On global optimizations of the rank and inertia of the matrix function $A_1 - B_1XB_1^*$ subject to a pair of matrix equations $[\, B_2XB_2^*,\, B_3XB_3^* \,] = [\, A_2,\, A_3 \,]$

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Abstract. For a given linear matrix function $A_1 - B_1XB_1^*$, where X is a variable Hermitian matrix, this paper derives a group of closed-form formulas for calculating the global maximum and minimum ranks and inertias of the matrix function subject to a pair of consistent matrix equations $B_2XB_2^* = A_2$ and $B_3XB_3^* = A_3$. As applications, we give necessary and sufficient conditions for the triple matrix equations $B_1XB_1^* = A_1$, $B_2XB_2^* = A_2$ and $B_3XB_3^* = A_3$ to have a common Hermitian solution. In addition, we discuss the global optimizations on the rank and inertia of the common Hermitian solution of the pair of matrix equations $B_2XB_2^* = A_2$ and $B_3XB_3^* = A_3$.

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1 Introduction

Throughout this paper, $\mathbb{C}^{m\times n}$ and $\mathbb{C}_{\rm H}^{m}$ stand for the sets of all $m \times n$ complex matrices and all $m \times m$ complex Hermitian matrices, respectively. The symbols A^T, A^*, $r(A)$, $\mathscr{R}(A)$ and $\mathscr{N}(A)$ stand for the transpose, conjugate transpose, rank, range (column space) and null space of a matrix $A \in \mathbb{C}^{m\times n}$, respectively; I_m denotes the identity matrix of order m; $[\, A,\, B \,]$ denotes a row block matrix consisting of A and B. We write $A > 0$ ($A \geqslant 0$) if A is Hermitian positive definite (nonnegative definite). Two Hermitian matrices A and B of the same size are said to satisfy the inequality $A > B$ ($A \geqslant B$) in the Löwner partial ordering if $A - B$ is positive definite (nonnegative definite). The Moore–Penrose inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A^{\dagger}$, is defined to be the unique solution X satisfying the four matrix equations

$$\text{(i) } AXA = A, \quad \text{(ii) } XAX = X, \quad \text{(iii) } (AX)^* = AX, \quad \text{(iv) } (XA)^* = XA.$$

If X satisfies (i), it is called a g-inverse of A and is denoted by A^-. A matrix X is called a Hermitian g-inverse of $A \in \mathbb{C}_{\rm H}^{m}$, denoted by $A^{\sim}$, if it satisfies both $AXA = A$ and $X = X^*$. Further, the symbols E_A and F_A stand for the two orthogonal projectors $E_A = I_m - AA^{\dagger}$ and $F_A = I_n - A^{\dagger}A$. The ranks of E_A and F_A are given by $r(E_A) = m - r(A)$ and $r(F_A) = n - r(A)$. A well-known property of the Moore–Penrose inverse is $(A^{\dagger})^* = (A^*)^{\dagger}$. In addition, $AA^{\dagger} = A^{\dagger}A$ if $A = A^*$. We shall repeatedly use them in the latter part of this paper. The inertia of A is defined to be the triplet $\operatorname{In}(A) = \{\, i_+(A),\, i_-(A),\, i_0(A) \,\}$, where $i_+(A)$, $i_-(A)$ and $i_0(A)$ are the numbers of the positive, negative and zero eigenvalues of A counted with multiplicities, respectively. Both $i_+(A)$ and $i_-(A)$, usually called the partial inertia, can easily be computed by elementary congruence matrix operations. For a Hermitian matrix A, the equality $r(A) = i_+(A) + i_-(A)$ holds.

In recent years, some optimization problems on the global maximum and minimum ranks and inertias of the following linear matrix functions (LMFs)

$$A - BXB^*, \quad A - BX - (BX)^*, \quad A - BXC - (BXC)^*, \tag{1.1}$$

and their applications were studied, where A is a given complex Hermitian matrix, B and C are given complex matrices, and X is a variable matrix of appropriate size; see [14, 15, 16, 17, 18, 28, 31, 32, 33]. As a generalization, we address in this paper the following optimization problem:

Problem 1.1 *For a given LMF $A_1 - B_1XB_1^*$ and a pair of matrix equations $[\, B_2XB_2^*,\, B_3XB_3^* \,] = [\, A_2,\, A_3]$ that have a a common Hermitian solution, give formulas for calculating the following global*

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maximum and minimum ranks and inertias

$$\max_{X \in \mathbb{C}_{\mathrm{H}}^{n}} r(A_1 - B_1XB_1^{*}) \quad s.t. \ [B_2XB_2^{*}, \, B_3XB_3^{*}] = [A_2, \, A_3], \tag{1.2}$$

$$\min_{X \in \mathbb{C}_{\mathrm{H}}^{n}} r(A_1 - B_1XB_1^{*}) \quad s.t. \ [B_2XB_2^{*}, \, B_3XB_3^{*}] = [A_2, \, A_3], \tag{1.3}$$

$$\max_{X \in \mathbb{C}_{\mathrm{H}}^{n}} i_{\pm}(A_1 - B_1XB_1^{*}) \quad s.t. \ [B_2XB_2^{*}, \, B_3XB_3^{*}] = [A_2, \, A_3], \tag{1.4}$$

$$\min_{X \in \mathbb{C}_{\mathrm{H}}^{n}} i_{\pm}(A_1 - B_1XB_1^{*}) \quad s.t. \ [B_2XB_2^{*}, \, B_3XB_3^{*}] = [A_2, \, A_3]. \tag{1.5}$$

We shall use some pure algebraic operations on matrices to derive a group of analytical formulas for calculating the global maximum and minimum values of the objective functions in (1.2)–(1.5), and then to present a variety of valuable consequences of these formulas. In particular, we shall use the minimum rank formula for (1.3) to derive necessary and sufficient conditions for the triple matrix equations

$$B_1XB_1^{*} = A_1, \quad B_2XB_2^{*} = A_2, \quad B_3XB_3^{*} = A_3 \tag{1.6}$$

to have a common Hermitian solution.

The rank and inertia of a Hermitian matrix are two basic concepts in matrix theory for describing the dimension of the row/column vector space and the sign distribution of the eigenvalues of the matrix, which are well understood and are easy to compute by the well-known elementary or congruent matrix operations. These two quantities play an essential role in characterizing algebraic properties of Hermitian matrices. Because the rank and inertia of a matrix are finite nonnegative integers, the global maximum and minimum values of the rank and inertia of a matrix expression always exist no matter what the domains of variable entries in the matrix expression are given. The extremal ranks and inertias of a matrix expression can directly be used to characterize some fundamental algebraic properties of the matrix expression, for example, (I) the maximum and minimum dimensions of the row and column spaces of the matrix expression; nonsingularity of the matrix expression when it is square; (III) solvability of the corresponding matrix equation; (IV) rank, inertia and range invariance of the matrix expression; (V) definiteness of the matrix expression when it is Hermitian; etc.

Since variable entries in a matrix function are often regarded as continuous variables in some constrained sets, while the objective functions—the rank and inertia of the matrix function take values only from a finite set of nonnegative integers, Hence, (1.2)–(1.5) can be regarded as continuous-integer optimization problems subject to equality constraints. This kind of nonsmooth optimization problems cannot be solved by using various optimization methods for solving continuous or discrete cases. There is no rigorous mathematical theory for solving a general rank and inertia optimization problem due to the discontinuity and nonconvexity of rank and inertia of matrix. In fact, it has been realized that rank and inertia optimization problems have deep connections with computational complexity, are regarded as NP-hard in general settings; see, e.g., [1, 2, 3, 4, 6, 7, 8, 11, 20, 23, 25].

The following are some known results for ranks and inertias of matrices and their usefulness, which will be used in the latter part of this paper.

Lemma 1.2 ([28]) *Let $\mathcal{H}$ be a set consisting of Hermitian matrices over $\mathbb{C}_{\mathrm{H}}^{m}$. Then,*

(a) *$\mathcal{H}$ has a matrix $X > 0$ $(X < 0)$ if and only if* $\max_{X \in \mathcal{H}} i_{+}(X) = m$ $\left(\max_{X \in \mathcal{H}} i_{-}(X) = m \right)$.

(b) *All $X \in \mathcal{H}$ satisfy $X > 0$ $(X < 0)$ if and only if* $\min_{X \in \mathcal{H}} i_{+}(X) = m$ $\left(\min_{X \in \mathcal{H}} i_{-}(X) = m \right)$.

(c) *$\mathcal{H}$ has a matrix $X \geqslant 0$ $(X \leqslant 0)$ if and only if* $\min_{X \in \mathcal{H}} i_{-}(X) = 0$ $\left(\min_{X \in \mathcal{H}} i_{+}(X) = 0 \right)$.

(d) *All $X \in \mathcal{H}$ satisfy $X \geqslant 0$ $(X \leqslant 0)$ if and only if* $\max_{X \in \mathcal{H}} i_{-}(X) = 0$ $\left(\max_{X \in \mathcal{H}} i_{+}(X) = 0 \right)$.

Lemma 1.3 ([19]) *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times p}$ and $C \in \mathbb{C}^{q\times n}$. Then, the following rank expansion formulas hold*

$$r[\,A,\, B\,] = r(A) + r(E_AB) = r(B) + r(E_BA), \tag{1.7}$$

$$r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(CF_A) = r(C) + r(AF_C), \tag{1.8}$$

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r(E_BAF_C). \tag{1.9}$$

Three useful rank expansion formulas derived from (1.9) are

$$r\begin{bmatrix} A & B & 0 \\ C & 0 & P \end{bmatrix} = r(P) + r\begin{bmatrix} A & B \\ E_PC & 0 \end{bmatrix}, \tag{1.10}$$

$$r\begin{bmatrix} A & B \\ C & 0 \\ 0 & Q \end{bmatrix} = r(Q) + r\begin{bmatrix} A & BF_Q \\ C & 0 \end{bmatrix}, \tag{1.11}$$

$$r\begin{bmatrix} A & B & 0 \\ C & 0 & P \\ 0 & Q & 0 \end{bmatrix} = r(P) + r(Q) + r\begin{bmatrix} A & BF_Q \\ E_PC & 0 \end{bmatrix}. \tag{1.12}$$

We shall use them in Section 2 to simplify ranks of block matrices involving E_P and F_Q.

