{j=l_j}^{T}\lambda_j^{l_j}\right)$$
$$-\sum_{j\in N^-}\left(\tau_j\lambda_j^{l_j+1}+\sum_{j=l_j+2}^{T}\lambda_j^{k}\right)\leq\left|C^+\right|-1 \tag{58}$$

is valid for P, *where* $C^+=C_1\cup C_2$, $l_j>2\ \forall j\in C_1$, $l_j=0\ \forall j\in N^--C^-$, $\alpha_j,\beta_j,\gamma_j$ *are given by Theorem 3, and*

$$\tau_j=\max\left\{1,\frac{u_j^{l_j+1}}{\rho}\right\}.$$

□

Theorem 11 can be proved similarly to Theorem 3 for the C^+ part and Theorem 5 in [21] for the C^- part. We then omit its proof.

Example 1 (Continued) We let $C = \{1, 2, 3\}$ with $C^+ = \{1, 2\}$, $C^- = \{3\}$, $l_1 = 3$, $l_2 = 2$, and $l_3 = 2$. We also let $C_1 = \{1\}$ and $C_2 = \{2\}$, and we have that $l_4 = 0$. It then follows that

$$-5\lambda_1^1 - \lambda_1^2 + \lambda_1^3 - 3\lambda_2^1 + \lambda_2^2 + \lambda_2^3 - 2\lambda_3^3 - 2\lambda_4^1 - \lambda_4^2 - \lambda_4^3 \le 1$$

is valid for P. □

5 Intersection with Semi-Continuous Constraints

In many applications, piecewise linear optimization arises with other combinatorial constraints, notoriously with *semi-continuous constraints* [11]. A variable x is semi-continuous when $x \in [a, b] \cup [c, d]$, where $b < c$. For simplicity of notation, and without loss of generality, we assume that the forbidden interval (b, c) is determined by two adjacent breakpoints of x. Specifically, let $i \in N$ and the breakpoints of x_i be $d_i^0(= 0), \ldots, d_i^T(= u_i)$. Then, $b = d_i^{k-1}$ and $c = d_i^k$ for some $k \in K$ (and of course $a = 0$ and $d = u_i$. Note that this allows for such semi-continuous constraints as $x_i \in \{0\} \cup [d_i^1, u_i]$, for $k = 1$, and $x_i \in [0, d_i^{T-1}] \cup \{u_i\}$, for $k = T$). Also for simplicity of notation, and without loss of generality, we assume that all variables x_j, $j \in N$, are semi-continuous. We denote the index $k \in K$ that defines the semi-continuous constraint of x_i as k_i^*, i.e. $x_i \in [0, d_i^{k_i^*-1}] \cup [d_i^{k_i^*}, u_i]$; and we denote the SPLO polytope with semi-continuous constraints P_{SC}.

In this section we strengthen the inequalities of the previous sections to cut off points that do not satisfy the semi-continuous constraints. Specifically, we strengthen inequalities (13), (19), (50), and (51). In order to do this we observe that the constraint $x_j \in [0, d_j^{k_j^*-1}] \cup [d_j^{k_j^*}, u_j]$ is equivalent to adding to the SOS2 $\{\lambda_j^0, \ldots \lambda_j^T\}$ an "artificial" variable, call it $\bar{\lambda}_j$, between $\lambda_j^{k_j^*-1}$ and $\lambda_j^{k_j^*}$, and fixing $\bar{\lambda}_j$ at 0. This forbids $\lambda_j^{k_j^*-1}$ and $\lambda_j^{k_j^*}$ from being simultaneously positive, thus enforcing $x_j \notin (d_j^{k_j^*-1}, d_j^{k_j^*})$. We illustrate this idea below with inequality (13).

