Think co(mpletely)positive !
Matrix properties, examples and a clustered bibliography on copositive optimization

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Dedicated to the memory of Reiner Horst

Abstract

Copositive optimization is a quickly expanding scientific research domain with wide-spread applications ranging from global nonconvex problems in engineering to NP-hard combinatorial optimization. It falls into the category of conic programming (optimizing a linear functional over a convex cone subject to linear constraints), namely the cone $\mathcal{C}$ of all completely positive symmetric $n \times n$ matrices (which can be factorized into $FF^\top$, where F is a rectangular matrix with no negative entry), and its dual cone $\mathcal{C}^*$, which coincides with the cone of all copositive matrices (those which generate a quadratic form taking no negative value over the positive orthant). We provide structural algebraic properties of these cones, and numerous (counter-)examples which demonstrate that many relations familiar from semidefinite optimization may fail in the copositive context, illustrating the transition from polynomial-time to NP-hard worst-case behaviour. In course of this development we also present a systematic construction principle for non-attainability phenomena, which apparently has not been noted before in an explicit way. Last but not least, also seemingly for the first time, a somehow systematic clustering of the vast and scattered literature is attempted in this paper.

1 Introduction

Copositive optimization (or copositive programming, coined in [47]) is a special case of conic optimization, which consists of optimizing a linear function over a cone subject to additional linear constraints.

It is well known that the simplest class of hard problems in continuous optimization is that of quadratic optimization problems [214] – to minimize a (possibly indefinite) quadratic form $\mathbf{x}^\top Q \mathbf{x}$ over a polyhedron $\left\{ \mathbf{x} \in \mathbb{R}^n_+ : A\mathbf{x} = \mathbf{b} \right\}$. Note that a linear term in the objective function can be removed by an affine transformation of the polyhedron. The number of local, non-global solutions to this problem may be exponential in the number of variables and/or constraints [42].

This class has a close connection to copositive optimization. The so–called *lifting* idea here is to linearize the quadratic form

$$\mathbf{x}^\top Q \mathbf{x} = \text{trace } (\mathbf{x}^\top Q \mathbf{x}) = \text{trace } (Q \mathbf{x} \mathbf{x}^\top) = Q \bullet \mathbf{x} \mathbf{x}^\top$$

by introducing the new symmetric matrix variable $X = \mathbf{x} \mathbf{x}^\top$ and Frobenius duality $X \bullet Y = \text{trace } (XY)$. This technique was mainly applied previously to semidefinite optimization [194]. If $A\mathbf{x} \in \mathbb{R}^m_+$ for all $\mathbf{x} \in \mathbb{R}^n_+$ and $\mathbf{b} \in \mathbb{R}^m_+$, then the linear constraints can be squared, to arrive in a similar way at constraints of the form $A_i \bullet X = b_i^2$.

Now the set of all these X generated by feasible $\mathbf{x}$ is non-convex since rank $(\mathbf{x}\mathbf{x}^\top) = 1$. The convex hull

$$\mathcal{C} = \text{conv } \left\{ \mathbf{x}\mathbf{x}^\top : \mathbf{x} \in \mathbb{R}^n_+ \right\},$$

results in a convex matrix cone called the cone of completely positive matrices since [133]; see [22]. Note that a similar construction dropping nonnegativity constraints leads to

$$\mathcal{P} = \text{conv } \left\{ \mathbf{x}\mathbf{x}^\top : \mathbf{x} \in \mathbb{R}^n \right\},$$

the cone of positive-semidefinite matrices, the basic set in semidefinite optimization (or semidefinite programming, SDP); see for instance [59].

The first account on copositive optimization goes back to [47], who established a copositive representation of a subclass of particular interest, namely in Standard Quadratic Optimization (StQP). Here the feasible polyhedron is the standard simplex $\Delta = \left\{ \mathbf{x} \in \mathbb{R}^n_+ : \sum_i x_i = 1 \right\}$: this subclass is also NP-hard from the worst-case complexity but allows for a polynomial-time approximation scheme [46]. There can be exponentially many local non-global solutions (see [61] for the lower bound $30^{\lfloor n/9 \rfloor}$.)

This phenomenon is typical for a problem of Global Optimization. While researchers were aware of this phenomenon since long, the field of Deterministic Global Optimization received a decisive impact by the seminal book [147] by R. Horst and H. Tuy. Soon after, R. Horst co-founded the journal at hand and was its first managing editor, so he can be seen as one of the most influential and driving personalities in Global Optimization.

Now, with J the $n \times n$ all-ones matrix, we have

$$\min \left\{ \mathbf{x}^\top Q \mathbf{x} : \mathbf{x} \in \Delta \right\} = \min \left\{ Q \bullet X : J \bullet X = 1 \,, X \in \mathcal{C} \right\} . \qquad (1)$$

Note that the right-hand problem is convex, so there are no more local, non-global solutions. In addition, the objective function is now linear, and there is just one linear equality constraint. The complexity has been completely pushed into the feasibility condition $X \in \mathcal{C}$, which also shows that there are indeed convex minimization problems which cannot be solved easily.

Duality theory for conic optimization problems requires the dual cone $\mathcal{C}^*$ of $\mathcal{C}$ w.r.t. the Frobenius inner product, which is

$$\mathcal{C}^* = \left\{ S \in \mathbf{S}^{n \times n} : S \bullet X \geq 0 \text{ for all } X \in \mathcal{C} \right\} ,$$

where $\mathbf{S}^{n \times n}$ is the set of symmetric $n \times n$ matrices. Here it can easily be shown that $\mathcal{C}^*$ coincides with the cone of copositive matrices, which justifies terminology:

$$\mathcal{C}^* = \left\{ S \in \mathbf{S}^{n \times n} : \mathbf{x}^\top S \mathbf{x} \geq 0 \text{ if } \mathbf{x} \in \mathbb{R}^n_+ \right\} ,$$

i.e., a matrix S is copositive [208] (most probably abbreviating "*conditionally positive*-semidefinite"), if S generates a quadratic form $\mathbf{x}^\top S \mathbf{x}$ taking no negative values over the positive orthant. The dual of the special program (1) over $\mathcal{C}$ above is then

$$\max \left\{ y \in \mathbb{R} : S = Q - yJ \in \mathcal{C}^* \right\} , \qquad (2)$$

a linear objective in just one variable y with the innocent-looking feasibility constraint $S \in \mathcal{C}^*$. This shows that checking membership of $\mathcal{C}^*$ (and, similarly, of $\mathcal{C}$) is already NP-hard, and there are many approaches to algorithmic copositivity detection, we refer to Section 4. More generally, a typical primal-dual pair in copositive optimization (COP) is of the following form:

$$\inf \left\{ C \bullet X : A_i \bullet X = b_i \,, i = 1, \ldots, m \,, X \in \mathcal{C} \right\}$$
$$\geq \quad \sup \left\{ \textstyle\sum_i b_i y_i : \mathbf{y} \in \mathbb{R}^m \,, S = C - \sum_i y_i A_i \in \mathcal{C}^* \right\} .$$

The inequality above is just standard weak duality, but observe we have to use inf and sup since – as in general conic optimization – there may be problems with attainability of either or both problems above, and likewise there could be a (finite or infinite) positive duality gap without any further conditions like strict feasibility (Slater's condition). For the above representation of Standard Quadratic Optimization problems, this is not the case:

$$\min \left\{ Q \bullet X : J \bullet X = 1 \,, X \in \mathcal{C} \right\} = \max \left\{ y \in \mathbb{R} : S = Q - yJ \in \mathcal{C}^* \right\} .$$

But for a similar class arising in many applications, the Multi-Standard Quadratic Optimization problems [57], dual attainability is not guaranteed while the duality gap is zero – an intermediate form between weak and strong duality [234]. We will discuss in detail these phenomena in Section 3. But

let us start with collecting a number of elementary properties and counterexamples illustrating the difference between the semidefinite cone $\mathcal{P}$ and the copositive/completely positive cone $\mathcal{C}^*/\mathcal{C}$. This is important for many copositivity detection procedures, and as we saw in (2), the feasibility constraint incorporates most of the hardness in copositive optimization.

