A Polyhedral Study of the Semi-Continuous Knapsack Problem

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Abstract

We study the convex hull of the feasible set of the *semi-continuous knapsack problem*, in which the variables belong to the union of two intervals. Besides being important in its own right, the semi-continuous knapsack problem arises in a number of other contexts, e.g. it is a relaxation of general mixed-integer programming. We show how strong inequalities valid for the *semi-continuous knapsack polyhedron* can be derived and used in a branch-and-cut scheme for problems with semi-continuous variables. To demonstrate the effectiveness of these inequalities, which we call collectively *semi-continuous cuts*, we present computational results on real instances of the unit commitment problem.

Keywords: semi-continuous variables, mixed-integer programming, disjunctive programming, polyhedral combinatorics, branch-and-cut, unit commitment problem

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1 Introduction

Let n be a positive integer, $N = \{1, \ldots, n\}$, N^+, N^+_∞, and N^- three disjoint subsets of N with $N^+ \cup N^+_\infty \cup N^- = N$, $a_j > 0\ \forall j \in N^+ \cup N^+_\infty$, $a_j < 0\ \forall j \in N^-$, $p_j, l_j \in \Re\ \forall j \in N$, and $u_j \in \Re\ \forall j \in N^+$. We study the inequality description of $P = \text{clconv}(S)$, the closure of the convex hull of S, where $S = \{x \in \Re^n : x \text{ satisfies } (1), (2), \text{ and } (3)\}$, and

$$\sum_{j \in N} a_j x_j \leq b, \tag{1}$$

$$x_j \in [0, p_j] \cup [l_j, u_j]\ \forall j \in N^+, \tag{2}$$

and

$$x_j \in [0, p_j] \cup [l_j, \infty)\ \forall j \in N^+_\infty \cup N^-. \tag{3}$$

(Bounded variables x_j with $a_j < 0$ are replaced with $u_j - x_j$.) When $p_j < l_j$, we call x_j a *semi-continuous variable*, and the corresponding constraint (2) or (3) a *semi-continuous constraint*. Note that our definition of semi-continuous variable is more general than the usual one (Beale [2]), in which $p_j = 0$. When $p_j \geq l_j$, we say that x_j is a continuous variable. The set P is the *semi-continuous knapsack polyhedron*. An optimization problem whose constraints are (1), (2), and (3) is a *semi-continuous knapsack problem* (SCKP).

Semi-continuous constraints appear in a number of applications, for example production scheduling (Beale [2], Biegler et al. [4]), portfolio optimization (Bienstock [5], Perold [34]), blending (Williams [39]), and the unit commitment problem (Takriti et al. [38]). They also appear in general mixed-integer programming (MIP), since $x_j \in Z \cap [0, u_j] \Rightarrow x_j \in [0, p_j] \cup [p_j + 1, u_j]$, where p_j is an integer.

We study the inequality description of P and the use of such inequalities as cuts within branch-and-cut to solve difficult optimization problems involving semi-continuous variables. Here we do not use binary variables to enforce (2) or (3). Rather, we perform our study in the space of the structural variables x_j. Note, in any case, that the cuts given here can be used regardless of whether (2) and (3) are modeled with or without binary variables (see e.g. Beale [2] and de Farias et al. [11]). This approach was successfully used on other combinatorial constraints, such as complementarity (de Farias et al. [12], Ibaraki [23], Ibaraki et al. [24], Jeroslow [25]), cardinality (Bienstock [5], de Farias and Nemhauser [13, 14], Laundy [28]), and special ordered sets (Beale and Tomlin [3], de Farias et al. [9, 10, 15], Keha et al. [26, 27], Martin et al. [31], and Zhao and de Farias [43]).

The main tool used in this paper to study the inequality representation of semi-continuous knapsack polyhedra is lifting, see Louveaux and Wolsey [29] for the basic theory of lifting and recent developments. As part of our lifting approach, we generalize the concepts of cover and cover inequality (Balas [1], Hammer et al. [20], Wolsey [40]), which have proven to be of great use in 0-1 programming, for semi-continuous constraints. We also show how to lift our cover inequalities to derive strong cuts to be used in branch-and-cut. We call our cuts collectively *semi-continuous cuts*. We test the effectiveness of our theory on real instances of the unit commitment problem.

Throughout the paper we denote $(d)^+ = \max\{0, d\}$, where $d \in \Re$. We adopt the convention that:

- $[d, e] = \emptyset$ when $d > e$, where $d, e \in \Re$, or when $d = \infty$ or $e = -\infty$;
- $\sum_{j \in T} d_j = 0$ when $T = \emptyset$, where $d_j \in \Re$;
- $\sup\{f(x) : x \in W\} = -\infty$ and $\inf\{f(x) : x \in W\} = \infty$ when $W = \emptyset$;
- $\sup\{f(x) : x \in W\} = \infty$ and $\inf\{f(x) : x \in W\} = -\infty$ when the problem is unbounded, where $f : \Re^n \to \Re$ and $W \subseteq \Re^n$.

We will repeatedly refer to Proposition 1, where the following notation is used. First, $u_j = \infty$ $\forall j \in N^+_\infty$. If

$$\sum_{j \in N^+ \cup N^+_\infty} a_j u_j > b, \tag{4}$$

let $t \in N^+ \cup N^+_\infty$ be the smallest index such that $\sum_{\{j \in N^+ \cup N^+_\infty : j \leq t\}} a_j u_j > b$, and $\hat{x}$ be given by $\hat{x}_j = u_j$ $\forall j \in \{k \in N^+ : k < t\}$,

$$\hat{x}_t = \frac{b - \sum_{\{j \in N^+ : j < t\}} a_j u_j}{a_t},$$

and $\hat{x}_j = 0$ otherwise. If $N^- \neq \emptyset$, let $r = \max\{j : j \in N^-\}$, and $\tilde{x}$ be given by $\tilde{x}_j = u_j$ $\forall j \in \{k \in N^+ : k < r\}$,

$$\tilde{x}_r = \left(\frac{b - \sum_{\{j \in N^+ : j < r\}} a_j u_j}{a_r} \right)^+,$$

and $\tilde{x}_j = 0$ otherwise. Finally, let $\bar{x}$ be given by $\bar{x}_j = u_j$ $\forall j \in N^+$, and $\bar{x}_j = 0$ otherwise.

Proposition 1 *Consider the continuous knapsack problem*

$$max \left\{ \sum_{j \in N} c_j x_j : \sum_{j \in N} a_j x_j \leq b, x_j \leq u_j \ \forall j \in N^+, \text{and } x_j \geq 0 \ \forall j \in N \right\}, \tag{5}$$

where $c_j \geq 0$ $\forall j \in N^+ \cup N^+_\infty$, $c_j \leq 0$ $\forall j \in N^-$, *and*

$$\frac{c_1}{a_1} \geq \cdots \geq \frac{c_n}{a_n}.$$

Then, (5) has an optimal solution iff it is feasible and $\forall j \in N^+_\infty, k \in N^-$,

$$\frac{c_j}{a_j} \leq \frac{c_k}{a_k}.$$

Assume that (5) has an optimal solution x^*. If

$$\frac{c_j}{a_j} > \frac{c_k}{a_k}$$

for some $j \in N^+, k \in N - \{j\}$, then $x_j^* = u_j$ whenever $k \in N^-$ or $x_k^* > 0$. Finally, suppose that

1. *$N^+ \cup N_\infty^+ = \emptyset$. Then $\tilde{x}$ is an optimal solution to (5);*
2. *$N^- = \emptyset$. If (4) holds, then $\hat{x}$ is an optimal solution to (5), otherwise $\bar{x}$ is an optimal solution to (5);*
3. *both $N^+ \cup N_\infty^+ \neq \emptyset$ and $N^- \neq \emptyset$. If (4) does not hold, then $\bar{x}$ is an optimal solution to (5). Assume that (4) holds. If $t < r$ and $t \in N^+$, then $\tilde{x}$ is an optimal solution to (5), otherwise $\hat{x}$ is an optimal solution to (5).* □

In Section 2 we introduce assumptions and we present a few simple results about P. We give the trivial facet-defining inequalities and a necessary and sufficient condition for them to describe P. We then discuss the nontrivial inequalities. In Section 3 we present the lifting technique and a few lifting results that hold for the nontrivial inequalities for P. We show that in some cases it is easy to obtain the exact lifting coefficients of several variables, and we show how, within time $O(n^2)$, the lifting coefficients of all variables can be calculated approximately. We also give the full inequality description of P when all variables are continuous with the exception of one semi-continuous variable. In Section 4 we extend the concepts of cover and cover inequality of 0-1 programming to our case. We show that when the cover is *simple*, the cover inequality is the only nontrivial facet-defining inequality of the semi-continuous knapsack polyhedron P_0 obtained by fixing at 0 all variables not indexed by the cover. We give the value of the lifting coefficient of a continuous variable x_k, $k \in N^-$, that is fixed at 0, when the cover inequality is lifted with respect to it first. Also, we show that the cover inequality defines a facet of P^0 when $l_k = u_k$ for all variables in the cover. In Section 5 we present results of our computational experience on the effectiveness of semi-continuous cuts on real instances of the stochastic unit commitment problem with linear costs. In Section 6 we present directions for further research. An earlier version of this paper was [7, 8].

2 The Semi-Continuous Knapsack Polyhedron

In this section we introduce assumptions and we present a few simple results about P. We establish the *trivial* facet-defining inequalities and when they suffice to describe P. We present a *nontrivial* facet-defining inequality that dominates (1) when $N = N^-$, and under certain conditions is the only nontrivial facet-defining inequality for P. Finally, we present a few relations that hold among the coefficients of the variables in a nontrivial inequality.