Example 2 (Continued) Suppose that (15) in x-space is

$$2x_1 + x_2 + 2x_3 + x_4 \le 10, \tag{59}$$

where the breakpoints are $\{0, 1, 3, 4\}$ for x_1, $\{0, 3, 7, 10\}$ for x_2, $\{0, 2, 4, 5\}$ for x_3, and $\{0, 5, 7, 9\}$ for x_4. The points $\tilde{\lambda}_1^2 = 1$, $\tilde{\lambda}_4^1 = \frac{4}{5}$, $\tilde{\lambda}_j^k = 0$ otherwise, and $\hat{\lambda}_1^2 = 1$, $\hat{\lambda}_2^1 = \frac{3}{4}$, $\hat{\lambda}_2^2 = \frac{1}{4}$, $\hat{\lambda}_j^k = 0$ otherwise, are vertices of P. We now add the semi-continuous constraints $x_1 \in \{0\} \cup [1, 4]$, $x_2 \in [0, 3] \cup [7, 10]$, $x_3 \in [0, 4] \cup \{5\}$, and $x_4 \in \{0\} \cup [5, 9]$ (here, $k_1^* = 1$, $k_2^* = 2$, $k_3^* = 3$, and $k_4^* = 1$). Note that $\tilde{\lambda}, \hat{\lambda} \notin P_{SC}$ (in the case of $\tilde{\lambda}$ we get $x_4 = 4$, and in the case of $\hat{\lambda}$ we get $x_2 = 4$).

We next strengthen (13) with $j = 1$ and $s = 2$ to cut off $\tilde{\lambda}$ and $\hat{\lambda}$ (the reason for choosing $j = 1$ and $s = 2$ is that $\tilde{\lambda}_1^2 = \hat{\lambda}_1^2 = 1$). We first observe that (13) is

$$\frac{1}{3}\lambda_1^1 + \lambda_1^2 + \lambda_1^3 - \frac{1}{6}\lambda_2^1 + \frac{1}{2}\lambda_2^2 + \lambda_2^3 + \frac{2}{3}\lambda_3^2 + \lambda_3^3 + \frac{1}{2}\lambda_4^2 + \frac{5}{6}\lambda_4^3 \leq 1, \tag{60}$$

(here $I = \{2, 3, 4\}$, $k_2 = 2$, $k_3 = 2$, and $k_4 = 1$) and that $\tilde{\lambda}$ and $\hat{\lambda}$ satisfy (60) at equality. We then add breakpoints between 0 and 1 for x_1, 3 and 7 for x_2, 4 and 5 for x_3 and 0 and 5 for x_4. Let $\bar{\lambda}_1$, $\bar{\lambda}_2$, $\bar{\lambda}_3$, and $\bar{\lambda}_4$ be the λ-variables corresponding to the breakpoints added to x_1, x_2, x_3, and x_4, respectively. Constraint (15) becomes

$$(a_1\bar{\lambda}_1 + 2\lambda_1^1 + 6\lambda_1^2 + 8\lambda_1^3) + (3\lambda_2^1 + a_2\bar{\lambda}_2 + 7\lambda_2^2 + 10\lambda_2^3) + \\ (4\lambda_3^1 + 8\lambda_3^2 + a_3\bar{\lambda}_3 + 10\lambda_3^3) + (a_4\bar{\lambda}_4 + 5\lambda_4^1 + 7\lambda_4^2 + 9\lambda_4^3) \leq 10. \tag{61}$$

By choosing $a_1 = 1$, $a_2 = a_4 = 4, a_3 = 9$, and invoking Theorem 1 on (61) with $j = 1$ and $s = 2$, we obtain

$$\frac{1}{3}\lambda_1^1 + \lambda_1^2 + \lambda_1^3 + \frac{1}{2}\lambda_2^2 + \lambda_2^3 + \frac{2}{3}\lambda_3^2 + \lambda_3^3 + \frac{1}{6}\lambda_4^1 + \frac{1}{2}\lambda_4^2 + \frac{5}{6}\lambda_4^3 \leq 1, \tag{62}$$

which is valid for P_{SC} and cuts off $\tilde{\lambda}$ and $\hat{\lambda}$. □

We note that the terms corresponding to x_1 and x_3 in (60) and (62) are the same, and $j = 1$ and $k_3 \neq k_3^*$; on the other hand, the terms corresponding to x_2 and x_4 are different, and $k_2 = k_2^*$ and $k_4 = k_4^*$. As it should be clear from the example, this is not a coincidence. In fact, we could have dispensed with the addition of breakpoints for x_1 and x_3 (and consequently with introducing $\bar{\lambda}_1$ and $\bar{\lambda}_3$) right from the beginning. We now formalize these ideas in Theorem 12.

