Integration formulas via the Legendre-Fenchel Subdifferential of nonconvex functions*

Rafael Correa[†], Yboon García[‡], Abderrahim Hantoute[§]

[†§]Universidad de Chile, Centro de Modelamiento Matemático (CMM)

Av. Blanco Encalada 2120 Piso 7, Santiago, Chile

[‡]Instituto de Matemáticas y Ciencias Afines, Calle Los Biolgos 245 Urb. San César - La Molina, Lima 12, Perú

Abstract

Starting from explicit expressions for the subdifferential of the conjugate function, we establish in the Banach space setting some integration results for the so-called epi-pointed functions. These results use the ε-subdifferential and the Legendre-Fenchel subdifferential of an appropriate weak lower semicontinuous (lsc) envelope of the initial function. We apply these integration results to the construction of the lsc convex envelope either in terms of the ε-subdifferential of the nominal function or of the subdifferential of its weak lsc envelope.

Key words. integration, lower semicontinuous convex envelope, epi-pointed functions, ε-subdifferential, conjugate function.

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1 Introduction

Determining a function from its first-order variations is a fundamental principle in nonlinear analysis. This question becomes more involved when the nominal function fails to be differentiable, so that the first problem to deal with concerns the choice of a convenient concept to quantify the variations of this function. The integration theory for lower semicontinuous (lsc, for short) convex proper functions in the Banach space setting was completely solved in the sixties [19, 20] by using the so-called Legendre-Fenchel subdifferential operator:

$$\partial f(x) \subset \partial g(x) \ \text{ for all } x \in X \Longleftrightarrow f = g + \text{constant}. \tag{1}$$

This concept of generalized differentiation has been shown to be very useful in the framework of convex analysis so that many classical results of differential calculus and linear operator theory are beneficially extended; see, e.g., [17, 18]. Beside this powerful property, this subdifferential operator behaves very badly outside the convex framework; just think of the (Lipschitz continuous) function $f(x) = -|x|$ defined on the real-line, which has an empty subdifferential at each point. This fact has led to the introduction of many tools of nonsmooth analysis which realize useful integration properties, like the Clarke, the Fréchet, the Ioffe, the Michel-Penot, the

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[†]e-mail: rcorrea@dim.uchile.cl

[‡]e-mail: yboon@imca.edu.pe

[§]e-mail: ahantoute@dim.uchile.cl (corresponding author)

Mordukhovich subdifferentials, among many others; we refer to the book [16] and the references therein.

Nevertheless, the Legendre-Fenchel subdifferential can also be useful for many purposes even when dealing with non-necessarily convex functions, namely, those verifying some kind of coercivity as the condition (5) below. In what concerns this paper, we provide some integration results by using the concept of ε-subdifferential. These results ensure, for a large class of non-necessarily convex functions defined on a Banach space, the coincidence up to an additive constant of the lsc convex envelopes rather than the functions themselves:

$$\partial_\varepsilon f(x) \subset \partial_\varepsilon g(x) \text{ for all } x \in X \text{ and } \varepsilon > 0 \text{ small enough} \Longrightarrow \overline{\text{co}} f = \overline{\text{co}} g + \text{constant}. \tag{2}$$

It is clear that both criteria in (1) and (2) are equivalent for proper lsc convex functions. In the setting of locally convex spaces, the above integration formula was established in [15] for proper lsc convex functions. Moreover, a criterion using only the exact subdifferential ($\varepsilon = 0$) will be given by means of an appropriate concept of weak lsc envelope:

$$\partial \bar{f}^{w^{**}}(x) \subset \partial \bar{g}^{w^{**}}(x) \text{ for all } x \in X \Longrightarrow \overline{\text{co}} f = \overline{\text{co}} g + \text{constant}.$$

Hence, in the line of [20] we construct the lsc convex envelope of a function by means of its ε-subdifferential, or, equivalently, in terms of the exact subdifferential of its weak lsc envelope. Since this approach easily breaks down for general functions we will limit ourselves to the useful and quite large family of epi-pointed functions; i.e., those whose conjugate functions are finite and continuous at some point.

To obtain the above results, we follow a natural idea which consists of passing through the conjugate function, which is by construction a (weak*) lsc proper convex function, and the classical integration formula of [20]. This explains why the first part of this work is dedicated to expressing the subdifferential of the conjugate function in terms of the subdifferential of the nominal function, when the dual space X^* is endowed with its norm topology. First results giving such expressions, dealing with the conjugate function, have been recently established in [11, 12, 14] for the general setting of two real locally convex Hausdorff topological vector spaces paired in duality.

1.1 Problem formulation and notation

Let us first fix some notations that are needed in the problem formulation below and throughout the paper; other ones will be given progressively. We work on a Banach space $(X, \|\cdot\|)$ whose dual and bidual spaces are denoted by X^* and X^{**}, respectively. We use $\sigma(X, X^*)$, $\sigma(X^*, X)$ and $\sigma(X^{**}, X^*)$ to refer to the weak, the weak* and the weak** topologies, respectively. We shall identify X to a subset of X^{**}, by the canonical embedding, and, unless otherwise expressed, endow X^{**} with the weak** topology $\sigma(X^{**}, X^*)$ which makes $(X^{**})^*$ isomorphic to X^*. A subset $U \subset X$ (or X^*) is said to be a θ-neighborhood if it is a convex symmetric neighborhood of the zero vector θ. If $f : X \to \mathbb{R} \cup \{+\infty\}$ is a given function and $\varepsilon \geq 0$, the ε-subdifferential of f at a point $x \in X$ is the (possibly empty) subset of X^* given by

$$\partial_\varepsilon f(x) := \{x^* \in X^* \mid \langle x^*, y - x\rangle \leq f(y) - f(x) + \varepsilon \text{ for all } y \in X\},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality product of X and X^*; we will omit referring to ε when it equals 0. We use $\text{dom}\, f$ to denote the (effective) *domain* of f, $\text{dom}\, f := \{x \in X \mid f(x) < +\infty\}$. We say that f is *proper* if $\text{dom}\, f \neq \emptyset$ and $f > -\infty$. We denote $\Gamma_0(X)$ the family of the lsc convex proper functions defined on X; $\Gamma_0(X^*)$ and $\Gamma_0(X^{**})$ are defined similarly. The *conjugate* of $f : X \to \overline{\mathbb{R}}$

is the function $f^* : X^* \to \overline{\mathbb{R}}$ given by

$$f^*(x^*) := \sup_X (x^* - f).$$

Similarly, the conjugate of f^* is the function $f^{**} : X^{**} \to \overline{\mathbb{R}}$ given by

$$f^{**}(x^{**}) = \sup_{X^*} (x^{**} - f^*).$$

In particular, provided that f^* is proper, the restriction of f^{**} on X coincides with the *lsc convex envelope* of f, $\overline{\text{co}}f : X \longrightarrow \overline{\mathbb{R}}$ defined by

$$\overline{\text{co}}f(x) := \sup\{g(x) \mid g \in \Gamma_0(X),\ g \leq f\}.$$

Equivalently, the ε-subdifferential mapping $\partial_\varepsilon f^* : X^* \rightrightarrows X^{**}$ is written as

$$\partial_\varepsilon f^*(x^*) = \{x^{**} \in X^{**} \mid f^{**}(x^{**}) + f^*(x^*) \leq \langle x^{**}, x^* \rangle + \varepsilon\},$$

where $\langle \cdot, \cdot \rangle$ is also used to denote the duality product of X^* and X^{**}. Then, the ε-subdifferential mapping $\partial_\varepsilon f^{**} : X^{**} \rightrightarrows X^*$ is written as

$$\partial_\varepsilon f^{**}(x^{**}) = \{x^* \in X^* \mid f^{**}(x^{**}) + f^*(x^*) \leq \langle x^{**}, x^* \rangle + \varepsilon\}.$$

The *indicator* and the *support* functions of a subset A ($\subset X, X^*$) are, respectively,

$$\mathrm{I}_A(x) := 0 \text{ if } x \in A;\ +\infty \text{ if } x \notin A,\ \sigma_A := \mathrm{I}_A^*.$$

The *inf-convolution* of two functions $f, g : X \to \overline{\mathbb{R}}$ is $f \square g := \inf_{x \in X}\{f(x) + g(\cdot - x)\}$. If $M : Y \rightrightarrows Z$ is a set-valued operator, for two sets Y, Z, we denote $M^{-1}(z) := \{y \in Y \mid z \in My\}$, $\operatorname{Im} M := \{My \mid y \in Y\}$ and $\operatorname{dom} M := \{y \in Y \mid My \neq \emptyset\}$. We shall write $(y, z) \in M$ when $z \in My$.