We will assume throughout the paper that:

Assumption 1 $n \geq 2$;

Assumption 2 $p_j \geq 0$ and $l_j > 0$ $\forall j \in N$, and $u_j \geq l_j$ $\forall j \in N^+$;

Assumption 3 when $N^- = \emptyset$, $N^+_\infty = \emptyset$ and $a_j u_j \leq b$ $\forall j \in N^+$;

Assumption 4 when $N^+_\infty = \emptyset$, $\sum_{j \in N^+} a_j u_j > b$.

When Assumption 1 does not hold, the problem is trivial. So, there is no loss of generality in Assumption 1. In addition, with Assumption 1 it is possible to simplify the presentation of several results that would otherwise have to consider separately the case $n = 1$. If $u_j = 0$ for some $j \in N^+$, x_j can be eliminated from the problem. Therefore, there is no loss of generality in Assumption 2. Assumption 2 implies that $p_j > 0$ whenever x_j is continuous. When $N^- = \emptyset$, x_j is bounded $\forall j \in N$. In addition, for $j \in N^+$, it is possible, in this case, to scale x_j so that $a_j u_j \leq b$, unless $b = 0$ or $x_j \in \{0\} \cup [l_j, u_j]$ with $a_j l_j > b$. In the first case the problem is trivial, and in the second case x_j can be eliminated from the problem. Thus, there is no loss of generality in Assumption 3. Assumption 3 implies that when $N^- = \emptyset$, $b > 0$. Finally, if $N^+_\infty = \emptyset$, (1) is redundant unless $\sum_{j \in N^+} a_j u_j > b$. This means that there is no loss of generality in Assumption 4 either. Assumption 4 implies that when $N^+ \cup N^+_\infty = \emptyset$, $b < 0$.

As a result of Assumptions 1, 2, 3, and 4, Propositions 2, 3, and 4 follow.

Proposition 2 *P is full-dimensional.* □

Proposition 3 *The inequality*

$$x_j \geq 0 \tag{6}$$

is facet-defining $\forall j \in N^+ \cup N^+_\infty$. For $j \in N^-$, (6) is facet-defining iff either

1. *$|N^-| > 1$ or*
2. *$b > 0$ and $\forall k \in N^+ \cup N^+_\infty$ either $p_k > 0$ or $a_k l_k \leq b$.*

□

Proposition 4 *When $N^- \neq \emptyset$,*

$$x_j \leq u_j \tag{7}$$

is facet-defining $\forall j \in N^+$. When $N^- = \emptyset$, (7) is facet-defining iff $a_j u_j < b$ and $\forall k \in N^+ - \{j\}$ either $p_k > 0$ or $a_j u_j + a_k l_k \leq b$. □

Example 1 Let $S = \{x \in \Re^3 : x$ satisfies (8), $x_1 \in [0,1] \cup [2,3]$, $x_2 \in \{0\} \cup [3,\infty)$, and $x_3 \geq 0\}$, where

$$2x_1 + 3x_2 \leq 4 + x_3. \tag{8}$$

Then, $x_1 \geq 0$, $x_1 \leq 3$, and $x_2 \geq 0$ are facet-defining for P. On the other hand, $x_3 \geq x_2$ $\forall x \in S$, and therefore $x_3 \geq 0$ is not facet-defining. □

Unlike inequalities (6) and (7), there do not seem to exist simple necessary and sufficient conditions to determine when (1) is facet-defining. We now present an inequality that, when $N = N^-$, is valid for P, is at least as strong as (1) (and possibly stronger), and under an additional condition gives, together with (6), a full inequality description for P.

Proposition 5 *Suppose that $N = N^-$, and let $N_0^- = \{j \in N^- : p_j = 0\}$. Then,*

$$\sum_{j \in N_0^-} \frac{a_j}{min\{b, a_j l_j\}} x_j + \frac{1}{b} \sum_{j \in N^- - N_0^-} a_j x_j \geq 1 \tag{9}$$

is valid for P. If $N^- = N_0^-$ or $\exists k \in N^- - N_0^-$ such that x_k is continuous, then (9) is facet-defining. If $N^- = N_0^-$, then $P = \{x \in \Re^n_+ : x$ satisfies (9)$\}$.

Proof Recall that due to Assumption 4, $b < 0$. Let $\tilde{x} \in S$. If $\tilde{x}_j > 0$ for some $j \in N_0^-$ with $a_j l_j < b$, then, since $\tilde{x}_j \geq l_j$, $\tilde{x}$ satisfies (9). If $\tilde{x}_j = 0$ for all $j \in N_0^-$ with $a_j l_j < b$, then (1) $\Rightarrow \tilde{x}$ satisfies (9). Clearly, if $N^- = N_0^-$ or $\exists k \in N^- - N_0^-$ such that x_k is continuous, then (9) is facet-defining.

Suppose that $N = N_0^-$. We prove that $P = \{x \in \Re^n_+ : x$ satisfies (9)$\}$ by showing that for an arbitrary nonzero vector $(c_1, \ldots, c_n)$ of objective function coefficients, one of the inequalities (6) or (9) is satisfied at equality by every optimal solution to SCKP. We assume WLOG that SCKP is a minimization problem.

If $c_j < 0$ for some $j \in N$, then SCKP is unbounded. So we assume that $c_j \geq 0$ $\forall j \in N$. Let $I = \{j \in N : c_j = 0\}$. If $I \neq \emptyset$, then for every optimal solution to SCKP, $x_j = 0$ $\forall j \in N - I$. So we assume that $c_j > 0$ $\forall j \in N$. Let x^* be an optimal solution to SCKP, $R = \{j \in N : x_j^* > 0\}$, and

$$s = \text{argmax}\left\{\frac{c_j}{a_j} : j \in R\right\}.$$

Suppose that x^* does not satisfy (9) at equality. Since

$$\sum_{j \in R} \frac{a_j}{\min\{b, a_j l_j\}} x_j^* > 1 \Rightarrow \sum_{j \in R} a_j x_j^* < b,$$

then $x_j^* = l_j$ $\forall j \in R$, and we have that

$$\sum_{j \in R} a_j l_j < b. \tag{10}$$

Note that $|R| \geq 2$. This is true because if $R = \{s\}$, then (10) $\Rightarrow \min\{a_s l_s, b\} = a_s l_s$, and (9) is satisfied at equality. Since $|R| \geq 2$, it is clear that $\forall j \in R$,

$$l_j < \frac{b}{a_j}. \tag{11}$$

Now, let z^* be the optimal value of SCKP. Then, (10) implies that

$$c_s \frac{b}{a_s} < c_s \frac{\sum_{j \in R} a_j l_j}{a_s} \leq \sum_{j \in R} c_j l_j = z^*.$$

However, due to (11), $\hat{x}$ given by

$$\hat{x}_j = \begin{cases} \frac{b}{a_s} & \text{if } j = s \\ 0 & \text{otherwise} \end{cases}$$

is a feasible solution to SCKP. Thus, x^* must satisfy (9) at equality. □

Example 2 Let $S = \{x \in \Re^5_+ : x$ satisfies (12), $x_1 \in \{0\} \cup [1, \infty)$, $x_2 \in \{0\} \cup [2, \infty)$, $x_3 \in [0, 1] \cup [2, \infty)$, and $x_5 \in [0, 1] \cup \{2\}\}$, where

$$2x_1 + 3x_2 + 3x_3 + x_4 \geq 4 + x_5. \tag{12}$$

Then, $P \cap \{x \in \Re^5 : x_3 = x_4 = x_5 = 0\} = \{x \in \Re^5_+ : x$ satisfies (13) and $x_3 = x_4 = x_5 = 0\}$, where

$$x_1 + x_2 \geq 2. \tag{13}$$

□

Let $PR = \{x \in \Re^n_+ : x$ satisfies (1) and (7) $\forall j \in N^+\}$, the feasible set of the LP relaxation of SCKP. The following proposition is easy to prove.

Proposition 6 *Let x be a vertex of PR. Then, with the possible exception of one, all components of x must satisfy*

$$x_j = 0, j \in N^- \cup N^+_\infty \tag{14}$$

and

$$x_j \in \{0, u_j\}, j \in N^+. \tag{15}$$

If one of conditions (14) or (15) is not satisfied by a component of x, then x must satisfy (1) at equality. □

We now give a necessary and sufficient condition for $P = PR$.

Proposition 7 *$P = PR$ iff $\forall T \subseteq N^+$, $i \in N^+ \cup N^+_\infty - T$, and $k \in N^-$, the following two conditions are satisfied*

1. $\sum_{j\in T} a_j u_j + a_i p_i \geq b$ *or* $\sum_{j\in T} a_j u_j + a_i l_i \leq b$
2. $\sum_{j\in T} a_j u_j + a_k p_k \leq b$ *or* $\sum_{j\in T} a_j u_j + a_k l_k \geq b$.