Lemma 1.4 ([28]) *Let $A \in \mathbb{C}_{\mathrm{H}}^{m}$, $B \in \mathbb{C}^{m\times n}$, $D \in \mathbb{C}^{m\times n}$, and let*

$$U = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \quad V = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix}.$$

Then, the inertias of U and V can be expanded as

$$i_{\pm}(U) = r(B) + i_{\pm}(E_BAE_B), \tag{1.13}$$

$$i_{\pm}(V) = i_{\pm}(A) + i_{\pm}\begin{bmatrix} 0 & E_AB \\ B^*E_A & D - B^*A^{\dagger}B \end{bmatrix}. \tag{1.14}$$

(a) *If $A \succcurlyeq 0$, then*

$$i_{+}(U) = r[\,A,\, B\,], \quad i_{-}(U) = r(B), \quad r(U) = r[\,A,\, B\,] + r(B). \tag{1.15}$$

(b) *If $A \preccurlyeq 0$, then*

$$i_{+}(U) = r(B), \quad i_{-}(U) = r[\,A,\, B\,], \quad r(U) = r[\,A,\, B\,] + r(B). \tag{1.16}$$

(c) *If $\mathscr{R}(B) \subseteq \mathscr{R}(A)$, then*

$$i_{\pm}(V) = i_{\pm}(A) + i_{\pm}(\,D - B^*A^{\dagger}B\,), \quad r(V) = r(A) + r(\,D - B^*A^{\dagger}B\,). \tag{1.17}$$

(d) *If $\mathscr{R}(B) \cap \mathscr{R}(A) = \{0\}$ and $\mathscr{R}(B^*) \cap \mathscr{R}(D) = \{0\}$, then*

$$i_{\pm}(V) = i_{\pm}(A) + i_{\pm}(D) + r(B), \quad r(V) = r(A) + 2r(B) + r(D). \tag{1.18}$$

Three expansion formulas derived from (1.13) are

$$i_{\pm}\begin{bmatrix} A & BF_P \\ F_PB^* & 0 \end{bmatrix} = i_{\pm}\begin{bmatrix} A & B & 0 \\ B^* & 0 & P^* \\ 0 & P & 0 \end{bmatrix} - r(P), \; r\begin{bmatrix} A & BF_P \\ F_PB^* & 0 \end{bmatrix} = r\begin{bmatrix} A & B & 0 \\ B^* & 0 & P^* \\ 0 & P & 0 \end{bmatrix} - 2r(P). \tag{1.19}$$

We shall use them to simplify the inertias of block Hermitian matrices that involve $F_P = I - P^{\dagger}P$.

Lemma 1.5 *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}_{\mathrm{H}}^{m}$ be given. Then,*

(a) [5, 9] *The matrix equation* $AXA^* = B$ *has a solution* $X \in \mathbb{C}_{\rm H}^{n}$ *if and only if* $\mathscr{R}(B) \subseteq \mathscr{R}(A)$, *or equivalently,* $AA^{\dagger}B = B$.

(b) [28] *Under* $AA^{\dagger}B = B$, *the general Hermitian solution of* $AXA^* = B$ *can be written in the following two forms*

$$X = A^{\dagger}B(A^{\dagger})^* + U - A^{\dagger}AUA^{\dagger}A, \tag{1.20}$$

$$X = A^{\dagger}B(A^{\dagger})^* + F_AV + V^*F_A, \tag{1.21}$$

where $U \in \mathbb{C}_{\rm H}^{n}$ *and* $V \in \mathbb{C}^{n\times n}$ *are arbitrary.*

More results on properties of solutions of $AXA^* = B$ can be found in [15, 18].

Lemma 1.6 *Let* $A_j \in \mathbb{C}^{m_j\times n}$, $B_j \in \mathbb{C}^{p\times q_j}$ *and* $C_j \in \mathbb{C}^{m_j\times q_j}$ *be given,* $j = 1, 2$. *Then,*

(a) [24] *The pair of matrix equations*

$$A_1XB_1 = C_1 \quad and \quad A_2XB_2 = C_2 \tag{1.22}$$

have a common solution for $X \in \mathbb{C}^{n\times p}$ *if and only if*

$$\mathscr{R}(C_j) \subseteq \mathscr{R}(A_j), \ \ \mathscr{R}(C_j^*) \subseteq \mathscr{R}(B_j^*), \ \ r\begin{bmatrix} C_1 & 0 & A_1 \\ 0 & -C_2 & A_2 \\ B_1 & B_2 & 0 \end{bmatrix} = r\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} + r[\, B_1, \, B_2\,], \quad j = 1, \, 2. \tag{1.23}$$

(b) [27] *Under* (1.27), *the general common solution to* (1.26) *can be written in the following parametric form*

$$X = X_0 + F_AV_1 + V_2E_B + F_{A_1}V_3E_{B_2} + F_{A_2}V_4E_{B_1}, \tag{1.24}$$

where $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$, $B = [\, B_1, \, B_2\,]$, *and the four matrices* $V_1, \dots, V_4 \in \mathbb{C}^{n\times p}$ *are arbitrary.*

Lemma 1.7 ([17]) *Let* $A \in \mathbb{C}_{\rm H}^{m}$, $B \in \mathbb{C}^{m\times p}$ *and* $C \in \mathbb{C}^{q\times m}$ *be given, and let*

$$M_1 = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} A & C^* \\ C & 0 \end{bmatrix}, \tag{1.25}$$

$$N = [\, A, \, B, \, C^*\,], \quad N_1 = \begin{bmatrix} A & B & C^* \\ B^* & 0 & 0 \end{bmatrix}, \quad N_2 = \begin{bmatrix} A & B & C^* \\ C & 0 & 0 \end{bmatrix}. \tag{1.26}$$

Then, the global maximum and minimum rank and inertias of $A - BXC - (BXC)^*$ *are given by*

$$\max_{X\in\mathbb{C}^{p\times q}} r[\, A - BXC - (BXC)^*\,] = \min\{r(N), \ \ r(N_1), \ \ r(N_2)\}, \tag{1.27}$$

$$\min_{X\in\mathbb{C}^{p\times q}} r[\, A - BXC - (BXC)^*\,] = 2r(N) + \max\{\, s_1, \ \ s_2, \ \ s_3, \ \ s_4\,\}, \tag{1.28}$$

$$\max_{X\in\mathbb{C}^{p\times q}} i_{\pm}[\, A - BXC - (BXC)^*\,] = \min\{i_{\pm}(M_1), \ \ i_{\pm}(M_2)\}, \tag{1.29}$$

$$\min_{X\in\mathbb{C}^{p\times q}} i_{\pm}[\, A - BXC - (BXC)^*\,] = r(N) + \max\{\, i_{\pm}(M_1) - r(N_1), \ \ i_{\pm}(M_2) - r(N_2)\,\}, \tag{1.30}$$

where

$$s_1 = r(M_1) - 2r(N_1), \quad s_2 = r(M_2) - 2r(N_2),$$

$$s_3 = i_{+}(M_1) + i_{-}(M_2) - r(N_1) - r(N_2), \quad s_4 = i_{+}(M_1) + i_{-}(M_2) - r(N_1) - r(N_2).$$

In particular, if $\mathscr{R}(C^*) \subseteq \mathscr{R}(B)$, *then*

$$\max_{X\in\mathbb{C}^{p\times q}} r[\, A - BXC - (BXC)^*\,] = \min\left\{ r[\, A, \, B\,], \ \ r\begin{bmatrix} A & C^* \\ C & 0 \end{bmatrix} \right\}, \tag{1.31}$$

$$\min_{X\in\mathbb{C}^{p\times q}} r[\, A - BXC - (BXC)^*\,] = 2r[\, A, \, B\,] + r\begin{bmatrix} A & C^* \\ C & 0 \end{bmatrix} - 2r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}, \tag{1.32}$$

$$\max_{X\in\mathbb{C}^{p\times q}} i_{\pm}[\, A - BXC - (BXC)^*\,] = i_{\pm}\begin{bmatrix} A & C^* \\ C & 0 \end{bmatrix}, \tag{1.33}$$

$$\min_{X\in\mathbb{C}^{p\times q}} i_{\pm}[\, A - BXC - (BXC)^*\,] = r[\, A, \, B\,] + i_{\pm}\begin{bmatrix} A & C^* \\ C & 0 \end{bmatrix} - r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}. \tag{1.34}$$

The matrices Xs that satisfy (1.27)–(1.30) (namely, the global maximizers and minimizers of the objective rank and inertia functions) are not necessarily unique and their expressions were also given in [17] by using certain simultaneous decomposition of the three given matrices. Observe that the right-hand sides of (1.27)–(1.30) are represented in analytical forms of the ranks and inertias of the five given block matrices, we can easily use them to derive extremal ranks and inertias of some general linear and nonlinear matrix functions. In these cases, combining the rank and inertia formulas obtained with the assertions in Lemma 1.1 may yield various conclusions on algebraic properties of linear and nonlinear matrix functions.

2 The global maximum and minimum ranks and inertias of $A_1 - B_1XB_1^*$ subject to a consistent matrix equation

Because of the noncommutativity of matrix multiplications, solving matrix equations has been a challenging topic of study in linear algebra. If some matrix equations of appropriate sizes are given together, it is natural to ask whether the equations have a possible common solution. For example, if two Hermitian matrix equations $B_1X_1B_1^* = A_1$ and $B_2X_2B_2^* = A_2$, are given, where X_1 and X_2 have the same size, and each of them has a solution. In such a case, it would be of interest to seek relations between the Hermitian solutions of the two equations. As mentioned in the previous section, the existence of solution of a matrix equation can be characterized by the minimum rank of the corresponding matrix expression. Further, relations between two matrix equations can also be characterized by rank/methods. Some recent work on the rank and inertia method in the investigation of Hermitian matrix equations can be found in [14, 15, 16, 17, 28, 33].