Problem formulation. The classical integration formula [20] states that two lsc convex proper functions $f, g : X \to \mathbb{R} \cup \{+\infty\}$ satisfying

$$\partial f(x) \subset \partial g(x) \quad \text{for all } x \in X, \tag{3}$$

coincide up to an additive constant c,

$$f = g + c.$$

This result readily breaks down outside the convex framework; for instance, the function $f(x) = -|x|$ recalled above satisfies (3) independently of the choice of the function g. Nevertheless, the convexity assumption on the second function g is not necessary and can be easily overcome by using the lsc convex envelope of g. Indeed, by observing that the inclusion $\partial g(x) \subset \partial(\overline{\text{co}}g)(x)$ always holds, (3) implies that $\partial f(x) \subset \partial(\overline{\text{co}}g)(x)$ for all $x \in X$. Therefore, provided that f is proper, lsc and convex, by the integration formula for convex functions we get

$$f = \overline{\text{co}}g + c,$$

which obviously covers (3). The question is then to what extent do formulas similar to this last one hold? In this paper, we will be interested in criteria like (3) which imply the validity of the following expression

$$\overline{\mathrm{co}}f = \overline{\mathrm{co}}g + c. \tag{4}$$

But, what kind of assumption would one introduce towards this aim? In view of the example of $f(x) = -|x|$ cited above, it follows that the worst situation occurs when ∂f is often empty-valued. So, a reasonable condition to guaranty (4) for non necessarily convex functions would be that $\partial f(x)$ is nonempty "for many points"; for instance, the conjugate function f^* satisfies

$$\mathrm{int}(\mathrm{dom}\, f^*) \neq \emptyset. \tag{5}$$

The functions f satisfying this dual condition, very recurrent in the literature [17, 21], are referred to as the (asymptotically) epi-pointed functions in [7] (see, also, [22] for an extension of this property). This condition (5) is also related to the behaviour at infinity of the initial function and, due to the current Banach space setting, it is equivalent to f^* being finite and continuous on $\mathrm{int}(\mathrm{dom}\, f^*)$. It is also worth recalling that, from a primal point of view, (5) reflects the strong coercivity of a linear translation of the initial function f; that is (see, e.g., [6]), there exists $x^* \in X^*$ such that

$$\liminf_{\|x\| \to +\infty} \frac{f(x) - \langle x^*, x\rangle}{\|x\|} > 0;$$

hence, $\theta \in \mathrm{int}(\mathrm{dom}\, f^*)$ if and only if f is strongly coercive.

We shall prove in this paper (Section 3) that under a slight modification of (3), namely

$$\partial_\varepsilon f(x) \subset \partial_\varepsilon g(x) \quad \text{for all } x \in X \text{ and all } \varepsilon > 0 \text{ small enough}, \tag{6}$$

the following variant of (4) holds

$$\overline{\mathrm{co}}f = (\overline{\mathrm{co}}g)\Box\sigma_{\mathrm{dom}\, f^*} + c; \tag{7}$$

hence, one can deduce (4) in many practical cases. Another criterion using the exact subdifferential will be given by means of the weak** lsc envelopes $\bar{f}^{w^{**}}$ and $\bar{g}^{w^{**}}$ (Definition1), namely

$$\partial \bar{f}^{w^{**}}(x) \subset \partial \bar{g}^{w^{**}}(x) \quad \text{for all } x \in X^{**}. \tag{8}$$

Conditions (6) and (8), together with (7), are somewhat natural since they are implicitly included in the integration statement given in the convex framework; see Remark 3.

We shall apply the previous results in the construction of the lsc convex envelope of epi-pointed functions. Namely, we show in Section 4 that for any function f the lsc convex envelope of f, $\overline{\mathrm{co}}f$, is obtained in the following way, for any given $x_0 \in \mathrm{dom}(\partial f)$,

$$\overline{\mathrm{co}}f(x) = f(x_0) + \sup\left\{\sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i\rangle + \langle x_n^*, x - x_n\rangle\right\},$$

where the supremum is taken over $n \in \mathbb{N}$, $(x_i, x_i^*) \in \partial \bar{f}^{w^{**}}$, $i = 1, \cdots n$, and $x_0^* \in \partial f(x_0)$. Equivalently, we obtain a relaxed formula which uses the ε-subdifferential, for any given $x_0 \in \mathrm{dom}(\partial f)$ and $\delta > 0$,

$$\overline{\mathrm{co}}f(x) = f(x_0) + \sup\left\{\sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i\rangle + \langle x_n^*, x - x_n\rangle - \sum_{i=1}^{n} \varepsilon_i\right\},$$

where the supremum is taken over $n \in \mathbb{N}$, $\varepsilon_i \in (0, \delta)$, $(x_i, x_i^*) \in \partial_{\varepsilon_i} f$, $i = 1, \cdots n$, and $x_0^* \in \partial f(x_0)$.

When the space X has the Radon-Nikodym property [18, §5], the last two formulas above are written by only calling to the pairs of ∂f, giving a different proof of a similar result given in [6].

Our main tools are formulas for expressing the subdifferential of the conjugate function with respect to the pair (X^*, X^{**}); i.e., $\partial f^*(x^*)$ is seen as a subset of X^{**} and, so, may contain points that are not in X (with the abuse of language). Thus, we will have to adapt to our current setting, when X^* is endowed with its norm topology, some similar formulas established in [11, 12, 14] for the duality pair (X^*, X).

The summary of the remainder of the paper is as follows. In Section 2, we gather the main tools of our analysis. Namely, we provide formulas for the subdifferential of the conjugate function with respect to the pair (X^*, X^{**}) : Proposition 3 uses an enlargement of the Legendre-Fenchel subdifferential; Proposition 4 uses the ε-subdifferential of the initial function; Proposition 5 concerns positively homogeneous functions; and Proposition 6 investigates the case when the conjugate is Fréchet-differentiable at the nominal point. In Section 3, the main integration formula is presented in Theorem 9 using the weak** lsc envelope of the associated functions. A version of this result using the ε-subdifferential is given in Corollary 10. Finally, in Section 4 we provide the construction of the lsc convex envelope either by means of the weak** lsc envelope (Theorem 14) or by the ε-subdifferential (Theorem 13).

2 Subdifferential of the conjugate function

In this section, we express the subdifferential set of the conjugate function in the Banach space X, $\partial f^* : X^* \rightrightarrows X^{**}$. In our setting, the recent results of [11] (and [14]) cannot be immediately applied unless the dual space X^* is associated with a topology which is compatible with the duality pair (X^*, X); for instance, the weak* topology $\sigma(X^*, X)$. Nevertheless, our analysis makes use of these results to overcome the current difficulty which occurs outside of reflexive spaces.

Here, and hereafter, we use the notation

$$\mathcal{F}(f) := \{L \subset X^* \text{ closed and convex} \mid f^*_{|\mathrm{ri}(L \cap \mathrm{dom}\, f^*)} \text{ is finite and continuous}\}, \tag{9}$$

where ri denotes the (topological) *relative interior* (i.e., the interior relative to the *affine envelope* when it is closed, and the empty set otherwise ([25])), and $f^*_{|\mathrm{ri}(L \cap \mathrm{dom}\, f^*)}$ denotes the restriction of f^* on $\mathrm{ri}(L \cap \mathrm{dom}\, f^*)$. If $x^* \in X^*$, then we set

$$\mathcal{F}(f, x^*) := \{L \in \mathcal{F}(f) \mid x^* \in L\}. \tag{10}$$

We first introduce the following enlargement of the subdifferential, given and studied in [11] (see, also, [12]).

Definition 1 *Given a function $f : X \to \overline{\mathbb{R}}$ and a subset $L \subset X^*$, a vector $x^* \in L$ is said to be a relative subgradient of f at $x \in X$ with respect to L, if $f^*(x^*) \in \mathbb{R}$ and there exists a net $(x_\alpha) \subset X$ such that*

$$\lim \langle x_\alpha - x, y^* \rangle = 0 \quad \forall y^* \in \overline{\mathrm{aff}}\,(L \cap \mathrm{dom}\, f^* - x^*)\,, \text{ and}$$

$$\lim (f(x_\alpha) - \langle x_\alpha, x^* \rangle) = -f^*(x^*).$$

The set of such relative subgradients, denoted by $\partial^r_L f(x)$, is called the relative subdifferential of f at x with respect to L. If $\mathrm{dom}\, f^* \subset L$, we omit the reference to L and simply write $\partial^r f(x) := \partial^r_L f(x)$.

For given A, $B \subset X$ (or X^*), we shall write

$$A + B := \{a + b \mid a \in A,\ b \in B\},\ A + \emptyset = \emptyset + A := \emptyset.$$

To denote the *normal cone* to A at x we use

$$\mathrm{N}_A(x) := \{x^* \in X^* \mid \langle x^*, y - x\rangle \leq 0 \ \ \forall y \in A\} \text{ if } \ x \in A;\ \emptyset \text{ if } x \in X \setminus A.$$

By $\operatorname{co} A$ we denote the *convex envelope* of A. We use τ-int A and $\mathrm{cl}^\tau A$ (or, indistinctly, $\overline{A}^\tau$) to respectively denote the *interior* and the *closure* of A with respect to a given topology τ; hence, $\overline{\mathrm{co}}^\tau A := \mathrm{cl}^\tau(\operatorname{co} A)$. For example, $\overline{\mathrm{co}}^{w^{**}}(A)$ stands for the weak** closed convex envelope of A. Unless otherwise expressed, when τ is the norm topology on X or X^* then we omit the superscript τ for convenience.

The following result allows the characterization of $X \cap \partial f^*(x^*)$ which is generally a proper subset of $\partial f^*(x^*)$ ($\subset X^{**}$). The characterization of the whole set $\partial f^*(x^*)$ will be given in Propositions 3 and 4 below. We use the notation $\bar{f}^w : X \to \overline{\mathbb{R}}$ to denote the *weak lsc envelope* of f given by

$$\bar{f}^w(x) := \liminf_{y \xrightarrow{w} x} f(y),$$

where $\xrightarrow{w}$ refers to the convergence in the weak topology $\sigma(X, X^*)$ on X.

Proposition 1 [11, Theorem 4] *We endow X^* with a locally convex topology τ compatible with the duality pair (X, X^*). Given a function $f : X \to \overline{\mathbb{R}}$, for every $x^* \in X^*$ we have that*

$$X \cap \partial f^*(x^*) = \bigcap_{L \in \mathcal{F}_\tau(f,x^*)} \overline{\mathrm{co}}\left\{(\partial_L^r f)^{-1}(x^*) + \mathrm{N}_{L \cap \operatorname{dom} f^*}(x^*)\right\},$$

where $\mathcal{F}_\tau(f, x^*)$ is defined as in (9)-(10) *but with τ instead of the norm topology. In particular, if τ-int(dom f^*) is nonempty and f^* is τ-continuous on τ-int(dom f^*), then*

$$X \cap \partial f^*(x^*) = \mathrm{N}_{\operatorname{dom} f^*}(x^*) + \overline{\mathrm{co}}\left\{(\partial \overline{f}^w)^{-1}(x^*)\right\}.$$

We shall need the following definition.