Proof The *if* part follows from Proposition 6. If $a_i p_i < b - \sum_{j\in T} a_j u_j < a_i l_i$ for some $T \subseteq N^+$ and $i \in N^+ \cup N^+_\infty - T$, then $\hat{x}$ given by

$$\hat{x}_j = \begin{cases} u_j & \text{if } j \in T \\ \frac{b - \sum_{v\in T} a_v u_v}{a_i} & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

is a vertex of PR that does not satisfy (2). If $a_k p_k > b - \sum_{j\in T} a_j u_j > a_k l_k$ for some $T \subseteq N^+$ and $k \in N^-$, then $\tilde{x}$ given by

$$\tilde{x}_j = \begin{cases} u_j & \text{if } j \in T \\ \frac{b - \sum_{v\in T} a_v u_v}{a_k} & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}$$

is a vertex of PR that does not satisfy (3). □

Inequalities (1), (6), (7), and any inequality dominated by one of them, is called *trivial*. For the remainder of the paper we will study the *nontrivial* inequalities for P. We will denote a generic nontrivial inequality valid for P as

$$\sum_{j\in N} \alpha_j x_j \leq \beta. \tag{16}$$

Note that $\alpha_j \leq 0\ \forall j \in N^-$, $\beta > 0$ whenever $N^- = \emptyset$, and $\beta < 0$ whenever $N^+ \cup N^+_\infty = \emptyset$. In addition, (16) remains valid if we replace α_j with $(\alpha_j)^+\ \forall j \in N^+ \cup N^+_\infty$, and we will then assume that $\alpha_j \geq 0\ \forall j \in N^+ \cup N^+_\infty$.

Proposition 8 *Let $k \in N^-$. If*

1. *$i \in N^+_\infty$ and (16) is satisfied at equality for at least one point $\tilde{x}$ of S, then*

$$\frac{\alpha_i}{a_i} \leq \frac{\alpha_k}{a_k}; \tag{17}$$

2. $i \in N^+$, x_k is continuous, and (16) is satisfied at equality for at least one point $\tilde{x}$ of S with $\tilde{x}_i < u_i$, then

$$\frac{\alpha_i}{a_i} \leq \frac{\alpha_k}{a_k}, \tag{18}$$

or in case $\tilde{x}_i > 0$, then

$$\frac{\alpha_i}{a_i} \geq \frac{\alpha_k}{a_k}; \tag{19}$$

3. $i \in N^-$, x_k is continuous, and (16) is satisfied at equality for at least one point $\tilde{x}$ of S with $\tilde{x}_i > 0$, then

$$\frac{\alpha_i}{a_i} \leq \frac{\alpha_k}{a_k}. \tag{20}$$

Proof We prove the proposition for statement 1, the proofs for statements 2 and 3 being similar. For ϵ sufficiently large, $\tilde{x}'$ given by

$$\tilde{x}'_j = \begin{cases} \tilde{x}_i + \epsilon & \text{if } j = i \\ \tilde{x}_k - \epsilon \frac{a_i}{a_k} & \text{if } j = k \\ \tilde{x}_j & \text{otherwise} \end{cases}$$

also belongs to S. Therefore, it satisfies (16), and

$$\alpha_i \epsilon - \alpha_k \epsilon \frac{a_i}{a_k} \leq 0,$$

which implies (17). □

Proposition 9 *Suppose that (16) is facet-defining. If $i \in N^+_\infty$ and x_i is continuous, then $\alpha_i = 0$.*

Proof Because (16) is nontrivial and facet-defining, there exists a point $x^* \in P$ that satisfies (16) at equality and (1) strictly at inequality. Thus, for $\epsilon > 0$ sufficiently small, x' given by

$$x'_j = \begin{cases} x^*_i + \epsilon & \text{if } j = i \\ x^*_j & \text{otherwise} \end{cases}$$

belongs to P. This means that $\alpha_i \leq 0$, and therefore, $\alpha_i = 0$. □

From statement 3 of Proposition 8 it follows that if (16) is facet-defining, and x_i and x_k, $i, k \in N^-$, are continuous,

$$\frac{\alpha_i}{a_i} = \frac{\alpha_k}{a_k}. \tag{21}$$

Because of (21) and Proposition 9 we will continue our polyhedral study under the following two assumptions:

Assumption 5 There is at most one continuous variable x_i with $i \in N^-$;

Assumption 6 $p_j < l_j \; \forall j \in N^+_\infty$.

3 Lifting

In this section we present the lifting technique and a few lifting results that hold for the nontrivial inequalities for P. We show that in some cases it is easy to obtain the exact lifting coefficients of several variables. We then apply the lifting technique to obtain a nontrivial family of facets of P when all variables are continuous with the exception of one semi-continuous variable. We show that, in this particular case, this family of facets, together with the trivial facets, gives P. Finally we show how, within time $O(n^2)$, all the lifting coefficients can be calculated approximately.

Let $\tilde{x} \in P$, $T^+ \subseteq N^+$, $T^+_\infty \subseteq N^+_\infty$, $T^- \subseteq N^-$, and $T = T^+ \cup T^+_\infty \cup T^-$. We will henceforth denote a generic nontrivial valid inequality for $P_T = P \cap \{x \in \Re^n : x_j = \tilde{x}_j \; \forall j \in N - T\}$ as

$$\sum_{j \in T} \alpha_j x_j \leq \beta. \tag{22}$$

We will also denote the resulting knapsack constraint (1) when $x_j = \tilde{x}_j \; \forall j \in N - T$ as

$$\sum_{j \in T} a_j x_j \leq b - \sum_{j \in N-T} a_j \tilde{x}_j. \tag{23}$$

In Lemma 1 we establish the lifting technique, see for example Louveaux and Wolsey [29].

Lemma 1 *Let $i \in N - T$,*

$$\alpha_i^{min} = sup\left\{\frac{\sum_{j \in T} \alpha_j x_j - \beta}{\tilde{x}_i - x_i} : x \in P_{T \cup \{i\}} \text{ and } x_i < \tilde{x}_i\right\} \tag{24}$$

and

$$\alpha_i^{max} = inf\left\{\frac{\beta - \sum_{j \in T} \alpha_j x_j}{x_i - \tilde{x}_i} : x \in P_{T \cup \{i\}} \text{ and } x_i > \tilde{x}_i\right\}. \tag{25}$$

Then

$$\sum_{j \in T} \alpha_j x_j + \alpha_i x_i \leq \beta + \alpha_i \tilde{x}_i \tag{26}$$

is a valid inequality for $P_{T \cup \{i\}}$ if and only if $\alpha_i \in [\alpha_i^{min}, \alpha_i^{max}]$. Suppose the sup (resp. inf) in (24) (resp. (25)) is attained at a point of $P_{T \cup \{i\}}$. If (22) defines a face of P_T of dimension t, and $\alpha_i = \alpha_i^{min}$ (resp. $\alpha_i = \alpha_i^{max}$), then (26) defines a face of $P_{T \cup \{i\}}$ of dimension at least $t + 1$. □

Note that when $\alpha_i^{\min} = \infty$ or $\alpha_i^{\max} = -\infty$, it is not possible to lift (22) with respect to x_i. In addition, when $\tilde{x}_i$ is neither a lower nor an upper bound to x_i, it may happen that $\alpha_i^{\min} > \alpha_i^{\max}$ (see [29]), in which case x_i cannot be lifted either. On the other hand, when $\tilde{x}_i$ is a lower (resp. upper) bound for x_i, $\alpha_i^{\min} = -\infty$ (resp. $\alpha_i^{\max} = \infty$), and it is possible to lift as long as $\alpha_i^{\max} \neq -\infty$ (resp. $\alpha_i^{\min} \neq -\infty$). Because of this, in the lifting theory presented in this paper, we will fix variables at their bounds. We leave as an open question what values variables can be fixed at before lifting. For the particular case of 0-1 mixed-integer programming this issue was settled in Richard et al. [36, 37].

As we establish in Propositions 10 and 11, in some cases it is easy to obtain the exact lifting coefficients of several variables. In Proposition 10 we establish that when $\tilde{x}_i = 0$, $i \in (N^+ \cup N^+_\infty) - T$, and $p_i > 0$, the lifting coefficient of x_i is equal to 0. We omit the proof of Proposition 10, which is similar to the proof of Proposition 9.

Proposition 10 *Let $i \in (N^+ \cup N^+_\infty) - T$, $\tilde{x}_i = 0$, and suppose that (22) defines a facet of P_T. If $p_i > 0$, the lifting coefficient of x_i is 0.* □

Example 2 (Continued) The lifting coefficient of x_5 in (13) is 0 (regardless of the lifting order), and therefore $P \cap \{x \in \Re^5 : x_3 = x_4 = 0\} = \{x \in \Re^5_+ : x$ satisfies (13) and $x_3 = x_4 = 0\}$. □

We now establish that when (22) contains a continuous variable x_k with $k \in N^-$, it is easy to obtain the lifting coefficient of a variable x_i with $i \in N^-$ and $p_i > 0$ that is fixed at 0, or with $i \in N^+$ and $l_i < u_i$ that is fixed at u_i. The proof of Proposition 11 follows from Proposition 8.

Proposition 11 *Let x_k, $k \in T^-$, be a continuous variable, and suppose that (22) defines a facet of P_T. If*

1. *$i \in N^- - T$, $p_i > 0$, and $\tilde{x}_i = 0$; or*
2. *$i \in N^+ - T$, $l_i < u_i$, and $\tilde{x}_i = u_i$,*

the lifting coefficient of x_i is

$$\alpha_i = \frac{a_i}{a_k} \alpha_k.$$

□

Example 2 (Continued) The lifting coefficient of x_4 in (13) is the optimal value of

$$\max \left\{ \frac{2 - x_1 - x_2}{x_4} : x \in S, x_4 > 0, \text{ and } x_3 = 0 \right\}. \tag{27}$$

Let x^* be an optimal solution to (27). Because $x_2 > 0 \Rightarrow x_2 \geq 2$, $x_2^* = 0$. In addition, it is clear that x^* must satisfy (12) at equality, i.e. $2x_1^* + x_4^* = 4$. Thus,

$$\max\left\{\frac{2 - x_1 - x_2}{x_4} : x \in S, x_4 > 0, \text{ and } x_3 = 0\right\} = \frac{2 - x_1^*}{4 - 2x_1^*} = \frac{1}{2}.$$

So,

$$2x_1 + 2x_2 + x_4 \geq 4 \tag{28}$$

defines a facet of $P \cap \{x \in \Re^5 : x_3 = 0\}$. From Proposition 11, if we now lift (28) with respect to x_3, the lifting coefficient is 3, and thus

$$2x_1 + 2x_2 + 3x_3 + x_4 \geq 4$$

defines a facet of P. □

We now apply the lifting technique to derive a family of nontrivial facet-defining inequalities for P when all variables are continuous, with the exception of one semi-continuous variable x_i with $i \in N^+$. Note that, due to Assumptions 5 and 6, in this case $|N^-| \leq 1$ and $N^+_\infty = \emptyset$.