Theorem 12 *Let $j \in N^+$, $N_1^- \subseteq N^-$, and $b' = b + \sum_{i \in N_1^-} a_i^{m_i}$, where $m_i \in K$ $\forall i \in N_1^-$. Let $s \in K$ be such that $a_j^s < b'$, $I = \left\{i \in N^+ - \{j\} : a_j^s + a_i^T > b'\right\}$, and $k_i = \min\left\{k \in K : a_j^s + a_i^k > b'\right\}$ $\forall i \in I$. Let $I^* = \{i \in I : k_i^* = k_i\}$. Suppose that $I^* \neq \emptyset$. Then,*

$$\frac{1}{a_j^s} \sum_{k=1}^{s-1} a_j^k \lambda_j^k + \sum_{k=s}^{T} \lambda_j^k + \sum_{i \in I^*} \sum_{k=k_i}^{T} \alpha_i^{*k} \lambda_i^k + \sum_{i \in I - I^*} \sum_{k=\max\{1, k_i - 1\}}^{T} \alpha_i^k \lambda_i^k \\ - \sum_{i \in N_1^-} \sum_{k=m_i+1}^{T} \beta_i^k \lambda_i^k - \sum_{i \in N^- - N_1^-} \sum_{k \in K} \frac{a_i^k}{a_j^s} \lambda_i^k \leq 1 \tag{63}$$

is valid for P_{SC}, where

$$\alpha_i^{*k} = \frac{a_j^s + a_i^k - b'}{a_j^s} \; \forall i \in I^*, k \in K,$$

$$\left(\alpha_i^{k_i - 1}, \alpha_i^{k_i}\right) \in \left\{(0, 0), \left(\frac{a_j^s + a_i^{k_i - 1} - b'}{a_j^s}, \frac{a_j^s + a_i^{k_i} - b'}{a_j^s}\right)\right\} \; \forall i \in I - I^*$$

with $k_i > 1$ and $a_j^s + a_i^{k_i - 1} < b'$,

$$\left(\alpha_i^{k_i-1}, \alpha_i^{k_i}\right) = \left(0, \frac{a_j^s + a_i^{k_i} - b'}{a_j^s}\right) \ \forall i \in I - I^* \ \textit{with} \ k_i > 1 \ \textit{and} \ a_j^s + a_i^{k_i-1} = b',$$

$$\alpha_i^{k_i} = 0 \ \forall i \in I - I^* \ \textit{with} \ k_i = 1,$$

$$\alpha_i^k = \frac{a_j^s + a_i^k - b'}{a_j^s} \ \forall i \in I - I^* \ \textit{with} \ k > k_i,$$

and

$$\beta_i^k = \frac{a_i^k - a_i^{m_i}}{a_j^s}.$$

Proof As remarked previously, the semi-continuous constraint $x_i \in [0, d_i^{k_i^*-1}] \cup [d_i^{k_i^*}, d_i^T]$ is equivalent to having a variable $\bar{\lambda}_i$ in the SOS2 of x_i located between $\lambda^{k_i^*-1}$ and $\lambda^{k_i^*}$ and fixed at 0. By adding such variables $\forall x_i$ with $i \in I^*$ with coefficients

$$a_i = b' - a_j^s,$$

and invoking Theorem 1, (63) follows. □

We now consider the semi-continuous constraints in Theorem 2. First, we give an example.

Example 3 (Continued) Suppose that (20) in x-space is

$$2x_1 + x_2 + x_3 \leq 10,$$

where the breakpoints are $\{0, 1, 3, 4\}$ for x_1, $\{0, 3, 7, 10\}$ for x_2, and $\{0, 5, 7, 9\}$ for x_3. The point $\hat{\lambda}_1^3 = 1$, $\hat{\lambda}_3^1 = \frac{2}{5}$, $\hat{\lambda}_j^k = 0$ otherwise is a vertex of P. We now add the semi-continuous constraints $x_1 \in \{0\} \cup [1, 4]$, $x_2 \in \{0\} \cup [3, 10]$, and $x_3 \in \{0\} \cup [5, 9]$ (here $k_1^* = k_2^* = k_3^* = 1$). Note that $\hat{\lambda} \notin P_{SC}$.