Therefore, this paper is organized as follows: Section 2 discusses some algebraic properties of matrices belonging to the copositive or related cones, some of them more or less well known, others apparently never noticed before. Section 3 provides a complete picture of possible attainability/duality gap constellations in primal-dual pairs of copositive programs. Here, apparently for the first time in the literature, we also propose a systematic construction principle for non-attainability phenomena. Finally, also seemingly for the first time, Section 4 strives to provide a rough literature survey by clustering a hopefully large part of copositivity-related publications.

2 Elementary algebraic properties and counterexamples

Here we collect some properties which the cones $\mathcal{C}$ and $\mathcal{C}^*$ share with the more ubiquitous cones $\mathcal{P}$ and $\mathcal{N}$ (of nonnegative matrices), and some other properties which distinguish $\mathcal{C}$ and $\mathcal{C}^*$ from the other cones (and from each other). Some of these properties apparently never have been noticed before in the literature. For the sake of conciseness, in this section (and only here) we will abuse notation by regarding $\mathcal{C}$ as the class of completely positive matrices *of any order*; likewise we use the symbols $\mathcal{C}^*$, $\mathcal{P}$ and $\mathcal{N}$ in this section. So let $\mathcal{N}$ consist of all symmetric matrices with nonnegative entries, let $\mathcal{P} \cap \mathcal{N}$ be the set of *doubly nonnegative* matrices, and finally $\mathcal{P} + \mathcal{N} = \{P + N : P \in \mathcal{P}, N \in \mathcal{N}, P, N$ of the same order$\}$. We have the inclusion $\mathcal{C} \subseteq \mathcal{P} \cap \mathcal{N} \subseteq \mathcal{P} + \mathcal{N} \subseteq \mathcal{C}^*$, with equalities $\mathcal{C} = \mathcal{P} \cap \mathcal{N}$ and $\mathcal{P} + \mathcal{N} = \mathcal{C}^*$, if and only if we restrict these classes to matrices of order at most 4, see [22, Thm. 2.4, Rem. 1.10], [96], [133].

One readily observes that taking a principal submatrix of a member of one of these six classes again yields a member of that class (this is called *completeness* in [74]), and also, for A from one of these six classes, every *permutation similar matrix* $P^{-1}AP$ (with P a permutation matrix) and every *positive diagonal congruence* $D^\top A D$ (with D a positive-definite diagonal matrix, see [165]) is again a member of that class.

These properties are all special cases of the following more general one:

Proposition 2.1 (Sandwiching property):

(a) *Let $\mathcal{K} \in \{\mathcal{P}, \mathcal{N}, \mathcal{C}, \mathcal{P} \cap \mathcal{N}, \mathcal{P} + \mathcal{N}, \mathcal{C}^*\}$.*
If $B \in \mathcal{K}$ and A is a rectangular matrix of fitting order with no negative entries, then we have $A^\top B A \in \mathcal{K}$.

(b) *Let $\mathcal{K} \in \{\mathcal{P}, \mathcal{N}, \mathcal{C}, \mathcal{P} \cap \mathcal{N}\}$. Then we have $\{A, B\} \subset \mathcal{K} \Rightarrow ABA \in \mathcal{K}$.*

(c) *The counterpart of (b) is not true for the classes $\mathcal{K} \in \{\mathcal{P}+\mathcal{N}, \mathcal{C}^*\}$.*

Proof. For the case of $\mathcal{K} = \mathcal{C}$, this is stated in [22, Prop. 2.2]. It is as obvious as for the class $\mathcal{K} = \mathcal{P}$, and it is trivial for $\mathcal{K} = \mathcal{N}$. Assertion (b) is a straightforward consequence of $\mathcal{K} \subseteq \mathcal{N}$ and (a), while the claim in (c) is demonstrated by the following example. $\qquad\square$

Example 2.1 *With* $A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \in \mathcal{P} \subset \mathcal{C}^*$, $B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \in \mathcal{N} \subset \mathcal{C}^*$,

we have $ABA^\top = \begin{bmatrix} -3 & 3 \\ 3 & -2 \end{bmatrix} \notin \mathcal{C}^*$. *Note that* $\mathcal{P}+\mathcal{N} = \mathcal{C}^*$ *when restricted to matrices of order* 2.

Similarly, for $\mathcal{K} \in \{\mathcal{P}, \mathcal{N}, \mathcal{C}, \mathcal{P} \cap \mathcal{N}\}$ and $n \in \mathbb{N}$ we have $A \in \mathcal{K} \Rightarrow A^n \in \mathcal{K}$. This follows inductively, for even n directly, for odd n by sandwiching. This is only true for even powers for the classes $\mathcal{P}+\mathcal{N}$ and $\mathcal{C}^*$, as can be shown directly, but not for odd powers, as the following example demonstrates.

Example 2.2 *Let*

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathcal{P}+\mathcal{N}.$$

Then

$$A^3 = \begin{bmatrix} 20 & 21 & -15 \\ 21 & 20 & -15 \\ -15 & -15 & 11 \end{bmatrix} \quad with \quad [0,2,3]\, A^3 \,[0,2,3]^\top = -1.$$

Thus $A^3 \notin \mathcal{P}+\mathcal{N}$. *Note that* $\mathcal{P}+\mathcal{N} = \mathcal{C}^*$ *when restricted to matrices of order* 3. *Also the famous* 5×5 *Horn matrix* $H \in \mathcal{C}^* \setminus (\mathcal{P}+\mathcal{N})$ *[133] satisfies* $\mathbf{x}^\top H^3 \mathbf{x} = -3 < 0$ *for* $\mathbf{x} = [2,0,0,2,3]^\top \in \mathbb{R}^5_+$.

The symmetrization $AB + BA$ of the product AB is less well behaved. Of course $A, B \in \mathcal{N} \Rightarrow AB + BA \in \mathcal{N}$, but there is no analogous result for the other matrix classes, as shown below.

Example 2.3 *Let* $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, *which are both in* $\mathcal{C}$,

resulting in $AB + BA = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$, *with eigenvalues* $1 \pm \sqrt{2}$. *Thus* $AB + BA \notin \mathcal{K}$ *for* $\mathcal{K} \in \{\mathcal{P}, \mathcal{C}, \mathcal{P} \cap \mathcal{N}\}$.

Example 2.4 *For a demonstration of* $A, B \in \mathcal{K}, AB + BA \notin \mathcal{K}$ *for* $\mathcal{K} \in \{\mathcal{P}+\mathcal{N}, \mathcal{C}^*\}$ *let* A *be the matrix from Example 2.2, and take* $B = A^2$.

Table 1: Closure properties of cones for sandwiching, symmetrized products, and posynomials $p(A) = \sum_k c_k A^k$ with $c_k \geq 0$, integer $k \geq 0$.