Definition 2 *Given a function $f : X \to \overline{\mathbb{R}}$, we call extension of f to X^{**} the function $\hat{f} : X^{**} \to \overline{\mathbb{R}}$ defined by*

$$\hat{f}(x^{**}) := f(x^{**}),\ \textit{if } x^{**} \in X;\ \ +\infty,\ \textit{otherwise}.$$

*We denote by $\bar{f}^{w^{**}} : X^{**} \to \overline{\mathbb{R}}$ the weak** lsc envelope of $\hat{f}$; that is,*

$$\bar{f}^{w^{**}}(x^{**}) = \liminf_{\substack{x \xrightarrow{w^{**}} x^{**} \\ x \in X}} f(x),$$

*where $x \underset{x\in X}{\xrightarrow{w^{**}}} x^{**}$ refers to the convergence of x ($\in X$) to x^{**} in the weak** topology $\sigma(X^{**}, X^*)$ on X^{**} .*

The following lemma provides some simple properties of the functions $\hat{f}$ and $\bar{f}^{w^{**}}$ defined above.

Lemma 2 *We endow X^{**} with the weak** topology $\sigma(X^{**}, X^*)$. For a given function $f : X \to \overline{\mathbb{R}}$, the following statements hold*:

(i) $\bar{f}^{w^{**}}$ *is weak** lsc.*

(ii) $\bar{f}^{w^{**}} = f^{**}$ *provided that* $f \in \Gamma_0(X)$.

(iii) $f^* = (\bar{f}^{w^{**}})^* = (\hat{f})^*$.

(iv) $\partial_\varepsilon \hat{f} = \partial_\varepsilon f$ *for every* $\varepsilon > 0$.

(v) *The subdifferential operator of* $\bar{f}^{w^{**}}$, $\partial \bar{f}^{w^{**}} : X^{**} \rightrightarrows X^*$, *is characterized by*

$$\partial \bar{f}^{w^{**}}(x^{**}) = \bigcap_{\substack{\varepsilon>0 \\ U\in\mathcal{N}}} \bigcup_{y\in x^{**}+U} \partial_\varepsilon f(y),$$

where $\mathcal{N}$ *denotes the collection of the θ-neighborhoods in* $(X^{**}, \sigma(X^{**}, X^*))$.

(vi) *For every* $x^* \in X^*$, $\left(\partial \bar{f}^{w^{**}}\right)^{-1}(x^*) = \bigcap_{\varepsilon>0} \overline{(\partial_\varepsilon f)^{-1}(x^*)}^{w^{**}}$.

Proof. The statements (i), (iii) and (iv) are immediate, while (ii) is asserted in the proof of [20, Proposition 1], and (vi) follows by inverting the formula in (v). Finally, (v) comes from Corollary 2.3 and Remark 2.4 in [24], applied to the function $\hat{f}$; that is,

$$\partial \bar{f}^{w^{**}}(x^{**}) = \bigcap_{\substack{\varepsilon>0 \\ U\in\mathcal{N}}} \bigcup_{y\in x^{**}+U} \partial_\varepsilon \hat{f}(y).$$

Thus, (v) follows in view of (iv). ■

The following result gives the subdifferential of f^* with respect to the pair (X^*, X^{**}). We recall that $\mathcal{F}(f, x^*)$ is defined in (9) and (10), and the extension $\hat{f}$ is introduced in Definition 2.

Proposition 3 *Let* X *be a Banach space. Given a function* $f : X \to \overline{\mathbb{R}}$, *for every* $x^* \in X^*$ *we have the formula*

$$\partial f^*(x^*) = \bigcap_{L\in\mathcal{F}(f,x^*)} \overline{\mathrm{co}}^{w^{**}} \left\{ (\partial_L^r \hat{f})^{-1}(x^*) + \mathrm{N}_{L\cap \mathrm{dom}\, f^*}(x^*) \right\}.$$

In particular, if $\mathrm{int}(\mathrm{dom}\, f^*) \neq \emptyset$, *then*

$$\partial f^*(x^*) = \mathrm{N}_{\mathrm{dom}\, f^*}(x^*) + \overline{\mathrm{co}}^{w^{**}} \left\{ (\partial \bar{f}^{w^{**}})^{-1}(x^*) \right\}.$$

Proof. We endow X^{**} with the weak** topology, $\sigma(X^{**}, X^*)$, and X^* with its norm topology so that the pair (X^{**}, X^*) becomes a dual topological pair. Then, by applying Proposition 1 to the extension function of f introduced in Definition 2, $\hat{f} : X^{**} \to \overline{\mathbb{R}}$, we get that

$$\partial (\hat{f})^*(x^*) = \bigcap_{L\in\mathcal{F}(\hat{f},x^*)} \overline{\mathrm{co}}^{w^{**}} \left\{ (\partial_L^r \hat{f})^{-1}(x^*) + \mathrm{N}_{L\cap \mathrm{dom} (\hat{f})^*}(x^*) \right\}.$$

Therefore, the first desired formula follows from Lemma 2(iii)-(iv) and the straightforward fact that $\mathcal{F}(\hat{f}, x^*) = \mathcal{F}(f, x^*)$. Similarly, the second formula follows from the second part of Proposition 1, by taking into account that $\bar{f}^{w^{**}}$ is the weak** lsc envelope of $\hat{f}$. ■

The following proposition expresses the subdifferential of the conjugate function by means of the ε-subdifferential. It is interesting to observe that the formulas given here use the initial function f rather than its extension $\hat{f}$.

Proposition 4 *With the notation of Proposition* 3, *for every $x^* \in X^*$ we have the formula*

$$\partial f^*(x^*) = \bigcap_{\substack{\varepsilon>0 \\ L\in\mathcal{F}(f,x^*)}} \overline{\mathrm{co}}^{w^{**}} \left\{ \mathrm{N}_{L\cap \mathrm{dom}\, f^*}(x^*) + (\partial_\varepsilon f)^{-1}(x^*) \right\}.$$

Moreover, provided that $\mathrm{int}(\mathrm{dom}\, f^*) \neq \emptyset$ *the formula above reduces to*

$$\partial f^*(x^*) = \mathrm{N}_{\mathrm{dom}\, f^*}(x^*) + \bigcap_{\varepsilon>0} \overline{\mathrm{co}}^{w^{**}} \left\{ (\partial_\varepsilon f)^{-1}(x^*) \right\},$$

or, equivalently,

$$\partial f^*(x^*) = \mathrm{N}_{\mathrm{dom}\, f^*}(x^*) + \overline{\mathrm{co}}^{w^{**}} \left\{ \bigcap_{\varepsilon>0} \overline{(\partial_\varepsilon f)^{-1}(x^*)}^{w^{**}} \right\}.$$

Proof. As in the proof of Proposition 3, we consider the dual pair (X^{**}, X^*) where X^{**} is endowed with the weak** topology $\sigma(X^{**}, X^*)$ and X^* with its norm topology. According to [14, Corollary 4.9], applied to the extension function of f to X^{**}, $\hat{f}$ (see Definition 2), we have that

$$\partial(\hat{f})^*(x^*) = \bigcap_{\substack{\varepsilon>0 \\ L\in\mathcal{F}(\hat{f},x^*)}} \overline{\mathrm{co}}^{w^{**}} \left\{ \mathrm{N}_{L\cap \mathrm{dom}(\hat{f})^*}(x^*) + (\partial_\varepsilon \hat{f})^{-1}(x^*) \right\}.$$

Thus, the first formula easily follows from Lemma 2(iii)-(iv). In the same manner, we prove the second formula by applying the result of [12, Proposition 7] to $\hat{f}$,

$$\partial(\hat{f})^*(x^*) = \mathrm{N}_{\mathrm{dom}(\hat{f})^*}(x^*) + \bigcap_{\substack{\varepsilon>0 \\ L\in\mathcal{F}(\hat{f},x^*)}} \overline{\mathrm{co}}^{w^{**}} \left\{ (\partial_\varepsilon \hat{f})^{-1}(x^*) \right\}.$$

Hence, the conclusion holds since, by Lemma 2(iii)-(iv), $f^* = (\hat{f})^*$, $\mathcal{F}(\hat{f}, x^*) = \mathcal{F}(f, x^*)$ and $\partial_\varepsilon \hat{f} = \partial_\varepsilon f$. The last formula follows by combining Proposition 3 and Lemma 2(vi). ■

The previous formulas simplify considerably for positively homogeneous functions: we recall that $f : X \to \overline{\mathbb{R}}$ is said to be positively homogeneous, if $f(\lambda x) = \lambda f(x)$ for all $\lambda \geq 0$ and all $x \in X$ such that $f(x) \in \mathbb{R}$.

