Inner approximations for polynomial matrix inequalities and robust stability regions

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Abstract

Following a polynomial approach, many robust fixed-order controller design problems can be formulated as optimization problems whose set of feasible solutions is modelled by parametrized polynomial matrix inequalities (PMI). These feasibility sets are typically nonconvex. Given a parametrized PMI set, we provide a hierarchy of linear matrix inequality (LMI) problems whose optimal solutions generate inner approximations modelled by a single polynomial sublevel set. Those inner approximations converge in a strong analytic sense to the nonconvex original feasible set, with asymptotically vanishing conservatism. One may also impose the hierarchy of inner approximations to be nested or convex. In the latter case they do not converge any more to the feasible set, but they can be used in a convex optimization framework at the price of some conservatism. Finally, we show that the specific geometry of nonconvex polynomial stability regions can be exploited to improve convergence of the hierarchy of inner approximations.

Keywords: polynomial matrix inequality, linear matrix inequality, robust optimization, robust fixed-order controller design, moments, positive polynomials.

1 Introduction

Linear system stability can be formulated semialgebraically in the space of coefficients of the characteristic polynomial. The region of stability is generally *nonconvex* in this space, and this is a major obstacle when solving fixed-order and/or robust controller design problems. Using the Hermite stability criterion, these problems can be formulated as parametrized polynomial matrix inequalities (PMIs) where parameters account for uncertainties and the decision variables are controller coefficients. Recent results on real algebraic geometry and generalized problems of moments can be used to build up a

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hierarchy of convex linear matrix inequality (LMI) *outer* approximations of the region of stability, with asymptotic convergence to its convex hull, see e.g. [5] for a software implementation and examples, and see [6] for an application to PMI problems arising from static output feedback design.

If outer approximations of nonconvex semialgebraic sets can be readily constructed with these LMI relaxations, *inner* approximations are much harder to obtain. However, for controller design purposes, inner approximations are essential since they correspond to sufficient conditions and hence guarantees of stability or robust stability. In the robust systems control literature, convex inner approximations of the stability region have been proposed in the form of polytopes [13], ellipsoids [3] or more general LMI regions [4, 8] derived from polynomial positivity conditions.

In this paper we provide a numerical scheme for approximating from inside the feasible set $\mathbf{P} \subset \mathbb{R}^n$ of a parametrized PMI $P(x,u) \succeq 0$ (for some matrix polynomial P), that is, the set of points x such that $P(x,u) \succeq 0$ for *all* values of the parameter u in some specified domain $\mathbf{U} \subset \mathbb{R}^p$ (assumed to be a basic compact semialgebraic set). This includes as a special case the approximation of the stability region (and the robust stability region) of linear systems. The particular case where $P(x,u)$ is affine in x covers parametrized LMIs with many applications in robust control, as surveyed e.g. in [14].

Given a compact set $\mathbf{B} \subset \mathbb{R}^n$ containing $\mathbf{P}$, this numerical scheme consists of building up a sequence of inner approximations $\mathbf{G}_d \subset \mathbf{P} \subset \mathbf{B}$, $d \in \mathbb{N}$, which fulfils two essential conditions:

1. The approximation is in a *strong* analytic sense (hence efficient);
2. Each set $\mathbf{G}_d$ is defined in a *simple* manner, as a level set of a single polynomial. In our mind, this feature is essential for a successful implementation in practical applications.

More precisely, we provide a hierarchy of inner approximations $(\mathbf{G}_d)$ of $\mathbf{P}$, where each $\mathbf{G}_d = \{x \in \mathbf{B} : g_d(x) \geq 0\}$ is a basic semi-algebraic set for some polynomial g_d of degree d. The vector of coefficients of the polynomial g_d is an optimal solution of an LMI problem. When d increases, the convergence of $(\mathbf{G}_d)$ to $\mathbf{P}$ is very strong. Indeed, the Lebesgue volume of $\mathbf{G}_d$ converges to the Lebesgue volume of $\mathbf{P}$. In fact, on any (a priori fixed) compact set $\mathbf{B}$, the sequence (g_d) converges for the L_1-norm on $\mathbf{B}$ to the function $x \mapsto \lambda_{\min}(x) = \min_{u \in \mathbf{U}} \lambda_{\min}(x,u)$ where $\lambda_{\min}(x,u)$ is the minimum eigenvalue of the matrix-polynomial $P(x,u)$ associated with the PMI. Consequently, $g_d \to \lambda_{\min}$ in (Lebesgue) measure on $\mathbf{B}$, and $g_{d_k} \to \lambda_{\min}$ almost everywhere and almost uniformly on $\mathbf{B}$, for a subsequence (g_{d_k}). In addition, if one defines the piecewise polynomial $\bar{g}_d := \max_{k \leq d} g_k$, then $\bar{g}_d \to \lambda_{\min}$ almost everywhere almost uniformly and in (Lebesgue) measure on $\mathbf{B}$.

Finally, we can easily enforce that the inner approximations $(\mathbf{G}_d)$ are nested and/or convex. Of course, for the latter convex approximations, convergence to $\mathbf{P}$ is lost if $\mathbf{P}$ is not convex. However, on the other hand, having a convex inner approximation of $\mathbf{P}$ may reveal to be very useful, e.g., for optimization purposes.

On the practical and computational sides, the quality of the approximation of $\mathbf{P}$ depends heavily on the chosen set $\mathbf{B} \supset \mathbf{P}$ on which to make the approximation of the function

$\lambda_{\min}$. The smaller $\mathbf{B}$, the better the approximation. In particular, it is worth emphasizing that when the set $\mathbf{P}$ to approximate is the stability or robust stability region of a linear system, then its particular geometry can be exploited to construct a tight bounding set $\mathbf{B}$. Therefore, a good approximation of $\mathbf{P}$ is obtained significantly faster than with an arbitrary set $\mathbf{B}$ containing $\mathbf{P}$.

The outline of the paper is as follows. In Section 2 we formally state the problem to be solved. In Section 3 we describe our hierarchy of inner approximations. In Section 4, we show that the specific geometry of the stability region can be exploited, as illustrated on several standard problems of robust control. The final section collects technical results and the proofs.

2 Problem statement

Let $\mathbb{R}[x]$ denote the ring or real polynomials in the variables $x = (x_1, \ldots, x_n)$, and let $\mathbb{R}[x]_d$ be the vector space of real polynomials of degree at most d. Similarly, let $\Sigma[x] \subset \mathbb{R}[x]$ denote the convex cone of real polynomials that are sums of squares (SOS) of polynomials, and $\Sigma[x]_d \subset \Sigma[x]$ its subcone of SOS polynomials of degree at most $2d$. Denote by $\mathbb{S}^m$ the space of $m \times m$ real symmetric matrices. For a given matrix $A \in \mathbb{S}^m$, the notation $A \succeq 0$ means that A is positive semidefinite, i.e., all its eigenvalues are real and nonnegative.

Let $P : \mathbb{R}[x, u] \to \mathbb{S}^m$ be a matrix polynomial, i.e. a matrix whose entries are scalar multivariate polynomials of the vector indeterminates x and u. Then

$$\mathbf{P} := \{x \in \mathbb{R}^n \; : \; \forall u \in \mathbf{U}, \; P(x, u) \succeq 0\} \tag{1}$$

defines a parametrized polynomial matrix inequality (PMI) set, where $x \in \mathbb{R}^n$ is a vector of decision variables, $u \in \mathbb{R}^p$ is a vector of uncertain parameters belonging to a compact semialgebraic set

$$\mathbf{U} := \{u \in \mathbb{R}^p \; : \; a_i(u) \geq 0, \; i = 1, \ldots, n_a\} \tag{2}$$

described by given polynomials $a_i(u) \in \mathbb{R}[u]$, and $P(x, u)$ is a given symmetric polynomial matrix of size m. As $\mathbf{U}$ is compact, without loss of generality we assume that for some $i = i^*$, $a_{i^*}(u) = M^2 - u^T u$, where M is sufficiently large.

We also assume that $\mathbf{P}$ is bounded and that we are given a compact set $\mathbf{B} \supset \mathbf{P}$ with explicitly known moments $y = (y_\alpha)$, $\alpha \in \mathbb{N}^n$, of the Lebesgue measure on $\mathbf{B}$, i.e.

$$y_\alpha := \int_{\mathbf{B}} x^\alpha dx \tag{3}$$

where $x^\alpha := \prod_{i=1}^n x_i^{\alpha_i}$. Typical choices for $\mathbf{B}$ are a box or a ball. To fix ideas, let

$$\mathbf{B} \; := \; \{x \in \mathbb{R}^n \; : \; b_j(x) \geq 0, \; j = 1, \ldots, n_b\}$$

for some polynomials $b_j \in \mathbb{R}[x]$. Again, with no loss of generality, we may and will assume that for some $j = j^*$, $b_{j^*}(x) = M^2 - x^T x$, where M is sufficiently large. Finally, denote by $\operatorname{vol} \mathbf{A}$ the Lebesgue volume of any Borel set $\mathbf{A} \subset \mathbf{B}$.

We are now ready to state our polynomial inner approximation problem.