We next strengthen (19) with $j = 1$ and $s = 3$ to cut off $\hat{\lambda}$ (the reason for choosing $j = 1$ and $s = 3$ is that $\hat{\lambda}_1^3 = 1$). We first observe that (19) is

$$\frac{2}{7}\lambda_1^1 + \frac{6}{7}\lambda_1^2 + \lambda_1^3 + \frac{4}{7}\lambda_2^2 + \lambda_2^3 \leq 1 \tag{64}$$

(here $I = \{2, 3\}$, $k_2 = 2$, $k_3 = 1$, $L = \{2\}$, and $a_L = 3$) and that $\hat{\lambda}$ satisfies (64) at equality. Because $j = 1$, $k_2 \neq k_2^*$, and $k_3 = k_3^*$, we only add $\bar{\lambda}_3$, between λ_3^0 and λ_3^1 in the SOS2 of x_3. We give for the coefficient of $\bar{\lambda}_3$ $a_3 = 2$ (this is the smallest possible value of a_3 that will include index $i = 3$ into the set L. Note that other values would also do the same, e.g.

$a_3 = 3$. However, choosing the smallest possible value for a_3 will increase the violation of $\hat{\lambda}$ in the cut). Now (20) becomes

$$2\lambda_1^1 + 6\lambda_1^2 + 8\lambda_1^3 + 3\lambda_2^1 + 7\lambda_2^2 + 10\lambda_2^3 + 2\bar{\lambda}_3 + 5\lambda_3^1 + 7\lambda_3^2 + 9\lambda_3^3 \leq 10. \tag{65}$$

Invoking Theorem 2 on (65) and fixing $\bar{\lambda}_3$ at 0, we obtain

$$\frac{1}{4}\lambda_1^1 + \frac{3}{4}\lambda_1^2 + \lambda_1^3 + \frac{5}{8}\lambda_2^2 + \lambda_2^3 + \frac{3}{8}\lambda_3^1 + \frac{5}{8}\lambda_3^2 + \frac{7}{8}\lambda_3^3 \leq 1, \tag{66}$$

which is valid for P_{SC} and cuts off $\hat{\lambda}$.

Note that, if instead of 3, the coefficient of λ_2^1 in (20) were, say 5, then Theorem 2 would yield no inequality for $j = 1$ and $s = 3$. The reason is that, in this case, $k_2 = 1$, and $L = \emptyset$. However, we can still obtain a cut for the semi-continuous constraint. Since now $k_1 = k_1^*$, we repeat the above procedure with $\bar{\lambda}_1$ being added to the SOS2 of x_1 between λ_1^0 and λ_1^1 with coefficient 2 to get

$$\frac{1}{4}\lambda_1^1 + \frac{3}{4}\lambda_1^2 + \lambda_1^3 + \frac{3}{8}\lambda_2^1 + \frac{5}{8}\lambda_2^2 + \lambda_2^3 + \frac{3}{8}\lambda_3^1 + \frac{5}{8}\lambda_3^2 + \frac{7}{8}\lambda_3^3 \leq 1, \tag{67}$$

which cuts off, besides $\hat{\lambda}$, $\tilde{\lambda}_1^3 = 1$, $\tilde{\lambda}_2^1 = \frac{2}{5}$, $\tilde{\lambda}_j^k = 0$ otherwise. □

Theorem 13 *Let $j \in N^+$, $N_1^- \subseteq N^-$, and $b' = b + \sum_{i \in N_1^-} a_i^{m_i}$, where $m_i \in K$ $\forall i \in N_1^-$. Let $s \in K - \{1\}$ and $I = \{i \in N^+ - \{j\} : a_j^{s-1} + a_i^T \geq b'\}$. Suppose that $I \neq \emptyset$. Let $k_i = \min\{k \in K : a_j^{s-1} + a_i^k \geq b'\}$, $i \in I$. Denote $I^* = \{i \in I : k_i = k_i^*\}$ and $L = \left\{i \in I - I^* : a_j^s + a_i^{k_i - 1} \geq b' \text{ and } k_i > 1\right\}$. Suppose that $I^* \neq \emptyset$. Then,*

