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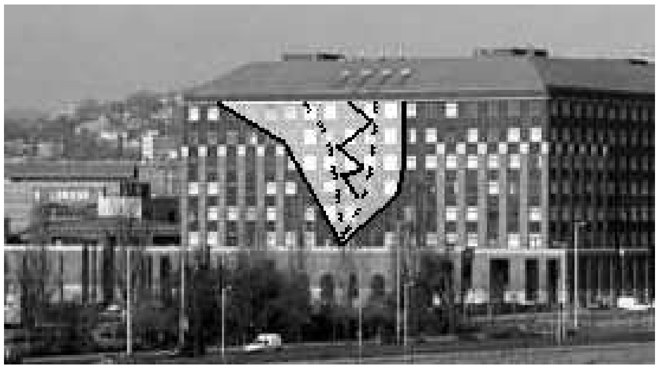

Applications of the inverse theta number in stable set problems

Miklós Ujvári

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Eötvös Loránd University of Sciences
Department of Operations Research

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Abstract

Recently the author introduced a semidefinite upper bound on the square of the stability number of a graph, the inverse theta number, which is proved here to be multiplicative with respect to the strong graph product, hence to be an upper bound for the square of the Shannon capacity of the graph. We also describe a heuristic algorithm for the stable set problem based on semidefinite programming, Cholesky factorization, and eigenvector computation.

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1 Introduction

An algorithm for the stable set problem is useful in many ways, e.g. it can be used for colouring a graph: find a stable set, remove it from the graph, and iterate the algorithm. (See [2] for further applications and approximation algorithms for the stable set problem.) The strength of the semidefinite programming approach for the stable set and colouring problems is shown by the algorithms of Grötschel–Lovász–Schrijver, Karger–Motwani–Sudan, and Alon–Kahale, see [4] for a summary of these results. In this paper we will describe a heuristic algorithm for the stable set problem based on semidefinite optimization. In contrast with the Karger–Motwani–Sudan and Alon–Kahale algorithms our algorithm does not use random vectors and its detailed analysis is lacking.

We start the paper with stating the main results. First we fix some notation.

Let $n \in \mathcal{N}$, and let $G = (V(G), E(G))$ be an undirected graph, with vertex set $V(G) = \{1, \dots, n\}$, and with edge set $E(G) \subseteq \{\{i,j\} : i \neq j\}$. Let $A(G)$

be the 0-1 adjacency matrix of the graph G, that is let

$$A(G) := (a_{ij}) \in \{0,1\}^{n \times n}, \text{ where } a_{ij} := \begin{cases} 0, & \text{if } \{i,j\} \notin E(G), \\ 1, & \text{if } \{i,j\} \in E(G). \end{cases}$$

The complementer graph $\overline{G}$ is the graph with adjacency matrix $A(\overline{G}) := J - I - A(G)$, where I is the identity matrix, and J denotes the matrix with all elements equal to one. The disjoint union of the graphs G_1 and G_2 is the graph $G_1 + G_2$ with adjacency matrix

$$A(G_1 + G_2) := \begin{pmatrix} A(G_1) & 0 \\ 0 & A(G_2) \end{pmatrix}.$$

We will use the notation K_n for the clique graph (defined by $A(K_n) = J - I$), and $K_{s_1,\dots,s_k}$ for the complete multipartite graph $\overline{K_{s_1} + \ldots + K_{s_k}}$. Also, we will denote by C_n the n-cycle, the polygon graph with n vertices.

By Rayleigh's Theorem (see [8]) for a symmetric matrix $M = M^T \in \mathcal{R}^{n \times n}$ the minimum and maximum eigenvalue, λ_M, resp. Λ_M can be expressed as

$$\lambda_M = \min_{||u||=1} u^T M u, \ \Lambda_M = \max_{||u||=1} u^T M u.$$

Attainment occurs if and only if $u \in \mathcal{R}^n$ is a unit eigenvector corresponding to λ_M and Λ_M, respectively.