Proposition 5 *With the notation of Proposition* 3, *if $f : X \to \overline{\mathbb{R}}$ is positively homogeneous and satisfies* $\mathrm{int}(\mathrm{dom}\, f^*) \neq \emptyset$, *then for every $x^* \in X^*$*

$$\partial f^*(x^*) = \overline{\mathrm{co}}^{w^{**}} \left\{ (\partial \bar{f}^{w^{**}})^{-1}(x^*) \right\} = \bigcap_{\varepsilon>0} \overline{\mathrm{co}}^{w^{**}} \left\{ (\partial_\varepsilon f)^{-1}(x^*) \right\} = \overline{\mathrm{co}}^{w^{**}} \left\{ \bigcap_{\varepsilon>0} \overline{(\partial_\varepsilon f)^{-1}(x^*)}^{w^{**}} \right\}.$$

Proof. We fix $x^* \in X^*$. As in the proof of Proposition 3 we consider the extension function of f, $\hat{f} : X^{**} \to \overline{\mathbb{R}}$ given in Definition 2. We also consider the dual pair (X^{**}, X^*) endowed with the topology $\sigma(X^{**}, X^*) \times \|\cdot\|_*$, with $\|\cdot\|_*$ being the norm of X^*. Then, according to [12, Theorem 6 and Proposition 7], with respect to the dual pair (X^{**}, X^*) the subdifferential of $(\hat{f})^*$ at x^* is given by

$$\partial(\hat{f})^*(x^*) = \overline{\text{co}}^{w^{**}}\left\{(\partial \bar{f}^{w^{**}})^{-1}(x^*)\right\} = \bigcap_{\varepsilon>0} \overline{\text{co}}^{w^{**}}\left\{(\partial_\varepsilon \hat{f})^{-1}(x^*)\right\}.$$

Thus, the first two desired equalities follow from Lemma 2(iii)-(iv) as in the proof of Proposition 3. Moreover, the remainder equality holds by invoking Lemma 2(vi). ■

The previous formulas are written in a very explicit way when the conjugate function is Fréchet-differentiable as shown in the following proposition, which is indeed a simple consequence of [8, Proposition 2.2] (see, also, [13]). Here, we give a slightly different proof which uses the general formula given in Proposition 3.

Proposition 6 *With the notation of Proposition* 3, *we assume that the conjugate function is Fréchet-differentiable at $x^* \in X^*$. If f is lsc, then*

$$\partial f^*(x^*) = (\partial f)^{-1}(x^*).$$

Proof. First, we observe that f^* is continuous at x^* so that $x^* \in \text{int}(\text{dom}\, f^*)$, $\text{N}_{\text{dom}\, f^*}(x^*) = \{\theta\}$ and $\partial f^*(x^*) = \{\nabla f^*(x^*)\} \subset X$ (see [25, Corollary 3.3.4]). Then, according to Proposition 3 we obtain

$$\partial f^*(x^*) = \overline{\text{co}}^{w^{**}}\left\{(\partial \bar{f}^{w^{**}})^{-1}(x^*)\right\} = (\partial \bar{f}^{w^{**}})^{-1}(x^*) = \{\nabla f^*(x^*)\} \subset X. \tag{11}$$

Now, we fix $x \in X \cap (\partial \bar{f}^{w^{**}})^{-1}(x^*)$. If U is an arbitrary neighborhood of θ in $(X, \sigma(X, X^*))$ then, by Lemma 2(vi), for every $n > 0$ there exists $x_n \in X$ such that

$$x_n \in x + U \quad \text{and} \quad x_n \in (\partial_{n^{-1}} f)^{-1}(x^*) \subset \partial_{n^{-1}} f^*(x^*).$$

Consequently, by [25, Theorem 3.3.2] (see, also, [8, Theorem 1.2]), the differentiability assumption on f^* implies that the sequence $(x_n)_n$ (norm-)converges to $\nabla f^*(x^*)$ in X. So, since U was arbitrarily chosen, we deduce that $x = \nabla f^*(x^*)$, entailing that $(x_n)_n$ also (norm-)converges to x. Moreover, by using the lsc of f we get $\bar{f}^{w^{**}}(x) = -f^*(x^*) + \langle x, x^*\rangle = \lim_n -f^*(x^*) + \langle x_n, x^*\rangle \geq \liminf_n f(x_n) - n^{-1} \geq f(x)$ so that $\bar{f}^{w^{**}}(x) = f(x) = -f^*(x^*) + \langle x, x^*\rangle$; hence, $x \in (\partial f)^{-1}(x^*)$. This shows that $(\partial \bar{f}^{w^{**}})^{-1}(x^*) \subset \{\nabla f^*(x^*)\} \cap (\partial f)^{-1}(x^*)$ which by convexification gives us $\overline{\text{co}}^{w^{**}}\{(\partial \bar{f}^{w^{**}})^{-1}(x^*)\} \subset \{\nabla f^*(x^*)\} \cap (\partial f)^{-1}(x^*)$ (this last set being a singleton). Finally, since the opposite inclusion $(\partial f)^{-1}(x^*) \subset (\partial \bar{f}^{w^{**}})^{-1}(x^*)$ holds trivially we conclude in view of (11). ■

Remark 1 (i) It is worth observing that the lsc of f in Proposition 6 is only needed to hold at the point $\nabla f^*(x^*)$, which was shown to belong to X.

(ii) The conclusion of Proposition 6 also reads, provided that f^* is Fréchet-differentiable at x^* and f is lsc at $\nabla f^*(x^*)$,

$$\partial f^*(x^*) = \left\{(\partial f)^{-1}(x^*)\right\} = \left\{(\partial \bar{f}^{w})^{-1}(x^*)\right\} = \left\{(\partial \bar{f}^{w^{**}})^{-1}(x^*)\right\} = \{\nabla f^*(x^*)\},$$

where $\bar{f}^w$ and $\bar{f}^{w^{**}}$ respectively denote the weak lsc and the weak** lsc envelopes of f. Moreover, these equalities also hold if in the definition of $\bar{f}^{w^{**}}$ (Definition 2) one takes sequences instead of nets.

(iii) The differentiability properties of the conjugate function are related to the stability properties of the optimization problems $\inf_X(f - x^*)$, $x^* \in X^*$; see, e.g., [2], [13], [25, Theorem 3.9.1].

Some consequences of Proposition 3 are in order, showing that the epi-pointed functions (those satisfying $\text{int}(\text{dom}\, f^*) \neq \emptyset$) share many useful properties with the convex ones. Let us recall for this aim the following facts which can be found in [18, §5]; some of them will be used in the next sections. The Banach space X is said to have the Radon-Nikodym property (RNP, for short) if every nonempty closed convex bounded subset can be written as the closed convex envelope of its strongly exposed points (see, e.g., [18, Theorem 5.21]). Moreover, by Collier's Theorem ([9]), if X has the RNP, then the dual space X^* is weak*-Asplund ([1]); that is, every weak* lsc convex continuous function defined on an open bounded convex set is Fréchet-differentiable in a G_δ dense subset of this set.

Corollary 7 *Let X be a Banach space, and let $f : X \to \overline{\mathbb{R}}$ be such that* $\text{int}(\text{dom}\, f^*) \neq \emptyset$*. Then,*

$$\text{dom}\, \partial f^* = \text{Im}\, \partial \bar{f}^{w^{**}} \text{ and } \overline{\text{dom}\, f^*} = \overline{\text{Im}\, \partial \bar{f}^{w^{**}}}.$$

In particular, the following statements hold:

(i) *if X has the Radon-Nikodym property and f is lsc, then*

$$\overline{\text{dom}\, f^*} = \overline{\text{Im}\, \partial f}.$$

(ii) *if X is reflexive and f is weakly lsc, then*

$$\text{dom}\, \partial f^* = \text{Im}\, \partial f.$$

Proof. If $x^* \in \text{dom}\, \partial f^*$ then, in view of Proposition 3, $(\partial \bar{f}^{w^{**}})^{-1}(x^*)$ is nonempty and, so, $x^* \in \text{Im}\, \partial \bar{f}^{w^{**}}$, showing that $\text{dom}\, \partial f^* \subset \text{Im}\, \partial \bar{f}^{w^{**}}$. Thus, since the converse inclusion is immediate from Lemma 2(iii), the first equality in the main statement of the corollary follows; the other equality holds by invoking Brøndsted-Rockafellar Theorem (see, e.g., [25]).

To show (i) we only need to check that $\overline{\text{dom}\, f^*} \subset \overline{\text{Im}\, \partial f}$; the other inclusion being straightforward. Indeed, since X^* is weak*-Asplund (by [9]), and f^* is weak* lsc, convex and continuous on $\text{int}(\text{dom}\, f^*)$ ($\neq \emptyset$, by assumption), there exists a subset $D \subset \text{int}(\text{dom}\, f^*)$ such that $\overline{D} = \overline{\text{int}(\text{dom}\, f^*)} = \overline{\text{dom}\, f^*}$ and f^* is Fréchet-differentiable on D. So, given $x^* \in \overline{\text{dom}\, f^*}$, we find a sequence $(x_k^*) \subset D$ such that $\|x_k^* - x^*\|_* \to 0$ and f^* is Fréchet-differentiable at each x_k^*. Consequently, by Proposition 6, we obtain

$$\partial f^*(x_k^*) = \{(\partial f)^{-1}(x_k^*)\} \quad \text{for all } k.$$

In other words, since $\partial f^*(x_k^*) = \{\nabla f^*(x_k^*)\}$ ($\in X$) we deduce that $x_k^* \in \text{Im}\, \partial f$. Hence, by taking the limit on k it follows that $x^* \in \overline{\text{Im}\, \partial f}$, as we wanted to prove.

To finish the proof of the corollary we observe that the last conclusion (ii) follows from the main conclusion, since in the reflexive case both the weak** and the weak lsc envelopes of f coincide. ■

The following corollary furnishes a simple and useful extension of similar results established in [11, 12] (dealing with (X^*, X) as a dual topological pair). We recall that for a given function $f : X \to \overline{\mathbb{R}}$, the notation ε-argmin f refers to the set of the *global ε-minima* of f; that is,

$$\varepsilon\text{-}\operatorname{argmin} f := \{x \in X \mid f(x) \leq \inf f + \varepsilon\};$$

if $\varepsilon = 0$, we simply write $\operatorname{argmin} f$.