$$\sum_{k=1}^{s-1} \gamma_j^k \lambda_j^k + \sum_{k=s}^{T} \lambda_j^k + \sum_{i \in I^* \cup L} \sum_{k=k_i}^{T} \alpha_i^k \lambda_i^k - \sum_{i \in N_1^-} \sum_{k=m_i+1}^{T} \beta_i^k \lambda_i^k - \sum_{i \in N^- - N_1^-} \sum_{k \in K} \gamma_i^k \lambda_i^k \leq 1 \tag{68}$$

is valid for P_{SC}, where

$$\alpha_i^k = \frac{a_i^k - a_L}{b' - a_j^s} \ \forall i \in I^* \cup L \text{ with } k \geq k_i,$$

$$\beta_i^k = \frac{a_i^k - a_i^{m_i}}{b' - a_j^s} \ \forall i \in N_1^- \text{ with } k \geq m_i + 1,$$

and

$$\gamma_i^k = \frac{a_i^k}{b' - a_j^s} \ \forall i \in N^- \cup \{j\} - N_1^- \text{ with } k \in K.$$

Proof We introduce for each $i \in I^*$ a variable $\bar{\lambda}_i$ in the SOS2 of x_i, between variables $\lambda_i^{k_i^*-1}$ and $\lambda_i^{k_i^*}$, with coefficient $a_i = b' - a_j^s$. We then invoke Theorem 2. □

Finally, we consider the semi-continuous constraints in Theorems 9 and 10. First, we give an example.

Example 2 (Continued) Remember that, in x-space, (15) is (59), and the breakpoints are $\{0,1,3,4\}$ for x_1, $\{0,3,7,10\}$ for x_2, $\{0,2,4,5\}$ for x_3, and $\{0,5,7,9\}$ for x_4. However, we now suppose that the semi-continuous constraints are $x_1 \in \{0\} \cup [1,4]$, $x_2 \in \{0\} \cup [3,10]$, $x_3 \in \{0\} \cup [2,5]$, and $x_4 \in \{0\} \cup [5,9]$ (here $k_1^* = k_2^* = k_3^* = k_4^* = 1$).

The points $\tilde{\lambda}_2^1 = 1$, $\tilde{\lambda}_3^3 = \frac{7}{10}$, $\tilde{\lambda}_j^k = 0$ otherwise, and $\hat{\lambda}_1^1 = \hat{\lambda}_3^1 = 1$, $\hat{\lambda}_4^1 = \frac{4}{5}$, $\hat{\lambda}_j^k = 0$ otherwise are vertices of P. Neither, though, belong to P_{SC}. Note that to form a cover C it is necessary that $l_j \geq 2 \; \forall j \in C$, see Definition 3. So it is not even possible to form a cover inequality with $C = \{2,3\}$ and $l_2 = 1$, $l_3 = 3$ (corresponding to $\tilde{\lambda}$), or $C = \{1,3,4\}$ and $l_1 = l_3 = l_4 = 1$ (corresponding to $\hat{\lambda}$). However, this restriction disappears if, in the first case, we add and fix at 0 the variable $\bar{\lambda}_2$ to the SOS2 of x_2 between λ_2^0 and λ_2^1 (here $l_2 = k_2^*$), and, in the second case, the variables $\bar{\lambda}_1, \bar{\lambda}_3, \bar{\lambda}_4$ in the SOS2 of x_1, x_3, x_4 (respectively) between λ_1^0 and λ_1^1, λ_3^0 and λ_3^1, λ_4^0 and λ_4^1 (respectively. Here $l_1 = k_1^*$, $l_3 = k_3^*$, $l_4 = k_4^*$).