The set of the n by n real symmetric positive semidefinite matrices will be denoted by $\mathcal{S}^n_+$, that is

$$\mathcal{S}^n_+ := \left\{ M \in \mathcal{R}^{n \times n} : M = M^T, \, u^T M u \geq 0 \ (u \in \mathcal{R}^n) \right\}.$$

It is well-known (see [8]), that the following statements are equivalent for a symmetric matrix $M = (m_{ij}) \in \mathcal{R}^{n \times n}$: a) $M \in \mathcal{S}^n_+$; b) $\lambda_M \geq 0$; c) M is Gram matrix, that is $m_{ij} = v_i^T v_j$ $(i,j = 1,\dots,n)$ for some vectors $v_1, \dots, v_n$. Furthermore, by Lemma 2.1 in [11], the set $\mathcal{S}^n_+$ can be described as

$$\mathcal{S}^n_+ = \left\{ \left(\frac{a_i^T a_j}{(a_i a_j^T)_{11}} - 1 \right)_{i,j=1}^n \;\middle|\; \begin{array}{l} d \in \mathcal{N}, \, a_i \in \mathcal{R}^d \ (1 \leq i \leq n) \\ a_i^T a_i = 1 \ (1 \leq i \leq n) \end{array} \right\}. \quad (1)$$

The stability number, $\alpha(G)$, is the maximum cardinality of the (so-called stable) sets $S \subseteq V(G)$ such that $\{i,j\} \subseteq S$ implies $\{i,j\} \notin E(G)$. The chromatic number, $\chi(G)$, is the minimum number of stable sets covering the vertex set $V(G)$.

In the seminal paper [5] L. Lovász proved the following result, now popularly called *sandwich theorem*:

$$\alpha(G) \le \vartheta(G) \le \chi(\overline{G}),$$

where $\vartheta(G)$ is the *Lovász number* of the graph G, defined as

$$\vartheta(G) := \inf \left\{ \max_{1 \le i \le n} \frac{1}{(a_i a_i^T)_{11}} \;\middle|\; \begin{array}{l} d \in \mathcal{N},\, a_i \in \mathcal{R}^d \ (i = 1, \dots, n) \\ a_i^T a_i = 1 \ (i = 1, \dots, n) \\ a_i^T a_j = 0 \ (\{i,j\} \in E(\overline{G})) \end{array} \right\}.$$

The feasible solutions (a_i) of the program defining $\vartheta(G)$ are called the *orthonormal representation*s of the graph G.

The Lovász number has several equivalent descriptions, see [5]. For example, by (1) and standard semidefinite duality theory (see e.g. [10]), it is the common optimal value of the Slater-regular primal-dual semidefinite programs

$$(TP) \qquad \min \lambda, \left\{ \begin{array}{l} x_{ii} = \lambda - 1 \ (i \in V(G)), \\ x_{ij} = -1 \ (\{i,j\} \in E(\overline{G})), \\ X = (x_{ij}) \in \mathcal{S}_+^n,\ \lambda \in \mathcal{R} \end{array} \right.$$

and

$$(TD) \qquad \max \operatorname{tr}(JY), \left\{ \begin{array}{l} \operatorname{tr}(Y) = 1, \\ y_{ij} = 0 \ (\{i,j\} \in E(G)), \\ Y = (y_{ij}) \in \mathcal{S}_+^n. \end{array} \right.$$

(Here tr stands for trace.) Reformulating the program (TD), Lovász derived the following description of the theta number (Theorem 5 in [5]):

$$\vartheta(G) = \sup \left\{ \sum_{i=1}^n (b_i b_i^T)_{11} \;\middle|\; \begin{array}{l} d \in \mathcal{N},\, b_i \in \mathcal{R}^d \ (i = 1, \dots, n) \\ b_i^T b_i = 1 \ (i = 1, \dots, n) \\ b_i^T b_j = 0 \ (\{i,j\} \in E(G)) \end{array} \right\}.$$

An important application of the theory of the theta number is described in Theorem 1 of [5], where it is proved that

$$\Theta(G) \le \vartheta(G),$$

with $\Theta(G)$ denoting the *Shannon capacity* of the graph, that is

$$\Theta(G) := \sup_{k \in \mathcal{N}} \sqrt[k]{\alpha(G^k)}.$$

(Here $G \cdot H$ denotes the strong graph product of the graphs G, H, the graph with vertex set

$$V(G \cdot H) := \{(i,j) : i \in V(G),\ j \in V(H)\}$$

and edge set

$$E(G \cdot H) := \left\{ \{(i_1,j_1),(i_2,j_2)\} \;\middle|\; \begin{array}{l} i_1 = i_2 \text{ or } \{i_1,i_2\} \in E(G) \\ j_1 = j_2 \text{ or } \{j_1,j_2\} \in E(H) \end{array} \right\}.$$

Also, G^k denotes the strong graph product of k copies of the graph G.)