Proposition 12 *Let $i \in N^+$ with $p_i < l_i$. Suppose that x_j is continuous $\forall j \in N - \{i\}$. Let $U \subseteq N^+ - \{i\}$ be such that*

$$a_i l_i + \sum_{j \in U} a_j u_j > b \tag{29}$$

and

$$a_i p_i + \sum_{j \in U} a_j u_j < b. \tag{30}$$

Then,

$$\Delta_U x_i + \sum_{j \in U} a_j x_j + \sum_{j \in N^-} a_j x_j \leq p_i \Delta_U + \sum_{j \in U} a_j u_j, \tag{31}$$

with

$$\Delta_U = \frac{\sum_{j \in U} a_j u_j - b + a_i l_i}{l_i - p_i},$$

is valid and facet-defining for P.

Proof Because of (29),

$$\sum_{j\in U} a_j x_j + \sum_{j\in N^-} a_j x_j \le b - a_i l_i \tag{32}$$

is facet-defining for $P \cap \{x : x_i = l_i \text{ and } x_j = 0\ \forall j \in N^+ - (U \cap \{i\})\}$. We derive (31) by lifting (32).

Let α_i be the lifting coefficient of x_i. Using the notation of Lemma 1,

$$\alpha_i^{\max} = \max\left\{\frac{\sum_{j\in U} a_j x_j + \sum_{j\in N^-} a_j x_j - b + a_i l_i}{l_i - x_i} : x \in S \text{ and } x_i \le p_i\right\}$$

and

$$\alpha_i^{\max} = \min\left\{\frac{b - a_i l_i - \sum_{j\in U} a_j x_j - \sum_{j\in N^-} a_j x_j}{x_i - l_i} : x \in S \text{ and } x_i > l_i\right\}. \tag{33}$$

Because of (30),

$$\max\left\{\frac{\sum_{j\in U} a_j x_j + \sum_{j\in N^-} a_j x_j - b + a_i l_i}{l_i - x_i} : x \in S \text{ and } x_i \le p_i\right\} = \frac{\sum_{j\in U} a_j u_j - b + a_i l_i}{l_i - p_i} = \Delta_U.$$

Now, let x^* be an optimal solution to (33). Clearly

$$\sum_{j\in U} a_j x_j^* + \sum_{j\in N^-} a_j x_j^* + a_i x_i^* = b,$$

otherwise, by decreasing x_j, $j \in N^-$, or increasing x_i or $x_j, j \in U$, it is possible to improve over x^*. This means that

$$\alpha_i^{\max} = \frac{b - a_i l_i - \sum_{j\in U} a_j x_j^* - \sum_{j\in N^-} a_j x_j^*}{x_i^* - l_i} = \frac{b - a_i l_i - b + a_i x_i^*}{x_i^* - l_i} = a_i.$$

Since $\Delta_U < a_i$, Lemma 1 implies that it is possible to lift (32) with respect to x_i, and by choosing $\alpha_i = \Delta_U$, the lifted inequality defines a nontrivial facet of $P \cap \{x \in \Re^n : x_j = 0\ \forall j \in N^+ - (U \cap \{i\})\}$. From Proposition 10, the lifting coefficient of x_j is $0\ \forall j \in N^+ - (U \cap \{i\})$. Thus, (31) defines a facet of P. □

As we show next, under the assumptions of Proposition 12, P is given by (31) and the trivial facet-defining inequalities. For general MIP, a result of this type has been given by Magnanti et al. [30].

Theorem 1 *Let $i \in N^+$ with $l_i > p_i$. Suppose that x_j is continuous $\forall j \in N - \{i\}$. Then $P = \{x \in \Re^n : x$ satisfies (1), (6), (7), and (31)$\}$.*

Proof As we did in Proposition 5, we prove the theorem by showing that for an arbitrary nonzero vector $(c_1, \ldots, c_n)$ of objective function coefficients, the set of all optimal solutions to SCKP is contained in the face defined by an inequality of one of the families (1), (6), (7), or (31). We assume that $N^- = \{k\}$, the proof for when $N^- = \emptyset$ being similar. We assume WLOG that SCKP is a maximization problem.

If $c_j < 0$ for some $j \in N^+$, then in all optimal solutions to SCKP $x_j = 0$, so we assume that $c_j \geq 0 \ \forall j \in N^+$. If $c_k > 0$, then SCKP is unbounded. So we assume that $c_k \leq 0$.

If $c_j = 0 \ \forall j \in N^+$, then since $(c_1, \ldots, c_n)$ is a nonzero vector, $c_k < 0$, and in all optimal solutions to SCKP (1) is satisfied at equality in case $b < 0$, and $x_k = 0$ in case $b \geq 0$. So let $I = \{j \in N^+ : c_j > 0\}$ and assume that $I \neq \emptyset$.

If

$$\frac{c_j}{a_j} > \frac{c_k}{a_k}$$

for some $j \in I$, then in all optimal solutions to SCKP, $x_j = u_j$. So we assume that $\forall j \in I$

$$\frac{c_j}{a_j} \leq \frac{c_k}{a_k}.$$

If $b \leq 0$, then in all optimal solutions to SCKP, (1) is satisfied at equality. So we assume that $b > 0$.

If $\sum_{j \in I} a_j u_j \leq b$, then in all optimal solutions to SCKP, $x_r = u_r \ \forall r \in I$. So we assume that

$$\sum_{j \in I} a_j u_j > b. \tag{34}$$

If $i \notin I$, then (34) implies that in all optimal solutions to SCKP (1) is satisfied at equality. So we assume that $i \in I$. If $\sum_{j \in I - \{i\}} a_j u_j + a_i p_i \geq b$ or $\sum_{j \in I - \{i\}} a_j u_j + a_i l_i \leq b$, then again (34) implies that in all optimal solutions to SCKP (1) is satisfied at equality. So we assume that

$$\sum_{j \in I - \{i\}} a_j u_j + a_i p_i < b \tag{35}$$

and

$$\sum_{j \in I - \{i\}} a_j u_j + a_i l_i > b. \tag{36}$$

Let $I' = I - \{i\}$. Inequalities (35) and (36) imply that

$$\Delta_{I'} x_i + \sum_{j \in I'} a_j x_j + a_k x_k \leq p_i \Delta_{I'} + \sum_{j \in I'} a_j u_j \tag{37}$$

is a facet-defining inequality of the family (31).

Suppose now that SCKP has an optimal solution $\hat{x}$ that does not satisfy (1) at equality. Because of (35) and (36), $\hat{x}_i = p_i$, $\hat{x}_j = u_j \ \forall j \in I - \{i\}$, and $\hat{x}_k = 0$, in which case $\hat{x}$ satisfies (37) at equality. Let $\bar{x}$ be an alternative optimal solution that satisfies (1) at equality. Then,

$$\bar{x}_i \geq l_i.$$

We show that $\bar{x}$ satisfies (37) at equality. Note that

$$c_i \bar{x}_i + \sum_{j \in I'} c_j \bar{x}_j + c_k \bar{x}_k = c_i p_i + \sum_{j \in I'} c_j u_j. \tag{38}$$

Suppose that $\bar{x}_i = l_i$. It follows that $a_i l_i + \sum_{j \in I'} a_j \bar{x}_j + a_k \bar{x}_k = b$, or

$$\frac{a_i l_i + \sum_{j \in I'} a_j u_j - b}{l_i - p_i}(l_i - p_i) + \sum_{j \in I'} a_j \bar{x}_j + a_k \bar{x}_k = \sum_{j \in I'} a_j u_j,$$

and so

$$\Delta_{I'} l_i + \sum_{j \in I'} a_j \bar{x}_j + a_k \bar{x}_k = \Delta_{I'} p_i + \sum_{j \in I'} a_j u_j.$$

Finally, suppose $\bar{x}_i > l_i$. We have that

$$\bar{x}_i = \frac{b - \sum_{j \in I'} a_j \bar{x}_j - a_k \bar{x}_k}{a_i}.$$

Because of (38),

$$c_i \frac{b - \sum_{j \in I'} a_j \bar{x}_j - a_k \bar{x}_k}{a_i} + \sum_{j \in I'} c_j \bar{x}_j + c_k \bar{x}_k = c_i p_i + \sum_{j \in I'} c_j u_j,$$

or

$$b - \sum_{j \in I'} a_j \bar{x}_j - a_k \bar{x}_k = a_i p_i + \sum_{j \in I'} \frac{a_i}{c_i} c_j (u_j - \bar{x}_j) - \frac{a_i}{c_i} c_k \bar{x}_k. \tag{39}$$