$\mathcal{K}$	$A, B \in \mathcal{K}$ $\Rightarrow ABA \in \mathcal{K}$	$A, B \in \mathcal{K}$ $\Rightarrow AB + BA \in \mathcal{K}$	$A \in \mathcal{K}$ $\Rightarrow p(A) \in \mathcal{K}$
$\mathcal{P}$	yes: Prop. 2.1	no: Ex. 2.3	yes: follows
$\mathcal{N}$	yes: evident	yes: evident	yes: evident
$\mathcal{P} \cap \mathcal{N}$	yes: Prop. 2.1	no: Ex. 2.3	yes: follows
$\mathcal{P} + \mathcal{N}$	no: Ex. 2.1	no: Ex. 2.4	no[a]: Ex. 2.2
$\mathcal{C}$	yes: Prop. 2.1	no: Ex. 2.3	yes: follows
$\mathcal{C}^*$	no: Ex. 2.1	no: Ex. 2.4	no[a]: Ex. 2.2

[a] but $A^k \in \mathcal{K}$ for $k \in \mathbb{N}$ even (evident)

Example 2.5 *Take* $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. *Then* $A \in \mathcal{C} = \mathcal{P} \cap \mathcal{N}$ *and* $B \in \mathcal{C}^* = \mathcal{P} + \mathcal{N}$, *but* $AB + BA = \begin{bmatrix} -2 & -1 \\ -1 & 4 \end{bmatrix} \notin \mathcal{C}^* = \mathcal{P} + \mathcal{N}$.

Now consider tensor (or Kronecker) products. Recall that for an $m \times m$ matrix A and an $n \times n$ matrix B, this product is defined as the block $mn \times mn$ matrix $A \otimes B = [a_{ij}B]_{i,j=1}^n$.

Proposition 2.2 (tensor product properties):
Let A and B be two symmetric matrices, not necessarily of same size.

(a) *Let* $\mathcal{K} \in \{\mathcal{P}, \mathcal{N}, \mathcal{C}, \mathcal{P} \cap \mathcal{N}\}$. *Then* $\{A, B\} \subset \mathcal{K} \Rightarrow A \otimes B \in \mathcal{K}$.

(b) *However, this implication is wrong if* $\mathcal{K} \in \{\mathcal{P} + \mathcal{N}, \mathcal{C}^*\}$.

(c) *Let* $\mathcal{K} \in \{\mathcal{C}, \mathcal{P} \cap \mathcal{N}\}$. *Then* $A \in \mathcal{K}$ *and* $B \in \mathcal{K}^*$ *imply* $A \otimes B \in \mathcal{K}^*$.

Proof. (a) is well known; for instance, closure under the tensor product of the completely positive cone has been established already in [22, Prop. 2.3]. We include a simple proof for the readers' convenience here. For $\mathcal{K} = \mathcal{C}$, consider $A = FF^\top$ and $B = GG^\top$, so that $A \otimes B = (FF^\top) \otimes (GG^\top) = (F \otimes G)(F \otimes G)^\top \in \mathcal{C}$, since $F \otimes G$ has no negative entry if neither F nor G have one. Along the same lines one can prove $A, B \in \mathcal{P} \Rightarrow A \otimes B \in \mathcal{P}$, and $A, B \in \mathcal{N} \Rightarrow A \otimes B \in \mathcal{N}$ is evident. Assertion (a) follows then also for $\mathcal{P} \cap \mathcal{N}$. Example 2.6 below illustrates claim (b). To establish (c), observe that (a) implies $A \otimes B \in \mathcal{P} + \mathcal{N}$, if $A \in \mathcal{P} \cap \mathcal{N}$ and $B \in \mathcal{P} + \mathcal{N}$. Finally, we have to show that $A \in \mathcal{C}$ and $B \in \mathcal{C}^*$ implies $A \otimes B \in \mathcal{C}^*$. Indeed, first observe that any $\mathbf{x} \in \mathbb{R}_+^{mn}$ can be written as $\mathbf{x} = \sum_{i=1}^m \mathbf{e}_i \otimes \mathbf{x}_i$ with $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ the standard basis of $\mathbb{R}^m$ and $\mathbf{x}_i \in \mathbb{R}_+^n$. Let $A = FF^\top \in \mathcal{C}$ with F an $m \times k$

matrix without negative entries, and $B \in \mathcal{C}^*$. Then

$$
\begin{aligned}
\mathbf{x}^\top (A \otimes B)\mathbf{x} &= \textstyle\sum_{i,j}(\mathbf{e}_i \otimes \mathbf{x}_i)^\top (FF^\top \otimes B)(\mathbf{e}_j \otimes \mathbf{x}_j) \\
&= \textstyle\sum_{i,j}(\mathbf{e}_i^\top \otimes \mathbf{x}_i^\top)(FF^\top \mathbf{e}_j \otimes B\mathbf{x}_j) \\
&= \textstyle\sum_{i,j}(F^\top \mathbf{e}_i)^\top (F^\top \mathbf{e}_j) \otimes \mathbf{x}_i^\top B\mathbf{x}_j .
\end{aligned}
$$

Now the latter Kronecker factors are scalars, so that the product is the usual scalar one. Hence

$$
\begin{aligned}
\mathbf{x}^\top (A \otimes B)\mathbf{x} &= \textstyle\sum_{i,j}(F^\top \mathbf{e}_i)^\top (F^\top \mathbf{e}_j)(\mathbf{x}_i^\top B\mathbf{x}_j) \\
&= \textstyle\sum_{i,j}(F^\top \mathbf{e}_j)^\top (F^\top \mathbf{e}_i)(\mathbf{x}_i^\top B\mathbf{x}_j) \\
&= \textstyle\sum_{i,j} \text{trace } [(F^\top \mathbf{e}_i)(\mathbf{x}_i^\top B\mathbf{x}_j)(F^\top \mathbf{e}_j)^\top] = \text{trace } [G^\top BG] ,
\end{aligned}
$$

where $G = \sum_j \mathbf{x}_j \mathbf{e}_j^\top F$ is an $n \times k$ matrix without negative entries. Thus

$$
\mathbf{x}^\top (A \otimes B)\mathbf{x} = B \bullet GG^\top \geq 0 \quad \text{for all } \mathbf{x} \in \mathbb{R}_+^{mn} ,
$$

where $A \bullet B = \text{trace } (AB)$ denotes the Frobenius inner product of symmetric $n \times n$ matrices A and B. □

Example 2.6 *Let* $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \in \mathcal{P} \subset \mathcal{C}^*$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathcal{N} \subset \mathcal{C}^*$.
Then

$$
A \otimes B = \begin{bmatrix} 0 & 0 & 1 & -1 \\ & \dots & \end{bmatrix} \notin \mathcal{C}^* .
$$

(See also Example 2.7.) It does not help if both factors are the same: Let $C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in \mathcal{C}^*$, *with A and B as before. Then $C \otimes C$ has $A \otimes B$ as one of its principle submatrices, therefore $C \otimes C \notin \mathcal{C}^*$.*

Finally we turn to Hadamard products. For square matrices A, B of the same size, the Hadamard product is defined as $A.B = [a_{ij}b_{ij}]_{i,j}$, so it is bilinear. Also $(A.B)^\top = A^\top.B^\top$ and $A.B = B.A$. We further define the Hadamard power $A^{(n)} = [a_{ij}^n]_{i,j}$, and call *Hadamard posynomial* a function that maps a matrix A to $[p(a_{ij})]_{i,j}$, where p is a polynomial with no negative coefficients.