Analogously, the *inverse theta number*, $\iota(G)$ of a graph G, defined in [12] as

$$\iota(G) := \inf \left\{ \sum_{i=1}^n \frac{1}{(a_i a_i^T)_{11}} \;\middle|\; \begin{array}{l} d \in \mathcal{N},\, a_i \in \mathcal{R}^d \ (i = 1, \dots, n) \\ a_i^T a_i = 1 \ (i = 1, \dots, n) \\ a_i^T a_j = 0 \ (\{i,j\} \in E(\overline{G})) \end{array} \right\},$$

equals the common optimal value of the Slater-regular primal-dual semidefinite programs

$$(TP^-) \qquad \min \operatorname{tr}(Z) + n,\ z_{ij} = -1 \ (\{i,j\} \in E(\overline{G})),\ Z = (z_{ij}) \in \mathcal{S}_+^n,$$

$$(TD^-) \qquad \max \operatorname{tr}(JM), \left\{ \begin{array}{l} m_{ii} = 1 \ (i = 1, \dots, n), \\ m_{ij} = 0 \ (\{i,j\} \in E(G)), \\ M = (m_{ij}) \in \mathcal{S}_+^n. \end{array} \right.$$

Moreover, rewriting the feasible solution M of the program (TD^-) as the Gram matrix $M = (b_i^T b_j)$ for some vectors $b_1, \dots, b_n \in \mathcal{R}^d$, we obtain the equivalent description

$$\iota(G) = \sup \left\{ \sum_{i,j=1}^n b_i^T b_j \;\middle|\; \begin{array}{l} d \in \mathcal{N},\, b_i \in \mathcal{R}^d \ (i = 1, \dots, n) \\ b_i^T b_i = 1 \ (i = 1, \dots, n) \\ b_i^T b_j = 0 \ (\{i,j\} \in E(G)) \end{array} \right\}. \qquad (2)$$

For the inverse theta number the inequality

$$\alpha(G) \le \sqrt{\iota(G)}$$

holds (see [12], Theorem 2.2) as an analogue of Lovász's sandwich theorem. In Section 2 we will prove also the stronger relation

$$\Theta(G) \le \sqrt{\iota(G)}.$$

It is known (see Proposition 2.2 in [12]) that e.g. for the cycle graphs C_n, $\sqrt{\iota(C_n)} > \vartheta(C_n)$ holds. Hence, the inverse theta number does not help in determining the Shannon capacity of the odd cycles $C_7, C_9, \dots$, which is still an open problem, though, using the theta number, Lovász determined the Shannon capacity of the 5-cycle and other graphs in [5]. However, we will see in Section 3, that orthonormal representations of the complementer graph $\overline{G}$ of high value in the dual description (2) of the inverse theta number, can be of use in a heuristic algorithm calculating large stable sets in any graph G.

2 Shannon capacity

In this section we will prove that the inverse theta function has the same multiplicativity properties as the theta function, consequently its square root is an upper bound for the Shannon capacity of the graph.

First, we will verify the submultiplicativity of the inverse theta function, an analogue of Lemma 2 in [5].

LEMMA 2.1. *For any graphs G, H, $\iota(G \cdot H) \le \iota(G) \cdot \iota(H)$.*

Proof. Let (a_i^G) and (a_j^H) be orthonormal representations of the graphs G and H, respectively. Then, by Lemma 1 in [5], $(a_i^G \otimes a_j^H)$ is an orthonormal representation of the graph $G \cdot H$. (Here $\otimes$ denotes Kronecker product of matrices, see [7] for the definition.) Thus,

$$\begin{aligned} \iota(G \cdot H) &\le \sum_{i,j} 1/((a_i^G \otimes a_j^H)(a_i^G \otimes a_j^H)^T)_{11} \\ &= \sum_i 1/(a_i^G a_i^{GT})_{11} \cdot \sum_j 1/(a_j^H a_j^{HT})_{11}, \end{aligned}$$

and, taking infimum in (a_i^G) and (a_j^H), we have the statement. □

Now, we will prove the skew-supermultiplicativity of the inverse theta function.