For $j \in I'$, if

$$\frac{c_j}{a_j} > \frac{c_i}{a_i},$$

then $\bar{x}$ can be an optimal solution only if $\bar{x}_j = u_j$. This means that $\forall j \in I'$,

$$\frac{a_i}{c_i} c_j (u_j - \bar{x}_j) \leq a_j (u_j - \bar{x}_j).$$

In the same way, if

$$\frac{c_k}{a_k} > \frac{c_i}{a_i},$$

then $\bar{x}$ can be an optimal solution only if $\bar{x}_k = 0$. This means that

$$\frac{a_i}{c_i} c_k \bar{x}_k \geq a_k \bar{x}_k.$$

So,

$$a_i p_i + \sum_{j \in I'} \frac{a_i}{c_i} c_j (u_j - \bar{x}_j) - \frac{a_i}{c_i} c_k \bar{x}_k \leq a_i p_i + \sum_{j \in I'} a_j (u_j - \bar{x}_j) - a_k \bar{x}_k. \tag{40}$$

Combining (39) and (40), we obtain $b \leq a_i p_i + \sum_{j \in I'} a_j u_j$, which is inconsistent with (35). Thus, it must be that $\bar{x}_i = l_i$, and in all optimal solutions to SCKP (31) is satisfied at equality. □

Example 3 Let $S = \{x \in \Re^3_+ : x$ satisfies (41), $x_1 \in [0, 1] \cup [2, 3]$, and $x_2 \leq 2\}$, where

$$2x_1 + 3x_2 \leq 9 + x_3. \tag{41}$$

Then, $P = \{x \in \Re^3_+ : x$ satisfies (41), (42), $x_1 \leq 3$, and $x_2 \leq 2\}$, where

$$x_1 + 3x_2 \leq 8 + x_3. \tag{42}$$

□

In 0-1 programming, the objective function denominator in either (24) or (25) is always equal to 1, and the lifting problem is a linear 0-1 knapsack problem. In practice, it is common, in this case, to solve the lifting problem approximately by solving its LP relaxation and to round the resulting optimal value down for (24) and up for (25); see Gu et al. [19], where an extensive computational study is presented that shows, among other things, that it is more practical to use the LP relaxation approximation to compute the lifting coefficients than to use dynamic programming to compute them exactly.

In the case of semi-continuous variables, however, the objective function denominator in (24) or (25) may not be a constant. We now show how to solve the continuous relaxation of (24) and (25) for this case to obtain approximate values for the lifting coefficients of x_j, $j \in N - T$, in (22). As in 0-1 programming, the procedure gives the approximate values of all lifting coefficients within time $O(n^2)$. We will discuss specifically the case where the next variable to be lifted is x_k, $k \in N^+ - T$, and $\tilde{x}_k = 0$. The other cases can be treated in a similar way. (We note that the complexity of exact lifting for semi-continuous variables is not known.)

First, following Proposition 10 and our earlier assumption that $\alpha_k \geq 0$ when $k \in N^+$, we take the lifting coefficient of x_k as 0 when $p_k > 0$. Now, we discuss the case $p_k = 0 < l_k < u_k$. The approximate lifting coefficient α_k of x_k is given by the optimal value of the continuous relaxation of (25), i.e.

$$\alpha_k = \min \left\{ \frac{\beta - \sum_{j \in T} \alpha_j x_j}{x_k} : x \in \bar{S}_k \right\}, \tag{43}$$

where

$$\begin{aligned} \bar{S}_k = \{x \in \Re^n_+ : \; & \textstyle\sum_{j \in T} a_j x_j + a_k x_k \leq b - \sum_{j \in N-T} a_j \tilde{x}_j, \\ & x_j \leq u_j \; \forall j \in N^+, x_j = \tilde{x}_j \; \forall j \in N - (T \cup \{k\}), \text{and } x_k \geq l_k\}. \end{aligned}$$

We assume that $\bar{S}_k \neq \emptyset$, otherwise x_k cannot be the next lifted variable. Rather than solving (43), we solve

$$\max \left\{ \sum_{j \in T} \alpha_j x_j + \tilde{\alpha}_k x_k : x \in \bar{S}_k, \tilde{\alpha}_k \in \Re, \text{ and } \sum_{j \in T} \alpha_j x_j + \tilde{\alpha}_k x_k \leq \beta \right\}. \tag{44}$$

Note that $\tilde{\alpha}_k$ is a variable and the optimal value of (44) is β. We now show that problems (43) and (44) are equivalent.

Proposition 13 *Let $(\tilde{\alpha}_k^*, x^*)$ be an optimal solution to (44). Then, $\alpha_k = \tilde{\alpha}_k^*$ and x^* is an optimal solution to (43).*

Proof Let $\hat{x}$ be an optimal solution to (43). Because $\hat{x} \in \bar{S}_k$, $(\tilde{\alpha}_k^*, \hat{x})$ is a feasible solution to (44). So, $\sum_{j \in T} \alpha_j \hat{x}_j + \tilde{\alpha}_k^* \hat{x}_k \leq \beta = \sum_{j \in T} \alpha_j \hat{x}_j + \alpha_k \hat{x}_k$. Since $\hat{x}_k \geq l_k > 0$, $\alpha_k \geq \tilde{\alpha}_k^*$. In the same way, since x^* is a feasible solution to (43), $\sum_{j \in T} \alpha_j x_j^* + \alpha_k x_k^* \leq \beta = \sum_{j \in T} \alpha_j x_j^* + \tilde{\alpha}_k^* x_k^*$ and $\alpha_k \leq \tilde{\alpha}_k^*$. □

Suppose WLOG that $T = \{1, \ldots, t\}$ and

$$\frac{\alpha_1}{a_1} \geq \cdots \geq \frac{\alpha_t}{a_t}. \tag{45}$$

We now show how to solve (44) in linear time after $\left\{\frac{\alpha_1}{a_1}, \cdots, \frac{\alpha_t}{a_t}\right\}$ is sorted as in (45). If we know whether

$$\frac{\tilde{\alpha}_k^*}{a_k} \geq \frac{\alpha_1}{a_1}, \tag{46}$$

or

$$\frac{\alpha_j}{a_j} \geq \frac{\tilde{\alpha}_k^*}{a_k} \geq \frac{\alpha_{j+1}}{a_{j+1}} \tag{47}$$

for some $j \in \{1, \ldots, t-1\}$, or else

$$\frac{\alpha_t}{a_t} \geq \frac{\tilde{\alpha}_k^*}{a_k}, \tag{48}$$

then using Proposition 1 it is possible to find x^* even if we do not know $\tilde{\alpha}_k^*$. But then,

$$\tilde{\alpha}_k^* = \frac{\beta - \sum_{j \in T} \alpha_j x_j^*}{x_k^*}. \tag{49}$$

We determine $\tilde{\alpha}_k^*$ by calculating the values that result from (49) under the cases (46)–(48). Specifically, let $\tilde{\alpha}_k^0$ be the value of $\tilde{\alpha}_k^*$ resulting from (46), $\tilde{\alpha}_k^j$, $j \in \{1, \ldots, t-1\}$, the values resulting from (47), and $\tilde{\alpha}_k^t$ the value resulting from (48). Then, $\tilde{\alpha}_k^*$ will be the value of $\tilde{\alpha}_k^j$, $j \in \{0, \ldots, t\}$, for which $\frac{\tilde{\alpha}_k^j}{a_k}$ satisfies the corresponding assumption (46) – (48).

Example 4 Let $S = \{x \in Z^3 : 4x_1 + 3x_2 + 3x_3 \leq 16,\ x_1 \in [0,1] \cup \{2\},\ x_2 \in [0,2] \cup [3,4],\ x_3 \in \{0\} \cup [1,3]\}$. As we will show in Section 4, $2x_1 + x_2 \leq 6$ defines a facet of conv(S) $\cap$ $\{x \in \Re^3 : x_3 = 0\}$. We now lift x_3 approximately. The approximate lifting problem is $\max\{2x_1 + x_2 + \tilde{\alpha}_3 x_3 : 4x_1 + 3x_2 + 3x_3 \leq 16,\ x_1 \in [0,2],\ x_2 \in [0,4],\ x_3 \in [1,3],\ \tilde{\alpha}_3 \in \Re,\ 2x_1 + x_2 + \tilde{\alpha}_3 x_3 \leq 6\}$.

If $\frac{1}{2} \leq \frac{\tilde{\alpha}_3^*}{3}$, the optimal x is $x_1^{(0)} = \frac{7}{4}$, $x_2^{(1)} = 0$, $x_3^{(1)} = 3$, and $\tilde{\alpha}_3^0 = \frac{5}{6}$. If $\frac{1}{3} \leq \frac{\tilde{\alpha}_3^*}{3} \leq \frac{1}{2}$, the optimal x is $x_1^{(1)} = 2$, $x_2^{(1)} = 0$, $x_3^{(1)} = \frac{8}{3}$, and $\tilde{\alpha}_3^1 = \frac{8}{3}$. If $\frac{\tilde{\alpha}_3^*}{3} \leq \frac{1}{3}$, the optimal x is $x_1^{(2)} = 2$, $x_2^{(2)} = \frac{5}{3}$, $x_3^{(2)} = 1$, and $\tilde{\alpha}_3^2 = \frac{1}{3}$.

Now note that $\frac{1}{2} > \frac{\tilde{\alpha}_3^0}{3}$, $\frac{\tilde{\alpha}_3^1}{3} > \frac{1}{2}$, and $\frac{\tilde{\alpha}_3^2}{3} \leq \frac{1}{3}$. Thus, the approximate lifting coefficient is $\tilde{\alpha}_3^* = \frac{1}{3}$. Incidentally, this is also the exact (as opposed to approximate) value of the lifting coefficient. □

By calculating $\tilde{\alpha}_k^t, \ldots, \tilde{\alpha}_k^0$ in this order it is possible to calculate all of them in time $O(n)$. Also, by maintaining a list of the variable indices sorted as in (45) every time a new lifting coefficient is calculated it will not be necessary to sort the indices, except for the initial inequality before any variable is lifted. Therefore, all approximate lifting coefficients can be calculated in time $O(n^2)$.