Problem 1 (Inner Approximations) *Given set* $\mathbf{P}$*, build up a sequence of basic closed semialgebraic sets* $\mathbf{G}_d = \{x \in \mathbf{B} \,:\, g_d(x) \geq 0\}$*, for some* $g_d \in \mathbb{R}[x]$*, such that*

$$\mathbf{G}_d \subseteq \mathbf{P} \quad \textit{and} \quad \lim_{d\to\infty} \operatorname{vol} \mathbf{G}_d \;=\; \operatorname{vol} \mathbf{P}.$$

In addition, we may want the sequence of inner approximations to satisfy additional nesting or convexity conditions.

Problem 2 (Nested Inner Approximations) *Solve Problem 1 with the additional constraint*

$$\mathbf{G}_d \;\subseteq\; \mathbf{G}_{d+1} \;\subseteq\; \mathbf{P}, \quad d = 1, 2, \ldots$$

Problem 3 (Convex Inner Approximations) *Given set* $\mathbf{P}$*, build up a sequence of nested basic closed convex semialgebraic sets* $\mathbf{G}_d = \{x \in \mathbf{B} \,:\, g_d(x) \geq 0\}$*, for some* $g_d \in \mathbb{R}[x]$*, such that*

$$\mathbf{G}_d \;\subseteq\; \mathbf{G}_{d+1} \;\subseteq\; \mathbf{P}, \quad d = 1, 2, \ldots$$

3 A hierarchy of semialgebraic inner approximations

Given a polynomial matrix $P(x,u)$ which defines the set $\mathbf{P}$ in (1), polynomials $h_i \in \mathbb{R}[u]$ which define the uncertain set $\mathbf{U}$ in (2), let $\mathbf{V} = \{v \in \mathbb{R}^m \,:\, v^T v \leq 1\}$ denote the unit Euclidean ball of $\mathbb{R}^m$ and let $\lambda_{\min} : \mathbf{B} \to \mathbb{R}$ be the function:

$$x \mapsto \lambda_{\min}(x) = \min_{u\in\mathbf{U}} \min_{v\in\mathbf{V}} v^T P(x,u) v \tag{4}$$

as the robust minimum eigenvalue function of $P(x,u)$. Function $\lambda_{\min}$ is algebraic, continuous, but not necessarily differentiable. It allows to define set $\mathbf{P}$ alternatively as the sublevel set

$$\mathbf{P} = \{x \in \mathbb{R}^n \,:\, \lambda_{\min}(x) \geq 0\}.$$

Let $a_0 \in \mathbb{R}[u]$ be the constant polynomial 1, $d_{a_i} := \lceil (\deg a_i)/2 \rceil$, $i = 0, 1, \ldots, n_a$ and $d_{b_j} := \lceil (\deg b_j)/2 \rceil$, $j = 1, \ldots, n_b$. Let $2d_0 \geq \max(2 + \deg P, \max_i \deg a_i, \max_j \deg b_j)$, and consider the hierarchy of convex optimization problems indexed by the parameter $d \in \mathbb{N}$, $d \geq d_0$:

$$\begin{aligned} \rho_d \;=\; \min_{g,r,s,t} \quad & \int_{\mathbf{B}} (\lambda_{\min}(x) - g(x))\, dx \\ \text{s.t.} \quad & v^T P(x,u) v - g(x) \;=\; r(x,u,v)(1 - v^T v) \\ & + \sum_{i=0}^{n_a} s_i(x,u,v) a_i(u) + \sum_{j=1}^{n_b} t_j(x,u,v) b_j(x) \quad \forall (x,u,v) \end{aligned} \tag{5}$$

where decision variables are coefficients of polynomials $g \in \mathbb{R}[x]_{2d}$, and coefficients of SOS polynomials $r \in \Sigma[x,u,v]_{d-1}$, $s_i \in \Sigma[x,u,v]_{d-d_{a_i}}$, and $t_j \in \Sigma[x,u,v]_{d-d_{b_j}}$.

For each $d \in \mathbb{N}$ fixed, the associated optimization problem (5) is a semidefinite programming (SDP) problem. Indeed, stating that the two polynomials in both sides of the equation in (5) are identical translates into linear equalities between the coefficients of polynomials $g, r, (s_i), (t_j)$ and stating that some of them are SOS translates into semidefiniteness of appropriate symmetric matrices. For more details, the interested reader is referred to e.g. [10, Chapter 2].

Before stating our main results, let us recall some standard notions of functional analysis. Let $g : \mathbf{B} \to \mathbb{R}$ be a function of x, and let (g_d) denote a sequence of functions of x indexed by $d \in \mathbb{N}$. Lebesgue space $L_1(\mathbf{B})$ is the Banach space of integrable functions on $\mathbf{B}$ equipped with the norm

$$\|g\|_1 = \int_{\mathbf{B}} |g| dx.$$

Regarding sequence (g_d), we use the following notions of convergence in $\mathbf{B}$ when $d \to \infty$:

- $g_d \to g$ in L_1 norm means $\lim_{d\to\infty} \|g - g_d\|_1 = 0$;
- $g_d \to g$ in Lebesgue measure means that for every $\varepsilon > 0$,
$$\lim_{d\to\infty} \operatorname{vol}\{x : |g(x) - g_d(x)| \geq \varepsilon\} = 0;$$
- $g_d \to g$ almost everywhere means that $\lim_{d\to\infty} g_d(x) = g(x)$ pointwise except possibly for $x \in \mathbf{A} \subset \mathbf{B}$ with $\operatorname{vol} \mathbf{A} = 0$;
- $g_d \to g$ almost uniformly means that given $\varepsilon > 0$, there is a set $\mathbf{A} \subset \mathbf{B}$ such that $\operatorname{vol} \mathbf{A} < \varepsilon$ and $g_d \to g$ uniformly on $\mathbf{B} \setminus \mathbf{A}$;
- finally, with the notation $g_d \uparrow g$ we mean that $g_d \to g$ while satisfying $g_d(x) \leq g_{d+1}(x)$ for all d.

For more details on these related notions of convergence, see [1, §2.5].

Lemma 1 *For every $d \geq d_0$, SDP problem (5) has an optimal solution $g_d \in \mathbb{R}[x]_{2d}$ and*

$$\rho_d = \int_{\mathbf{B}} (\lambda_{\min}(x) - g_d(x))\, dx = \|\lambda_{\min} - g_d\|_1. \tag{6}$$

A detailed proof of Lemma 1 can be found in §6.2.

For every $d \geq d_0$, let $\bar{g}_d : \mathbf{B} \to \mathbb{R}$ be the piecewise polynomial

$$x \mapsto \bar{g}_d(x) := \max_{d_0 \leq k \leq d} g_k(x). \tag{7}$$

We are now in position to prove our main result.

Theorem 1 *Let $g_d \in \mathbb{R}[x]_{2d}$ be an optimal solution of SDP problem (5) and consider the associated sequence $(g_d) \subset L_1(\mathbf{B})$ for $d \geq d_0$. Then:*

(a) $g_d \to \lambda_{\min}$ *in L_1 norm and in Lebesgue measure.*

(b) $\bar{g}_d \uparrow \lambda_{\min}$ *almost everywhere, almost uniformly and in Lebesgue measure.*

A proof can be found in §6.3.

3.1 Polynomial and piecewise polynomial inner approximations

Corollary 1 *For every $d \geq d_0$, let $g_d \in \mathbb{R}[x]_{2d}$ be an optimal solution of SDP problem (5), let $\bar{g}_d$ be the piecewise polynomial defined in (7), and let*

$$\mathbf{G}_d := \{x \in \mathbf{B} : g_d(x) \geq 0\}, \quad \bar{\mathbf{G}}_d := \{x \in \mathbf{B} : \bar{g}_d(x) \geq 0\}. \tag{8}$$

Then

$$\mathbf{G}_d \subset \mathbf{P} \quad \forall\, d \geq d_0 \quad \textit{and} \quad \lim_{d\to\infty} \mathrm{vol}(\mathbf{P} \setminus \mathbf{G}_d) = 0. \tag{9}$$

$$\bar{\mathbf{G}}_{d_0} \subseteq \bar{\mathbf{G}}_1 \subseteq \cdots \subseteq \bar{\mathbf{G}}_d \subseteq \cdots \subset \mathbf{P} \quad \textit{and} \quad \lim_{d\to\infty} \mathrm{vol}(\mathbf{P} \setminus \bar{\mathbf{G}}_d) = 0. \tag{10}$$

That is, sequence $(\mathbf{G}_d)$ solves Problem 1 and sequence $(\bar{\mathbf{G}}_d)$ solves Problem 2 if piecewise polynomials are allowed.

A proof can be found in §6.4.

3.2 Nested polynomial inner approximations

We now consider Problem 2 where g_d is constrained to be a polynomial instead of a piecewise polynomial. We need to slightly modify SDP problem (5). Suppose that at step $d-1$ in the hierarchy we have already obtained an optimal solution $g_{d-1} \in \mathbb{R}[x]_{2d-2}$, such that $g_{d-1} \geq g_{d_0}$ on $\mathbf{B}$, for all $d_0 \leq d-1$. At step d we now solve SDP problem (5) with the additional constraint

$$g(x) - g_{d-1}(x) \;=\; c_0(x) + \sum_{j=1}^{n_b} c_j(x) b_j(x), \quad \forall x \tag{11}$$

with unknown SOS polynomials $c_0 \in \Sigma[x]_d$ and $c_j \in \Sigma[x]_{d-d_{b_j}}$.