In [17], the extremal ranks and inertias of $A_1 - B_1XB_1^*$ subject to a consistent matrix equation $B_2XB_2^* = A_2$ were studied and the following results were obtained.

Lemma 2.1 ([17]) *Let $A_i \in \mathbb{C}_{\rm H}^{m_i}$ and $B_i \in \mathbb{C}^{m_i \times n}$ be given for $i = 1, 2$, and suppose that the matrix equation $B_2XB_2^* = A_2$ has a Hermitian solution. Also, let*

$$\mathcal{S}_2 = \{\, X \in \mathbb{C}_{\rm H}^{n} \mid B_2XB_2^* = A_2\}, \quad M = \begin{bmatrix} A_1 & 0 & B_1 \\ 0 & -A_2 & B_2 \\ B_1^* & B_2^* & 0 \end{bmatrix}. \tag{2.1}$$

Then, the global maximum and minimum rank and inertias of $A_1 - B_1XB_1^*$ s.t. $X \in \mathcal{S}$ are given by

$$\max_{X\in\mathcal{S}_2} r(\, A_1 - B_1XB_1^*\,) = \min\{r[\, A_1,\, B_1\,], \quad r(M) - 2r(B_2)\}, \tag{2.2}$$

$$\min_{X\in\mathcal{S}_2} r(\, A_1 - B_1XB_1^*\,) = 2r[\, A_1,\, B_1\,] - 2r\begin{bmatrix} A_1 & B_1 & 0 \\ B_1^* & 0 & B_2^* \end{bmatrix} + r(M), \tag{2.3}$$

$$\max_{X\in\mathcal{S}_2} i_{\pm}(\, A_1 - B_1XB_1^*\,) = i_{\pm}(M) - r(B_2), \tag{2.4}$$

$$\min_{X\in\mathcal{S}_2} i_{\pm}(\, A_1 - B_1XB_1^*\,) = r[\, A_1,\, B_1\,] - r\begin{bmatrix} A_1 & B_1 & 0 \\ B_1^* & 0 & B_2^* \end{bmatrix} + i_{\pm}(M). \tag{2.5}$$

Hence,

(a) *There exists an $X \in \mathbb{C}_{\rm H}^{n}$ such that $B_2XB_2^* = A_2$ and $A_1 - B_1XB_1^*$ is nonsingular if and only if $r[\, A_1,\, B_1\,] = m_1$ or $r(M) = 2r(B_2) + m_1$.*

(b) *There exists an $X \in \mathbb{C}_{\rm H}^{n}$ such that $B_1XB_1^* = A_1$ and $B_2XB_2^* = A_2$ if and only if*

$$\mathscr{R}(A_1) \subseteq \mathscr{R}(B_1) \quad and \quad \mathscr{R}(A_2) \subseteq \mathscr{R}(B_2) \quad and \quad r(M) = 2r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}. \tag{2.6}$$

(c) *There exists an $X \in \mathbb{C}_{\rm H}^{n}$ such that $B_2XB_2^* = A_2$ and $A_1 - B_1XB_1^* > 0$ if and only if $i_+(M) = r(B_2) + m_1$.*

(d) *There exists an $X \in \mathbb{C}_{\rm H}^{n}$ such that $B_2XB_2^* = A_2$ and $A_1 - B_1XB_1^* < 0$ if and only if $i_-(M) = r(B_2) + m_1$.*

(e) *There exists an* $X \in \mathbb{C}_{\mathrm{H}}^{n}$ *such that* $B_2XB_2^* = A_2$ *and* $A_1 - B_1XB_1^* \geqslant 0$ *if and only if*

$$\mathscr{R}(A_2) \subseteq \mathscr{R}(B_2) \quad \textit{and} \quad i_{+}(M) = r\begin{bmatrix} A_1 & B_1 & 0 \\ B_1^* & 0 & B_2^* \end{bmatrix} - r[\, A_1,\, B_1\,]. \tag{2.7}$$

(f) *There exists an* $X \in \mathbb{C}_{\mathrm{H}}^{n}$ *such that* $B_2XB_2^* = A_2$ *and* $A_1 - B_1XB_1^* \leqslant 0$ *if and only if*

$$\mathscr{R}(A_2) \subseteq \mathscr{R}(B_2) \quad \textit{and} \quad i_{-}(M) = r\begin{bmatrix} A_1 & B_1 & 0 \\ B_1^* & 0 & B_2^* \end{bmatrix} - r[\, A_1,\, B_1\,]. \tag{2.8}$$

Corollary 2.2 *Let* $A_i \in \mathbb{C}_{\mathrm{H}}^{m_i}$ *and* $B_i \in \mathbb{C}^{m_i \times n}$ *be given with* $B_i \neq 0$ *for* $i = 1,\, 2,$ *and suppose that each of the two matrix equations* $B_1XB_1^* = A_1$ *and* $B_2XB_2^* = A_2$ *has a Hermitian solution. Also denote the sets of all solutions of* $B_1XB_1^* = A_1$ *and* $B_2XB_2^* = A_2$ *by* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *respectively. Then, the global maximum and minimum ranks and inertias of* $A_1 - B_1XB_1^*$ *subject to* $X \in \mathcal{S}_2$ *are given by*

$$\max_{X \in \mathcal{S}_2} r(\, A_1 - B_1XB_1^*\,) = \min\{r(B_1), \quad r(M) - 2r(B_2)\}, \tag{2.9}$$

$$\min_{X \in \mathcal{S}_2} r(\, A_1 - B_1XB_1^*\,) = r(M) - 2r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \tag{2.10}$$

$$\max_{X \in \mathcal{S}_2} i_{\pm}(\, A_1 - B_1XB_1^*\,) = i_{\pm}(M) - r(B_2), \tag{2.11}$$

$$\min_{X \in \mathcal{S}_2} i_{\pm}(\, A_1 - B_1XB_1^*\,) = i_{\pm}(M) - r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \tag{2.12}$$

where M *is as given in* (2.1). *In particular,*

(a) $\mathcal{S}_2 \subseteq \mathcal{S}_1$ *if and only if* $r(M) = 2r(B_2)$.

(b) $\mathcal{S}_2 = \mathcal{S}_1$ *if and only if* $r(M) = 2r(B_1) = 2r(B_2)$.

The above results can be used to derive algebraic properties of the submatrices in a solution to the matrix equation $BXB^* = A$. Rewrite $BXB^* = A$ as

$$[\, B_1,\, B_2\,]\begin{bmatrix} X_1 & X_2 \\ X_2^* & X_3 \end{bmatrix}\begin{bmatrix} B_1^* \\ B_2^* \end{bmatrix} = A, \tag{2.13}$$

where $B_1 \in \mathbb{C}^{m \times n_1}$, $B_2 \in \mathbb{C}^{m \times n_2}$, $X_1 \in \mathbb{C}_{\mathrm{H}}^{n_1}$, $X_2 \in \mathbb{C}^{n_1 \times n_2}$ and $X_3 \in \mathbb{C}_{\mathrm{H}}^{n_2}$ with $n_1 + n_2 = n$. We next derive the global maximum and minimum ranks and inertias of the submatrices X_1 and X_3 in a Hermitian solution to (2.13). Note that X_1, X_2, X_3 in (2.13) can be rewritten as

$$X_1 = P_1XP_1^*, \quad X_2 = P_1XP_2^*, \quad X_3 = P_2XP_2^*, \tag{2.14}$$

where $P_1 = [\, I_{n_1},\, 0\,]$ and $P_2 = [\, 0,\, I_{n_2}\,]$. For convenience, we adopt the following notation for the collections of the submatrices X_1 and X_3 in (2.13):

$$\mathcal{T}_1 = \left\{ X_1 \in \mathbb{C}_{\mathrm{H}}^{n_1} \;\middle|\; [\, B_1,\, B_2\,]\begin{bmatrix} X_1 & X_2 \\ X_2^* & X_3 \end{bmatrix}\begin{bmatrix} B_1^* \\ B_2^* \end{bmatrix} = A \right\} = \{X_1 = P_1XP_1^* \mid BXB^* = A\}\,, \tag{2.15}$$

$$\mathcal{T}_3 = \left\{ X_3 \in \mathbb{C}_{\mathrm{H}}^{n_2} \;\middle|\; [\, B_1,\, B_2\,]\begin{bmatrix} X_1 & X_2 \\ X_2^* & X_3 \end{bmatrix}\begin{bmatrix} B_1^* \\ B_2^* \end{bmatrix} = A \right\} = \{X_3 = P_2XP_2^* \mid BXB^* = A\}\,. \tag{2.16}$$

Applying Corollary 2.2 to (2.15) and (2.16) gives the following results.