We now invoke Theorem 9 with $C = \{2,3\}$, $l_2 = 1$, $l_3 = 3$, $C_1 = \{3\}$, and $C_2 = \{1\}$ to obtain

$$-\lambda_1^3 + \frac{1}{3}\lambda_3^2 + \lambda_3^3 + \lambda_2^1 + \lambda_2^2 + \lambda_2^3 \leq 1,$$

which is valid for P_{SC} and cuts off $\tilde{\lambda}$. Finally, we invoke Theorem 10 with $C = \{1,3,4\}$ and $l_1 = l_3 = l_4 = 1$ to obtain

$$\lambda_1^1 + \lambda_1^2 + \lambda_1^3 + \lambda_3^1 + \lambda_3^2 + \lambda_3^3 + \lambda_4^1 + \lambda_4^2 + \lambda_4^3 + \lambda_2^3 + \leq 2$$

(here $\bar{C} = \{2\}$ and $t_2 = 3$), which is valid for P_{SC} and cuts off $\hat{\lambda}$. □

As seen in the above example, by adding the artificial variables $\bar{\lambda}_j$, new elements may be added to the cover (as defined in Definition 3) in case $k_j^* = 1$ for some j. Before giving the new lifted cover inequalities, we then adapt the definition of cover accordingly.

Definition 5 *Let $C \subseteq N$ and $l_j \in \{2, \ldots, T\} \cup \{k_j^*\} \; \forall j \in C$ be such that*

$$\sum_{j \in C} a_j^{l_j} = b + \rho$$

with $\rho > 0$. The set C is a cover. □

Because the proofs of Theorems 14 and 15 follow essentially the ideas previously discussed in this section, we omit their proofs.

Theorem 14 *Let C_1, C_2, l_j, α_j, β_j, and γ_j be as in Theorem 3. Let $C_2^* = \{j \in C_2 : l_j = k_j^*\}$. Then,*

$$\sum_{j \in C_1} (\alpha_j \lambda_j^{l_j - 2} + \beta_j \lambda_j^{l_j - 1} + \sum_{j = l_j}^{T} \lambda_j^{l_j}) + \sum_{j \in C_2^*} \sum_{j = l_j}^{T} \lambda_j^{l_j} + \sum_{j \in C_2 - C_2^*} (\gamma_j \lambda_j^{l_j - 1} + \sum_{j = l_j}^{T} \lambda_j^{l_j}) \leq |C| - 1 \quad (69)$$

is valid for P_{SC}. □

Theorem 15 *Let $C^* = \{j \in C : l_j = k_j^*\}$, $a_C = max\ \{a_i^{l_i} : \forall i \in C\}$, $\bar{C} = \{j \in N - C : a_j^T \geq a_C$ and $a_j^{T-1} \geq a_C - \rho\}$, and $t_j = min\ \{k : a_j^k \geq a_C$ and $a_j^{k-1} \geq a_C - \rho\}$ $\forall j \in \bar{C}$. Suppose that $\bar{C} \neq \emptyset$. The lifted cover inequality*

$$\sum_{j \in C^*} \sum_{k = l_j}^{T} \lambda_j^k + \sum_{j \in C - C^*} (\gamma_j \lambda_j^{l_j - 1} + \sum_{k = l_j}^{T} \lambda_j^k) + \sum_{j \in \bar{C}} \sum_{k = t_j}^{T} \lambda_j^k \leq |C| - 1 \quad (70)$$

is valid for P_{SC}. □

6 Further Research

The most obvious direction to be taken at this point is to verify whether the cuts presented here are computationally efficient in solving SPLO by branch-and-cut. This is done in [12]. Another obvious direction is to examine further the inequality description of P. Additionally, we believe it is worth investigating the polytope of non-separable piecewise linear optimization, see D'Ambrosio et al. [7] and Fügenschuh et al. [17] for recent work on nonseparable piecewise linear optimization.

As mentioned in Section 5, in many applications piecewise linear optimization arises with other combinatorial constraints. Here we gave initial results for when semi-continuous constraints are present. Much more needs to be investigated in this direction, even in the case of the SPLO knapsack polytope. For example, we could have considered semi-continuous constraints on Theorem 11, i.e. for the case $N^- \neq \emptyset$, just by extending the results of Section 5 (that considered the case $N^- = \emptyset$) to it in the obvious way. However, we believe that this subject deserves a study deeper than that. Finally, due to its frequent presence in applications and difficulty in being handled computationally, we suggest studying *cardinality constraints* [4, 15] in the context of piecewise linear optimization.

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