Corollary 2 *Let $g_d \in \mathbb{R}[x]_{2d}$ be an optimal solution of SDP problem (5) with the additional constraint (11) and let $\mathbf{G}_d$ be as in (8) for $d \geq d_0$. Then the sequence $(\mathbf{G}_d)$ solves Problem 2.*

For a proof see §6.5.

3.3 Convex nested polynomial inner approximations

Finally, for $g \in \mathbb{R}[x]_{2d}$, denote by $\nabla^2 g(x)$ the Hessian matrix of g at x, and consider SDP problem (5) with the additional constraint

$$v^T \nabla^2 g(x) v = c_0(x,v) + \sum_{j=1}^{n_b} c_j(x,v) b_j(x) + c_{n_b+1}(x,v)(1 - v^T v), \tag{12}$$

for some SOS polynomials $c_0 \in \Sigma[x,v]_d$, $c_j \in \Sigma[x,v]_{d-d_{b_j}}$ and $c_{n_b+1} \in \Sigma[x,v]_{d-1}$.

Corollary 3 *Let $g \in \mathbb{R}[x]_{2d}$ be an optimal solution of SDP problem (5) with the additional constraint (12) and let $\mathbf{G}_d$ be as in (8) for $d \geq d_0$. Then the sequence $(\mathbf{G}_d)$ solves Problem 3.*

The proof follows along the same lines as the proof of Corollary 2.

3.4 Example

Consider the nonconvex planar PMI set

$$\mathbf{P} = \{x \in \mathbb{R}^2 \,:\, P(x) = \begin{bmatrix} 1 - 16x_1x_2 & x_1 \\ x_1 & 1 - x_1^2 - x_2^2 \end{bmatrix} \succeq 0\}$$

which is Example II-E in [6] scaled to fit within the unit box

$$\mathbf{B} = \{x \in \mathbb{R}^2 \,:\, \|x\|_\infty \leq 1\}$$

whose moments (3) are readily given by

$$y_\alpha = \frac{4}{(\alpha_1 + 1)(\alpha_2 + 1)}.$$

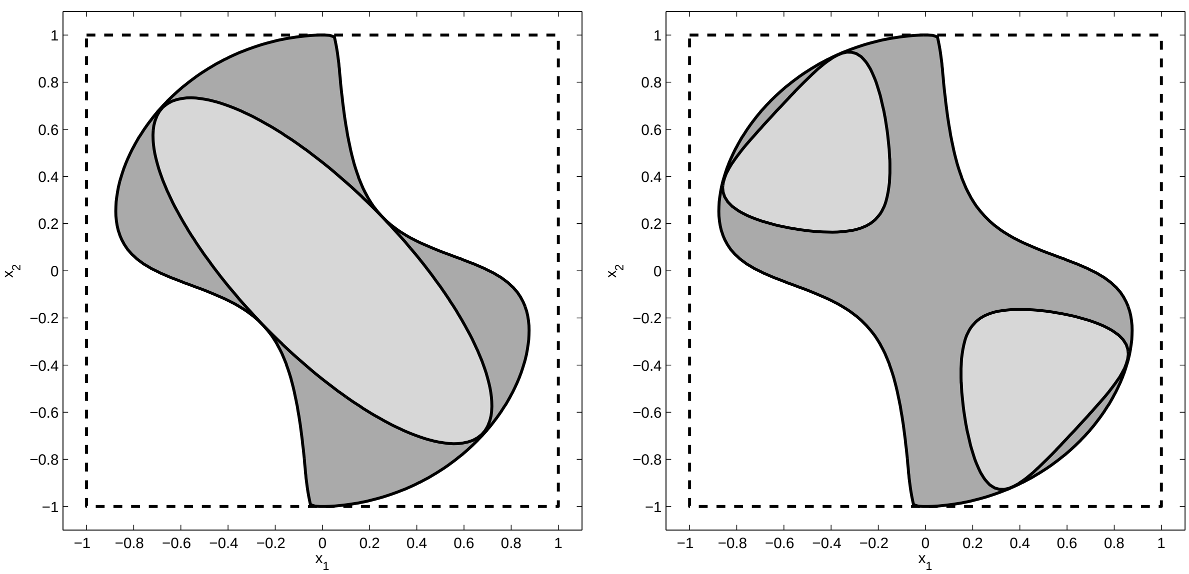


Figure 1: Degree two (left) and four (right) inner approximations (light gray) of PMI set (dark gray) embedded in unit box (dashed).

On Figure 1 we represent the degree two and degree four solutions to SDP problem (5), modelled by YALMIP 3 and solved by SeDuMi 1.3 under a Matlab environment. We see in particular that the degree four polynomial sublevel set $\mathbf{G}_2$ is somewhat smaller than expected. This is due to the fact that the objective function in problem (5) is the integral of $g(x)$ over the whole box $\mathbf{B}$, not only over PMI set $\mathbf{P}$. There is a significant role played by the components of the integral on complement set $\mathbf{B}\backslash\mathbf{P}$, and this deteriorates the inner approximation.

This issue can be address partly by embedding $\mathbf{P}$ in a tighter set $\mathbf{B}$, for example here the unit disk

$$\mathbf{B} = \{x \in \mathbb{R}^2 \ : \ \|x\|_2 \leq 1\}$$

whose moments (3) are given by

$$y_\alpha = \frac{\Gamma(\frac{\alpha_1+1}{2})\Gamma(\frac{\alpha_2+1}{2})}{\Gamma(2+\frac{\alpha_1+\alpha_2}{2})}$$

where Γ is the gamma function such that $\Gamma(k) = (k-1)!$ for integer k. See [11, Theorem 3.1] for the general expression[1] of moments of the unit disk in $\mathbb{R}^n$.

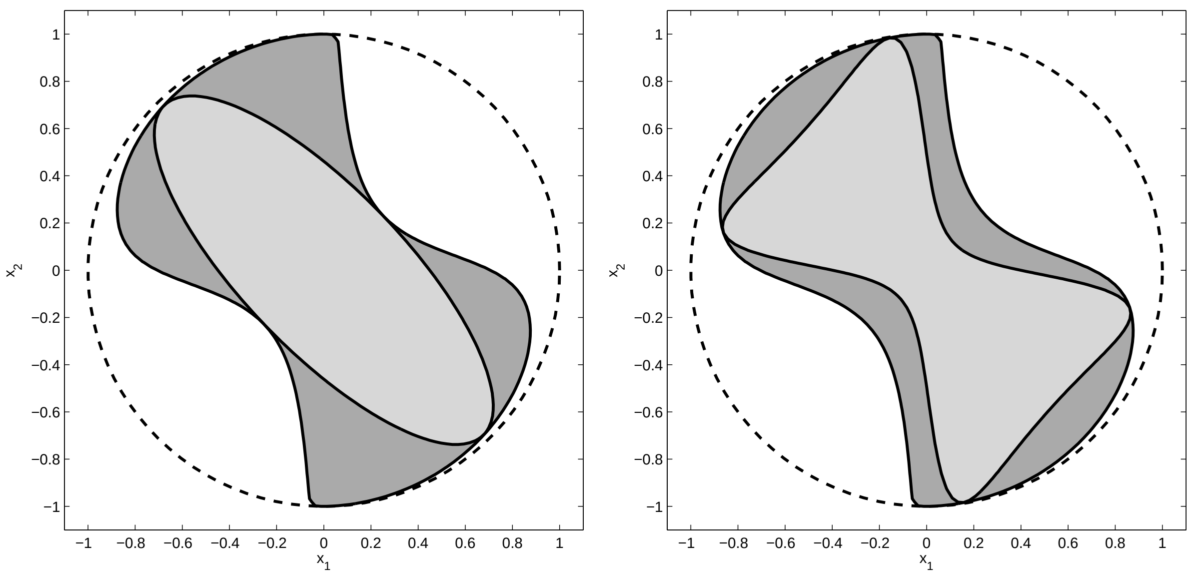


Figure 2: Degree two (left) and four (right) inner approximations (light gray) of PMI set (dark gray) embedded in unit disk (dashed).

On Figure 2 we represent the degree two and degree four solutions to SDP problem (5). Comparing with Figure 1, we see that the approximations embedded in the unit disk are much tighter than the approximations embedded in the unit box. Finally, on Figure 3 we represent the tighter degree six and degree eight inner approximations within the unit disk.

4 Geometry of control problems

As explained in the introduction, inner approximations of the stability regions are essential for fixed-order controller design. The PMI regions arising from parametric stability

[1]Note however that there is an incorrect factor 2^{-n} in the right handside of equation (3.3) in [11].