Theorem 2.3 *Suppose that the matrix equation* (2.13) *is consistent, and let* $\mathcal{T}_1$ *and* $\mathcal{T}_3$ *be of the forms* (2.15) *and* (2.16). *Then,*

$$\max_{X_1 \in \mathcal{T}_1} r(X_1) = \min\left\{ n_1, \quad r\begin{bmatrix} A & B_2 \\ B_2^* & 0 \end{bmatrix} - 2r(B) + 2n_1 \right\}, \tag{2.17}$$

$$\min_{X_1 \in \mathcal{T}_1} r(X_1) = r\begin{bmatrix} A & B_2 \\ B_2^* & 0 \end{bmatrix} - 2r(B_2), \tag{2.18}$$

$$\max_{X_1 \in \mathcal{T}_1} i_{\pm}(X_1) = i_{\pm}\begin{bmatrix} C & B_2 \\ A_2^* & 0 \end{bmatrix} - r(A) + n_1, \tag{2.19}$$

$$\min_{X_1 \in \mathcal{T}_1} i_{\pm}(X_1) = i_{\pm}\begin{bmatrix} C & B_2 \\ A_2^* & 0 \end{bmatrix} - r(B_2), \tag{2.20}$$

and

$$\max_{X_3\in\mathcal{T}_3} r(X_3) = \min\left\{ n_2,\ \ r\begin{bmatrix} A & B_1 \\ B_1^* & 0 \end{bmatrix} - 2r(B) + 2n_2 \right\}, \tag{2.21}$$

$$\min_{X_3\in\mathcal{T}_3} r(X_3) = r\begin{bmatrix} A & B_1 \\ B_1^* & 0 \end{bmatrix} - 2r(B_1), \tag{2.22}$$

$$\max_{X_3\in\mathcal{T}_3} i_{\pm}(X_3) = i_{\pm}\begin{bmatrix} C & B_1 \\ A_1^* & 0 \end{bmatrix} - r(A) + n_2, \tag{2.23}$$

$$\min_{X_3\in\mathcal{T}_3} i_{\pm}(X_3) = i_{\pm}\begin{bmatrix} C & B_1 \\ A_1^* & 0 \end{bmatrix} - r(B_1). \tag{2.24}$$

Hence,

(c) *Eq.* (2.13) *has a solution in which* X_1 *is nonsingular if and only if* $r\begin{bmatrix} C & B_2 \\ A_2^* & 0 \end{bmatrix} \geqslant 2r(A) - n_1$.

(d) *The submatrix* X_1 *in any solution to* (2.13) *is nonsingular if and only if* $r\begin{bmatrix} C & B_2 \\ A_2^* & 0 \end{bmatrix} = 2r(B_2) + n_1$.

(e) *Eq.* (2.13) *has a solution in which* $X_1 = 0$ *if and only if* $r\begin{bmatrix} C & B_2 \\ A_2^* & 0 \end{bmatrix} = 2r(B_2)$.

(f) *The submatrix* X_1 *in any solution to* (2.13) *satisfies* $X_1 = 0$ *if and only if* $r\begin{bmatrix} C & B_2 \\ A_2^* & 0 \end{bmatrix} = 2r(A) - 2n_1$.

(g) *Eq.* (2.13) *has a solution in which* $X_1 > 0$ $(X_1 < 0)$ *if and only if*

$$i_{+}\begin{bmatrix} C & B_2 \\ A_2^* & 0 \end{bmatrix} = r(A) \ \ \left(i_{-}\begin{bmatrix} C & B_2 \\ A_2^* & 0 \end{bmatrix} = r(A) \right).$$

(h) *The submatrix* X_1 *in any solution to* (2.13) *satisfies* $X_1 > 0$ $(X_1 < 0)$ *if and only if*

$$i_{+}\begin{bmatrix} C & B_2 \\ A_2^* & 0 \end{bmatrix} = n_1 + r(B_2) \ \ \left(i_{-}\begin{bmatrix} C & B_2 \\ A_2^* & 0 \end{bmatrix} = n_1 + r(B_2) \right).$$

(i) *Eq.* (2.13) *has a solution satisfying* $X_1 \geqslant 0$ $(X_1 \leqslant 0)$ *if and only if*

$$i_{-}\begin{bmatrix} C & B_2 \\ A_2^* & 0 \end{bmatrix} = r(B_2) \ \ \left(i_{+}\begin{bmatrix} C & B_2 \\ A_2^* & 0 \end{bmatrix} = r(B_2) \right).$$

(j) *The submatrix* X_1 *in any solution to* (2.13) *satisfies* $X_1 \geqslant 0$ $(X_1 \leqslant 0)$ *if and only if*

$$i_{-}\begin{bmatrix} C & B_2 \\ A_2^* & 0 \end{bmatrix} = r(A) - n_1 \ \ \left(i_{-}\begin{bmatrix} C & B_2 \\ A_2^* & 0 \end{bmatrix} = r(A) - n_1 \right).$$

(k) *The positive signature of* X_1 *in* (2.13) *is invariant* $\Leftrightarrow$ *the negative signature of* X_1 (2.13) *is invariant* $\Leftrightarrow$ $\mathscr{R}(B_1) \cap \mathscr{R}(B_2) = \{0\}$ *and* $r(B_1) = n_1$.

The definiteness of the Hermitian solutions of a given consistent matrix equation is attractive topic in matrix theory and applications; see, e.g., [13, 9, 33, 35]. From Corollary 2.2, we now can derive the existences of Hermitian solutions of $AXA^* = B$ satisfying some inequalities.

Corollary 2.4 ([31]) *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}_{\mathrm{H}}^{m}$ *and* $A_1 \in \mathbb{C}_{\mathrm{H}}^{n}$ *be given, and assume that the matrix equation* $AXA^* = B$ *has a solution for* $X \in \mathbb{C}_{\mathrm{H}}^{n}$ *and define* $\mathcal{S} = \{\, X \in \mathbb{C}_{\mathrm{H}}^{n} \mid AXA^* = B\}$. *Then,*

$$\max_{X\in\mathcal{S}} i_{\pm}(\, X - A_1 \,) = n + i_{\pm}(\, B - AA_1A^* \,) - r(A), \tag{2.25}$$

$$\min_{X\in\mathcal{S}} i_{\pm}(\, X - A_1 \,) = i_{\pm}(\, B - AA_1A^* \,), \tag{2.26}$$

$$\max_{X\in\mathcal{S}} i_{\pm}(X) = n + i_{\pm}(B) - r(A), \tag{2.27}$$

$$\min_{X\in\mathcal{S}} i_{\pm}(X) = i_{\pm}(B). \tag{2.28}$$

Hence,

(a) $AXA^* = B$ *has a solution* $X > A_1$ $(X < A_1)$ *if and only if* $i_+(B - AA_1A^*) = r(A)$ $(i_-(B - AA_1A^*) = r(A))$.

(b) $AXA^* = B$ *has a solution* $X \geqslant A_1$ $(X \leqslant A_1)$ *if and only if* $B \geqslant AA_1A^*$ $(B \leqslant AA_1A^*)$.

(c) $AXA^* = B$ *has a solution* $X > 0$ $(X < 0)$ *if and only if* $B \geqslant 0$ *and* $r(A) = r(B)$ $(B \leqslant 0$ *and* $r(A) = r(B))$.

(d) $AXA^* = B$ *has a solution* $X \geqslant 0$ $(X \leqslant 0)$ *if and only if* $B \geqslant 0$ $(B \leqslant 0)$.

3 The global maximum and minimum ranks and inertias of $A - B_1XB_1^*$ subject to a pair of matrix equations

We first derive a parametric form for the general common Hermitian solution of the pair of matrix equations in (1.2)–(1.5).

Lemma 3.1 ([31]) *Let* $A_i \in \mathbb{C}_{\mathrm{H}}^{m_i}$, $B_i \in \mathbb{C}^{m_i \times n}$ *be given for* $i = 2,\, 3$, *and suppose that each of the two matrix equations*

$$B_2XB_2^* = A_2 \quad and \quad B_3XB_3^* = A_3 \tag{3.1}$$

has a solution, i.e., $\mathscr{R}(A_i) \subseteq \mathscr{R}(B_i)$ *for* $i = 2,\, 3$. *Then,*

(a) *The pair of matrix equations have a common Hermitian solution if and only if*

$$r\begin{bmatrix} A_2 & 0 & B_2 \\ 0 & -A_3 & B_3 \\ B_2^* & B_3^* & 0 \end{bmatrix} = 2r\begin{bmatrix} B_2 \\ B_3 \end{bmatrix}. \tag{3.2}$$

(b) *Under* (3.2), *the general common Hermitian solution of the pair of equations can be written in the following parametric form*

$$X = X_0 + VF_B + F_BV^* + F_{B_2}UF_{B_3} + F_{B_3}U^*F_{B_2}, \tag{3.3}$$

where X_0 *is a special Hermitian common solution to the pair of equations,* $B = \begin{bmatrix} B_2 \\ B_3 \end{bmatrix}$, *and* $U,\, V \in \mathbb{C}^{n \times n}$ *are arbitrary.*

Substituting (3.3) into $A_1 - B_1XB_1^*$ gives

$$A_1 - B_1XB_1^* = A_1 - B_1X_0B_1^* - B_1VF_BB_1^* - B_1F_BV^*B_1^* - B_1F_{B_2}UF_{B_3}B_1^* - B_1F_{B_3}U^*F_{B_2}B_1^*, \tag{3.4}$$

which is a matrix expression involving two variable matrices V and U. Thus, the constrained matrix expression in (1.2) is equivalently converted to the unconstrained matrix expression in (3.4). To find the global maximum and minimum ranks and inertias of (3.4), we need the following result.