An important observation is that $A.B$ is a principal submatrix of $A \otimes B$ [145, p.304]. Thus from Proposition 2.2 we conclude

$$
A, B \in \mathcal{K} \Rightarrow A.B \in \mathcal{K} \text{ for } \mathcal{K} \in \{\mathcal{P}, \mathcal{N}, \mathcal{P} \cap \mathcal{N}, \mathcal{C}\} . \tag{3}
$$

In case $\mathcal{K} = \mathcal{P}$, this is known as Schur's theorem, see, e.g., [145, p.309] or [22, Prop.1.7]. The case $\mathcal{K} = \mathcal{C}$ is treated in [22, Cor.2.2]. The following is a counterexample for the case $\mathcal{K} \in \{\mathcal{P} + \mathcal{N}, \mathcal{C}^*\}$:

Table 2: Closure properties of cones for Kronecker and Hadamard products, and Hadamard posynomials $p(A) = \sum_k c_k A^{(k)}$ with $c_k \geq 0$.

$\mathcal{K}$	$A, B \in \mathcal{K}$ $\Rightarrow A \otimes B \in \mathcal{K}$	$A, B \in \mathcal{K}$ $\Rightarrow A.B \in \mathcal{K}$	$A \in \mathcal{K}$ $\Rightarrow p(A) \in \mathcal{K}$
$\mathcal{P}$	yes: Prop 2.2	yes: (3)	yes: follows
$\mathcal{N}$	yes: evident	yes: (3)	yes: evident
$\mathcal{P} \cap \mathcal{N}$	yes: Prop 2.2	yes: (3)	yes: follows
$\mathcal{P} + \mathcal{N}$	no[b]: Ex. 2.6	no[c]: Ex. 2.7	yes: Prop. 2.3
$\mathcal{C}$	yes: Prop 2.2	yes: (3)	yes: follows
$\mathcal{C}^*$	no[b]: Ex. 2.6	no[c]: Ex. 2.7	unclear

[b] not even if $A = B$
[c] but $A.A \in \mathcal{K}$ (evident)

Example 2.7 *Both matrices A, B from Example 2.6 belong to $\mathcal{C}^*$, however*
$$A.B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \notin \mathcal{C}^*.$$

Remark 2.1 *There is no way for concluding $A \otimes B \in \mathcal{C}^*$ from $A.B \in \mathcal{C}^*$, in particular the implication $\{A, B, A.B\} \subset \mathcal{C}^* \Rightarrow A \otimes B \in \mathcal{C}^*$ does not hold. Just take $A = B = C$, with C from Example 2.6.*

Proposition 2.3 (odd Hadamard powers)*:*
If $A \in \mathcal{P} + \mathcal{N}$, and $n = 2k + 1$ with $k \in \mathbb{N}$ is odd, then $A^{(n)} \in \mathcal{P} + \mathcal{N}$.

Proof. Assume $A = P + N$ with $P \in \mathcal{P}$ and $N \in \mathcal{N}$. Then by Schur's theorem, $P^{(n)} \in \mathcal{P}$, and by the monotonicity of odd power functions, $A^{(n)} - P^{(n)} \in \mathcal{N}$. $\qquad\square$

We conclude this section by observations on inversion and Schur complements. These two operations leave (the interior of) $\mathcal{P}$ invariant, but it is well known that this is not true for $\mathcal{N}$.

Example 2.8 *The example*
$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \in \mathcal{N} \subset \mathcal{C}^* \quad \text{with} \quad A^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} \notin \mathcal{C}^*$$

shows that the Schur complement of a positive-definite principal submatrix in a (co)positive matrix need not be (co)positive. Further,
$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \in \mathcal{P} \cap \mathcal{N} \quad \text{but} \quad A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \notin \mathcal{N} \supset \mathcal{C}.$$

Note that $\mathcal{P} \cap \mathcal{N} = \mathcal{C}$ when restricted to matrices of order 2.

Table 3: Further closure properties of cones for several products

$\mathcal{K}$	$A \in \mathcal{K}, B \in \mathcal{K}^*$ $\Rightarrow A \otimes B \in \mathcal{K} + \mathcal{K}^*$	$A \in \mathcal{K}, B \in \mathcal{K}^*$ $\Rightarrow A.B \in \mathcal{K} + \mathcal{K}^*$	$A \in \mathcal{K}, B \in \mathcal{K}^*$ $\Rightarrow AB + BA \in \mathcal{K} + \mathcal{K}^*$
$\mathcal{C}$	yes: Prop 2.2(c)	yes: follows	no: Ex. 2.5
$\mathcal{P} \cap \mathcal{N}$	yes: Prop 2.2(c)	yes: follows	no: Ex. 2.5

On the other hand, if the operator norm $\|A\| < 1$ and $A \in \mathcal{C}$, then from Table 1 and closedness of $\mathcal{C}$ we get $(I - A)^{-1} = \sum_{k=0}^{\infty} A^k \in \mathcal{C}$ while $I - A \notin \mathcal{N}$ unless A is diagonal.

3 Duality and attainability in copositive programs

We consider a primal/dual pair of copositive programs, whose primal consists of optimizing a linear function over the intersection of an affine subspace with the completely positive cone: for m symmetric matrices A_i of the order of X, let $\mathbf{A}X = [A_1 \bullet X, \ldots, A_m \bullet X]^\top \in \mathbb{R}^m$, if $X \in \mathcal{C}$, and $\mathbf{b} \in \mathbb{R}^m$. Then

$$p^* = \inf \{C \bullet X : \mathbf{A}X = \mathbf{b}, \ X \in \mathcal{C}\} \tag{4}$$

and with multipliers $\mathbf{y} = [y_1, \ldots y_m]^\top \in \mathbb{R}^m$, the Lagrangian dual function reads

$$\begin{aligned} \Theta_p(\mathbf{y}) &= \inf \{C \bullet X + \sum_{i=1}^m y_i[b_i - A_i \bullet X] : X \in \mathcal{C}\} \\ &= \inf \{(C - \mathbf{A}^\top \mathbf{y}) \bullet X + \mathbf{b}^\top \mathbf{y} : X \in \mathcal{C}\}, \end{aligned}$$

where $\mathbf{A}^\top \mathbf{y} = \sum_i y_i A_i$ is again a symmetric matrix of the order of X. Obviously, $\Theta_p(\mathbf{y}) > -\infty$ if and only if $C - \mathbf{A}^\top \mathbf{y} \in \mathcal{C}^*$, and then $\Theta_p(\mathbf{y}) = \mathbf{b}^\top \mathbf{y}$. Hence the Lagrangian dual problem reads

$$d^* = \sup \{\Theta_p(\mathbf{y}) : \mathbf{y} \in \mathbb{R}^m\} = \sup \{\mathbf{b}^\top \mathbf{y} : C - \mathbf{A}^\top \mathbf{y} \in \mathcal{C}^*, \ \mathbf{y} \in \mathbb{R}^m\}, \tag{5}$$

which can be rewritten as a linear optimization problem over the intersection of an affine subspace with the cartesian product cone $\mathbb{R}^m \times \mathcal{C}^*$:

$$d^* = \sup \{\mathbf{b}^\top \mathbf{y} : S + \mathbf{A}^\top \mathbf{y} = C, \ (\mathbf{y}, S) \in \mathbb{R}^m \times \mathcal{C}^*\}. \tag{6}$$