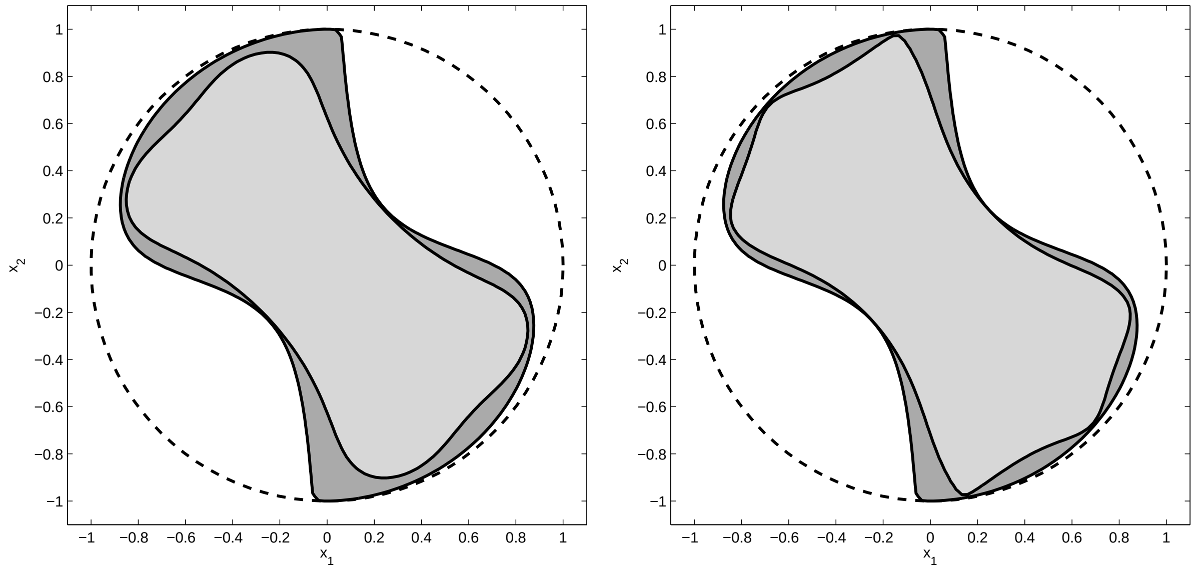


Figure 3: Degree six (left) and eight (right) inner approximations (light gray) of PMI set (dark gray) embedded in unit disk (dashed).

conditions have a specific geometry that can be exploited to improve the convergence of the hierarchy of inner approximations. In this section, we first recall Hermite's PMI formulation of (discrete-time) stability conditions. Then we recall that the PMI stability region is the image of a unit box through a multi-affine mapping, which allows to derive explicit expressions for the moments of the full-dimensional stability region, as well as tight polytopic outer approximations of low-dimensional affine sections of the stability region. Numerical examples illustrate these techniques for fixed-order nominal and robustly stabilizing controller design.

4.1 Hermite's PMI

Derived by the French mathematician Charles Hermite in 1854, the Hermite matrix criterion is a symmetric version of the Routh-Hurwitz criterion for assessing stability of a polynomial. Originally it was derived for locating the roots of a polynomial in the open upper half of the complex plane, but with a fractional transform it can be readily transposed to the open unit disk and discrete-time stability. The criterion says that a polynomial $x(z) = z^n + x_1 z^{n-1} + \cdots + x_{n-1} z + x_n$ has all its roots in the open unit disk if and only if its Hermite matrix $P(x) = T_1^T(x)T_1(x) - T_2^T(x)T_2(x)$ is positive definite, where

$$T_1(x) = \begin{bmatrix} 1 & x_1 & x_2 & \\ 0 & 1 & x_1 & \\ 0 & 0 & 1 & \\ & & & \ddots \end{bmatrix} \quad T_2(x) = \begin{bmatrix} x_n & x_{n-1} & x_{n-2} & \\ 0 & x_n & x_{n-1} & \\ 0 & 0 & x_n & \\ & & & \ddots \end{bmatrix}$$

are n-by-n upper-right triangular Toeplitz matrices, see e.g. the entrywise formulas of [2, Theorem 3.13] or the construction explained in [3]. The Hermite matrix is n-by-n, symmetric and quadratic in coefficients $x = (x_1, x_2, \ldots, x_n)$, so that the interior of the

PMI set

$$\mathbf{P} = \{x \in \mathbb{R}^n \,:\, P(x) \succeq 0\}$$

is the parametric stability domain which is bounded, connected but nonconvex for $n \geq 3$. Optimal controller design amounts to optimizing over semialgebraic set $\mathbf{P}$.

4.2 Multiaffine mapping of the unit box

As explained e.g. in [13] or [15, §3.5] and references therein, stability domain $\mathbf{P}$ can also be constructed as the image of the unit box (in the space of so-called reflection coefficients) through a multiaffine mapping. More explicitly $\mathbf{P} = f(\mathbf{K})$ where $\mathbf{K} = \{k \in \mathbb{R}^n \,:\, \|k\|_\infty \leq 1\}$ and multiaffine mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$\begin{aligned} f(k) &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & k_3 \\ 0 & 1 & k_3 & 0 \\ 0 & k_3 & 1 & 0 \\ k_3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & k_2 \\ 0 & 1+k_2 & 0 \\ k_2 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & k_1 \\ k_1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} k_2k_3 + k_1(1+k_2) \\ k_2 + k_1k_3(1+k_2) \\ k_3 \end{bmatrix} \end{aligned}$$

in the case $n = 3$. The general expression of f for other values of n is not given here for space reasons, but it follows readily from the construction outlined above.

Using this mapping we can obtain moments (3) of $\mathbf{B} = \mathbf{P}$ analytically:

$$y_\alpha = \int_{\mathbf{P}} dx = \int_{\mathbf{K}} (k_2k_3 + k_1(1+k_2))^{\alpha_1}(k_2 + k_1k_3(1+k_2))^{\alpha_2} k_3^{\alpha_3} \det \nabla f(k) dk \qquad (13)$$

where $\det \nabla f(k) = (1+k_2)(1-k_3^2)$ is the determinant of the Jacobian of f, in the case $n = 3$. For space reasons we do not give here the explicit value of y_α as a function of α, but it can be obtained by integration by parts.

Finally, let us mention a well-known geometric property of $\mathbf{P}$: its convex hull is a polytope whose vertices correspond to the $n+1$ polynomials with roots equal to -1 or $+1$. For example, when $n = 3$, we have

$$\operatorname{conv} \mathbf{P} = \operatorname{conv}\{(-3,3,-1),\ (-1,-1,1),\ (1,-1,-1),\ (3,3,1)\}. \qquad (14)$$

4.3 Third degree stability region

Consider the problem of approximating from the inside the nonconvex stability region $\mathbf{P}$ of a discrete-time third degree polynomial $z \mapsto z^3 + x_1z^2 + x_2z + x_3$. An ellipsoidal inner approximation was proposed in [3]. The Hermite polynomial matrix defining $\mathbf{P}$ as in (1) is given by

$$P(x) = \begin{bmatrix} 1 - x_3^2 & x_1 - x_2x_3 & x_2 - x_1x_3 \\ x_1 - x_2x_3 & 1 + x_1^2 - x_2^2 - x_3^2 & x_1 - x_2x_3 \\ x_2 - x_1x_3 & x_1 - x_2x_3 & 1 - x_3^2 \end{bmatrix}.$$

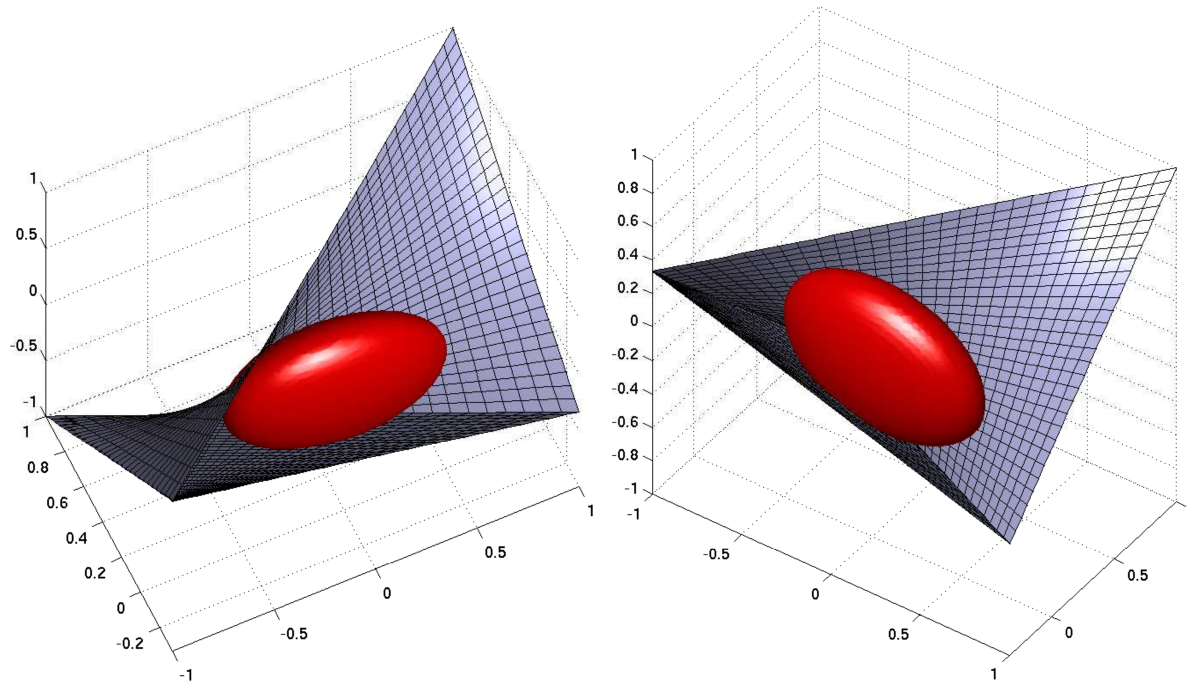


Figure 4: Two views of a degree two inner approximation (red) of nonconvex third-degree stability region (gray).