Lemma 3.2 *Let*

$$p(X_1,\, X_2) = A - B_1X_1C_1 - (B_1X_1C_1)^* - B_2X_2C_2 - (B_2X_2C_2)^*, \tag{3.5}$$

where $A \in \mathbb{C}_{\mathrm{H}}^{m}$, $B_i \in \mathbb{C}^{m \times p_i}$ *and* $C_i \in \mathbb{C}^{q_i \times m}$ *are given, and* $X_i \in \mathbb{C}^{p_i \times q_i}$ *are variable matrices for* $i = 1, 2$, *and assume that*

$$\mathscr{R}(B_2) \subseteq \mathscr{R}(B_1), \quad \mathscr{R}(C_1^*) \subseteq \mathscr{R}(B_1), \quad \mathscr{R}(C_2^*) \subseteq \mathscr{R}(B_1). \tag{3.6}$$

Also let

$$N = \begin{bmatrix} A & B_2 & C_1^* & C_2^* \\ C_1 & 0 & 0 & 0 \end{bmatrix}, \quad N_1 = \begin{bmatrix} A & B_2 & C_1^* & C_2^* \\ B_2^* & 0 & 0 & 0 \\ C_1 & 0 & 0 & 0 \end{bmatrix}, \quad N_2 = \begin{bmatrix} A & B_2 & C_1^* & C_2^* \\ C_1 & 0 & 0 & 0 \\ C_2 & 0 & 0 & 0 \end{bmatrix},$$

$$M = \begin{bmatrix} A & B_1 \\ C_1 & 0 \end{bmatrix}, \quad M_1 = \begin{bmatrix} A & B_2 & C_1^* \\ B_2^* & 0 & 0 \\ C_1 & 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} A & C_1^* & C_2^* \\ C_1 & 0 & 0 \\ C_2 & 0 & 0 \end{bmatrix}.$$

Then, the global maximum and minimum ranks and inertias of $p(X_1,\, X_2)$ *are given by*

$$\max_{X_1\in\mathbb{C}^{p_1\times q_1},\, X_2\in\mathbb{C}^{p_2\times q_2}} r[\,p(X_1,\, X_2)\,] = \min\{r[\,A,\, B_1\,],\quad r(N),\quad r(M_1),\quad r(M_2)\}, \tag{3.7}$$

$$\min_{X_1\in\mathbb{C}^{p_1\times q_1},\, X_2\in\mathbb{C}^{p_2\times q_2}} r[\,p(X_1,\, X_2)\,] = 2r[\,A,\, B_1\,] - 2r(M) + 2r(N) + \max\{\, s_1,\quad s_2,\quad s_3,\quad s_4\,\}, \tag{3.8}$$

$$\max_{X_1\in\mathbb{C}^{p_1\times q_1},\, X_2\in\mathbb{C}^{p_2\times q_2}} i_{\pm}[\,p(X_1,\, X_2)\,] = \min\{i_{\pm}(M_1),\quad i_{\pm}(M_2)\}, \tag{3.9}$$

$$\begin{aligned}\min_{X_1\in\mathbb{C}^{p_1\times q_1},\, X_2\in\mathbb{C}^{p_2\times q_2}} i_{\pm}[\,p(X_1,\, X_2)\,] &= r[\,A,\, B_1\,] - r(M) + r(N)\\ &\quad + \max\{\, i_{\pm}(M_1) - r(N_1),\quad i_{\pm}(M_2) - r(N_2)\,\},\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}&s_1 = r(M_1) - 2r(N_1), \quad s_2 = r(M_2) - 2r(N_2),\\ &s_3 = i_{+}(M_1) + i_{-}(M_2) - r(N_1) - r(N_2),\\ &s_4 = i_{+}(M_1) + i_{-}(M_2) - r(N_1) - r(N_2).\end{aligned}$$

Proof Under (3.6), applying Lemma 1.7 to the variable matrix X_1 in (3.5) and simplifying, we obtain

$$\begin{aligned}\max_{X_1} r[\,p(X_1,\, X_2)\,] &= \min\left\{ r[\,A - B_2X_2C_2 - (B_2X_2C_2)^*,\, B_1\,],\quad r\begin{bmatrix} A - B_2X_2C_2 - (B_2X_2C_2)^* & C_1^* \\ C_1 & 0\end{bmatrix}\right\}\\ &= \min\left\{ r[\,A,\, B_1\,],\quad r\begin{bmatrix} A - B_2X_2C_2 - (B_2X_2C_2)^* & C_1^* \\ C_1 & 0\end{bmatrix}\right\},\end{aligned} \tag{3.11}$$

$$\begin{aligned}\min_{X_1} r[\,p(X_1,\, X_2)\,] &= 2r[\,A - B_2X_2C_2 - (B_2X_2C_2)^*,\, B_1\,] + r\begin{bmatrix} A - B_2X_2C_2 - (B_2X_2C_2)^* & C_1^* \\ C_1 & 0\end{bmatrix}\\ \max_{X_1} i_{\pm}[\,p(X_1,\, X_2)\,] &= i_{\pm}\begin{bmatrix} A - B_2X_2C_2 - (B_2X_2C_2)^* & C_1^* \\ C_1 & 0\end{bmatrix},\end{aligned} \tag{3.12}$$

$$\begin{aligned}&- 2r\begin{bmatrix} A - B_2X_2C_2 - (B_2X_2C_2)^* & B_1 \\ C_1 & 0\end{bmatrix}\\ &= 2r[\,A,\, B_1\,] + r\begin{bmatrix} A - B_2X_2C_2 - (B_2X_2C_2)^* & C_1^* \\ C_1 & 0\end{bmatrix} - 2r\begin{bmatrix} A & B_1 \\ C_1 & 0\end{bmatrix},\end{aligned} \tag{3.13}$$

$$\begin{aligned}\min_{X_1} i_{\pm}[\,p(X_1,\, X_2)\,] &= r[\,A - B_2X_2C_2 - (B_2X_2C_2)^*,\, B_1\,] + i_{\pm}\begin{bmatrix} A - B_2X_2C_2 - (B_2X_2C_2)^* & C_1^* \\ C_1 & 0\end{bmatrix}\\ &\quad - r\begin{bmatrix} A - B_2X_2C_2 - (B_2X_2C_2)^* & B_1 \\ C_1 & 0\end{bmatrix}\\ &= r[\,A,\, B_1\,] + i_{\pm}\begin{bmatrix} A - B_2X_2C_2 - (B_2X_2C_2)^* & C_1^* \\ C_1 & 0\end{bmatrix} - r\begin{bmatrix} A & B_1 \\ C_1 & 0\end{bmatrix}.\end{aligned} \tag{3.14}$$

Notice that

$$\begin{aligned}\begin{bmatrix} A - B_2X_2C_2 - (B_2X_2C_2)^* & C_1^* \\ C_1 & 0\end{bmatrix} &= \begin{bmatrix} A & C_1^* \\ C_1 & 0\end{bmatrix} - \begin{bmatrix} B_2 \\ 0\end{bmatrix} X_2[\,C_2,\, 0\,] - \begin{bmatrix} C_2^* \\ 0\end{bmatrix} X_2^*[\,B_2^*,\, 0\,]\\ &:= q(X_2).\end{aligned} \tag{3.15}$$

Applying Lemma 1.6 to this expression gives

$$\max_{X_2\in\mathbb{C}^{m\times p_2}} r[q(X_2)] = \min\{\, r(N),\quad r(M_1),\quad r(M_2)\,\}, \tag{3.16}$$

$$\min_{X_2\in\mathbb{C}^{m\times p_2}} r[q(X_2)] = 2r(N) + \max\{\, s_1,\quad s_2,\quad s_3,\quad s_4\,\}, \tag{3.17}$$

$$\max_{X_2\in\mathbb{C}^{m\times p_2}} i_{\pm}[q(X_2)] = \min\{\, i_{\pm}(M_1),\quad i_{\pm}(M_2)\,\}, \tag{3.18}$$

$$\min_{X_2\in\mathbb{C}^{m\times p_2}} i_{\pm}[q(X_2)] = r(N) + \max\{\, i_{\pm}(M_1) - r(N_1),\quad i_{\pm}(M_2) - r(N_2)\,\}, \tag{3.19}$$

where

$$s_1 = r(M_1) - 2r(N_1), \quad s_2 = r(M_2) - 2r(N_2),$$

$$s_3 = i_+(M_1) + i_-(M_2) - r(N_1) - r(N_2), \quad s_4 = i_-(M_1) + i_+(M_2) - r(N_1) - r(N_2).$$

Substituting these results into (3.11)–(3.14) yields (3.9)–(3.2). $\square$

It is obviously of great importance to be able to give analytical formulas for calculating the global maximum and minimum ranks and inertias of the matrix expression in (3.5) under the assumptions in (3.7). However, it is not easy to find the global maximum and minimum ranks and inertias ranks and inertias of a general $p(X_1, X_2)$ as given in (3.5). For convenience of representation, we rewrite (3.4) as

$$A_1 - B_1XB_1^* = A - G_1VG_2 - (G_1VG_2)^* - G_3UG_4 - (G_3UG_4)^*, \tag{3.20}$$

where

$$A = A_1 - B_1X_0B_1^*, \quad G_1 = B_1, \quad G_2 = F_BB_1^*, \quad G_3 = B_1F_{B_2}, \quad G_4 = F_{B_3}B_1^*. \tag{3.21}$$

It is easy to verify that the above matrices satisfy the conditions

$$(G_2^*) \subseteq \mathscr{R}(G_1), \quad \mathscr{R}(G_3) \subseteq \mathscr{R}(G_1), \quad \mathscr{R}(G_4^*) \subseteq \mathscr{R}(G_1), \quad \mathscr{R}(G_2^*) \subseteq \mathscr{R}(G_3), \quad \mathscr{R}(G_2^*) \subseteq \mathscr{R}(G_4^*). \tag{3.22}$$

In this case, applying Lemma 3.2 to (3.22) yields the main results of this section.