LEMMA 2.2. *For any graphs G, H, $\iota(\overline{\overline{G} \cdot \overline{H}}) \ge \iota(G) \cdot \iota(H)$.*

Proof. Let (b_i^G) and (b_j^H) be orthonormal representations of the complementer graphs $\overline{G}$ and $\overline{H}$, respectively. Then, by Lemma 1 in [5], $(b_i^G \otimes b_j^H)$ is an orthonormal representation of the graph $\overline{G} \cdot \overline{H}$. Thus, by (2),

$$\begin{aligned} \iota(\overline{\overline{G} \cdot \overline{H}}) &\ge \sum_{i_1,i_2,j_1,j_2} (b_{i_1}^G \otimes b_{j_1}^H)^T (b_{i_2}^G \otimes b_{j_2}^H) \\ &= \sum_{i_1,i_2} b_{i_1}^{GT} b_{i_2}^G \cdot \sum_{j_1,j_2} b_{j_1}^{HT} b_{j_2}^H, \end{aligned}$$

and, taking supremum in (b_i^G) and (b_j^H), the statement is proved. □

Summarizing, we obtain the following analogue of Theorem 7 in [5].

THEOREM 2.1. *The inequalities in Lemmas 2.1 and 2.2 hold with equalities: for any graphs G, H,*

a) $\iota(G \cdot H) = \iota(G) \cdot \iota(H)$;

b) $\iota(\overline{\overline{G} \cdot \overline{H}}) = \iota(G) \cdot \iota(H)$.

Proof. It is enough to notice that the graph $G \cdot H$ is a subgraph of $\overline{\overline{G} \cdot \overline{H}}$, so

$$\iota(G \cdot H) \ge \iota(\overline{\overline{G} \cdot \overline{H}}).$$

Applying Lemmas 2.1 and 2.2, the proof is completed. □

A submultiplicative upper bound for the stability number of a graph is also an upper bound for the Shannon capacity of the graph, see Theorem 1 in [5]. Consequently,

THEOREM 2.2. *For any graph G, $\Theta(G) \le \sqrt{\iota(G)}$ holds.*

Proof. By Theorem 2.2 in [12], for any graph G, $\alpha(G) \le \sqrt{\iota(G)}$. Hence, from Lemma 2.1,

$$\alpha(G^k) \le \sqrt{\iota(G^k)} \le \left(\sqrt{\iota(G)}\right)^k$$

follows for $k \in \mathcal{N}$; the proof is finished. □

Summarizing Theorem 1 in [5] and Theorem 2.2 we obtain

$$\Theta(G) \le \min\left\{\vartheta(G), \sqrt{\iota(G)}\right\}.$$

Can $\sqrt{\iota(G)}$ be less than $\vartheta(G)$ for some graph G? Juhász's Theorem (see [3]) states that $\vartheta(G)$ is typically "around" $n^{1/2}$ in the following sense:

THEOREM 2.3. (Juhász) *Let G be a random graph with edge probability $p = 1/2$. Then, with probability $1 - o(1)$ for $n \to \infty$,*

$$\frac{1}{2}\sqrt{n} + O(n^{1/3}\log n) \le \vartheta(G) \le 2\sqrt{n} + O(n^{1/3}\log n).$$

Hence, the value $\sqrt{\iota(G)}$ (which is between $n \Big/ \sqrt{\vartheta(\overline{G})}$ and $\sqrt{n\vartheta(G)}$ by Proposition 2.2 in [12]) is typically "around" $n^{3/4}$. Thus, the graphs G, with $\sqrt{\iota(G)} < \vartheta(G)$, if they exist at all, are rare. However, we will see in the following section, that the fact that $\iota(G)$ with high probability is large, can be an advance, too.