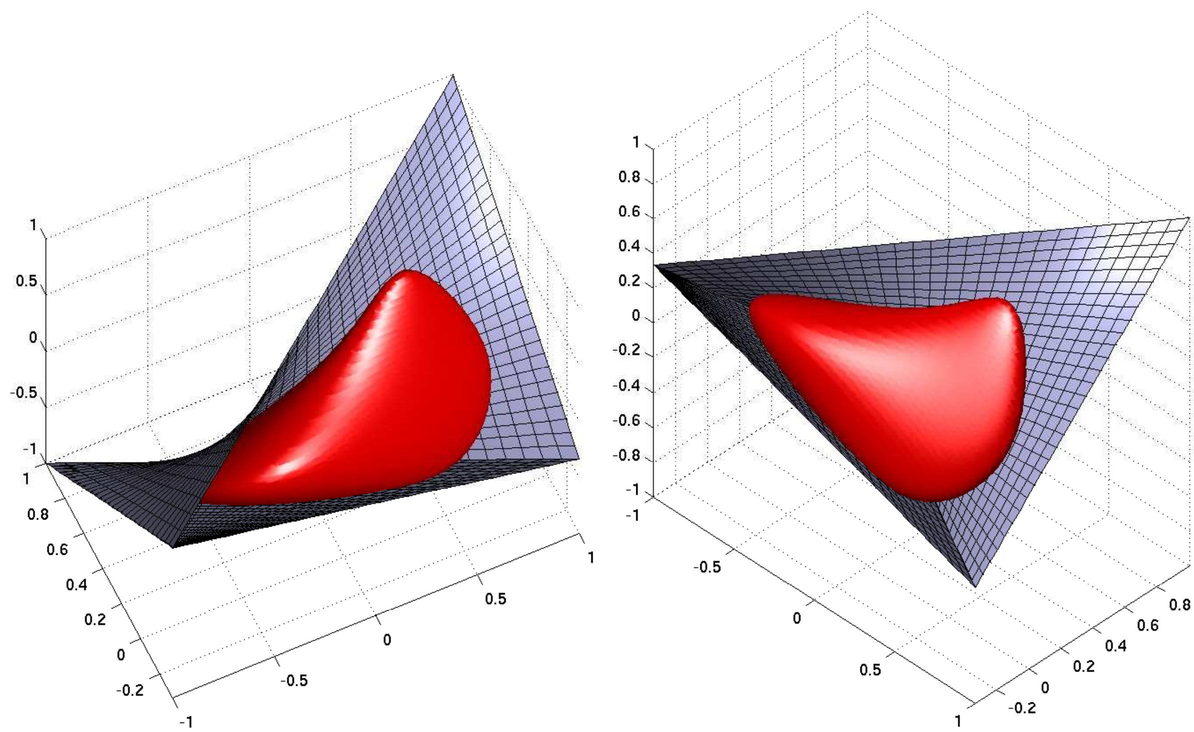
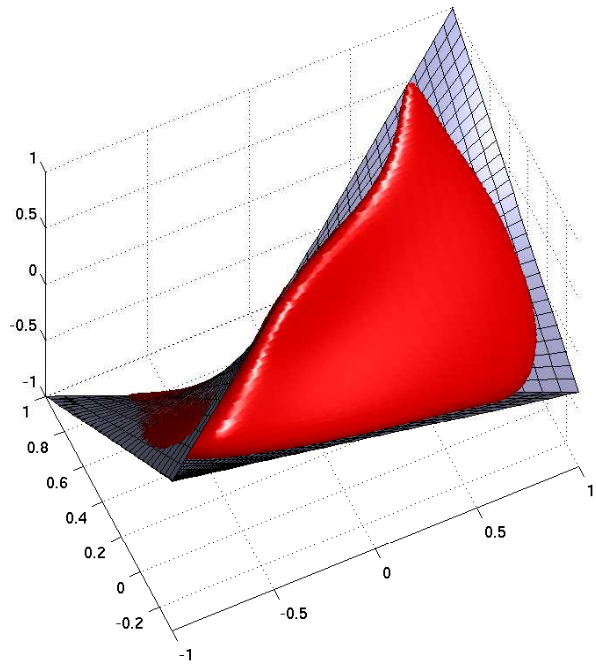


Figure 5: Two views of a degree four inner approximation (red) of nonconvex third-degree stability region (gray).



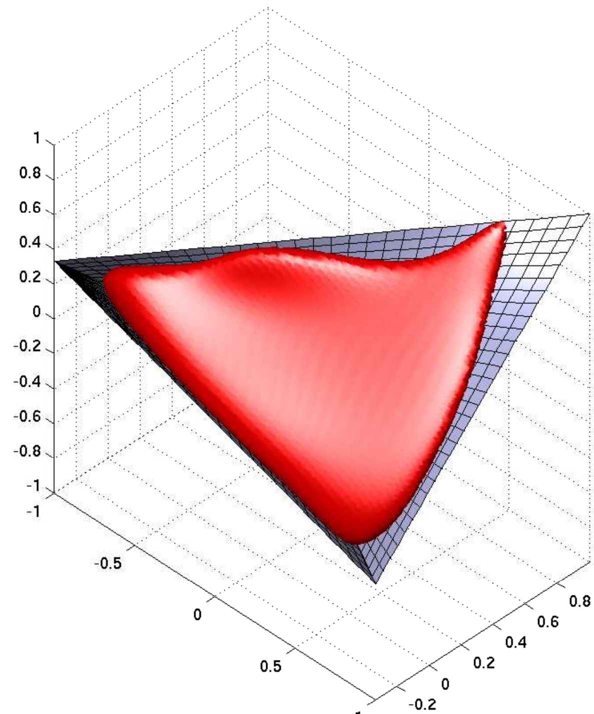


Figure 6: Two views of a degree six inner approximation (red) of nonconvex third-degree stability region (gray).

The boundary of $\mathbf{P}$ consists of two triangles and a hyperbolic paraboloid. The convex hull of $\mathbf{P}$ is the simplex described in (14). We have analytic expressions (13) for the moments (3) of $\mathbf{B} = \mathbf{P}$.

On Figures 4, 5 and 6 we respectively represent the degree two, four and six inner approximations of $\mathbf{P}$, scaled within the unit box for visualization purposes. We observe that the degree six approximation is very tight, thanks to the availability of the moments of the Lebesgue measure on $\mathbf{P}$.

4.4 Fixed-order controller design

Consider the linear discrete-time system with characteristic polynomial $z \mapsto z^4 - (2x_1 + x_2)z^3 + 2x_1 z + x_2$ depending affinely on two real design parameters x_1 and x_2. It follows from Hermite's stability criterion that this polynomial has its roots in the open unit disk if and only if

$$P(x) = \begin{bmatrix} 1-x_2^2 & -2x_1-x_2-2x_1x_2 & 0 & 2x_1+2x_1x_2+x_2^2 \\ -2x_1-x_2-2x_1x_2 & 1+4x_1x_2 & -2x_1-x_2-2x_1x_2 & 0 \\ 0 & -2x_1-x_2-2x_1x_2 & 1+4x_1x_2 & -2x_1-x_2-2x_1x_2 \\ 2x_1+2x_1x_2+x_2^2 & 0 & -2x_1-x_2-2x_1x_2 & 1-x_2^2 \end{bmatrix}$$

is positive definite. As recalled in (4.2), the convex hull of the four-dimensional stability domain of a degree four polynomial is the simplex with vertices $(-4, 6, -4, 1)$, $(-2, 0, 2, -1)$, $(0, -2, 0, 1)$, $(2, 0, -2, -1)$, $(4, 6, 4, 1)$ corresponding to the five polynomials with zeros equal to -1 or $+1$. Using elementary linear algebra, we find out that the image of this simplex through the affine mapping $(-(2x_1 + x_2), 0, 2x_1, x_2)$ parametrized by $x \in \mathbb{R}^2$ is the two-dimensional simplex

$$\mathbf{B} = \text{conv}\{(-\frac{1}{4}, 1), (\frac{7}{8}, -\frac{1}{2}), (-\frac{5}{8}, -\frac{1}{2})\}.$$

The (closure of the) stability region $\mathbf{P} = \{x \in \mathbb{R}^2 : P(x) \succeq 0\}$ is therefore included in $\mathbf{B}$, whose moments (3) are readily obtained e.g. by the explicit formulas of [9].