Corollary 8 *Let X be a Banach space, and let $f : X \to \overline{\mathbb{R}}$ be such that* $\operatorname{int}(\operatorname{dom} f^*) \neq \emptyset$. *Then,*

$$\begin{aligned}
\operatorname{argmin}(\overline{\operatorname{co}} f) &= X \bigcap \left(\mathrm{N}_{\operatorname{dom} f^*}(\theta) + \overline{\operatorname{co}}^{w^{**}} \left\{ \operatorname{argmin} \bar{f}^{w^{**}} \right\} \right) \\
&= X \bigcap \left(\mathrm{N}_{\operatorname{dom} f^*}(\theta) + \overline{\operatorname{co}}^{w^{**}} \left\{ \bigcap_{\varepsilon > o} \overline{(\varepsilon\text{-}\operatorname{argmin} f)}^{w^{**}} \right\} \right) \\
&= X \bigcap \left(\mathrm{N}_{\operatorname{dom} f^*}(\theta) + \bigcap_{\varepsilon > o} \overline{\operatorname{co}}^{w^{**}} \left\{ \varepsilon\text{-}\operatorname{argmin} f \right\} \right);
\end{aligned}$$

Moreover, if f is positively homogeneous then

$$\begin{aligned}
\operatorname{argmin}(\overline{\operatorname{co}} f) &= X \bigcap \overline{\operatorname{co}} \left\{ \operatorname{argmin} \bar{f}^{w^{**}} \right\} \\
&= X \bigcap \overline{\operatorname{co}} \left\{ \bigcap_{\varepsilon > o} \overline{(\varepsilon\text{-}\operatorname{argmin} f)}^{w^{**}} \right\} \\
&= X \bigcap \bigcap_{\varepsilon > o} \overline{\operatorname{co}}^{w^{**}} \left\{ \varepsilon\text{-}\operatorname{argmin} f \right\}.
\end{aligned}$$

In particular, the following statements hold:

(i) *If f^* is Fréchet-differentiable at θ and f is lsc at $\nabla f^*(\theta)$, then*

$$\operatorname{argmin}(\overline{\operatorname{co}} f) = \operatorname{argmin} f = \{\nabla f^*(\theta)\}.$$

(ii) *If X has the RNP, f is lsc and $\theta \in \operatorname{dom} f^*$, then for every $\varepsilon > 0$ there exists $x^* \in$* $\operatorname{int}(\operatorname{dom} f^*)$ *such that $\|x^*\|_* \leq \varepsilon$, f^* is Fréchet-differentiable at x^* and*

$$\operatorname{argmin}((\overline{\operatorname{co}} f) - x^*) = \operatorname{argmin}(f - x^*) = \{\nabla f^*(x^*)\}.$$

Proof. We may suppose without loss of generality that $\theta \in \operatorname{dom} f^*$. So, the main statement follows from Propositions 3 and 4, in view of the relationship $\operatorname{argmin}(\overline{\operatorname{co}} f) = X \cap \partial f^*(\theta)$ which holds under the current epi-pointedness property. Similarly, the statement concerning the positively homogeneous case holds by invoking Proposition 5. Since the assertion (i) directly follows from Proposition 6, it suffices to prove (ii). We fix $\varepsilon > 0$. By the RNP assumption, as $\theta \in \operatorname{dom} f^*$ there exists $x^* \in \operatorname{int}(\operatorname{dom} f^*)$ such that $\|x^*\|_* \leq \varepsilon$ and the Fréchet derivative $\nabla f^*(x^*)$ of f^* at x^* exists. Hence, by invoking Proposition 6 once again we write

$$\begin{aligned}
\operatorname{argmin}((\overline{\operatorname{co}} f) - x^*) = \operatorname{argmin}\left(\overline{\operatorname{co}}(f - x^*)\right) = \partial f^*(x^*) &= \{\nabla f^*(x^*)\} \\
&= (\partial f)^{-1}(x^*) \\
&= (\partial(f - x^*))^{-1}(\theta) = \operatorname{argmin}(f - x^*),
\end{aligned}$$

as we wanted to prove. ■

3 Integration formulas using the Legendre-Fenchel subdifferential

In this section, we apply the results of the previous section to establish the desired integration formulas using the Legendre-Fenchel subdifferential of non necessarily convex functions, defined on the Banach space X.

We recall that the bidual space X^{**} is endowed with the weak** topology $\sigma(X^{**}, X^*)$ so that $(X^{**})^*$ is isomorphic to X^*; hence, we write $\partial \bar{f}^{w^{**}} : X^{**} \rightrightarrows X^*$ for the subdifferential of the weak** lsc envelope of f; that is, the function $\bar{f}^{w^{**}}$ given in Definition 2.

Theorem 9 *Let X be a Banach space, and $f, g : X \to \overline{\mathbb{R}}$ be given functions. We assume that* $\operatorname{int}(\operatorname{dom} f^*) \neq \emptyset$. *If for every $x \in X^{**}$ we have that*

$$\partial \bar{f}^{w^{**}}(x) \subset \partial \bar{g}^{w^{**}}(x),$$

then there exists a constant $c \in \mathbb{R}$ such that

$$\overline{\text{co}} f = (\overline{\text{co}} g) \square \sigma_{\operatorname{dom} f^*} + c.$$

If, moreover, one of the following conditions (i)–(iii) *holds, then $\overline{\text{co}} f$ and $\overline{\text{co}} g$ coincide up to an additive constant,*

(i) $g \leq f$;

(ii) $\operatorname{int}(\operatorname{dom} g^*) \subset \overline{\operatorname{dom} f^*}$;

(iii) f *is positively homogeneous.*

Proof. Let us first observe that in view of the relationship $\partial \bar{g}^{w^{**}} \subset \partial g^{**}$, together with the epi-pointedness property of f ($\operatorname{int}(\operatorname{dom} f^*) \neq \emptyset$), we may assume without loss of generality that $\bar{g}^{w^{**}} \in \Gamma_0(X^{**})$. Moreover, taking into account Lemmas 2(iii) and Corollary 7, together with the current assumption ($\partial \bar{f}^{w^{**}} \subset \partial \bar{g}^{w^{**}}$), we get

$$\operatorname{int}(\operatorname{dom} f^*) \subset \operatorname{dom} \partial f^* \subset \operatorname{Im} \partial \bar{f}^{w^{**}} \subset \operatorname{Im} \partial \bar{g}^{w^{**}} \subset \operatorname{dom} g^*,$$

which leads us to

$$\emptyset \neq \operatorname{int}(\operatorname{dom} f^*) \subset \operatorname{int}(\operatorname{dom} g^*), \tag{12}$$

showing that g is also an epi-pointed function. On the other hand, using again Lemma 2(iii), since $\operatorname{Im} \partial \bar{f}^{w^{**}} \subset \operatorname{dom}(\bar{f}^{w^{**}})^* = \operatorname{dom} f^*$ the current assumption implies that

$$\partial \bar{f}^{w^{**}}(x) \subset \partial \bar{g}^{w^{**}}(x) \cap \overline{\operatorname{dom} f^*} \qquad \text{for every } x \in X^{**}. \tag{13}$$

Moreover, by observing that (see, e.g., [25, Corollary 2.4.7, p. 89]), whenever $\partial \bar{f}^{w^{**}}(x) \cap \overline{\operatorname{dom} f^*} \neq \emptyset$,

$$\partial \bar{f}^{w^{**}}(x) \cap \overline{\operatorname{dom} f^*} \subset \partial \bar{g}^{w^{**}}(x) \cap \overline{\operatorname{dom} f^*} = \partial \bar{g}^{w^{**}}(x) \cap \partial \sigma_{\operatorname{dom} f^*}(\theta) = \partial(\bar{g}^{w^{**}} \square \sigma_{\operatorname{dom} f^*})(x), \tag{14}$$

the condition (13) reads

$$\partial \bar{f}^{w^{**}}(x) \subset \partial(\bar{g}^{w^{**}} \square \sigma_{\operatorname{dom} f^*})(x) \qquad \text{for every } x \in X^{**}.$$

Now, we denote $h := \bar{g}^{w^{**}} \Box \sigma_{\operatorname{dom} f^*}$. Then, since the functions $\bar{g}^{w^{**}}$ and $\sigma_{\operatorname{dom} f^*}$ are proper we write

$$h^* = g^* + \mathrm{I}_{\overline{\operatorname{dom} f^*}}, \ \operatorname{dom} h^* = \operatorname{dom} g^* \cap \overline{\operatorname{dom} f^*} \subset \overline{\operatorname{dom} f^*}, \text{ and} \tag{15}$$

$$\operatorname{int}(\operatorname{dom} f^*) \subset \operatorname{int}(\operatorname{dom} g^* \cap \overline{\operatorname{dom} f^*}) = \operatorname{int}(\operatorname{dom} h^*), \tag{16}$$

where in (16) we used (12); that is, h is an epi-pointed function too. Hence, invoking (12) and the current assumption ($\bar{g}^{w^{**}} \in \Gamma_0(X^{**})$), it follows that

$$h^{**} = g^{**} \Box \sigma_{\operatorname{dom} f^*} = \bar{g}^{w^{**}} \Box \sigma_{\operatorname{dom} f^*} = h,$$

showing that $h \in \Gamma_0(X^{**})$. In particular, h is a weak** lower semicontinuous convex proper function on X^{**} so that (14), together with (13), reads

$$\partial \bar{f}^{w^{**}}(x) \subset \partial h(x) = \partial \bar{h}^{w^{**}}(x) \qquad \text{for every } x \in X^{**}.$$

But, by (15) we have that

$$\mathrm{N}_{\operatorname{dom} f^*}(x^*) \subset \mathrm{N}_{\operatorname{dom} h^*}(x^*) \text{ for all } x^* \in \operatorname{dom} f^* \cap \operatorname{dom} h^*$$

and so, by applying Proposition 3,

$$\partial f^*(x^*) \subset \partial h^*(x^*) \ \text{ for all } x^* \in \operatorname{dom} f^* \cap \operatorname{dom} h^*.$$