Theorem 3.3 *Let $A_i \in \mathbb{C}_{\rm H}^{m_i}$ and $B_i \in \mathbb{C}^{m_i\times n}$ be given for $i = 1, 2, 3$, and assume that the pair of matrix equations*

$$B_2XB_2^* = A_2 \quad and \quad B_3XB_3^* = A_3 \tag{3.23}$$

have a common solution $X \in \mathbb{C}_{\rm H}^n$. Also denote the set of all their common Hermitian solutions by

$$\mathcal{S} = \{ X \in \mathbb{C}_{\rm H}^n \mid B_2XB_2^* = A_2, \quad B_3XB_3^* = A_3 \}. \tag{3.24}$$

and let

$$P_1 = \begin{bmatrix} A_1 & B_1 & 0 & 0 \\ B_1^* & 0 & B_2^* & B_3^* \end{bmatrix}, \quad P_2 = \begin{bmatrix} A_1 & 0 & B_1 \\ 0 & -A_2 & B_2 \\ B_1^* & B_2^* & 0 \end{bmatrix}, \quad P_3 = \begin{bmatrix} A_1 & 0 & B_1 \\ 0 & -A_3 & B_3 \\ B_1^* & B_3^* & 0 \end{bmatrix}, \tag{3.25}$$

$$Q_1 = \begin{bmatrix} A_1 & 0 & 0 & B_1 & B_1 \\ 0 & -A_2 & 0 & B_2 & 0 \\ 0 & 0 & -A_3 & 0 & B_3 \\ B_1^* & B_2^* & B_3^* & 0 & 0 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} A_1 & 0 & B_1 & B_1 \\ 0 & -A_2 & B_2 & 0 \\ B_1^* & B_2^* & 0 & 0 \\ 0 & 0 & 0 & B_3 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} A_1 & 0 & B_1 & B_1 \\ 0 & -A_3 & B_3 & 0 \\ B_1^* & B_3^* & 0 & 0 \\ 0 & 0 & 0 & B_2 \end{bmatrix}. \tag{3.26}$$

Then,

(a) *The global maximum rank of $A_1 - B_1XB_1^*$ subject to (3.24) is*

$$\max_{X\in\mathcal{S}} r(A_1 - B_1XB_1^*)$$
$$= \min\left\{ r[\, A_1,\, B_1\,],\ r(Q_1) - r\begin{bmatrix} B_2 \\ B_3 \end{bmatrix} - r(B_2) - r(B_3),\ r(P_2) - 2r(B_2),\ r(P_3) - 2r(B_3) \right\}. \tag{3.27}$$

(b) *The global minimum rank of $A_1 - B_1XB_1^*$ subject to (3.24) is*

$$\min_{X\in\mathcal{S}} r(A_1 - B_1XB_1^*)$$
$$= 2r[\, A_1,\, B_1\,] - 2r(P_1) + 2r(Q_1) + \max\{\, r(P_2) - 2r(Q_2),\ r(P_3) - 2r(Q_3),\ u_1,\ u_2 \,\}, \tag{3.28}$$

where

$$u_1 = i_+(P_2) + i_-(P_3) - r(Q_2) - r(Q_3), \quad u_2 = i_-(P_2) + i_+(P_3) - r(Q_2) - r(Q_3).$$

(c) *The global maximum partial inertia of $A_1 - B_1XB_1^*$ subject to (3.24) is*

$$\max_{X\in\mathcal{S}} i_{\pm}(A_1 - B_1XB_1^*) = \min\left\{ i_{\pm}(P_2) - r(B_2), \quad i_{\pm}(P_3) - r(B_3) \right\}. \tag{3.29}$$

(d) *The global minimum partial inertia of* $A_1 - B_1XB_1^*$ *subject to* (3.24) *is*

$$\min_{X\in\mathcal{S}} i_{\pm}(\, A_1 - B_1XB_1^*\,) = r[\, A_1,\, B_1\,] - r(P_1) + r(Q_1) \tag{3.30}$$
$$+ \max\{i_{\pm}(P_2) - r(Q_2),\ i_{\pm}(P_3) - r(Q_3)\}. \tag{3.31}$$

Proof Under (3.22), we find by Lemma 3.2 that

$$\max_{X\in\mathcal{S}} r(\, A_1 - B_1XB_1^*\,) = \max_{V,\,U} r\,[A - G_1VG_2 - (G_1VG_2)^* - G_3UG_4 - (G_3UG_4)^*]$$
$$= \min\left\{ r[\, A,\, G_1\,],\ r\begin{bmatrix} A & G_3 & G_4^* \\ G_2 & 0 & 0 \end{bmatrix},\ r\begin{bmatrix} A & G_3 \\ G_3^* & 0 \end{bmatrix},\ r\begin{bmatrix} A & G_4^* \\ G_4 & 0 \end{bmatrix} \right\}, \tag{3.32}$$
$$\min_{X\in\mathcal{S}} r(\, A_1 - B_1XB_1^*\,) = \min_{V,\,U} r\,[\, A - G_1VG_2 - (G_1VG_2)^* - G_3UG_4 - (G_3UG_4)^*\,]$$
$$= 2r[\, A,\, G_1\,] - 2r\begin{bmatrix} A & G_1 \\ G_2 & 0 \end{bmatrix} + 2r\begin{bmatrix} A & G_3 & G_4^* \\ G_2 & 0 & 0 \end{bmatrix}$$
$$+ \max\{s_1,\quad s_2,\quad s_3,\quad s_4\,\}, \tag{3.33}$$
$$\max_{X\in\mathcal{S}} i_{\pm}(\, A_1 - B_1XB_1^*\,) = \max_{V,\,U} i_{\pm}\,[A - G_1VG_2 - (G_1VG_2)^* - G_3UG_4 - (G_3UG_4)^*]$$
$$= \min\left\{ i_{\pm}\begin{bmatrix} A & G_3 \\ G_3^* & 0 \end{bmatrix},\ i_{\pm}\begin{bmatrix} A & G_4^* \\ G_4 & 0 \end{bmatrix} \right\}, \tag{3.34}$$
$$\min_{X\in\mathcal{S}} i_{\pm}(\, A_1 - B_1XB_1^*\,) = \min_{V,\,U} i_{\pm}\,[\, A - G_1VG_2 - (G_1VG_2)^* - G_3UG_4 - (G_3UG_4)^*\,]$$
$$= r[\, A,\, G_1\,] - r\begin{bmatrix} A & G_1 \\ G_2 & 0 \end{bmatrix} + r\begin{bmatrix} A & G_3 & G_4^* \\ G_2 & 0 & 0 \end{bmatrix} + \max\{t_1,\quad t_2\,\}, \tag{3.35}$$

where

$$s_1 = r\begin{bmatrix} A & G_3 \\ G_3^* & 0 \end{bmatrix} - 2r\begin{bmatrix} A & G_3 & G_4^* \\ G_3^* & 0 & 0 \end{bmatrix},$$
$$s_2 = r\begin{bmatrix} A & G_4^* \\ G_4 & 0 \end{bmatrix} - 2r\begin{bmatrix} A & G_3 & G_4^* \\ G_4 & 0 & 0 \end{bmatrix},$$
$$s_3 = i_{+}\begin{bmatrix} A & G_3 \\ G_3^* & 0 \end{bmatrix} + i_{-}\begin{bmatrix} A & G_4^* \\ G_4 & 0 \end{bmatrix} - r\begin{bmatrix} A & G_3 & G_4^* \\ G_3^* & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A & G_3 & G_4^* \\ G_4 & 0 & 0 \end{bmatrix},$$
$$s_4 = i_{-}\begin{bmatrix} A & G_3 \\ G_3^* & 0 \end{bmatrix} + i_{+}\begin{bmatrix} A & G_4^* \\ G_4 & 0 \end{bmatrix} - r\begin{bmatrix} A & G_3 & G_4^* \\ G_3^* & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A & G_3 & G_4^* \\ G_4 & 0 & 0 \end{bmatrix},$$
$$t_1 = i_{\pm}\begin{bmatrix} A & G_3 \\ G_3^* & 0 \end{bmatrix} - r\begin{bmatrix} A & G_3 & G_4^* \\ G_3^* & 0 & 0 \end{bmatrix},$$
$$t_2 = i_{\pm}\begin{bmatrix} A & G_4 \\ G_4^* & 0 \end{bmatrix} - r\begin{bmatrix} A & G_3 & G_4^* \\ G_4^* & 0 & 0 \end{bmatrix}.$$

Applying (1.10)–(1.12) and (1.19), and simplifying by $(B_2X_0B_2^*,\, B_3X_0B_3^*) = (A_2,\, A_3)$, elementary matrix operations and congruence matrix operations, we obtain

$$r[\,A,\, G_1\,] = r[\,A_1 - B_1X_0B_1^*,\, B_1\,] = r[\,A_1,\, B_1\,], \tag{3.36}$$

$$\begin{aligned} r\begin{bmatrix} A & G_3 & G_4^* \\ G_2 & 0 & 0 \end{bmatrix} &= r\begin{bmatrix} A_1 - B_1X_0B_1^* & B_1F_{B_2} & B_1F_{B_3} \\ F_BB_1^* & 0 & 0 \end{bmatrix} \\ &= r\begin{bmatrix} A_1 - B_1X_0B_1^* & B_1 & B_1 & 0 \\ B_1^* & 0 & 0 & B^* \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & B_3 & 0 \end{bmatrix} - r(B) - r(B_2) - r(B_3) \\ &= r\begin{bmatrix} A_1 & B_1 & B_1 & B_1X_0B^* \\ B_1^* & 0 & 0 & B^* \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & B_3 & 0 \end{bmatrix} - r(B) - r(B_2) - r(B_3) \\ &= r\begin{bmatrix} A_1 & B_1 & B_1 & 0 & 0 \\ B_1^* & 0 & 0 & B_2^* & B_3^* \\ 0 & B_2 & 0 & -A_2 & 0 \\ 0 & 0 & B_3 & 0 & -A_3 \end{bmatrix} - r(B) - r(B_2) - r(B_3) \\ &= r(Q_1) - r(B) - r(B_2) - r(B_3), \end{aligned} \tag{3.37}$$

$$\begin{aligned} r\begin{bmatrix} A & G_1 \\ G_2 & 0 \end{bmatrix} &= r\begin{bmatrix} A_1 - B_1X_0B_1^* & B_1 \\ F_BB_1^* & 0 \end{bmatrix} = r\begin{bmatrix} A_1 & B_1 & 0 \\ B_1^* & 0 & B^* \end{bmatrix} - r(B) \\ &= r(P_1) - r(B), \end{aligned} \tag{3.38}$$