4 Cover Inequalities

In this section we extend the concepts of cover and cover inequality, commonly used in 0-1 programming, to our case. We consider two special cases: *simple* and *minimal* covers. We show that when the cover is simple, the cover inequality is the only nontrivial facet-defining inequality of the semi-continuous knapsack polyhedron P^0 obtained by fixing at 0 all variables not indexed by the cover. We then give the value of the lifting coefficient of a continuous variable x_k, $k \in N^-$, that is fixed at 0, when the cover inequality is lifted with respect to it first. Then, we show that when the cover is not simple, it still defines a facet of P^0 if it is minimal and $l_j = u_j$ for all variables in the cover.

We now define cover, minimal cover, and simple cover for semi-continuous knapsack polyhedra.

Definition 1 *Let $C \subseteq N^+$ with $p_j < l_j \ \forall j \in C$. We say that C is a cover if*

$$\sum_{j \in C} a_j l_j > b. \tag{50}$$

If in addition

$$\sum_{j \in C - \{i\}} a_j l_j \leq b \ \forall i \in C, \tag{51}$$

we say that the cover is minimal. Finally, if

$$\sum_{j \in C - \{i\}} a_j u_j + a_i p_i \leq b \ \forall i \in C, \tag{52}$$

we say that the cover is simple. □

Note that our definitions of cover, minimal cover, and simple cover coincide with the definitions of cover and minimal cover of 0-1 programming, where $p_j = 0$ and $l_j = u_j = 1$ $\forall j \in N$.

We now introduce cover inequalities.

Proposition 14 *Let C be a cover, $\Delta = \sum_{j \in C} a_j l_j - b$, and $\delta_j = a_j(u_j - p_j) \ \forall j \in C$. Then,*

$$\sum_{j \in C} a_j \frac{u_j - x_j}{max\{\delta_j, \Delta\}} \geq 1 \tag{53}$$

is valid for P.

Proof Let $\bar{x} \in S$. Since C is a cover, $\bar{x}_i \leq p_i$ for some $i \in C$. If $\delta_i \geq \Delta$,

$$\sum_{j \in C} a_j \frac{u_j - \bar{x}_j}{\max\{\delta_j, \Delta\}} \geq a_i \frac{u_i - \bar{x}_i}{\delta_i} \geq a_i \frac{u_i - p_i}{\delta_i} = 1.$$

Let $C_1 = \{j \in C : \bar{x}_j \leq p_j\}$. Suppose $\delta_j < \Delta \ \forall j \in C_1$. Because $\bar{x}_j \geq l_j \ \forall j \in C - C_1$, $\sum_{j \in C_1} a_j \bar{x}_j + \sum_{j \in C - C_1} a_j l_j \leq b$, and so $\sum_{j \in C_1} a_j \bar{x}_j + \sum_{j \in C - C_1} a_j l_j \leq b + \sum_{j \in C_1} a_j (u_j - l_j)$. This implies that $\sum_{j \in C_1} a_j (u_j - \bar{x}_j) \geq \Delta$. Thus, (53) holds for $\bar{x}$. □

We call (53) a *cover inequality*. When $p_j = 0 \ \forall j \in C$, i.e. for the "usual" semi-continuous variables, minimal and simple cover inequalities coincide and become

$$\sum_{j \in C} \frac{x_j}{u_j} \leq |C| - 1.$$

Particularly, for the case of 0-1 variables, they give $\sum_{j\in C} x_j \leq |C|-1$.

We now consider simple covers. Note that in this case $\delta_j \geq \Delta\ \forall j \in C$. We show that when the cover is simple, (53) is the only nontrivial facet-defining inequality for $P^0 = P \cap \{x \in \Re^n : x_j = 0\ \forall j \in N-C\}$.

Proposition 15 *If C is a simple cover, then (53) defines a facet of P^0, and $P^0 = \{x \in \Re^n : x$ satisfies (6), (7), and (53)$\}$.*

Proof Let C be a simple cover. The points $x^{(i)}, i \in C$, given by

$$x_j^{(i)} = \begin{cases} p_i & \text{if } j = i \\ u_j & \text{if } j \in C - \{i\} \\ 0 & \text{otherwise} \end{cases}$$

belong to P^0 and satisfy (53) at equality. Because the set $\{x^{(j)} : j \in C\}$ is linearly independent, (53) defines a facet for P^0. Now, let $t \in C$ and

$$d = \sum_{j\in C-\{t\}} \frac{u_j}{u_j - p_j} + \frac{p_t}{u_t - p_t}$$

(note that the value of d does not depend on t.) Since the maximal vertices of the polytope

$$\{y \in \Re^{|C|} : \sum_{j=1}^{|C|} x_{k_j}^{(i)} y_j \leq 1\ \forall i \in C \text{ and } y \geq 0\}$$

are $(\frac{1}{u_{k_1}}, 0, \ldots, 0)$, $(0, \frac{1}{u_{k_2}}, \ldots, 0)$, …, $(0, 0, \ldots, \frac{1}{u_{k_{|C|}}})$, and $\frac{1}{d}(\frac{1}{u_{k_1}-p_{k_1}}, \frac{1}{u_{k_2}-p_{k_2}}, \ldots, \frac{1}{u_{k_{|C|}}-p_{k_{|C|}}})$, it follows from antiblocking theory [17, 18, 32] that the only nontrivial facet-defining inequality for P^0 is

$$\sum_{j\in C} \frac{x_j}{u_j - p_j} \leq d,$$

which is the same as (53). Thus, $P^0 = \{x \in \Re^n : x$ satisfies (6), (7), (53), and $x_j = 0$ $\forall j \in N - C\}$. □

Example 5 Let $N = \{1, \ldots, 5\}$, $x_1 \in \{0\} \cup [2,3]$, $x_2 \in [0,1] \cup [3,4]$, $x_3 \in [0,2] \cup [4,5]$, $x_4 \in \{0\} \cup [1,2]$, and $x_5 \in [0,1] \cup \{2\}$. Consider the knapsack inequality

$$2x_1 + 3x_2 + 4x_3 + x_4 + x_5 \leq 33.$$

Note that $2l_1 + 3l_2 + 4l_3 + l_4 + l_5 = 32 < 33$, and therefore N is not a cover. However, by fixing $x_1 = 3$, $x_2 = 4$, and $x_4 = x_5 = 0$, $\{3\}$ becomes a simple cover, and

$$x_3 \leq 2 \tag{54}$$

is valid and facet-defining for $P \cap \{x \in \Re^5 : x_1 = 3, x_2 = 4, x_4 = x_5 = 0\}$. □

We now give the lifting coefficient of a continuous variable x_k, $k \in N^-$, that is fixed at 0, when the cover is simple and the cover inequality is lifted with respect to it first.

Proposition 16 *Let C be a simple cover and x_k, $k \in N^-$, a continuous variable that is fixed at 0. Then,*

$$\sum_{j\in C} \frac{u_j - x_j}{u_j - p_j} - \frac{a_k}{\sum_{j\in C} a_j l_j - b}\left(1 - \sum_{j\in C} \frac{u_j - l_j}{u_j - p_j}\right) x_k \geq 1 \tag{55}$$

is facet-defining for $P_{C\cup\{k\}}$.

Proof Let

$$\sum_{j\in C} \frac{u_j - x_j}{u_j - p_j} + \alpha_k x_k \geq 1$$

be the facet-defining inequality that results from lifting (53) with respect to x_k. Then,

$$\alpha_k = \max\left\{\frac{1}{x_k}\left(1 - \sum_{j\in C}\frac{u_j - x_j}{u_j - p_j}\right) : \sum_{j\in C} a_j x_j + a_k x_k \leq b, x_j \text{ satisfies (2) } \forall j \in C, \text{ and } x_k > 0\right\}. \tag{56}$$

Note that $\alpha_k > 0$, since $\bar{x}_j = u_j \; \forall j \in C$ and

$$\bar{x}_k = -\frac{\sum_{j\in C} a_j u_j - b}{a_k}$$

is a feasible solution to (56) with objective function value

$$\bar{\alpha}_k = -\frac{a_k}{\sum_{j\in C} a_j u_j - b}.$$

On the other hand, when $x_j < l_j$ for some $j \in C$, the objective function value is at most 0. Therefore, $x_j \geq l_j \; \forall j \in C$ in an optimal solution to (56), and clearly $\sum_{j\in C} a_j x_j + a_k x_k = b$. Problem (56) is then equivalent to

$$\alpha_k = \max\left\{\frac{-a_k}{\sum_{j\in C} a_j x_j - b}\left(1 - \sum_{j\in C}\frac{u_j - x_j}{u_j - p_j}\right) : x_j \in [l_j, u_j] \; \forall j \in C\right\}. \tag{57}$$

Consider now the solution to (57) $x_j^* = l_j \; \forall j \in C$ with objective function value

$$\alpha_k^* = -\frac{a_k}{\sum_{j\in C} a_j l_j - b}\left(1 - \sum_{j\in C}\frac{u_j - l_j}{u_j - p_j}\right).$$

We prove the proposition by showing that none of the vertices of $(\times[l_j, u_j])_{j\in C}$, has an objective function value greater than α_k^*.