Arranging the multipliers of the constraints in a symmetric matrix U and observing $(\mathbf{A}^\top \mathbf{y}) \bullet U = (\mathbf{A}U)^\top \mathbf{y}$, the Lagrangian dual function of this problem (6) is

$$\begin{aligned} \Theta_d(U) &= \sup \{\mathbf{b}^\top \mathbf{y} + (C - S - \mathbf{A}^\top \mathbf{y}) \bullet U : (\mathbf{y}, S) \in \mathbb{R}^m \times \mathcal{C}^*\} \\ &= \sup \{C \bullet U + (\mathbf{b} - \mathbf{A}U)^\top \mathbf{y} - S \bullet U : (\mathbf{y}, S) \in \mathbb{R}^m \times \mathcal{C}^*\}. \end{aligned}$$

Again, $\Theta_d(U) < +\infty$ if and only if $U \in (\mathcal{C}^*)^* = \mathcal{C}$ and $\mathbf{A}U = \mathbf{b}$, so that the bidual of (4)

$$\inf \left\{ \Theta_d(U) : U = U^\top \right\} = \inf \left\{ C \bullet U : \mathbf{A}U = \mathbf{b}, \ U \in \mathcal{C} \right\} = p^*,$$

coincides indeed with the primal, as it should be. Of course, weak duality $d^* \leq p*$ always holds for the pair (4) and (5), and Slater's condition applies to guarantee full strong duality (i.e., $d^* = \mathbf{b}^\top \mathbf{y}^*$ is attained for some dually feasible $(\mathbf{y}^*, S^*) \in \mathbb{R}^m \times \mathcal{C}^*$ and coincides with p^*). However, unlike the LP case, primal attainability is not guaranteed for general conic programs. This is well known for semidefinite programs, and there are examples of all sorts of phenomena like positive finite duality gap and/or non-attainability for either the primal or the dual or both. Looking at one such example [139, Ex. 2.2.1], taken from [252], we see that we cannot simply replace the semidefinite cone $\mathcal{P}$ by either $\mathcal{C}$ or $\mathcal{C}^*$, to arrive at suitable examples for copositive programs, which is the main purpose of this section.

First, exclude the standard infeasible/unbounded cases where $p^* = -\infty$ or $d^* = +\infty$.

Next let us examine the case of zero duality gap, i.e., $d^* = p^*$. By above exclusion, we are left with a common finite value. Full strong duality (attainability of both) holds of course under Slater's condition:

$$\{X : \mathbf{A}X = \mathbf{b}\} \cap \operatorname{int} \mathcal{C} \neq \emptyset$$

implies zero duality gap and dual attainability, and

$$\left\{ S : S + \mathbf{A}^\top \mathbf{y} = C \text{ for some } \mathbf{y} \in \mathbb{R}^m \right\} \cap \operatorname{int} \mathcal{C}^* \neq \emptyset$$

implies zero duality gap and primal attainability. But full strong duality also holds for the copositive reformulation of Standard Quadratic Problems (StQPs), as was already observed in [47]. By contrast, failure of dual attainability with $d^* = p^*$ can happen in the general case of reformulation of Multi-StQPs [57]. We complement these observations by two more examples, one where p^* is not attained but d^* is, and a second where both are not attained.

Example 3.1 *This is an adaptation of [139, Ex. 2.2.8] from $\mathcal{P}$ to $\mathcal{C}$ which works: let $n = 2$, $m = 1$, $C \bullet X = x_{11}$, $A_1 \bullet X = x_{12} + x_{21}$ and $b_1 = 2$. Then*

$$d^* = \sup \left\{ 2y_1 : \begin{bmatrix} 1 & -y_1 \\ -y_1 & 0 \end{bmatrix} \in \mathcal{C}^* \right\} = 0$$

is attained for $y_1^* = 0$. Observe that $y_1 = -1$ is also dually feasible, but not optimal. On the other hand, the choice of $x_{11} = \frac{1}{k}$ and $x_{22} = k$ with $x_{12} = 1$ gives a primally feasible X_k with $C \bullet X_k = \frac{1}{k} \searrow 0$ as $k \nearrow \infty$, so that $p^* = d^*$. Obviously, p^* cannot be attained since $x_{11} = 0$ conflicts with $x_{12} = 1$ and $X \in \mathcal{C} \subset \mathcal{P}$.

The next example results from a general principle of constructing non-attainability, starting from a given instance $(\mathbf{A}, \mathbf{b}, C)$ of a copositive program (4) and (5).

Theorem 3.1 (constructing failure in dual attainability):
Let $\mathcal{T}_{\mathrm{d}}(\mathbf{A}, \mathbf{b}, C)$ denote the following new copositive primal problem: augment the $n \times n$ variable matrices X by appending two more rows and columns, to arrive at $(n+2) \times (n+2)$ variable matrices $\bar{X}$; further, define the objective and $m + 2$ constraints as follows: $\bar{\mathbf{b}} = [\mathbf{b}^\top, 1, 0]^\top$ and, with $\mathbf{o} \in \mathbb{R}^n$ the zero vector and O the $n \times n$ zero matrix,

$$\bar{C} = \begin{bmatrix} C & \mathbf{o} & \mathbf{o} \\ \mathbf{o}^\top & 0 & -1 \\ \mathbf{o}^\top & -1 & 0 \end{bmatrix} \quad and \quad \bar{A}_i = \begin{bmatrix} A_i & \mathbf{o} & \mathbf{o} \\ \mathbf{o}^\top & 0 & 0 \\ \mathbf{o}^\top & 0 & 0 \end{bmatrix}, \ 1 \le i \le m,$$

while

$$\bar{A}_{m+1} = \begin{bmatrix} O & \mathbf{o} & \mathbf{o} \\ \mathbf{o}^\top & 1 & 0 \\ \mathbf{o}^\top & 0 & 0 \end{bmatrix} \quad and \quad \bar{A}_{m+2} = \begin{bmatrix} O & \mathbf{o} & \mathbf{o} \\ \mathbf{o}^\top & 0 & 0 \\ \mathbf{o}^\top & 0 & 1 \end{bmatrix}.$$

Then $\mathcal{T}_{\mathrm{d}}(\mathbf{A}, \mathbf{b}, C)$ is feasible if and only if (4) is feasible, in which case p^* is attained in (4) for $(\mathbf{A}, \mathbf{b}, C)$ if and only if

$$\bar{p}^* = \inf \left\{ \bar{C} \bullet \bar{X} : \bar{A}_i \bullet \bar{X} = \bar{b}_i, \ 1 \le i \le m+2, \ \bar{X} \in \mathcal{C} \right\} \qquad (7)$$

is attained in the primal of $\mathcal{T}_{\mathrm{d}}(\mathbf{A}, \mathbf{b}, C)$. Furthermore, $\mathcal{T}_{\mathrm{d}}(\mathbf{A}, \mathbf{b}, C)$ has the same primal and dual optimal values p^* and d^*, but d^* is not attained: $\bar{p}^* = p^*$ and

$$\bar{d}^* := \sup \left\{ \bar{\mathbf{b}}^\top \bar{\mathbf{y}} : \bar{C} - \sum_{i=1}^{m+2} \bar{y}_i \bar{A}_i \in \mathcal{C}^*, \ \bar{\mathbf{y}} \in \mathbb{R}^{m+2} \right\} = d^*, \qquad (8)$$

but $d^* \notin \left\{ \bar{\mathbf{b}}^\top \bar{\mathbf{y}} : \bar{C} - \sum_{i=1}^{m+2} \bar{y}_i \bar{A}_i \in \mathcal{C}^*, \ \bar{\mathbf{y}} \in \mathbb{R}^{m+2} \right\}.$