In other words, since f^* is continuous on the nonempty set $\operatorname{int}(\operatorname{dom} f^*)$ ($\subset \operatorname{int}(\operatorname{dom} h^*)$, by (16)), using successively the classical chain rule for the sum of lsc convex functions (see, e.g., [25]) we obtain, for all $x^* \in X^*$,

$$\partial(f^* + h^*)(x^*) = \partial f^*(x^*) + h^*(x^*) \subset \partial h^*(x^*) + \partial h^*(x^*) = \partial(2h^*)(x^*) = 2\partial h^*(x^*).$$

Now, by invoking the classical (convex) integration formula [20] we find a constant $c \in \mathbb{R}$ such that

$$f^*(x^*) + h^*(x^*) = 2h^*(x^*) + c \quad \text{ for all } x^* \in \operatorname{dom} h^*,$$

or, equivalently (as $\operatorname{dom} h^* = \operatorname{dom} g^* \cap \overline{\operatorname{dom} f^*}$),

$$f^*(x^*) = g^*(x^*) + \mathrm{I}_{\overline{\operatorname{dom} f^*}}(x^*) + c \quad \text{ for all } x^* \in \operatorname{dom} g^* \cap \overline{\operatorname{dom} f^*}. \tag{17}$$

Moreover, this last equality also holds on $\overline{\operatorname{dom} g^*} \cap \overline{\operatorname{dom} f^*}$. Indeed, we pick $x_0^* \in \operatorname{int}(\operatorname{dom} f^*)$ ($\subset \operatorname{int}(\operatorname{dom} g^*)$, by (12)). Then, given $x^* \in \overline{\operatorname{dom} g^*} \cap \overline{\operatorname{dom} f^*}$ we find a sequence $(x_k^*)_{k \geq 1}$ which belongs to the segment (x_0^*, x^*) ($\subset \operatorname{int}(\operatorname{dom} f^*) \cap \operatorname{int}(\operatorname{dom} g^*)$, by the accessibility lemma) such that $g^*(x_k^*)$ and $f^*(x_k^*)$ converge to $g^*(x^*)$ and $f^*(x^*)$, respectively. Then, for each $k \geq 1$, by (17) we write

$$f^*(x_k^*) = g^*(x_k^*) + \mathrm{I}_{\overline{\operatorname{dom} f^*}}(x_k^*) + c = g^*(x_k^*) + c$$

and, so, by taking the limit on k we get the desired equality; that is,

$$f^*(x^*) = g^*(x^*) + \mathrm{I}_{\overline{\operatorname{dom} f^*}}(x^*) + c \quad \text{ for all } x^* \in \overline{\operatorname{dom} g^*} \cap \overline{\operatorname{dom} f^*}.$$

Moreover, since this last equality obviously holds when $x^* \notin \overline{\operatorname{dom} f^*}$, by using again the fact that $\overline{\operatorname{dom} f^*} \subset \overline{\operatorname{dom} g^*}$ (by (16)) we get that

$$f^*(x^*) = g^*(x^*) + \mathrm{I}_{\overline{\operatorname{dom} f^*}}(x^*) + c \ \text{ for all } x^* \in X^*. \tag{18}$$

Finally, by taking the conjugate on both sides, and observing that the qualification condition holds (by (16)), we obtain the first conclusion.

Now, we will prove the second part of the theorem: (i) if $g \leq f$, then $f^* \leq g^*$ and, so, $\operatorname{dom} g^* \subset \operatorname{dom} f^*$. Hence, by the first part of the proof we find a constant $c \in \mathbb{R}$ such that

$$\overline{\text{co}}f = (\overline{\text{co}}g)\Box\sigma_{\operatorname{dom} f^*} + c \geq (\overline{\text{co}}g)\Box\sigma_{\operatorname{dom} g^*} + c = \overline{\text{co}}g + c. \tag{19}$$

Thus, we are done because the other inequality $\overline{\text{co}}f = (\overline{\text{co}}g)\Box\sigma_{\operatorname{dom} f^*} \leq \overline{\text{co}}g$ always holds.

(ii) From (12), we deduce that $\overline{\operatorname{dom} f^*} = \overline{\operatorname{dom} g^*}$. Thus, we conclude as in (19).

(iii) This last statement follows just by using Proposition 5 together with the classical integration formula of convex functions [20]. ■

Remark 2 (i) The main point in the proof of Theorem 9 was to check that

$$\partial f^*(x^*) \subset \partial h^*(x^*) \quad \text{for all } x^* \in X^*, \tag{20}$$

by using the formulas given in Propositions 3. Indeed, it was sufficient to establish that

$$X \cap \partial f^*(x^*) \subset X \cap \partial h^*(x^*) \quad \text{for all } x^* \in X^*,$$

since this last criterion is equivalent to (20), as it can be easily deduced from [20, Proposition 1]. However, this last condition is nothing else but the comparison between the subdifferentials of the conjugate functions f^* and h^*, with respect to the dual pair (X^*, X). So, an alternative for the proof of Theorem 9 would be to directly use the main formula in Proposition 1, but not the simplified one which appears there. This is because the epi-pointedness condition used in Proposition 1 is given with respect to a topology on X^* compatible with the duality pair (X^*, X). Hence, the use of Proposition 1 instead of Proposition 3 would cause more technical difficulties.

(ii) It would also be sufficient in the proof of Theorem 9 to show that (20) holds on a dense subset of $\operatorname{int}(\operatorname{dom} f^*)$. Moreover, inside $\operatorname{int}(\operatorname{dom} f^*)$ the formulas of the subdifferential of the conjugate function are considerably simplified; for instance, there, the normal cone to $\operatorname{dom} f^*$ reduces to θ. However, this observation would not have a notable change in the proof.

(iii) It follows from the proof of Theorem 9 (see (18)) that the conclusion of that theorem can be presented in the following equivalent form,

$$f^{**} = g^{**}\Box\sigma_{\operatorname{dom} f^*} + c,$$

giving an inequality in X^{**}.

In the following remark we make a comparison between Theorem 9 and the classical integration formula of the convex framework [20]. This is to explain the presence of the extra terms within the statement of Theorem 9; i.e., the support function of $\operatorname{dom} f^*$ and the weak** lsc envelopes $\bar{f}^{w^{**}}$ and $\bar{g}^{w^{**}}$.

Remark 3 The statement of Theorem 9 is somewhat natural. Indeed, if the functions f and g are taken in $\Gamma_0(X)$, then the standard condition of the integration formula,

$$\partial f(x) \subset \partial g(x) \quad \text{for all } x \in X, \tag{21}$$

implies both the assumption and the conclusion of Theorem 9. To see this, we easily observe that (21) gives rise to

$$\partial f(x) \subset \partial g(x) \subset \partial g\Box\sigma_{\operatorname{dom} f^*}(x) \quad \text{for all } x \in X,$$

which in turns yields, by [20],

$$f = g + c = g\square\sigma_{\operatorname{dom} f^*} + c,$$

for some constant $c \in \mathbb{R}$. In other words, the assertion of Theorem 9 follows. On the other hand, (21) is equivalent to

$$X \cap \partial f^*(x^*) \subset X \cap \partial g^*(x^*) \quad \text{for all } x^* \in X^*,$$

which, invoking once again [20, Proposition 1], implies that

$$\partial f^*(x^*) \subset \partial g^*(x^*) \quad \text{for all } x^* \in X^*,$$

or, equivalently, using Lemma 2(ii) (we recall that X^{**} is endowed with the $\sigma(X^{**}, X^*)$ topology),

$$\partial \bar{f}^{w^{**}}(x) = \partial f^{**}(x) \subset \partial g^{**}(x) = \partial \bar{g}^{w^{**}}(x) \quad \text{for all } x \in X^{**},$$

which is nothing else but the assumption of Theorem 9.

The use of the ε-subdifferential instead of the subdifferential allows overcoming the requirement to the weak** lsc envelopes in Theorem 9, as we show in the following corollary. It is the place here to mention that integration criteria invoking the ε-subdifferential have been used by many authors; for instance, [15, 23] deal with convex functions. See, also, [3, 10] for other purposes dealing with approximate mean value theorems.

Corollary 10 *With the notation of Theorem* 9, *we assume that* $\operatorname{int}(\operatorname{dom} f^*) \neq \emptyset$. *If there exists* $\alpha > 0$ *such that*

$$\partial_\varepsilon f(x) \subset \partial_\varepsilon g(x) \quad \text{for every } x \in X \text{ and } 0 < \varepsilon \leq \alpha,$$

then

$$\overline{\operatorname{co}} f = (\overline{\operatorname{co}} g)\square\sigma_{\operatorname{dom} f^*} + c,$$

for some constant c. Moreover, the functions $\overline{\operatorname{co}} f$ *and* $\overline{\operatorname{co}} g$ *coincide up to an additive constant provided that one of the conditions* (i)–(iii) *in Theorem* 9 *hold.*

Proof. Taking into account Lemma 2(vi), by the current assumption we write, for every $x^* \in X^*$,

$$\begin{aligned}\left(\partial \bar{f}^{w^{**}}\right)^{-1}(x^*) = \bigcap_{\varepsilon>0} \overline{(\partial_\varepsilon f)^{-1}(x^*)}^{w^{**}} &= \bigcap_{\alpha\geq\varepsilon} \overline{(\partial_\varepsilon f)^{-1}(x^*)}^{w^{**}} \\ &\subset \bigcap_{\alpha\geq\varepsilon} \overline{(\partial_\varepsilon g)^{-1}(x^*)}^{w^{**}} = \left(\partial \bar{g}^{w^{**}}\right)^{-1}(x^*).\end{aligned}$$

Therefore, the conclusion immediately follows from Theorem 9. ■

The following simple example shows that, in general, $\overline{\operatorname{co}} f$ and $\overline{\operatorname{co}} g$ do not coincide up to an additive constant under only the main condition of Theorem 9.