Let i =argmin$\{a_j(u_j - p_j) : j \in C\}$. Let $T \subseteq C$ and suppose that $T \neq \emptyset$ and $i \notin T$. Consider the solution $\tilde{x}_j = l_j \; \forall j \in C - T$ and $\tilde{x}_j = u_j \; \forall j \in T$. The objective function value is

$$\tilde{\alpha} = \frac{-a_k}{\sum_{j \in C-T} a_j l_j + \sum_{j \in T} a_j u_t - b} \left(1 - \sum_{j \in C-T} \frac{u_j - l_j}{u_j - p_j}\right).$$

We now show that $\alpha_k^* \geq \tilde{\alpha}_k$.

Condition (52) implies that

$$b \geq \sum_{j \in C-T} a_j l_j + \sum_{j \in T} a_j u_j + \sum_{j \in C-T \cup \{i\}} a_j(u_j - l_j) - a_i(l_i - p_i). \tag{58}$$

Clearly,

$$\sum_{j \in C-T \cup \{i\}} a_j(u_j - l_j) - a_i(l_i - p_i) < 0$$

and

$$\sum_{j \in T} \frac{a_j(u_j - l_j)}{a_i(u_i - p_i)} \geq \sum_{j \in T} \frac{u_j - l_j}{u_j - p_j}.$$

Therefore, (58) implies that

$$b \geq \sum_{j \in C-T} a_j l_j + \sum_{j \in T} a_j u_j + \frac{1}{\sum_{j \in T} \frac{u_j - l_j}{u_j - p_j}} \sum_{j \in T} \frac{a_j(u_j - l_j)}{a_i(u_i - p_i)} \left(\sum_{j \in C-T \cup \{i\}} \frac{a_i(u_i - p_i)}{a_j(u_j - p_j)} a_j(u_j - l_j) - a_i(l_i - p_i) \right)$$

$$= \sum_{j \in C-T} a_j l_j + \sum_{j \in T} a_j u_j + \frac{1}{\sum_{j \in T} \frac{u_j - l_j}{u_j - p_j}} \sum_{j \in T} a_j(u_j - l_j) \left(\sum_{j \in C-T \cup \{i\}} \frac{u_j - l_j}{u_j - p_j} - \frac{l_i - p_i}{u_i - p_i} \right)$$

$$= \sum_{j \in C-T} a_j l_j + \sum_{j \in T} a_j u_j - \frac{1}{\sum_{j \in T} \frac{u_j - l_j}{u_j - p_j}} \sum_{j \in T} a_j(u_j - l_j) \left(1 - \sum_{j \in C-T} \frac{u_j - l_j}{u_j - p_j}\right).$$

Thus,

$$b \sum_{j \in T} \frac{u_j - l_j}{u_j - p_j} \geq \left(\sum_{j \in C-T} a_j l_j + \sum_{j \in T} a_j u_j \right) \sum_{j \in T} \frac{u_j - l_j}{u_j - p_j} - \sum_{j \in T} a_j(u_j - l_j) \left(1 - \sum_{j \in C-T} \frac{u_j - l_j}{u_j - p_j}\right),$$

or,

$$\sum_{j\in T} a_j u_j \left(1 - \sum_{j\in C} \frac{u_j - l_j}{u_j - p_j}\right) \geq \left(\sum_{j\in C-T} a_j l_j - b\right) \sum_{j\in T} \frac{u_j - l_j}{u_j - p_j} + \sum_{j\in T} a_j l_j \left(1 - \sum_{j\in C-T} \frac{u_j - l_j}{u_j - p_j}\right). \tag{59}$$

By adding

$$\left(\sum_{j\in C-T} a_j l_j - b\right) \left(1 - \sum_{j\in C} \frac{u_j - l_j}{u_j - p_j}\right)$$

to both sides of (59),we obtain

$$\left(\sum_{j\in C-T} a_j l_j + \sum_{j\in T} a_j u_j - b\right) \left(1 - \sum_{j\in C} \frac{u_j - l_j}{u_j - p_j}\right) \geq \left(\sum_{j\in C} a_j l_j - b\right) \left(1 - \sum_{j\in C-T} \frac{u_j - l_j}{u_j - p_j}\right),$$

or

$$\frac{1}{\sum_{j\in C} a_j l_j - b} \left(1 - \sum_{j\in C} \frac{u_j - l_j}{u_j - p_j}\right) \geq \frac{1}{\sum_{j\in C-T} a_j l_j + \sum_{j\in T} a_j u_j - b} \left(1 - \sum_{j\in C-T} \frac{u_j - l_j}{u_j - p_j}\right),$$

and so, $\alpha_k^* \geq \tilde{\alpha}_k$. The case $i \in T$ can be treated similarly. □

Example 6 Let $S_1 = \{(x, s) \in Z_+^2 \times \Re_+ : (x, s)$ satisfies (60), $x_1 \leq 3$, and $x_2 \leq 2\}$ and $S_2 = \{(x, s) \in Z_+^2 \times \Re_+ : (x, s)$ satisfies (60), $x_1 \leq 4$, and $x_2 \leq 2\}$, where

$$3x_1 + 4x_2 \leq 16 + s. \tag{60}$$

The point

$$(x^*, s^*) = \left(\frac{8}{3}, 2, 0\right)$$

is a fractional vertex of the LP relaxation for both sets. (It is the only fractional vertex in the case of S_2.)

Consider S_1. The Gomory mixed-integer cut for the basis of (x^*, s^*) is

$$x_1 + 2x_2 \leq 6 + s. \tag{61}$$

Inequality (55) for $x_1 \in [0, 2] \cup \{3\}$ and $x_2 \in [0, 1] \cup \{2\}$ is

$$x_1 + x_2 \leq 4 + s, \tag{62}$$

which is facet-defining and stronger than (61), since

$$(x_1 + 2x_2 \leq 6 + s) = (x_1 + x_2 \leq 4 + s) + (x_2 \leq 2).$$

Note that in this case (62) coincides with the cover inequality of Ceria et al. [6].

Consider now the case of S_2. In [6] the following cover inequality is proposed

$$x_1 + x_2 \leq 5 + \frac{1}{4}s. \tag{63}$$

Inequality (63) does not cut off (x^*, s^*). On the other hand, inequality (55) for $x_1 \in [0, 2] \cup [3, 4]$ and $x_2 \in [0, 1] \cup \{2\}$ is

$$x_1 + 2x_2 \leq 6 + \frac{1}{2}s, \tag{64}$$

which is facet-defining and cuts off (x^*, s^*). Note that in this case (64) coincides with the Gomory mixed-integer cut for the basis of (x^*, s^*). □

Finally, we give a sufficient condition for (53) to be facet-defining for P^0 that holds even when the cover is not simple.

Proposition 17 *Let C be a minimal cover and suppose that $l_j = u_j$ $\forall j \in C$. Then, (53) is facet-defining for P^0.*

Proof For $i \in C$, consider the point $x^{(i)}$ given by $x_j^{(i)} = u_j$ $\forall j \in C - \{i\}$ and

$$x_i^{(i)} = \begin{cases} p_i & \text{if } \delta_i > \Delta \\ \frac{b - \sum_{j \in C - \{i\}} a_j u_j}{a_i} & \text{otherwise.} \end{cases}$$

These points are linearly independent and satisfy (53) at equality. □

5 Computational Experience

In this section we present the results of our computational experience on the use of semi-continuous cuts to solve 32 real instances of the stochastic unit commitment problem with linear costs through branch-and-cut. The unit commitment problem is to determine when to startup and shutdown power generating units over a period of time, while satisfying client demand and technological constraints, to minimize total operating cost, see for example [33] for a survey on it.

Let T be the number of time periods, m the number of generating units, F_i the no-load cost of unit i's offer, S_i unit i's startup cost, c_i the unit output cost of unit i, q_i and Q_i the minimum and maximum (respectively) generating capacities of unit i, P_s the probability of scenario s, and d_t^s the demand in period t under scenario s. We tackle the model:

$$
\begin{array}{llll}
\min & \sum_s P_s \left(\sum_{t=1}^{T} \sum_{i=1}^{m} \left(F_i z_{it}^s + S_i r_{it}^s + c_i g_{it}^s\right)\right) & & \\
s.t. & \sum_{i=1}^{m} g_{it}^s \geq d_t^s & \forall t, s & (65) \\
 & q_i z_{it}^s \leq g_{it}^s \leq Q_i z_{it}^s & \forall i, t, s & (66) \\
 & \text{Minimum up/down constraints} & & \\
 & \text{Nonanticippativity constraints} & & \\
 & z_{it}^s - z_{i,t-1}^s \leq r_{it}^s & \forall i, t \geq 2, s & \\
 & r_{it}^s, z_{it}^s \in \{0,1\}, & \forall i, t, s. &
\end{array}
$$

Here, g_{it}^s is the output of unit i in period t under scenario s, $r_{it}^s = 1$ if unit i is started in period t under scenario s (= 0 otherwise), and $z_{it}^s = 1$ if unit i is up in period t under scenario s (= 0 otherwise). We model the minimum up/down constraints as in Rajan and Takriti [35], i.e.

$$
\sum_{\tau=t-\Lambda_i+1}^{t} r_{i,\tau}^s \leq z_{i,t}^s \quad \forall i, t \geq \Lambda_i + 1, s
$$

$$
\sum_{\tau=t-\lambda_i+1}^{t} r_{i,\tau}^s \leq 1 - z_{i,t-\lambda_i}^s \quad \forall i, t \geq \lambda_i + 1, s,
$$

where Λ_i and λ_i are the minimum up and down times, respectively. The nonanticippativity constraints are

$$
z_{it}^{s_o} = z_{it}^{s_w} \ \forall i, t, s_o, s_w \text{ with } s_o \neq s_w, \text{ whenever } d_t^{s_o} = d_t^{s_w} \ \forall t, s_o, s_w \text{ with } s_o \neq s_w.
$$