We conclude this section with an open problem: With minor modification of the proof of Theorem 2.2 in [12] it can be proved that

$$\alpha(G)^2 \le \iota(G) - n + \alpha(G).$$

From this inequality we obtain the bound

$$\alpha(G) \le \frac{1}{2}\left(1 + \sqrt{4(\iota(G) - n) + 1}\right), \tag{3}$$

which is tighter than $\alpha(G) \le \sqrt{\iota(G)}$. It is an open problem, whether the bound in (3) is submultiplicative (and, thus, is an upper bound for the Shannon capacity $\Theta(G)$), or not.

3 Heuristic algorithm

In this section we will describe a heuristic algorithm for the stable set problem.

The key observation for the algorithm is the following simple

LEMMA 3.1. *Let the vectors $b_1, \ldots, b_n \in \mathcal{R}^d$ form an orthonormal representation of the complementer graph $\overline{G}$, and let $u \in \mathcal{R}^d$, $u^T u = 1$. Then,*

$$S := \left\{ i \in \{1, \ldots, n\} : (u^T b_i)^2 > \frac{1}{2} \right\} \tag{4}$$

is a stable set in the graph G.

Proof. Let us suppose indirectly that for some $i, j \in S$, $\{i, j\} \in E(G)$. Then, as $(b_1, \ldots, b_n)$ is an orthonormal representation of $\overline{G}$, so $b_i^T b_j = 0$, and $||b_i + b_j|| = \sqrt{2}$. By $i, j \in S$, we have $(u^T b_i)^2 > 1/2 < (u^T b_j)^2$. Let us consider for example the case when $u^T b_i > \sqrt{2}/2 < u^T b_j$. Then,

$$\sqrt{2} < u^T(b_i + b_j) \le ||u|| \cdot ||b_i + b_j|| = \sqrt{2},$$

which is a contradiction. The cases, when $u^T b_i < -\sqrt{2}/2$ or $u^T b_j < -\sqrt{2}/2$ can be dealt with similarly. This completes the proof. □

Taking into account Lemma 3.1 we could search for large stable sets as follows: We compute an orthonormal representation (b_i) of the complementer graph $\overline{G}$ and a unit vector u so that $\sum_i (u^T b_i)^2$ is maximal. The output stable set S will be the one in (4).

Unfortunately, this idea leads to an untractable procedure. Really, it follows from Rayleigh's Theorem that finding such a representation (b_i) and vector u means solving the programs

$$(P_d) \quad \sup \Lambda_{BB^T}, \begin{cases} (B^T B)_{ii} = 1 \ (i = 1, \ldots, n) \\ (B^T B)_{ij} = 0 \ (\{i, j\} \in E(G)), \end{cases}$$

where $B = (b_1, \ldots, b_n) \in \mathcal{R}^{d \times n}$. In other words, using the obvious equality $\Lambda_{BB^T} = \Lambda_{B^T B}$ and the variable transformation $M = B^T B$, we have to solve the program

$$(P) \quad \sup \Lambda_M, \begin{cases} m_{ii} = 1 \ (i = 1, \ldots, n) \\ m_{ij} = 0 \ (\{i, j\} \in E(G)) \\ M = (m_{ij}) \in \mathcal{S}_+^n. \end{cases}$$

As the maximum eigenvalue function $M \mapsto \Lambda_M$ is convex (a well-known, simple consequence of Rayleigh's Theorem), so the program (P) means convex maximization, which is not tractable. (Though the optimal value of program (P) is known, it is $\vartheta(G)$ by Theorem 5 in [5].)

Hence, we modify the original idea, and instead of (P) we solve the program (TD^-) for M, and from this matrix we compute B, u and the stable set S. The algorithm derived this way is as follows:

Algorithm: 1. Solve to optimality (or with $\varepsilon > 0$ additive error) the program (TD^-). Denote the solution by M^*. (The ε-optimal solution M^* can be determined in polynomial time using interior-point methods for semidefinite optimization, see e.g. [6], [1], [9].)

2. Determine a matrix $B = (b_1, \ldots, b_n) \in \mathcal{R}^{d \times n}$ such that $M^* = B^T B$. (An appropriate matrix B can be determined in polynomial time using algorithms from [8], e.g. Cholesky factorization.)

3. Compute a vector $u \in \mathcal{R}^d$, $u^T u = 1$ such that $\Lambda_{BB^T} = u^T BB^T u$ holds. In other words compute a unit eigenvector of the matrix BB^T corresponding to its maximum eigenvalue Λ_{BB^T}. (This can be accomplished in polynomial time using algorithms from [8].)