Example 1 We consider the function $f : \mathbb{R} \to \overline{\mathbb{R}}$ given by

$$f(x) := e^{-x}, \text{ if } x \geq 1;\ x, \text{ if } 0 \leq x < 1;\ +\infty, \text{ if } x < 0.$$

Then, we easily check that $\operatorname{dom} f^* = \mathbb{R}_-$ and, so, f^* is epi-pointed ($\operatorname{int}(\operatorname{dom} f^*) \neq \emptyset$). Moreover, direct calculus gives us

$$\partial f(x) = \emptyset, \text{ if } x \neq 0;\ \mathbb{R}_-, \text{ if } x = 0.$$

Then, the function

$$g(x) := x, \text{ if } x \geq 0;\ +\infty, \text{ if } x < 0,$$

satisfies $\partial f(x) \subset \partial g(x)$ for all $x \in \mathbb{R}$, but $\overline{\text{co}} f$ and $\overline{\text{co}} g$ never coincide up to an additive constant.

In the next proposition, we shall use the following lemma which is a recurrent argument in the theory of maximal monotone operators. For the reader's convenience, we give a proof in the particular case dealing with the subdifferential of lsc convex proper functions.

Lemma 11 *Given $f, g \in \Gamma_0(X)$ and a nonempty open subset $V \subset \operatorname{dom} \partial f$, if D is a dense subset of V such that $\partial f(x) \subset \partial g(x)$, for all $x \in D$, then*

$$\partial f(x) \subset \partial g(x) \quad \textit{for all } x \in V.$$

Proof. We fix $x \in V$. Proceeding by contradiction we assume that there exists $w^* \in \partial f(x) \setminus \partial g(x)$. Then, using a separation argument in $(X^*, \sigma(X^*, X))$, there exist $z \in X \setminus \{\theta\}$ and $\alpha \in \mathbb{R}$ such that

$$\langle w^*, z \rangle < \alpha < \langle x^*, z \rangle \quad \text{for all } x^* \in \partial g(x). \tag{22}$$

We denote $W := \{x^* \in X^* : \langle x^*, z \rangle > \alpha\}$. Then, W is open in $(X^*, \sigma(X^*, X))$ and obviously satisfies $\partial g(x) \subset W$. So, by the $\|\cdot\| \times \sigma(X^*, X)$-upper semicontinuity of the operator $\partial g : X \rightrightarrows X^*$, from (22) we infer the existence of $\varepsilon > 0$ such that

$$\partial g(y) \subset W \quad \text{for all } y \in B_\varepsilon(x), \tag{23}$$

where $B_\varepsilon(x)$ denotes the ball of radius ε centred at x; moreover, since V is open we may suppose that $B_\varepsilon(x) \subset V$. We let $\delta > 0$ be small enough such that $y := x - \delta z \in B_\varepsilon(x)$ $(\subset V)$ and, by the density of D in V, let $\{y_n\} \subset D$ be such that $y_n \to y$. Without loss of generality on n and δ, there exist a sequence $(y_n^*)_n \subset X^*$ and $y^* \in X^*$ such that $y_n^* \xrightarrow{w^*} y^*$ and $y_n^* \in \partial f(y_n)$ for all n. Hence, by the current assumption we also have that $y_n^* \in \partial g(y_n)$ for all n. Consequently, invoking once again the $\|\cdot\| \times \sigma(X^*, X)$ upper semicontinuity of ∂f and ∂g, we deduce that $y^* \in \partial f(y) \bigcap \partial g(y)$ $(\subset \partial f(y) \bigcap W$, by (23)). Thus, (22) leads us to

$$\langle w^* - y^*, x - y \rangle = \delta \langle w^* - y^*, z \rangle < 0,$$

which is a contradiction with the monotonicity of the operator ∂f. This finishes the proof. ■

The following result gives an integration result in the setting of Banach spaces with the Radon-Nikodym property (RNP); for the definition, we refer to the paragraph prior to Corollary 7. The condition we use here is slightly different from those of [6, Theorem 9], although both lead to a common property given next in Corollary 15.

Proposition 12 *Let X be a Banach space with the RNP, and $f, g : X \to \overline{\mathbb{R}}$ be given functions. We assume that f satisfies* $\operatorname{int}(\operatorname{dom} f^*) \neq \emptyset$. *If for every $x \in X$ we have that*

$$\partial f(x) \subset \partial g(x),$$

then

$$\overline{\text{co}} f = (\overline{\text{co}} g) \square \sigma_{\operatorname{dom} f^*} + c,$$

for some constant $c \in \mathbb{R}$. Moreover, the functions $\overline{\text{co}} f$ and $\overline{\text{co}} g$ coincide up to an additive constant provided that one of the conditions (i)–(iii) *in Theorem 9 hold.*

Proof. By the RNP assumption the dual space X^* is a weak*-Asplund space and, so, the weak* lsc convex continuous (on $\operatorname{int}(\operatorname{dom} f^*)$) function f^* is Fréchet-differentiable on a dense subset D of $\operatorname{int}(\operatorname{dom} f^*)$; that is, $D \subset \operatorname{int}(\operatorname{dom} f^*) \subset \operatorname{int}(\operatorname{dom} \partial f^*) \subset \overline{\operatorname{int}(\operatorname{dom} \partial f^*)}$. Let us denote $h := g\Box\sigma_{\mathrm{dom}\, f^*}$. Then, as in the proof of Theorem 9, by invoking Proposition 6 instead of Proposition 3, we obtain

$$\partial f^*(x^*) \subset \partial h^*(x^*) \ \text{ for all } x^* \in D.$$

Since D is dense in $\mathrm{int}(\mathrm{dom}\, f^*)$, by Lemma 11 we deduce that

$$\partial f^*(x^*) \subset \partial h^*(x^*) \ \text{ for all } x^* \in \mathrm{int}(\mathrm{dom}\, f^*).$$

Hence, using the integration formula of [20] we infer that f^* and h^* coincide up to an additive constant on $\mathrm{int}(\mathrm{dom}\, f^*)$. Moreover, taking into account the accessibility lemma [25], by the epi-pointedness assumption $(\mathrm{int}(\mathrm{dom}\, f^*) \neq \emptyset)$ it follows that this coincidence property of f^* and h^* holds on the whole set $\overline{\mathrm{dom}\, f^*}$. The remainder of the proof follows the same arguments as Theorem 9. ■

4 Application to the construction of the lsc convex envelope

In this section, we apply the results of Section 3 to give explicit constructive formulas for the lsc convex envelope of functions defined on the Banach space X.

We begin by the following theorem which provides a formula by means of the ε-subdifferential of the initial function.

Theorem 13 *Let X be a Banach space, $f : X \to \overline{\mathbb{R}}$ be a given function, and $x_0 \in \mathrm{dom}(\partial f)$. We assume that $\mathrm{int}(\mathrm{dom}\, f^*) \neq \emptyset$. Then, for every $\delta > 0$ we have that*

$$\overline{\mathrm{co}} f(x) = f(x_0) + \sup \left\{ \sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle + \langle x_n^*, x - x_n \rangle - \sum_{i=1}^{n} \varepsilon_i \right\},$$

where the supremum is taken over $n \in \mathbb{N}$, $\varepsilon_i \in (0, \delta)$, $(x_i, x_i^*) \in \partial_{\varepsilon_i} f$, $i = 1, \cdots n$, and $x_0^* \in \partial f(x_0)$ (with the convention that $\sum_{i=0}^{-1} = \sum_{i=1}^{0} = 0$).

Proof. We fix $\delta > 0$ and denote the function in the right-hand side by f_δ, which by construction belongs to $\Gamma_0(X)$. Let us first observe by the definition of the ε-subdifferential that f_δ is dominated by f; that is, $f_\delta \leq f$ and, so, $f_\delta \leq \overline{\mathrm{co}} f$. Thus, in view of Corollary 10, it suffices to establish the following inclusion for every given $x \in X$ and $\varepsilon \in (0, \delta)$,

$$\partial_\varepsilon f(x) \subset \partial_\varepsilon f_\delta(x). \tag{24}$$

Indeed, we pick $x^* \in \partial_\varepsilon f(x)$ so that $-\infty < f_\delta(x) \leq f(x) < +\infty$ (since $f_\delta \leq f$ and $f_\delta \in \Gamma_0(X)$). Then, if α is an arbitrary real number such that $f_\delta(x) > \alpha$, by the definition of f_δ there exist $n \in \mathbb{N}$, $(\varepsilon_i, x_i, x_i^*) \in \mathbb{R} \times X \times X^*$ with $x_i^* \in \partial_{\varepsilon_i} f(x_i)$, for $i = 1, \cdots, n$, and $x_0^* \in \partial f(x_0)$ such that

$$f(x_0) + \sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle + \langle x_n^*, x - x_n \rangle - \sum_{i=1}^{n} \varepsilon_i > \alpha. \tag{25}$$

Now, we fix $y \in X$. Since $(x_{n+1}, x_{n+1}^*) := (x, x^*) \in \partial_\varepsilon f$, by taking $\varepsilon_{n+1} := \varepsilon$ we deduce from the definition of f_δ that

$$f_\delta(y) \geq f(x_0) + \sum_{i=0}^{n} \langle x_i^*, x_{i+1} - x_i \rangle + \langle x^*, y - x \rangle - \sum_{i=1}^{n+1} \varepsilon_i$$

and, so, by (25),

$$f_\delta(y) > \alpha + \langle x^*, y - x \rangle - \varepsilon.$$

Hence, as α approaches $f_\delta(x)$ we obtain

$$f_\delta(y) \geq f_\delta(x) + \langle x^*, y - x \rangle - \varepsilon.$$

Therefore, by the arbitrariness of $y \in X$ we deduce that $x^* \in \partial_\varepsilon f_\delta(x)$. Thus, (24) holds and, so, by Corollary 10 there exists a constant $c \in \mathbb{R}$ such that

$$\overline{\text{co}} f = f_\delta + c.$$

Finally, the desired conclusion follows by the obvious fact that $c = \overline{\text{co}} f(x_0) - f_\delta(x_0) = 0$. ■

The following result gives the counterpart of Theorem 13 when the exact subdifferential is used instead of the ε-subdifferential in the construction of the lsc convex envelope. The proof is very similar to the one of Theorem 13 with only small details changing; for the reader's convenience we give a sketch of it.