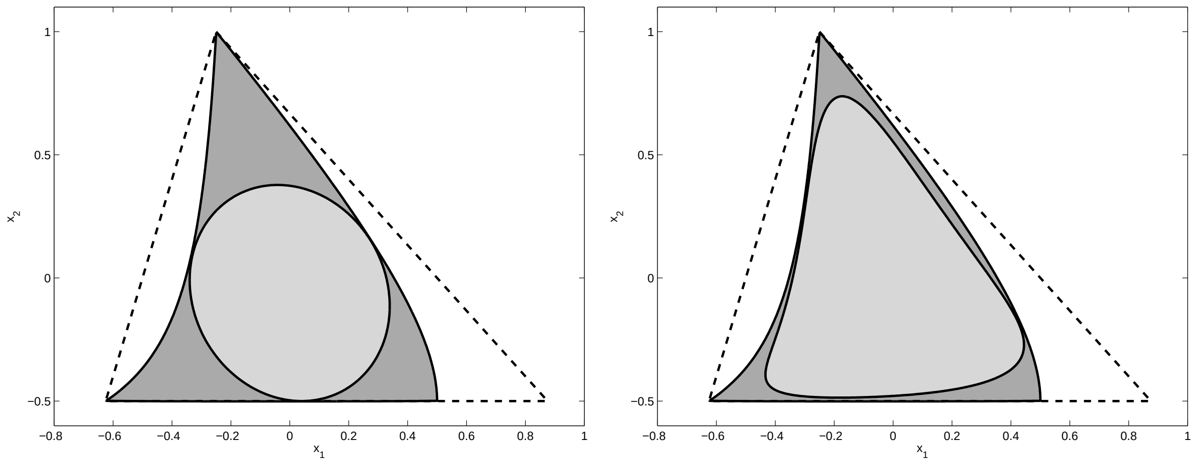


Figure 7: Degree two (left) and four (right) inner approximations (light gray) of PMI stability region (dark gray) embedded in simplex (dashed).

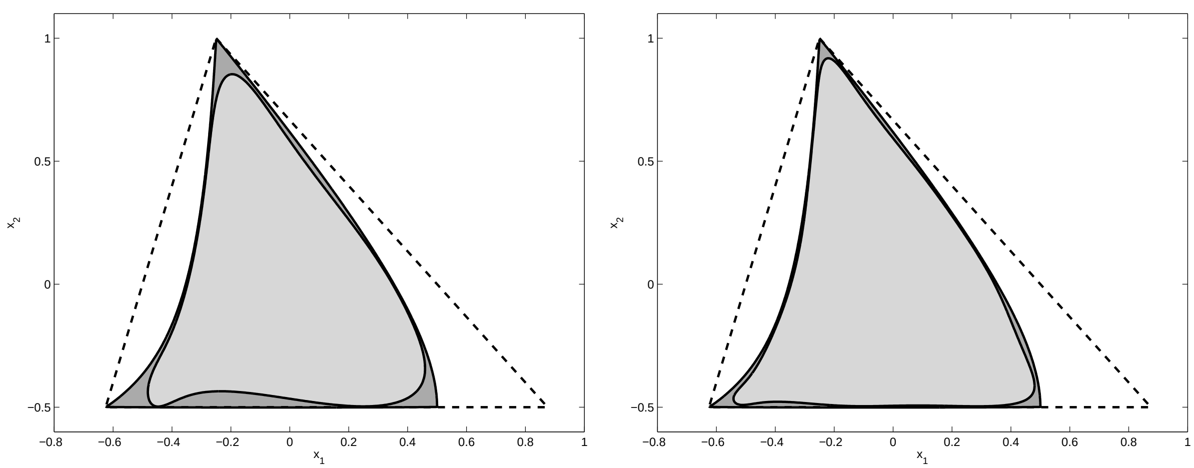


Figure 8: Degree six (left) and eight (right) inner approximations (light gray) of PMI stability region (dark gray) embedded in simplex (dashed).

On Figures 7 and 8 we represent the degree two, four, six and eight inner approximations to $\mathbf{P}$, corresponding to stability regions for the linear system. We observe that the approximations become tight rather quickly. This is due to the fact that $\mathbf{B}$ is a good outer approximation of $\mathbf{P}$ with known moments. Tighter outer approximations $\mathbf{B}$ would result in tighter inner approximations of $\mathbf{P}$, but then the moments of $\mathbf{B}$ can be hard to compute, see [7].

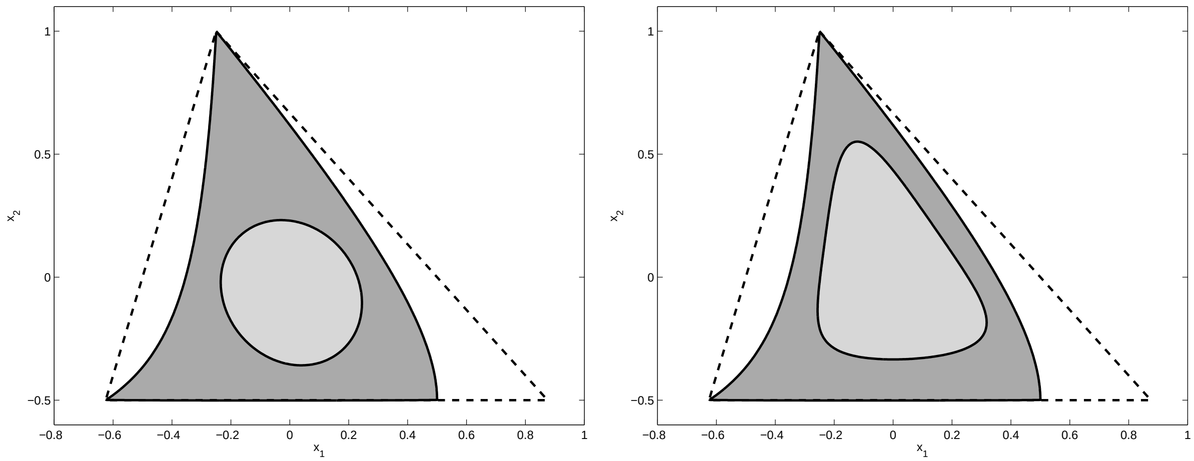


Figure 9: Degree two (left) and four (right) inner approximations (light gray) of robust PMI stability region (dark gray) embedded in simplex (dashed).

4.5 Robust controller design

Now consider the uncertain polynomial $z \mapsto x_2 + u + 2x_1 z - (2x_1 + x_2)z^3 + z^4$ with $u \in \mathbf{U} = \{u \in \mathbb{R} : u^2 \leq \frac{1}{16}\}$ with uncertain Hermite matrix $P(x, u)$ and the corresponding parametrized PMI stability region $\mathbf{P}$ in (1). Let us use the same bounding set $\mathbf{B}$ as in §4.4.

On Figure 9 we represent the degree two and degree four inner approximations to $\mathbf{P}$, corresponding to robust stability regions for the linear system. Comparing with Figure 7 we see that the approximations are smaller, and in particular they do not touch the stability boundary to cope with the robustness requirements.

5 Conclusion

We have constructed a hierarchy of inner approximations of feasible sets defined by parametrized or uncertain polynomial matrix inequalities (PMI). Each inner approximation is computed by solving a convex linear matrix inequality (LMI) problem. The hierarchy converges in a strong analytic sense, so that conservatism of the approximation is guaranteed to vanish asymptotically. In addition, the inner approximations are simple polynomial or piecewise-polynomial sublevel sets, so that optimization over these sets is significantly simpler than optimization over the original parametrized PMI set. One may also impose the hierarchy of inner approximations to be nested. Finally, one may also impose the inner approximations to be convex. In this latter case they do not converge any more to the feasible set but, on the other hand, optimization over the parametrized PMI set can be reformulated as a convex polynomial optimization problem (of course at the price of some conservatism).

The tradeoff to be found is between tightness of the inner approximation and degree of the defining polynomials. A satisfactory inner approximation can be possibly computed off-line, and then used afterwards on-line in a feedback control setup.

Our methodology is valid for general parametrized PMI problems. However, in the case of parametrized PMI problems coming from fixed-order robust controller design problems, geometric insight can be exploited to improve convergence of the hierarchy. The key information is the knowledge of the moments of the Lebesgue measure on a compact set which tightly contains the parametrized PMI set we want to approximate from the inside. In turns out that for robust control problems this knowledge is available easily, as illustrated in the paper by several examples.

The main limitation of the approach lies in the ability of solving primal moment and dual polynomial sum-of-squares LMI problems. State-of-the-art general-purpose semidefinite programming solvers can currently address problems of relatively moderate dimensions, but problem structure and data sparsity can be exploited for larger problems.

Acknowledgements

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6 Appendix

6.1 Moment and localizing matrices

With a sequence $y = (y_\alpha)$, $\alpha \in \mathbb{N}^n$, let $L_y : \mathbb{R}[x] \to \mathbb{R}$ be the linear functional

$$f \quad (= \sum_\alpha f_\alpha \, x^\alpha) \quad \mapsto \quad L_y(f) \, = \, \sum_\alpha f_\alpha \, y_\alpha, \quad f \in \mathbb{R}[x].$$

With $d \in \mathbb{N}$, the moment matrix of order d associated with y is the real symmetric matrix $M_d(y)$ with rows and columns indexed in $\mathbb{N}^n_d$, and defined by

$$M_d(y)(\alpha, \beta) \, := \, L_y(x^{\alpha+\beta}) \, = \, y_{\alpha+\beta}, \qquad \forall \alpha, \beta \in \mathbb{N}^n_d. \tag{15}$$

A sequence $y = (y_\alpha)$ has a representing measure if there exists a finite Borel measure μ on $\mathbb{R}^n$, such that $y_\alpha = \int x^\alpha d\mu$ for every $\alpha \in \mathbb{N}^n$.