Proof. Choose a (4)-feasible sequence $X_k \in \mathcal{C}$ such that $C \bullet X_k \to p^*$ and augment X_k by

$$\bar{X}_k = \begin{bmatrix} X_k & \mathbf{o} & \mathbf{o} \\ \mathbf{o}^\top & 1 & 0 \\ \mathbf{o}^\top & 0 & 0 \end{bmatrix}$$

to get a (7)-feasible sequence with $\bar{C} \bullet \bar{X}_k = C \bullet X_k \to p^*$. Of course, any (7)-feasible $\bar{X}$ contains a leading $n \times n$ principal submatrix X which is (4)-feasible. Further, $\bar{X} \in \mathcal{C}$ with $\bar{X}_{n+2,n+2} = 0$ enforces $\bar{X}_{n+1,n+2} = 0$ so that $\bar{C} \bullet \bar{X} = C \bullet X$. This proves $\bar{p}^* = p^*$, and also the assertions

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about equivalence of primal feasibility/attainability. Turning towards dual feasibility, and putting $\bar{\mathbf{y}} = [\mathbf{y}^\top, \bar{y}_{m+1}, \bar{y}_{m+2}]^\top$ with $\mathbf{y} \in \mathbb{R}^m$, we have

$$\bar{C} - \sum_{i=1}^{m+2} \bar{y}_i \bar{A}_i = \begin{bmatrix} C - \sum\limits_{i=1}^{m} y_i A_i & \mathbf{o} & \mathbf{o} \\ \mathbf{o}^\top & -\bar{y}_{m+1} & -1 \\ \mathbf{o}^\top & -1 & -\bar{y}_{m+2} \end{bmatrix},$$

so that $\bar{C} - \sum\limits_{i=1}^{m+2} \bar{y}_i \bar{A}_i \in \mathcal{C}^*$ entails $C - \sum\limits_{i=1}^{m} y_i A_i \in \mathcal{C}^*$ and $\bar{y}_{m+1} < 0$, which in turn implies

$$\bar{\mathbf{b}}^\top \bar{\mathbf{y}} = \mathbf{b}^\top \mathbf{y} + \bar{y}_{m+1} < \mathbf{b}^\top \mathbf{y} \le d^*, \tag{9}$$

hence $\bar{d}^* \le d^*$. On the other hand, select a sequence $\mathbf{y}_k \in \mathbb{R}^m$ such that $C - \mathbf{A}^\top \mathbf{y}_k \in \mathcal{C}^*$ and $\mathbf{b}^\top \mathbf{y}_k \to d^*$. Then the sequence $\bar{\mathbf{y}}_k = [\mathbf{y}_k^\top, -\frac{1}{k}, -k]^\top \in \mathbb{R}^{m+2}$ clearly satisfies $\bar{C} - \overline{\mathbf{A}^\top} \bar{\mathbf{y}}_k \in \mathcal{C}^*$ since its lower-right 2×2 block is even positive-semidefinite, and

$$\bar{\mathbf{b}}^\top \bar{\mathbf{y}}_k = \mathbf{b}^\top \mathbf{y}_k - \frac{1}{k} \to d^* \quad \text{as } k \to \infty,$$

which shows $\bar{d}^* \ge d^*$. The strict inequality in (9) implies dual non-attainability. $\square$

Clearly, $\mathcal{T}_d(\mathbf{A}, \mathbf{b}, C)$ violates Slater's condition even if $(\mathbf{A}, \mathbf{b}, C)$ satisfies it. Any such instance therefore generates, by Theorem 3.1 above, an example of zero duality gap with dual non-attainability. Even more:

Example 3.2 *Choose $(\mathbf{A}, \mathbf{b}, C)$ as in Example 3.1. Then $\mathcal{T}_d(\mathbf{A}, \mathbf{b}, C)$ has zero duality gap but neither of the (finite) optimal values are attained.*

Next we turn to the case to a finite positive duality gap:

$$-\infty < d^* < p^* < +\infty$$

where we need four examples for (non-)attainability.

Example 3.3 *Again, this is an adaptation of an example in [139], now by switching dual with primal. For $n = 3$ and $m = 2$ matrices, let C be such that $C \bullet X = x_{33}$ whereas $A_1 \bullet X = x_{33} + 2x_{12}$ and $A_2 \bullet X = x_{22}$. Further, let $\mathbf{b} = [1, 0]^\top \in \mathbb{R}^2$. Then*

$$p^* = \inf \{x_{33} : x_{33} + 2x_{12} = 1, \, x_{22} = 0, \, X \in \mathcal{C}\} = 1,$$

attained for an $X^* \in \mathcal{C}$ with all $x_{ij}^* = 0$ except $x_{33}^* = 1$. The dual reads

$$d^* = \sup \left\{ y_1 : \begin{bmatrix} 0 & -y_1 & 0 \\ -y_1 & -y_2 & 0 \\ 0 & 0 & 1 - y_1 \end{bmatrix} \in \mathcal{C}^* \right\} = 0,$$

attained for $\mathbf{y}^* = [0, 0]^\top$.

Note that the remedy via eigenspaces for SDPs as suggested in [139] does not seem to apply to copositive programs, the situation appears to be much more difficult.

Anyhow, by Theorem 3.1 we immediately have

Example 3.4 *With* $(\mathbf{A}, \mathbf{b}, C)$ *as in Example 3.3,* $(\tilde{A}, \tilde{\mathbf{b}}, \tilde{C}) = \mathcal{T}_{\mathrm{d}}(\mathbf{A}, \mathbf{b}, C)$ *has a positive duality gap, and* p^* *is attained but* d^* *is not.*

We now provide a direct primal construction as a counterpart to $\mathcal{T}_{\mathrm{d}}$, which is slightly simpler:

Theorem 3.2 (constructing failure in primal attainability):
Let $\mathcal{T}_{\mathrm{p}}(\mathbf{A}, \mathbf{b}, C)$ *denote the following new copositive primal problem: augment the* $n \times n$ *variable matrices* X *by appending two more rows and columns, to arrive at* $(n+2) \times (n+2)$ *variable matrices* $\bar{X}$*; further, define the objective and* $m + 1$ *constraints as follows: augment* A_i *by two more zero rows and columns to* $\bar{A}_i$ *and put* $\bar{\mathbf{b}} = [\mathbf{b}^\top, 2]^\top \in \mathbb{R}^{m+1}$ *while, with* $\mathbf{o} \in \mathbb{R}^n$ *the zero vector and* O *the* $n \times n$ *zero matrix,*

$$
\bar{C} = \begin{bmatrix} C & \mathbf{o} & \mathbf{o} \\ \mathbf{o}^\top & 1 & 0 \\ \mathbf{o}^\top & 0 & 0 \end{bmatrix} \quad and \quad \bar{A}_{m+1} = \begin{bmatrix} O & \mathbf{o} & \mathbf{o} \\ \mathbf{o}^\top & 0 & 1 \\ \mathbf{o}^\top & 1 & 0 \end{bmatrix}.
$$

Then $\mathcal{T}_{\mathrm{p}}(\mathbf{A}, \mathbf{b}, C)$ *is dually feasible if and only if (5) is feasible, in which case* d^* *is attained in (5) if and only if*

$$
\bar{d}^* := \sup \left\{ \bar{\mathbf{b}}^\top \bar{\mathbf{y}} : \bar{C} - \sum_{i=1}^{m+1} \bar{y}_i \bar{A}_i \in \mathcal{C}^* , \ \bar{\mathbf{y}} \in \mathbb{R}^{m+2} \right\} \tag{10}
$$

is attained. Furthermore, $\mathcal{T}_{\mathrm{p}}(\mathbf{A}, \mathbf{b}, C)$ *has the same primal and dual optimal values* p^* *and* d^**, but* p^* *is not attained:* $\bar{d}^* = d^*$ *and*

$$
\bar{p}^* := \inf \left\{ \bar{C} \bullet \bar{X} : \bar{A}_i \bullet \bar{X} = \bar{b}_i , \ 1 \le i \le m+1 , \ \bar{X} \in \mathcal{C} \right\} = p^* , \tag{11}
$$

but $p^* \notin \left\{ \bar{C} \bullet \bar{X} : \bar{A}_i \bullet \bar{X} = \bar{b}_i , \ 1 \le i \le m+1 , \ \bar{X} \in \mathcal{C} \right\}$.