4. The output stable set is S in (4).

We have some evidence that our algorithm finds large stable sets. Note that the following theorem implies, by Juhász's Theorem, that $\sum_i (u^T b_i)^2$ is typically "around" $\sqrt{n}$ for the modified algorithm, similarly as in the case of its original, untractable version.

THEOREM 3.1. *The algorithm described above computes an orthonormal representation $B = (b_1, \ldots, b_n) \in \mathcal{R}^{d \times n}$ of the complementer graph $\overline{G}$, and a* *unit vector $u \in \mathcal{R}^d$ such that the inequalities*

$$\vartheta(G) \geq \sum_{i=1}^{n}(u^T b_i)^2 \geq \frac{\iota(G)}{n} \geq \frac{n}{\vartheta(\overline{G})}$$

hold.

Proof. The first inequality is the immediate consequence of Theorem 5 in [5]. Let us prove the second inequality. Obviously,

$$\sum_{i=1}^{n}(u^T b_i)^2 = \Lambda_{BB^T} = \Lambda_{B^T B} = \Lambda_{M^*}.$$

On the other hand, by Rayleigh's Theorem,

$$\Lambda_{M^*} \geq \frac{\mathbf{1}^T}{\sqrt{n}} M^* \frac{\mathbf{1}}{\sqrt{n}} = \frac{\mathrm{tr}\,(JM^*)}{n} = \frac{\iota(G)}{n},$$

where $\mathbf{1}$ denotes the n-vector with all elements equal to one. This way we have verified the inequality $\sum_i (u^T b_i)^2 \geq \iota(G)/n$. Finally, the last inequality follows from Proposition 2.2 in [12]. □

Note that the following corollary of Theorem 3.1 implies the relation

$$\alpha(G) \geq \frac{2\iota(G)}{n} - n.$$

COROLLARY 3.1. *The algorithm finds a stable set S with cardinality $|S| \geq (2\iota(G)/n) - n$.*

Proof. The statement is an easy consequence of the inequality

$$\sum_{i \in S}(u^T b_i)^2 + \sum_{i \notin S}(u^T b_i)^2 \geq \frac{\iota(G)}{n},$$

as for $i \notin S$ we have $(u^T b_i)^2 \leq 1/2$ by the definition of the stable set S in (4). □

We conclude this section with a simple example. Let us consider the graph $G = K_{s_1,\ldots,s_k}$. Then, the output matrix M^* (the optimal solution of the program (TD^-)) is the block-diagonal matrix made up of the matrices $J \in \mathcal{R}^{s_1 \times s_1}$, ..., $J \in \mathcal{R}^{s_k \times s_k}$ as diagonal blocks, zero otherwise. The matrix

$B \in \mathcal{R}^{k \times n}$ such that $M^* = B^T B$ is made up of the column vectors of the identity matrix $I \in \mathcal{R}^{k \times k}$ with multiplicity $s_1, \ldots, s_k$, respectively. Then, $BB^T \in \mathcal{R}^{k \times k}$ is the diagonal matrix with diagonal elements $s_1, \ldots, s_k$. Let us suppose that $s_1 \geq s_2, \ldots, s_k$. Then, the vector $u \in \mathcal{R}^k$ equals the first column vector of the identity matrix $I \in \mathcal{R}^{k \times k}$; and $S = \{1, \ldots, s_1\}$ is the output stable set.

We can see that our heuristic algorithm in the case of the graph $G = K_{s_1,\ldots,s_k}$ finds a maximum stable set (and, iterating the algorithm, we obtain a minimum colouring). Generally, estimating from below the factor of the algorithm, the infimum ratio of the cardinality of the output stable set and the stability number for a graph with n vertices, is an unsolved problem.

Conclusion. In this paper we studied the multiplicativity properties of the inverse theta function, and as a consequence we proved that the square root of this function is an upper bound for the Shannon capacity of the graph. Though the square root of the inverse theta number, as compared to Lovász's theta number, is typically a weak upper bound, this fact could be exploited in a heuristic algorithm for the stable set problem.

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Miklós Ujvári
H-2600 Vác, Szent János utca 1. HUNGARY

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