$$\begin{aligned} i_{\pm}\begin{bmatrix} A & G_3 \\ G_3^* & 0 \end{bmatrix} &= i_{\pm}\begin{bmatrix} A_1 - B_1X_0B_1^* & B_1F_{B_2} \\ F_{B_2}B_1^* & 0 \end{bmatrix} = i_{\pm}\begin{bmatrix} A_1 - B_1X_0B_1^* & B_1 & 0 \\ B_1^* & 0 & B_2^* \\ 0 & B_2 & 0 \end{bmatrix} - r(B_2) \\ &= i_{\pm}\begin{bmatrix} A_1 & B_1 & B_1X_0B_2^*/2 \\ B_1^* & 0 & B_2^* \\ B_1X_0B_2^*/2 & B_2 & 0 \end{bmatrix} - r(B_2) = i_{\pm}\begin{bmatrix} A_1 & B_1 & 0 \\ B_1^* & 0 & B_2^* \\ 0 & B_2 & -A_2 \end{bmatrix} - r(B_2) \\ &= i_{\pm}(P_2) - r(B_2), \end{aligned} \tag{3.39}$$

$$\begin{aligned} r\begin{bmatrix} A & G_3 & G_4^* \\ G_3^* & 0 & 0 \end{bmatrix} &= r\begin{bmatrix} A_1 - B_1X_0B_1^* & B_1F_{B_2} & B_1F_{B_3} \\ F_{B_2}B_1^* & 0 & 0 \end{bmatrix} \\ &= r\begin{bmatrix} A_1 - B_1X_0B_1^* & B_1 & B_1 & 0 \\ B_1^* & 0 & 0 & B_2^* \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & B_3 & 0 \end{bmatrix} - 2r(B_2) - r(B_3) \\ &= r\begin{bmatrix} A_1 & B_1 & B_1 & B_1X_0B_2^* \\ B_1^* & 0 & 0 & B_2^* \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & B_3 & 0 \end{bmatrix} - 2r(B_2) - r(B_3) \\ &= r\begin{bmatrix} A_1 & B_1 & B_1 & 0 \\ B_1^* & 0 & 0 & B_2^* \\ 0 & B_2 & 0 & -A_2 \\ 0 & 0 & B_3 & 0 \end{bmatrix} - 2r(B_2) - r(B_3) \\ &= r(Q_2) - 2r(B_2) - r(B_3). \end{aligned} \tag{3.40}$$

By a similar approach, we can obtain

$$i_{\pm}\begin{bmatrix} A & G_4 \\ G_4^* & 0 \end{bmatrix} = i_{\pm}(P_3) - r(B_3), \quad r\begin{bmatrix} A & G_3 & G_4^* \\ G_4 & 0 & 0 \end{bmatrix} = r(Q_3) - r(B_2) - 2r(B_3). \tag{3.41}$$

Substituting (3.36)–(3.41) into (3.32)–(3.35) yields (3.27)–(3.31). □

Some direct consequences of the previous theorem are given below.

Corollary 3.4 *Let $A_i \in \mathbb{C}_{\mathrm{H}}^{m_i}$ and $B_i \in \mathbb{C}^{m_i\times n}$ be given for $i = 1,\ 2,\ 3,$ and suppose that each pair of $B_1XB_1^* = A_1$, $B_2XB_2^* = A_2$ and $B_3XB_3^* = A_3$ have a common Hermitian solution. Also let $\mathcal{S}$ be of the form* (3.23)*. Then,*

$$\max_{X\in\mathcal{S}} r(\,A_1 - B_1XB_1^*\,) = \min\left\{r(B_1),\ r(Q_1) - r\begin{bmatrix} B_2 \\ B_3 \end{bmatrix} - r(B_2) - r(B_3),\right.$$
$$\left. 2r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} - 2r(B_2),\ 2r\begin{bmatrix} B_1 \\ B_3 \end{bmatrix} - 2r(B_3)\right\}, \tag{3.42}$$

$$\min_{X\in\mathcal{S}} r(\,A_1 - B_1XB_1^*\,) = 2r(Q_1) - 2r\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} - 2r\begin{bmatrix} B_1 & B_1 \\ B_2 & 0 \\ 0 & B_3 \end{bmatrix}, \tag{3.43}$$

$$\max_{X\in\mathcal{S}} i_{\pm}(\,A_1 - B_1XB_1^*\,) = \min\left\{r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} - r(B_2),\quad r\begin{bmatrix} B_1 \\ B_3 \end{bmatrix} - r(B_3)\right\}, \tag{3.44}$$

$$\min_{X\in\mathcal{S}} i_{\pm}(\,A_1 - B_1XB_1^*\,) = r(Q_1) - r\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} - r\begin{bmatrix} B_1 & B_1 \\ B_2 & 0 \\ 0 & B_3 \end{bmatrix}, \tag{3.45}$$

where Q_1 is of the form (3.26).

Proof Under the given conditions, the ranks/inertias of the block matrices in (3.25) and (3.26) are given by

$$r(P_1) = r(B_1) + r\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix},\quad r(P_2) = 2r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix},\ r(P_3) = 2r\begin{bmatrix} B_1 \\ B_3 \end{bmatrix},\ i_{\pm}(P_2) = r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix},\ i_{\pm}(P_3) = r\begin{bmatrix} B_1 \\ B_3 \end{bmatrix},$$

$$r(Q_2) = r\begin{bmatrix} B_1 & B_1 \\ B_2 & 0 \\ 0 & B_3 \end{bmatrix} + r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix},\quad r(Q_3) = r\begin{bmatrix} B_1 & B_1 \\ B_2 & 0 \\ 0 & B_3 \end{bmatrix} + r\begin{bmatrix} B_1 \\ B_3 \end{bmatrix}.$$

Hence (3.27)–(3.31) reduce to (3.42)–(3.45). □

Corollary 3.5 *Let $A_i \in \mathbb{C}_{\mathrm{H}}^{m_i\times m_i}$ and $B_i \in \mathbb{C}^{m_i\times n}$ be given for $i = 1,\ 2,\ 3,$ and suppose that each pair of the triple matrix equations*

$$B_1XB_1^* = A_1,\quad B_2XB_2^* = A_2,\quad B_3XB_3^* = A_3 \tag{3.46}$$

have a common Hermitian solution. Then, there exists a Hermitian X such that (3.46) *holds if and only if*

$$r\begin{bmatrix} A_1 & 0 & 0 & B_1 & B_1 \\ 0 & -A_2 & 0 & B_2 & 0 \\ 0 & 0 & -A_3 & 0 & B_3 \\ B_1^* & B_2^* & B_3^* & 0 & 0 \end{bmatrix} = r\begin{bmatrix} B_1 & B_1 \\ B_2 & 0 \\ 0 & B_3 \end{bmatrix} + r[\,B_1^*,\ B_2^*,\ B_3^*\,]. \tag{3.47}$$

Proof It follows from (3.43). □

A challenging open problem on the triple matrix equations in (3.46) is to give a parametric form for their general common Hermitian solution.

Setting $B_1 = I_n$ in Theorem 3.3 may yield a group of results on the extremal ranks/inertias of $A_1 - X$ subject to (3.24). In particular, we have the following consequences.

Corollary 3.6 *Let $A_i \in \mathbb{C}_{\mathrm{H}}^{m_i}$ and $B_i \in \mathbb{C}^{m_i\times n}$ be given for $i = 2,\ 3,$ and assume that* (3.23) *has a common solution. Also let $\mathcal{S}$ be of the form* (3.24)*. Then,*

(a) *The global maximum rank of the solution of* (3.24) *is*

$$\max_{X\in\mathcal{S}} r(X) = \min\{\,n,\quad s_1,\quad s_2,\quad s_3\,\}, \tag{3.48}$$

where

$$s_1 = 2n + r\begin{bmatrix} A_2 & 0 & B_2 \\ 0 & A_3 & B_3 \end{bmatrix} - r\begin{bmatrix} B_2 \\ B_3 \end{bmatrix} - r(B_2) - r(B_3),$$
$$s_2 = 2n + r(A_2) - 2r(B_2),\quad s_3 = 2n + r(A_3) - 2r(B_3).$$

(b) *The global minimum rank of the solution of* (3.24) *is*

$$\min_{X\in\mathcal{S}} r(X) = 2r\begin{bmatrix} A_2 & 0 & B_2 \\ 0 & A_3 & B_3 \end{bmatrix} + \max\{\, t_1, \quad t_2, \quad t_3, \quad t_4 \,\}, \tag{3.49}$$

where

$$t_1 = r(A_2) - 2r\begin{bmatrix} A_2 & B_2 \\ 0 & B_3 \end{bmatrix}, \quad t_2 = r(A_3) - 2r\begin{bmatrix} 0 & B_2 \\ A_3 & B_3 \end{bmatrix},$$
$$t_3 = i_+(A_2) + i_-(A_3) - r\begin{bmatrix} A_2 & B_2 \\ 0 & B_3 \end{bmatrix} - r\begin{bmatrix} 0 & B_2 \\ A_3 & B_3 \end{bmatrix},$$
$$t_4 = i_-(A_2) + i_+(A_3) - r\begin{bmatrix} A_2 & B_2 \\ 0 & B_3 \end{bmatrix} - r\begin{bmatrix} 0 & B_2 \\ A_3 & B_3 \end{bmatrix}.$$

(c) *The global maximum partial inertia of the solution of* (3.24) *is*

$$\max_{X\in\mathcal{S}} i_{\pm}(X) = \min\{\, n + i_{\pm}(A_2) - r(B_2), \quad n + i_{\pm}(A_3) - r(B_3) \,\}. \tag{3.50}$$

(d) *The global minimum partial inertia of the solution of* (3.24) *is*

$$\min_{X\in\mathcal{S}} i_{\pm}(X) = r\begin{bmatrix} A_2 & 0 & B_2 \\ 0 & A_3 & B_3 \end{bmatrix} + \max\left\{ i_{\pm}(A_2) - r\begin{bmatrix} A_2 & B_2 \\ 0 & B_3 \end{bmatrix},\ i_{\pm}(A_3) - r\begin{bmatrix} 0 & B_2 \\ A_3 & B_3 \end{bmatrix} \right\}. \tag{3.51}$$