With y as above and $h \in \mathbb{R}[x]$, the localizing matrix of order d associated with y and h is the real symmetric matrix $M_d(h\,y)$ with rows and columns indexed by $\mathbb{N}^n_d$, and whose entry (α, β) is given by

$$M_d(y)(h\,y)(\alpha, \beta) \, := \, L_y(h(x)\, x^{\alpha+\beta}) \, = \, \sum_\gamma h_\gamma \, y_{\alpha+\beta+\gamma}, \quad \forall \alpha, \beta \in \mathbb{N}^n_d. \tag{16}$$

6.2 Proof of Lemma 1

Proof: The dual of polynomial SOS SDP problem (5) is the moment SDP problem

$$\begin{array}{rll} \rho_d^* = \displaystyle\int_{\mathbf{B}} \lambda_{\min}(x)dx \; - \; \min\limits_{y} & L_y(v^T P(x,u)v) & \\ \text{s.t.} & M_d(y) \succeq 0,\; M_{d-1}(r\,y) \succeq 0 & \\ & M_{d-d_{a_i}}(a_i\,y) \succeq 0, \quad i = 0,1,\ldots,n_a & \\ & M_{d-d_{b_j}}(b_j\,y) \succeq 0, \quad j = 1,\ldots,n_b & \\ & L_y(x^\alpha) \;=\; \int_{\mathbf{B}} x^\alpha\, dx, \quad \forall \alpha \in \mathbb{N}^n_{2d} & \end{array} \tag{17}$$

where $y \in \mathbb{N}^{n+p+m}_{2d}$. It turns out that Slater's condition holds for SDP problem (17), i.e., it has a strictly feasible solution $\hat{y}$. Indeed, let $\hat{y}$ be the sequence of moments of the Lebesgue measure μ on $\mathbf{B} \times \mathbf{U} \times \mathbf{V}$, and scaled so that for all $(\alpha, \beta, \gamma) \in \mathbb{N}^{n+p+m}_{2d}$

$$\hat{y}_{\alpha\beta\gamma} \;=\; \int_{\mathbf{B}\times\mathbf{U}\times\mathbf{V}} x^\alpha\, u^\beta v^\gamma\, d\mu(x,u,v) \;=\; \frac{1}{\operatorname{vol}\mathbf{U}\times\mathbf{V}} \int_{\mathbf{B}}\int_{\mathbf{U}}\int_{\mathbf{V}} x^\alpha\, u^\beta v^\gamma\, \underbrace{dx\, du\, dv}_{d\mu(x,u,v)}.$$

Therefore, for every $\alpha \in \mathbb{N}^n_{2d}$

$$\hat{y}_{\alpha 00} \;=\; L_y(x^\alpha) \;=\; \int_{\mathbf{B}\times\mathbf{U}\times\mathbf{V}} x^\alpha\, d\mu(x,u,v) \;=\; \int_{\mathbf{B}} x^\alpha dx.$$

As $\mathbf{B} \times \mathbf{U} \times \mathbf{V}$ has nonempty interior, it follows that $M_d(\hat{y}) \succ 0$, $M_{d-1}(r\,\hat{y}) \succ 0$, $M_{d-d_{a_i}}(a_i\,\hat{y}) \succ 0$ and $M_{d-d_{b_j}}(b_j\,\hat{y}) \succ 0$. Therefore, $\hat{y}$ is a strictly feasible solution of (17). Hence by a standard result of convex optimization, there is no duality gap between (5) and its dual (17), i.e. $\rho_d = \rho_d^*$. If $\rho_d < \infty$ then (5) is guaranteed to have an optimal solution.

We next prove that ρ_d is bounded. For any feasible solution y of (5), $y_0 \leq \operatorname{vol}\mathbf{B}$, and

$$L_y(x_i^{2d}) \;\leq\; M^{2d} y_0^d\,; \quad L_y(u_j^{2d}) \;\leq\; M^{2d} y_0^d\,; \quad L_y(v_k^{2d}) \;\leq\; y_0^d, \tag{18}$$

for all $i = 1,\ldots,n$, $j = 1,\ldots,p$, $k = 1,\ldots,m$. This follows from $M_{d-d_{a_{i^*}}}(a_{i^*}\,y) \succeq 0$, $M_{d-d_{b_{j^*}}}(b_{j^*}\,y) \succeq 0$ and $M_{d-1}(r\,y) \succeq 0$, where $a_{i^*}(x) \;=\; M^2 - x^T x$, $b_{j^*}(x) \;=\; M^2 - u^T u$, and $r(v) \;=\; 1 - v^T v$, see the comments after (2) and (3). Then by [12, Lemma 4.3], one obtains $|y_\alpha| \leq M^{2d}(\operatorname{vol}\mathbf{B})^d$, for all $\alpha \in \mathbb{N}^n_{2d}$, which shows that the feasible set of (17) is compact. Hence (17) has an optimal solution and ρ_d is finite; therefore its dual (5) also has an optimal solution, the desired result. □

6.3 Proof of Theorem 1

Proof: (a) Let $\mathbf{K} := \mathbf{B} \times \mathbf{U} \times \mathbf{V} \subset \mathbb{R}^{n+p+m}$ and consider the infinite-dimensional optimization problem

$$\begin{array}{rcll} \rho & = & \min\limits_{\mu \in M(\mathbf{K})} & \displaystyle\int_{\mathbf{K}} v^T P(x,u)v\, d\mu(x,u,v) \\ & & \text{s.t.} & \displaystyle\int_{\mathbf{K}} x^\alpha d\mu \;=\; \int_{\mathbf{B}} x^\alpha\, dx,\; \alpha \in \mathbb{N}^n \end{array} \tag{19}$$

where $M(\mathbf{K})$ is the space of finite Borel measures on $\mathbf{K}$. Problem (19) has an optimal solution $\mu^* \in M(\mathbf{K})$. Indeed, $\rho \geq \int_{\mathbf{B}} \lambda_{\min}(x)dx$ because for every $(x,u,v) \in \mathbf{K}$, $v^T P(x,u)v \geq \lambda_{\min}(x)$; and so for every feasible solution $\mu \in M(\mathbf{K})$,

$$\int_{\mathbf{K}} v^T P(x,u,v)v \, d\mu(x,u,v) \;\geq\; \int_{\mathbf{K}} \lambda_{\min}(x)\, d\mu(x,u,v) \;=\; \int_{\mathbf{B}} \lambda_{\min}(x)\, dx$$

because $\int_{\mathbf{K}} x^\alpha d\mu = \int_{\mathbf{B}} x^\alpha dx$ for all $x \in \mathbf{B}$. On the other hand, observe that for every $x \in \mathbf{B}$, $\lambda_{\min}(x) = v_x^T P(x,u_x)v_x$ for some $(u_x, v_x) \in \mathbf{U} \times \mathbf{V}$. Therefore, let $\mu^* \in M(\mathbf{K})$ be the Borel measure concentrated on (x, u_x, v_x) for all $x \in \mathbf{B}$, i.e.

$$\mu^*(\mathbf{A} \times \mathbf{B} \times \mathbf{C}) \;:=\; \int_{\mathbf{A}\cap\mathbf{B}} 1_{\mathbf{B}\times\mathbf{C}}(u_x, v_x)\, dx, \qquad \forall (\mathbf{A}, \mathbf{B}, \mathbf{C}) \in B(\mathbb{R}^n) \times B(\mathbb{R}^p) \times B(\mathbb{R}^m)$$

where $x \mapsto 1_{\mathbf{A}}(x)$ denotes the indicator function of set $\mathbf{A}$ and $B(\mathbb{R}^n)$ denotes the Borel σ-algebra of subsets of $\mathbb{R}^n$. Then μ^* is feasible for problem (19) with value

$$\int_{\mathbf{K}} v^T P(x,u)v \, d\mu^*(x,u,v) = \int_{\mathbf{B}} \lambda_{\min}(x)\, dx$$

which proves that $\rho = \int_{\mathbf{B}} \lambda_{\min}(x)\, dx$.