Proof. The constraint $\bar{A}_{m+1} \bullet \bar{X} = 2$ means $x_{n+1,n+2} = 1$ which forces $x_{n+1,n+1} > 0$, so that for any (11)-feasible $\bar{X}$ we have

$$
\bar{C} \bullet \bar{X} = C \bullet X + x_{n+1,n+1} > C \bullet X \ge p^* , \tag{12}
$$

and hence $\bar{p}^* \ge p^*$. If, on the other hand, a sequence $X_k \in \mathcal{C}$ with $\mathbf{A} X_k = \mathbf{b}$ satisfies $C \bullet X_k \to p^*$ and we form $\bar{X}_k$ by

$$
\bar{X}_k = \begin{bmatrix} X_k & \mathbf{o} & \mathbf{o} \\ \mathbf{o}^\top & \frac{1}{k} & 1 \\ \mathbf{o}^\top & 1 & k \end{bmatrix},
$$

then this is a (11)-feasible sequence such that $\lim_{k\to\infty} \bar{C} \bullet \bar{X}_k = p^*$, which shows $\bar{p}^* \le p^*$, and non-attainability of $\bar{p}^*$ follows from the strict inequality in (12). Turning towards dual feasibility, and putting $\bar{\mathbf{y}} = [\mathbf{y}^\top, \bar{y}_{m+1}]^\top$ with $\mathbf{y} \in \mathbb{R}^m$, we have

$$\bar{C} - \sum_{i=1}^{m+1} \bar{y}_i \bar{A}_i = \begin{bmatrix} C - \sum_{i=1}^{m} y_i A_i & \mathbf{o} & \mathbf{o} \\ \mathbf{o}^\top & 1 & -\bar{y}_{m+1} \\ \mathbf{o}^\top & -\bar{y}_{m+1} & 0 \end{bmatrix},$$

so that $\bar{C} - \sum_{i=1}^{m+1} \bar{y}_i \bar{A}_i \in \mathcal{C}^*$ entails $C - \sum_{i=1}^{m} y_i A_i \in \mathcal{C}^*$ and $\bar{y}_{m+1} \le 0$, which in turn implies

$$\bar{\mathbf{b}}^\top \bar{\mathbf{y}} = \mathbf{b}^\top \mathbf{y} + 2\bar{y}_{m+1} \le \mathbf{b}^\top \mathbf{y} \le d^*,$$

and thus $\bar{d}^* \le d^*$. On the other hand, for any $\mathbf{y} \in \mathbb{R}^m$ such that $C - \mathbf{A}^\top \mathbf{y} \in \mathcal{C}^*$ the point $\bar{\mathbf{y}} = [\mathbf{y}^\top, 0]^\top \in \mathbb{R}^{m+1}$ is (10)-feasible with $\bar{\mathbf{b}}^\top \bar{\mathbf{y}} = \mathbf{b}^\top \mathbf{y}$, so that $\bar{d}^* \ge d^*$, and also equivalence of dual feasibility/attainability follows. $\qquad\square$

Example 3.5 *Let $(\mathbf{A}, \mathbf{b}, C)$ be as in Example 3.3 and consider $\mathcal{T}_{\mathrm{p}}(\mathbf{A}, \mathbf{b}, C)$. Then Theorem 3.2 gives an instance with positive finite duality gap where p^* is not attained but d^* is attained. Applying Theorem 3.1, we finally see that $\mathcal{T}_{\mathrm{d}}(\mathcal{T}_{\mathrm{p}}(\mathbf{A}, \mathbf{b}, C))$ has a positive finite duality gap where neither of p^* and d^* are attained.*

Example 3.6 *The previous effect is also generated by $\mathcal{T}_{\mathrm{p}}(\tilde{\mathbf{A}}, \tilde{\mathbf{b}}, \tilde{C})$, where $(\tilde{\mathbf{A}}, \tilde{\mathbf{b}}, \tilde{C})$ is the instance from Example 3.4, in other words by considering $\mathcal{T}_{\mathrm{p}}(\mathcal{T}_{\mathrm{d}}(\mathbf{A}, \mathbf{b}, C))$ with $(\mathbf{A}, \mathbf{b}, C)$ from Example 3.3.*

Now we have to deal with infeasibility of exactly one problem and boundedness of the other one. In this case, the duality gap is infinite, but we still can have attainability or non-attainability in the bounded problem.

Example 3.7 *For $n = 3$, consider the constraints $A_1 \bullet X = 2x_{22} + 2x_{23} = 0 = b_1$ and $A_2 \bullet X = 2x_{12} - 2x_{33} = 2 = b_2$. If $X \in \mathcal{C}$, then $x_{23} \ge 0$ and $x_{22} \ge 0$ imply $x_{22} = 0$, hence $x_{12} = 0$, hence $x_{33} = -1 < 0$, which is absurd. Hence the primal problem is infeasible, $p^* = +\infty$. Now choose $C = O$. Since*

$$C - \mathbf{A}^\top \mathbf{y} = \begin{bmatrix} 0 & -y_2 & 0 \\ -y_2 & -2y_1 & -y_1 \\ 0 & -y_1 & 2y_2 \end{bmatrix}$$

has a zero in its top-left corner, we infer $y_2 \le 0$ for any $\mathbf{y} \in \mathbb{R}^2$ such that $C - \mathbf{A}^\top \mathbf{y} \in \mathcal{C}^*$. Of course, $\mathbf{y}^* = \mathbf{o}$ is dually feasible, thus optimal, and $d^* = 0$ is attained.

Table 4: Possible attainability/duality gap constellations in primal-dual pairs of copositive programs. Only the doubly infeasible case $d^* = -\infty, p^* = \infty$ is omitted.

duality gap attained	zero $d^* = p^* \in \mathbb{R}$	finite positive $-\infty < d^* < p^* < \infty$	infinite $-\infty < d^* < p^* = \infty$	infinite $-\infty = d^* < p^* < \infty$
both attained	StQP [47], strong duality, Slater for both	Ex.3.3	impossible	impossible
p^* attained, d^* not attained	MStQP [57], Slater for dual	Ex.3.4	impossible	Ex.3.9
p^* not attained, d^* attained	Ex.3.1, Slater for primal	Ex.3.5	Ex.3.7	impossible
neither attained	Ex.3.2	Ex.3.6	Ex.3.8	Ex.3.10

Example 3.8 *An application of Theorem 3.1 yields* $\mathcal{T}_{\mathrm{d}}(\mathbf{A}, \mathbf{b}, C)$ *from the instance* $(\mathbf{A}, \mathbf{b}, C)$ *in Example 3.7, an instance with* $d^* = 0 < p^* = +\infty$ *where* d^* *is not attained.*

To conclude, we need an instance $(\mathbf{A}, \mathbf{b}, C)$ with infeasible dual and bounded primal, where p^* is attained:

Example 3.9 *We keep* $\mathbf{A}$ *from the instance in Example 3.7 but change* $\mathbf{b} = \mathbf{o}$ *now. Then any feasible* X *satisfies* $x_{33} = 0$. *Now choose* C *with all zero entries except* $c_{33} = -1$. *Then* $X^* = O \in \mathcal{C}$ *is optimal, so* $p^* = 0$ *is attained. However,*

$$C - \mathbf{A}^\top \mathbf{y} = \begin{bmatrix} 0 & -y_2 & 0 \\ -y_2 & -2y_1 & -y_1 \\ 0 & -y_1 & -1 + 2y_2 \end{bmatrix} \in \mathcal{C}^*$$

is impossible, as $y_2 \leq 0$ *must still hold, which implies the absurd* $-1 + 2y_2 \leq -1 < 0$ *for the lower-right corner entry. Hence* $d^* = -\infty$.