Inequalities (65) play the role of (1), i.e. an individual knapsack set for semi-continuous cut generation is defined by one of the inequalities (65) and the respective semi-continuous constraints (66). To conform with the notation of the previous sections, we now make the change of variables $\bar{g}_{it}^s = Q_i - g_{it}^s \ \forall i, t, s$. We then have $\sum_{i=1}^{m} \bar{g}_{it}^s \leq \sum_{i=1}^{m} Q_i - d_t^s$ and $\bar{g}_{it}^s \in [0, Q_i - q_i] \cup \{Q_i\} \ \forall i, t, s$. Given a pair (t, s), $C \subseteq \{1, \ldots, m\}$ is a cover if $\sum_{j \in C} Q_j > d_t^s$. Given a cover C, $\Delta = d_t^s - \sum_{j \notin C} Q_j$ and the cover inequality is

$$
\sum_{j \in C} \frac{Q_j - \bar{g}_{jt}^s}{\max\{q_j, \Delta\}} \geq 1. \tag{67}
$$

From Proposition (17), it follows that (67) defines a facet of P^0 when C is minimal. But then, in this case, it follows from Proposition 11 that if the variables not indexed by C were fixed at 0, (67) defines a facet of P. For this reason, we expect that (67) will be a strong cut in a great number of occasions, and we will then restrict ourselves to using (67) as cuts.

So, given an optimal solution $(\bar{g}', z', r')$ to an LP relaxation at a certain node of the enumeration tree that does not satisfy one of the constraints (66), we find a valid inequality

(67) to separate $(\bar{g}', z', r')$. We proceed as follows. We consider pairs (t, s) for which (65) is satisfied at equality by $(\bar{g}', z', r')$. For each pair we try to obtain a violated cover inequality by collecting in C the indices i of the variables $\bar{g}^s_{it}$ for which $(\bar{g}')^s_{it} \geq 0.8Q_i$. The reason is that the inclusion of variables in C with values close to their upper bounds will potentially increase the chances that the resulting cover inequality will be violated by $(\bar{g}', z', r')$ (the choice of 0.8 was determined by our initial computational testing). If the resulting inequality (67) is violated by $(\bar{g}', z', r')$, we add it to the cutpool.

We performed our computational tests with the Texas Tech High Performance Computing Center's Intel Xeon E5450 3.0GHz CPU with 16GB RAM nodes (two CPUs on a single board for each node) [22]. We used CPLEX 12 Callable Library, running on a single thread, as background solver. We limited computational time to 1 hour of CPU time. The data are plant data from a utility company operating in the electricity market. We tested 8 different combinations of number of units (m), number of time periods (T), and number of scenarios (S). For each of the eight triples (m, T, S), we tested 4 instances.

CPLEX performs enumeration in two different modes, dynamic search (DS) and branch-and-cut (B&C). However, it only allows user cut separation in B&C mode. Because we want to compare the performance of CPLEX with and without semi-continuous cuts, we report our results for CPLEX in B&C mode.

Initially, CPLEX without semi-continuous cuts solved to proven optimality only 4 of the 32 instances tested. Also, the computational time required by the 4 instances solved was close to the limit of 1 hour. To our surprise, these figures improved tremendously after we turned off 0-1 cover cuts and flow cover cuts. We are not sure, at this point, why 0-1 cover cuts and flow cover cuts were so harmful. One possible explanation is that they turned out to be dense. In any case, we report our computational results with 0-1 cover cuts and flow cover cuts turned off.

The number of enumeration nodes and computational time (in seconds) for CPLEX with and without the use of semi-continuous cuts, as well as their percentage reduction obtained by using the semi-continuous cuts, is given in Table 1. Of the 32 instances tested, CPLEX was not capable of solving 2 without the use of the semi-continuous cuts; however, all instances were solved to proven optimality by using the semi-continuous cuts. Also, the reduction in number of nodes and computational time by using the semi-continuous cuts was considerable. Overall, the reduction in number of nodes obtained by using the semi-continuous cuts was of 76%; additionally, in 23 instances, or 72% of them, the reduction in number of nodes was of 50% or more. The overall reduction in computational time by using the semi-continuous cuts was of 34%; additionally, in 11 instances, or 34% of them, the reduction in computational time was of 50% or more. For all instances, the use of semi-continuous cuts reduced the computational time.

In Table 2 we give the number of CPLEX MIP cuts separated with and without the use of semi-continuous cuts, and the number of semi-continuous cuts separated (entries with value 0 were left blank; also, note that 0-1 cover cuts and flow cover cuts were turned off). CPLEX, in both cases, separated MIR, Gomory, and $0 - \frac{1}{2}$ cuts. Overall, MIR cuts were the ones used the most, followed (at a distance) by $0 - \frac{1}{2}$ cuts, followed by Gomory cuts. More importantly, the number of MIP cuts separated was nearly the same regardless of whether the semi-continuous cuts were used or not. This indicates that the use of the semi-continuous cuts did not interfere with the separation of the CPLEX MIP cuts.

In conclusion, as our computational results indicate, semi-continuous cuts have the potential of enhancing our ability to solve difficult combinatorial optimization problems in practice, whenever semi-continuous constraints are present.

6 Further Research

We are currently conducting a comprehensive computational study of semi-continuous cuts. In particular, we are testing different lifting strategies for semi-continuous cover cuts. We are studying the use of semi-continuous cuts on applications other than the unit commitment problem. We are studying the use of semi-continuous cuts on a number of combinatorial structures where it arises implicitly, such as integer programming and other combinatorial constraints (see e.g. Hooker [21]).

Many times semi-continuous constraints arise together with other constraints. Examples include piecewise linear optimization and cardinality constraints. We are studying sets that include, at the same time, semi-continuous and other combinatorial constraints. Initial results on piecewise linear optimization with semi-continuous constraints were given by de Farias et al. [9, 43].

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Table 1: Number of enumeration nodes, time, and % reduction

$m \times T \times S$	Nodes			Time		
	W/O (67)	W/ (67)	% Red.	W/O (67)	W/ (67)	% Red.
$20 \times 36 \times 10$	2,784	425	85	341	142	58
	964	124	87	186	111	40
	975	306	69	228	126	45
	2,561	362	86	143	94	34
$20 \times 36 \times 15$	7,319	952	87	1,194	360	70
	12,133	781	94	1,587	358	77
	14,191	1,271	91	1,533	344	78
	15,707	2,706	83	1,240	978	21
$20 \times 48 \times 25$	1,982	3	99	1,017	429	58
	1,000	219	78	1,351	521	61
	64	10	84	825	368	55
	30	7	77	612	506	17
$32 \times 48 \times 15$	193	69	64	564	385	31
	74	62	16	540	220	59
	250	36	86	656	290	56
	59	10	83	548	307	44
$32 \times 48 \times 20$	608	537	12	1,378	1,079	22
	406	155	62	1,363	782	43
	111	250	−125	1,079	710	34
	210	7	97	1,260	392	69
$32 \times 60 \times 15$	280	405	−45	1,156	1,103	5
	282	30	89	1,219	635	48
	195	110	44	1,307	967	26
	297	131	56	1,464	703	52
$32 \times 60 \times 20$	1,890	970	49	3,066	2,420	21
	900	292	68	3,514	2,151	39
	696	385	45	3,283	2,449	25
	667	283	58	3,270	2,237	32
$32 \times 72 \times 10$	1,897	1,793	5	**3,600**	3,188	11
	1,686	1,663	1	2,817	2,248	20
	1,559	700	55	2,951	2,588	12
	1,250	2,296	−83	**3,600**	3,005	17

Table 2: Number of enumeration nodes, time, and % reduction

	Without (67)			With (67)			
$m \times T \times S$	MIR	Gomory	0 - $\frac{1}{2}$	MIR	Gomory	0 - $\frac{1}{2}$	(67)
$20 \times 36 \times 10$	985		3	1,002		4	66
	899		5	1,035		6	18
	893		4	1,010		7	29
	1,007		2	1,118		8	8
$20 \times 36 \times 15$	1,370		2	1,436		5	20
	1,512		1	1,609		11	40
	1,852		24	1,567		5	21
	1,467			1,529		8	47
$20 \times 48 \times 25$	2,181	22	10	2,515	20	5	43
	1,986	19	6	2,942	21	13	32
	2,462	18	2	2,376	18	3	10
	2,194	19	8	2,421	21	10	1
$32 \times 48 \times 15$	860	7	23	973	6	27	80
	950	8	27	1,076	6	12	51
	875	10	27	976	9	18	48
	973	6	25	1,014	4	18	47
$32 \times 48 \times 20$	1,439	11	50	1,373	13	33	50
	989	9	30	1,279	9	30	50
	1,181	6	28	1,398	5	38	23
	1,349	7	42	1,353	9	28	24
$32 \times 60 \times 15$	1,265		25	1,204		30	50
	1,186		60	1,321	2	35	15
	1,444		55	1,309		25	5
	1,207	1	40	1,130	1	33	50
$32 \times 60 \times 20$	1,951		60	1,835		56	20
	1,785	2	50	1,748		39	40
	1,811		41	1,760		47	38
	1,710		57	1,582		50	52
$32 \times 72 \times 10$	1,422	9	48	1,524	4	43	25
	1,539	18	60	1,537	10	52	46
	1,546	5	32	1,506	2	31	76
	1,425	4	35	1,495	4	48	20