Next, $\lambda_{\min}$ being continuous on compact set $\mathbf{B}$, by the Stone-Weierstrass theorem [1, §A7.5], for every $\varepsilon > 0$ there exists a polynomial $h_\varepsilon \in \mathbb{R}[x]$ such that

$$\sup_{x \in \mathbf{B}} |\lambda_{\min}(x) - h_\varepsilon(x)| < \frac{\varepsilon}{2}.$$

Hence the polynomial $p_\varepsilon := h_\varepsilon - \varepsilon$ satisfies $\lambda_{\min} - p_\varepsilon > 0$ on $\mathbf{B}$ and so $v^T P(x,u)v - p_\varepsilon > 0$ on $\mathbf{B} \times \mathbf{U} \times \mathbf{V}$. By Putinar's Positivstellensatz, see e.g [10, Section 2.5], there exists SOS polynomials $r_\varepsilon, s_{i\varepsilon}, t_{j\varepsilon} \in \Sigma[x,u,v]$ such that equation (5) is satisfied. Hence for d sufficiently large, say $d \geq d_\varepsilon$, $(p_\varepsilon, r_\varepsilon, s_{i\varepsilon}, t_{j\varepsilon})$ is a feasible solution of (5) with associated value

$$\int_{\mathbf{B}} (\lambda_{\min}(x) - p_\varepsilon(x))\, dx \;\leq\; \frac{3\varepsilon}{2} \int_{\mathbf{B}} dx.$$

Hence $0 \leq \rho_d \leq \frac{3\varepsilon}{2} \int_B dx$ whenever $d \geq d_\varepsilon$ where ρ_d is defined in (6). As $\varepsilon > 0$ was arbitrary, we obtain the desired result

$$\lim_{d\to\infty} \rho_d = 0.$$

Observe that since $g_d \leq \lambda_{\min}$ for all d,

$$\rho_d = \int_{\mathbf{B}} (\lambda_{\min}(x) - g_d(x))\, dx = \int_{\mathbf{B}} |\lambda_{\min}(x) - g_d(x)|\, dx$$

so that the convergence $\rho_d \to 0$ is just the convergence $g_d \to \lambda_{\min}$ for the L_1 norm on $\mathbf{B}$. Finally the convergence $g_d \to \lambda_{\min}$ in Lebesgue measure on $\mathbf{B}$ follows from [1, Theorem 2.5.1].

(b) For each $x \in \mathbf{B}$, fixed and arbitrary, the sequence $(\bar{g}_d)$ is monotone nondecreasing and bounded above by $\lambda_{\min}$. Therefore there exists $g^* : \mathbf{B} \to \mathbb{R}$ such that for every $x \in \mathbf{B}$,

$\bar{g}_d(x) \uparrow g^*(x) \leq \lambda_{\min}(x)$ as $d \to \infty$. By Lebesgue's Dominated Convergence Theorem [1, §1.6.9]

$$\int_{\mathbf{B}} g^*(x)dx \;=\; \lim_{d\to\infty} \int_{\mathbf{B}} \bar{g}_d(x)dx \;=\; \int_{\mathbf{B}} \lambda_{\min}(x)dx,$$

and so from $g^*(x) \leq \lambda_{\min}(x)$ we deduce that $g^*(x) = \lambda_{\min}(x)$ for almost all $x \in \mathbf{B}$. Combining the latter with $\bar{g}_d \uparrow g^*$, we obtain that $\bar{g}_d \to \lambda_{\min}$ almost everywhere in $\mathbf{B}$. But then since the Lebesgue measure is finite on $\mathbf{B}$, by Egoroff's theorem [1, Theorem 2.5.5], $\bar{g}_d \to \lambda_{\min}$ almost uniformly in $\mathbf{B}$. Finally, convergence in Lebesgue measure on $\mathbf{B}$ also follows from [1, Theorem 2.5.2]. □

6.4 Proof of Corollary 1

Proof: By Theorem 1, $\lim_{d\to\infty} \|\lambda_{\min} - g_d\|_1 = 0$. Therefore, by [1, Theorem 2.5.1] the sequence (g_d) converges to $\lambda_{\min}$ in Lebesgue measure, i.e. for every $\varepsilon > 0$,

$$\lim_{d\to\infty} \mathrm{vol}\{x \,:\, |\lambda_{\min}(x) - g_d(x)| \,\geq\, \varepsilon\} = 0. \tag{20}$$

Let $\varepsilon > 0$ be fixed, arbitrary, and let $\mathbf{P}_\varepsilon := \{x \in \mathbf{B} \,:\, \lambda_{\min}(x) \geq \varepsilon\}$, so that $\lim_{\varepsilon\to 0} \mathrm{vol}\,\mathbf{P}_\varepsilon = \mathrm{vol}\,\mathbf{P}$. By (20), $\lim_{d\to\infty} \mathrm{vol}(\mathbf{P}_\varepsilon \cap \{x \in \mathbf{B} \,:\, g_d(x) < 0\}) = 0$. Next, for all $d \in \mathbb{N}$,

$$\mathrm{vol}\,\mathbf{P}_\varepsilon \;=\; \mathrm{vol}(\mathbf{P}_\varepsilon \cap \{x \in \mathbf{B} \,:\, g_d(x) < 0\}) + \mathrm{vol}(\mathbf{P}_\varepsilon \cap \{x \in \mathbf{B} \,:\, g_d(x) \geq 0\}).$$

Therefore, taking the limit as $d \to \infty$ yields

$$\begin{aligned}
\mathrm{vol}\,\mathbf{P}_\varepsilon &= \underbrace{\lim_{d\to\infty} \mathrm{vol}(\mathbf{P}_\varepsilon \cap \{x \in \mathbf{B} \,:\, g_d(x) < 0\})}_{=0 \text{ by } (20)} + \lim_{d\to\infty} \mathrm{vol}(\mathbf{P}_\varepsilon \cap \underbrace{\{x \in \mathbf{B} \,:\, g_d(x) \geq 0\}}_{=\mathbf{G}_d}) \\
&= \lim_{d\to\infty} \mathrm{vol}(\mathbf{P}_\varepsilon \cap \mathbf{G}_d) \;\leq\; \lim_{d\to\infty} \mathrm{vol}\,\mathbf{G}_d.
\end{aligned}$$

As $\varepsilon > 0$ was arbitrary and $\mathbf{G}_d \subset \mathbf{P}$, we obtain the desired result (9). The proof of (10) is similar. □

6.5 Proof of Corollary 2

Proof: Let $0 < \varepsilon < \frac{1}{3}$ be fixed, arbitrary. As in the proof of Theorem 1, for every $k \in \mathbb{N}$ there exists a polynomial $h_k \in \mathbb{R}[x]$ such that $\sup_{x\in\mathbf{B}} |\lambda_{\min}(x) - h_k(x)| < \varepsilon^k$. Hence for all $x \in \mathbf{B}$ and all $k \geq 1$,

$$\lambda_{\min}(x) - 3\varepsilon^k < h_k(x) - 2\varepsilon^k < \lambda_{\min}(x) - \varepsilon^k < \lambda_{\min}(x) - 3\varepsilon^{k+1} < h_{k+1}(x) - 2\varepsilon^{k+1} < \lambda_{\min}(x) - \varepsilon^{k+1}$$

and so the polynomial $x \mapsto p_k(x) := h_k(x) - 2\varepsilon^k$ satisfies $p_{k+1}(x) > p_k(x)$ and $\lambda_{\min}(x) > p_k(x)$ for all $x \in \mathbf{B}$. Again, by Putinar's Positivstellensatz, see e.g [10, Section 2.5], p_k is feasible for (5) with the additional constraint (11), provided that d is sufficiently large, and with associated value

$$\int_{\mathbf{B}} |\lambda_{\min}(x) - p_k(x)|dx \;=\; \int_{\mathbf{B}} (\lambda_{\min}(x) - p_k(x))dx \;<\; 3\varepsilon^k \int_{\mathbf{B}} dx \quad \to 0 \quad \text{as } k \to \infty.$$

□

References

[1] R. Ash. Real analysis and probability. Academic Press, Boston, USA, 1972.

[2] S. Barnett. Polynomials and linear control systems. Marcel Dekker, New York, USA, 1983.

[3] D. Henrion, D. Peaucelle, D. Arzelier, M. Šebek. Ellipsoidal approximation of the stability domain of a polynomial. IEEE Trans. Autom. Control, 48(12):2255-2259, 2003.

[4] D. Henrion, M. Šebek, V. Kučera. Positive polynomials and robust stabilization with fixed-order controllers. IEEE Trans. Autom. Control, 48(7):1178-1186, 2003.

[5] D. Henrion, J. B. Lasserre. Solving nonconvex optimization problems - How GloptiPoly is applied to problems in robust and nonlinear control. IEEE Control Syst. Mag., 24(3):72-83, 2004.

[6] D. Henrion, J. B. Lasserre. Convergent relaxations of polynomial matrix inequalities and static output feedback. IEEE Trans. Autom. Control, 51(2):192-202, 2006.

[7] D. Henrion, J. B. Lasserre, C. Savorgnan. Approximate volume and integration for basic semialgebraic sets. SIAM Review, 51(4):722-743, 2009.

[8] A. Karimi, H. Khatibi, R. Longchamp. Robust control of polytopic systems by convex optimization. Automatica, 43(6):1395-1402, 2007.

[9] J. B. Lasserre, K. E. Avrachenkov. The multi-dimensional version of $\int_a^b x^p dx$. Amer. Math. Monthly, 108:151–154, 2001.

[10] J. B. Lasserre. Moments, positive polynomials and their applications. Imperial College Press, London, UK, 2009.

[11] J. B. Lasserre, E. S. Zeron. Solving a class of multivariate integration problems via Laplace techniques. Applic. Math., 28(4):391-405, 2001.

[12] J.B. Lasserre, T. Netzer. SOS approximations of nonnegative polynomials via simple high degree perturbations. Math. Z., 256:99–112, 2006.

[13] U. Nurges. Robust pole assignment via reflection coefficients of polynomials. Automatica, 42(7):1223-1230, 2006.

[14] C. W. Scherer. LMI relaxations in robust control. Europ. J. Control, 12(1):3-29, 2006.

[15] P. Shcherbakov, F. Dabbene. On the generation of random stable polynomials. Europ. J. Control, 17(2):145-159, 2011.