Hence,

(e) *Eq.* (3.23) *has a solution* $X > 0$ *if and only if*

$$A_2 \geqslant 0, \quad A_3 \geqslant 0, \ \mathscr{R}(A_2) = \mathscr{R}(B_2), \quad \mathscr{R}(A_3) = \mathscr{R}(B_3).$$

(f) *All solutions of* (3.23) *satisfy* $X > 0$ *if and only if* $A_2 \geqslant 0$, $A_3 \geqslant 0$ *and one of*

$$r(A_2) = r(B_2) = n, \quad r(A_3) = r(B_3) = n.$$

(g) *Eq.* (3.23) *has a solution* $X < 0$ *if and only if*

$$A_2 \leqslant 0, \quad A_3 \leqslant 0, \quad \mathscr{R}(A_2) = \mathscr{R}(B_2), \quad \mathscr{R}(A_3) = \mathscr{R}(B_3).$$

(h) *All solutions of* (3.23) *satisfy* $X < 0$ *if and only if* $A_2 \leqslant 0$, $A_3 \leqslant 0$ *and one of*

$$r(A_2) = r(B_2) = n, \quad r(A_3) = r(B_3) = n.$$

(i) *Eq.* (3.23) *has a solution* $X \geqslant 0$ *if and only if*

$$A_2 \geqslant 0, \quad A_3 \geqslant 0, \quad \mathscr{R}\begin{bmatrix} A_2 \\ 0 \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} 0 & B_2 \\ A_3 & B_3 \end{bmatrix}, \quad \mathscr{R}\begin{bmatrix} 0 \\ A_3 \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} A_2 & B_2 \\ 0 & B_3 \end{bmatrix}.$$

(j) *All solutions of* (3.23) *satisfy* $X \geqslant 0$ *if and only if* $A_2 \geqslant 0$, $A_3 \geqslant 0$ *and one of*

$$r(B_2) = n \ \ \text{and} \ \ r(B_3) = n.$$

(k) *Eq.* (3.23) *has a solution* $X \leqslant 0$ *if and only if*

$$A_2 \leqslant 0, \quad A_3 \leqslant 0, \quad \mathscr{R}\begin{bmatrix} A_2 \\ 0 \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} 0 & B_2 \\ A_3 & B_3 \end{bmatrix}, \quad \mathscr{R}\begin{bmatrix} 0 \\ A_3 \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} A_2 & B_2 \\ 0 & B_3 \end{bmatrix}.$$

(l) *All solutions of* (3.23) *satisfy* $X \leqslant 0$ *if and only if* $A_2 \leqslant 0$, $A_3 \leqslant 0$ *and one of*

$$r(B_2) = n \ \ \text{and} \ \ r(B_3) = n.$$

Proof Set $A_1 = 0$ and $B_1 = I_n$ in Theorem 3.3 and simplifying, we obtain (a)–(d). Applying Lemma 1.2 to (3.49) and (3.50), we obtain (e)–(l). $\square$

Rewrite $B_2XB_2^* = A_2$ and $B_3XB_3^* = A_3$ as

$$[\, B_{21},\, B_{22}\,]\begin{bmatrix} X_1 & X_2 \\ X_2^* & X_3 \end{bmatrix}\begin{bmatrix} B_{21}^* \\ B_{22}^* \end{bmatrix} = A_2, \quad [\, B_{31},\, B_{32}\,]\begin{bmatrix} X_1 & X_2 \\ X_2^* & X_3 \end{bmatrix}\begin{bmatrix} B_{31}^* \\ B_{32}^* \end{bmatrix} = A_3, \tag{3.52}$$

where $B_{i1} \in \mathbb{C}^{m_i \times n_1}$, $B_{i2} \in \mathbb{C}^{m_i \times n_2}$, $i = 2,\, 3$, $X_1 \in \mathbb{C}_{\rm H}^{n_1}$, $X_2 \in \mathbb{C}^{n_1 \times n_2}$ and $X_3 \in \mathbb{C}_{\rm H}^{n_2}$ with $n_1 + n_2 = n$. We next derive the extremal ranks and inertias of the submatrices X_1 and X_3 in a Hermitian solution of (3.52). Note that X_1, X_2, X_3 in (3.52) can be rewritten as

$$X_1 = P_1XP_1^*, \quad X_2 = P_1XP_2^*, \quad X_3 = P_2XP_2^*, \tag{3.53}$$

where $P_1 = [\, I_{n_1},\, 0\,]$ and $P_2 = [\, 0,\, I_{n_2}\,]$. For convenience, we adopt the following notation for the collections of the submatrices X_1 and X_3 in (3.52):

$$\mathcal{S}_1 = \{ X_1 = P_1XP_1^* \mid B_2XB_2^* = A_2,\ B_3XB_3^* = A_3,\ X = X^* \}, \tag{3.54}$$

$$\mathcal{S}_3 = \{ X_3 = P_2XP_2^* \mid B_2XB_2^* = A_2,\ B_3XB_3^* = A_3,\ X = X^* \}. \tag{3.55}$$

The maximum and minimum ranks and inertias of the submatrices X_1 and X_3 in (3.52) can easily be derived from Theorem 3.3. The details are omitted.

If each of the triple matrix equations in (1.6) is not consistent, people may alternatively seek its common approximation solutions under various given optimal criteria. One of the most useful approximation solutions of $BXB^* = A$ is the least-squares Hermitian solution, which is defined to be a Hermitian matrix X that minimizes the norm of the difference $A - BXB^*$:

$$\| \, A - BXB^* \, \|^2 = {\rm tr}[\,(\, A - BXB^* \,)(\, A - BXB^* \,)^* \,] = \min. \tag{3.56}$$

The normal equation corresponding to the norm minimization problem is given by

$$B^*BXB^*B = B^*AB. \tag{3.57}$$

This equation is always consistent. Concerning the common least-squares Hermitian solution of (1.6), we have the following result.

Corollary 3.7 *Let $A_i \in \mathbb{C}_{\rm H}^{m_i \times m_i}$ and $B_i \in \mathbb{C}^{m_i \times n}$ be given for $i = 1,\, 2,\, 3$. Then, triple matrix equations have a common least-squares Hermitian solution, namely, there exists an $X \in \mathbb{C}_{\rm H}^{n \times n}$ such that*

$$\| \, A_i - B_iXB_i^* \, \| = \min, \quad i = 1,\, 2,\, 3, \tag{3.58}$$

if and only if

$$r\begin{bmatrix} B_i^*A_iB_i & 0 & B_i^*B_i \\ 0 & -B_j^*A_jB_j & B_j^*B_j \\ B_i^*B_i & B_j^*B_j & 0 \end{bmatrix} = 2r\begin{bmatrix} B_i \\ B_j \end{bmatrix}, \quad i \neq j, \ i,\, j = 1,\, 2,\, 3, \tag{3.59}$$

$$r\begin{bmatrix} B_1^*A_1B_1 & 0 & 0 & B_1^*B_1 & B_1^*B_1 \\ 0 & -B_2^*A_2B_2 & 0 & B_2^*B_2 & 0 \\ 0 & 0 & -B_3^*A_3B_3 & 0 & B_3^*B_3 \\ B_1^*B_1 & B_2^*B_2 & B_3^*B_3 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} B_1 & B_1 \\ B_2 & 0 \\ 0 & B_3 \end{bmatrix} + r\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}. \tag{3.60}$$

Proof It follows from Lemma 3.1, Corollary 3.5 and (3.57). $\square$

Some further research problems can be proposed. For example, a challenging task is to give the closed-form for the general common solution of $B_2XB_2^* = A_2$ and $B_3XB_3^* = A_3$ that satisfies $X > 0$ ($< 0,\ \geqslant 0,\ \leqslant 0$), which is equivalent to solving the inequalities

$$X_0 + VF_B + F_BV^* + F_{B_2}UF_{B_3} + F_{B_3}U^*F_{B_2} > 0\,(< 0,\ \geqslant 0,\ \leqslant 0). \tag{3.61}$$

Further, it would be of interest to consider the following optimization problems:

(a) Maximize and minimize the rank and inertia of the Hermitian matrix expression $A_1 - B_1XB_1^*$ subject to $k-1$ consistent Hermitian matrix equations $(\, B_2XB_2^*, \ldots, B_kXB_k^* \,) = (\, A_2, \ldots, A_k \,)$, and to give necessary and sufficient condition for the set of matrix equations $(\, B_1XB_1^*, \ldots, B_kXB_k^* \,) = (\, A_1, \ldots, A_k \,)$ to have a common Hermitian solution.

(b) Maximize and minimize the rank and inertia of the Hermitian matrix expression $A_1 - B_1XB_1^*$ subject to the matrix inequality $B_2XB_2^* \geqslant A_2$.

4 Conclusions

In this paper, we studied the problems of maximizing/minimizing the rank/inertia of the constrained matrix expression in (1.2), and obtained some closed-form formulas for the extremal ranks/inertias of (1.2) by pure algebraic operations of matrices and their generalized inverses. As direct applications, we gave necessary and sufficient conditions for the existence of X satisfying the triple matrix equations in (1.6), as well as some matrix inequalities. Although the problems of maximizing/minimizing the rank/inertia are regarded as NP-hard, the results obtained in this paper and [12, 14, 15, 16, 17, 28, 29, 30, 32, 33] show that closed-form formulas for the extremal ranks/inertias of some simpler matrix expressions can be derived, while these closed-form formulas can be used to solve some fundamental problems in matrix theory, as mentioned in the beginning of this paper. All the results obtained in these papers are brand-new and beyond our conventional understanding on linear matrix expressions. This series of researches show that for many basic or classic problems like solvability of matrix equations and matrix inequalities, we are still able to develop some new methods and use them to derive a variety of innovative results.

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