Example 3.10 *The last example is generated by* $\mathcal{T}_{\mathrm{p}}(\mathbf{A}, \mathbf{b}, C)$ *from the instance* $(\mathbf{A}, \mathbf{b}, C)$ *in Example 3.9. Here* $-\infty = d^* < p^* < +\infty$ *and* p^* *is not attained.*

Finally, already in the LP domain we also are familiar with the case $d^* = -\infty$ and $p^* = +\infty$ where both problems are infeasible. Hence, all possible combinations are demonstrated.

4   A clustered bibliography

Of course, any collection of literature references is doomed incomplete (and outdated by the appearance date of printed issues). Nevertheless, we tried to cluster the following lists somehow systematically.

4.1   Surveys, reviews, entries, book chapters

Copositive optimization receives increasing interest in the Operations Research community, and is a rapidly expanding and fertile field of research. While the time may not yet be ripe for writing up the final standard text book in this domain, several authors nonetheless bravely took the challenge of providing an overview, thereby aiming at a rapidly moving target. Recent surveys on copositive optimization are offered by [108] and [43], while [152] and [141] provide reviews on copositivity with less emphasis on optimization. [41] and [71] are entries in the most recent edition of the Encyclopedia of Optimization. Recent book chapters with some character of a survey on copositivity from an optimization viewpoint are [42, Section 1.4] and [68].

4.2   Copositivity checking and properties

To check whether a given matrix is copositive is NP-hard, see [214]; for the completely positive side, see [99], cf. also [21]. There are several algorithmic approaches for this problem, among them recursive methods [2, 7, 25, 69, 82, 83, 88, 105, 172, 188].

On the dual side, an explicit certificate for complete copositivity is given by a non-negative factorization [11, 19, 22, 21, 23, 40, 102, 171, 174, 189, 193, 200, 233, 238, 255, 262, 264].

Other approaches for copositivity checking can be found in [10, 12, 13, 14, 15, 16, 17, 18, 20, 24, 29, 30, 33, 36, 39, 48, 64, 65, 63, 76, 77, 75, 80, 96, 97, 98, 100, 104, 103, 106, 109, 111, 115, 116, 119, 123, 124, 128, 132, 133, 134, 135, 138, 140, 142, 144, 143, 149, 151, 160, 162, 163, 164, 165, 169, 170, 173, 176, 183, 184, 185, 192, 196, 197, 198, 199, 203, 205, 208, 209, 210, 213, 212, 211, 218, 224, 225, 243, 244, 245, 247, 248, 249, 250, 251, 254, 256, 257, 258, 259, 260]. These references also include investigations of (mainly algebraic, but sometimes also geometric) properties of copositive and/or completely positive matrices.

4.3   Role of copositivity in optimization theory

Copositivity occurs in many different optimization contexts. A meanwhile classical role of copositivity is connected to the linear complementarity problem (LCP) [60, 136, 148, 166, 175, 204, 213, 235]. This list also includes references which address feasibility and/or attainability issues.

Other papers dealing with the role of copositivity in theory and application of optimization are [1, 6, 8, 9, 26, 27, 44, 45, 34, 35, 47, 38, 54, 50, 53,

41, 49, 57, 58, 73, 75, 78, 74, 79, 81, 85, 84, 86, 87, 91, 89, 100, 113, 114, 117, 121, 124, 125, 150, 161, 168, 167, 184, 190, 191, 195, 202, 214, 219, 220, 222, 223, 230, 234, 237, 240, 244, 253, 254, 263].

4.4   Copositive programming algorithms

According to Franz Rendl, nobody knows how to solve a copositive program (personal communication). To put it in less categorical terms: there is no state-of-the-art algorithm for copositive optimization. Various attempts have been made to tackle this problem [6, 24, 28, 44, 45, 31, 32, 34, 47, 38, 37, 46, 50, 53, 41, 52, 49, 51, 70, 68, 85, 84, 86, 93, 94, 103, 107, 110, 112, 118, 120, 121, 125, 127, 137, 162, 184, 185, 186, 216, 219, 220, 221, 226, 227, 228, 231, 253, 266]. The apparently most successful procedures up to now employ adaptive simplicial subdivision (see [146] for a good survey on this topic including convergence results): see [65, 66, 261, 265]; related variants for testing copositivity are [48, 243].

4.5   Copositive reformulations and relaxations for hard optimization problems

A considerable part of the success of copositive optimization lies in the versatility of this model, which allows for reformulating many hard problems from several, seemingly unrelated optimization domains. This includes graph theoretical and other discrete optimization models, mixed-integer, (fractional) quadratic problems [1, 28, 55, 34, 35, 47, 38, 37, 46, 50, 53, 41, 49, 67, 68, 93, 91, 94, 89, 107, 111, 112, 126, 129, 130, 131, 149, 185, 186, 202, 216, 217, 219, 220, 221, 226, 227, 228, 230, 231, 232, 239, 240, 241, 242, 244, 253, 261].

4.6   Applications of copositivity

The copositivity concept plays an already classical role in (Linear) Complementarity Problems and connected feasibility questions [106, 113, 114, 132, 213]. An interesting application to Simpson's paradox can be found in [129], while the connection of copositivity with conic geometry and angles is discussed at length in [8, 122, 153, 154, 155, 156, 157, 158, 159, 205, 222, 223, 235, 236].

The central solution concepts in (evolutionary) game theory, evolutionarily and/or neutrally stable strategies, are both refinements of the Nash equilibrium concept. They are closely connected to copositivity, as shown in [56, 28, 38].

Copositive formulations of robust optimization and/or uncertainty modeling are addressed in [125, 216].

Friction and contact problems in rigid body mechanics are treated from a copositive perspective in [3, 4, 5, 168, 167, 182]. Network (stability) problems in queueing, traffic, and reliability are tackled along this approach in [150, 177, 178, 179, 180, 187, 206, 207].

Also in the domain of dynamical systems and optimal control, copositive matrices play an important role, see, e.g. [60, 66, 72, 81, 160, 181, 201, 229, 257], while the articles [62, 191, 215] deal with questions of majorization, under/overestimation and tight bounding.

A few applications of complete positivity can be found in the book [22].

Finally, graph theory application aspects of copositivity (and closely related domains), and those of more general combinatorial optimization character can be found, e.g., in [92, 95, 90, 101, 107, 126, 130, 131, 227, 265], and in the following articles which provide a partial survey in this domain: [186, 246].

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