Theorem 14 *With the notation of Theorem* 13, *we assume that* $\text{int}(\text{dom} f^*) \neq \emptyset$. *Then, we have that, for every* $x \in X^{**}$,

$$f^{**}(x) = f(x_0) + \sup \left\{ \sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle + \langle x_n^*, x - x_n \rangle \right\},$$

where the supremum is taken over $n \in \mathbb{N}$, $(x_i, x_i^*) \in \partial \bar{f}^{w^{**}}$, $i = 1, \cdots n$, *and* $x_0^* \in \partial f(x_0)$ *(with the convention that* $\sum_{i=0}^{-1} = 0$*). Consequently, for every* $x \in X$,

$$\overline{\text{co}} f(x) = f(x_0) + \sup \left\{ \sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle + \langle x_n^*, x - x_n \rangle \right\},$$

where the supremum is taken over the same parameters as in the formula of f^{**} *above.*

Proof. It suffices to establish the first formula expressing f^{**}. For this aim we denote by g the function of the right-hand side. Then, g is weak** lsc convex proper function on X^{**} and, so, it can be easily checked that $g \leq f^{**} \leq \bar{f}^{w^{**}}$ and $\text{dom}\, g^* \subset \text{dom} f^*$ (using Lemma 2(iii)). Then, the proof consists of showing that, for every $x \in X^{**}$,

$$\partial \bar{f}^{w^{**}}(x) \subset \partial g(x). \tag{26}$$

For this aim, we fix $x \in X^{**}$ and take $x^* \in \partial \bar{f}^{w^{**}}(x)$. So, from the inequality $g \leq f^{**}$, together with the fact that $g \in \Gamma_0(X^{**})$, we deduce that $g(x) \in \mathbb{R}$. Now, we fix $y \in X^{**}$ and let α be such that $g(x) > \alpha$. Then, there exist $n \in \mathbb{N}$, $(x_i, x_i^*) \in X^{**} \times X^*$ with $x_i^* \in \partial \bar{f}^{w^{**}}(x_i)$, for $i = 1, \cdots, n$, and $x_0^* \in \partial f(x_0)$ such that

$$f(x_0) + \sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle + \langle x_n^*, x - x_n \rangle > \alpha.$$

Hence, denoting $(x_{n+1}, x_{n+1}^*) := (x, x^*) \in \partial \bar{f}^{w^{**}}$, we deduce that

$$g(y) \geq f(x_0) + \sum_{i=0}^{n} \langle x_i^*, x_{i+1} - x_i \rangle + \langle x^*, y - x \rangle > \alpha + \langle x^*, y - x \rangle,$$

which, as $\alpha \to g(x)$, in view of the arbitrariness of $y \in X^{**}$ establishes that $x^* \in \partial g(x)$. Therefore, (26) holds and, so, by Theorem 9, together with Remark 2(iii), there exists a constant $c \in \mathbb{R}$ such that

$$f^{**} = g^{**} \square \sigma_{\mathrm{dom}\, f^*} + c = g \square \sigma_{\mathrm{dom}\, f^*} + c.$$

Thus, arguing as in the proof of Theorem 13 we show that $c = 0$ and $g \square \sigma_{\mathrm{dom}\, f^*} = g$. This finishes the proof of the theorem. ■

We close this paper with the following result which has already been established in [6, Corollary 17]; see, also, [4, 5] for the finite-dimensional case. Here, we use a different approach based on the analysis of the subdifferential of the conjugate function.

Corollary 15 *Let X be a Banach space with the Radon-Nikodym property, $f : X \to \overline{\mathbb{R}}$ be a given function, and $x_0 \in \mathrm{dom}(\partial f)$. We assume that $\mathrm{int}(\mathrm{dom}\, f^*) \neq \emptyset$. Then, we have that*

$$\overline{\mathrm{co}} f(x) = f(x_0) + \sup \left\{ \sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle + \langle x_n^*, x - x_n \rangle \right\},$$

where the supremum is taken over $n \in \mathbb{N}$, $(x_i, x_i^*) \in \partial f$, $i = 1, \cdots, n$, and $x_0^* \in \partial f(x_0)$.

Proof. If g denotes the right-hand side, then as in Theorems 13 we easily show that $g \in \Gamma_0(X)$, $g \leq f$ and $\partial f \subset \partial g$. Then, the conclusion follows by Proposition 12 by taking into account that $\mathrm{dom}\, g^* \subset \mathrm{dom}\, f^*$ and, so, $\overline{\mathrm{co}} f = g \square \sigma_{\mathrm{dom}\, g^*} = g$. ■

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References

[1] Asplund, E. *Fréchet-differentiability of convex functions.* Acta Math. 121 (1968) 31–47.

[2] Asplund, E., Rockafellar, R. T. *Gradients of convex functions.* Trans. Amer. Math. Soc. 139 (1969) 443–467.

[3] Aussel, D., Corvellec, J.-N., Lassonde, M. *Mean value property and subdifferential criteria for lower semicontinuous functions.* Trans. Amer. Math. Soc. 347 (1995), no. 10, 4147–4161.

[4] Bachir, M., Daniilidis, A., Penot, J.-P. *Lower subdifferentiability and integration.* Set-Valued Anal. 10 (2002), no. 1, 89–108.

[5] Benoist, J., Daniilidis, A. *Integration of Fenchel subdifferentials of epi-pointed functions.* SIAM J. Optim. 12 (2002), no. 3, 575–582.

[6] Benoist, J., Daniilidis, A. *Subdifferential representation of convex functions: refinements and applications.* J. Convex Anal. 12 (2005), no. 2, 255–265.

[7] Benoist, J., Hiriart-Urruty, J.-B. *What is the subdifferential of the closed convex hull of a function?* SIAM J. Math. Anal. 27 (1996), no. 6, 1661–1679.

[8] Borwein, J., Vanderwerff, J. *Differentiability of Conjugate Functions and Perturbed Minimization Principles.* J. Convex Anal. 16 (2009), no. 3, 707–711.

[9] Collier, J. B. *The dual of a space with the Radon-Nikodym property.* Pacific. J. Math. 64 (1976), no. 1, 103–106.

[10] Correa, R., Gajardo, P., Thibault, L. *Subdifferential representation formula and subdifferential criteria for the behavior of nonsmooth functions.* Nonlinear Anal. 65 (2006), no. 4, 864–891.

[11] Correa, R., Hantoute, A. *New formulas for the Fenchel subdifferential of the conjugate function.* Set-Valued Var. Anal. 18 (2010), no. 3-4, 405-422.

[12] Correa, R., Hantoute, A. *Subdifferential of the Fenchel-Legendre Conjugate Function and Argmin Set of the Lower Semicontinuous Convex Hull Via Asymptotic Analysis.* Preprint 2010.

[13] Lassonde, M. *Asplund spaces, Stegall variational principle and the RNP.* Set-Valued Var. Anal. 17 (2009), no. 2, 183–193.

[14] López, M. A., Volle, M. *A formula for the set of optimal solutions of a relaxed minimization problem. Applications to subdifferential calculus*, Journal of Convex Anal. 17 no. 3&4, 1057–1075 (2010).

[15] Marcellin, S., Thibault, L. *Integration of ε-Fenchel subdifferentials and maximal cyclic monotonicity.* J. Global Optim. 32 (2005), no. 1, 83–91.

[16] Mordukhovich, B. S. Variational Analysis and Generalized Differentiation. Vol. I: Basic Theory, Vol. II: Applications. Springer, Berlin, 2006.

[17] Moreau, J. J. Fonctionnelles convexes. Séminaire sur les équations aux dérivées partielles. Collège de France, 1966.

[18] Phelps, R. R. Convex functions, monotone operators and differentiability. Second edition. Springer-Verlag, Berlin, 1993.

[19] Rockafellar, R. T. *Characterization of the subdifferentials of convex functions.* Pacific J. Math. 17 (1966), 497–510.

[20] Rockafellar, R. T. *On the maximal monotonicity of subdifferential mappings.* Pacific J. Math. 33 (1970), 209–216.

[21] Rockafellar, R. T., Wets, R. J.-B. Variational Analysis, Springer, Berlin, 1998.

[22] Seeger, A. *Degree of pointedness of a convex function.* Bull. Austral. Math. Soc. 53 (1996), no. 1, 159–167.

[23] Thibault, L., Zlateva, N. *Integrability of subdifferentials of directionally Lipschitz functions.* Proc. Amer. Math. Soc. 133 (2005), no. 10, 2939–2948.

[24] Volle, M. *Complements on subdifferential calculus.* Pac. J. Optim. 4 (2008), no. 3, 621–628.

[25] Zălinescu, C. *Convex analysis in general vector spaces.* World Scientific Publishing Co., Inc., River Edge, NJ, 2002.