# On Penalty and Gap Function Methods for Bilevel Equilibrium Problems

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**Abstract.** We consider bilevel pseudomonotone equilibrium problems. We use a penalty function to convert a bilevel problem into one-level ones. We generalize a pseudo $\nabla$-monotonicity concept from $\nabla$-monotonicity and prove that under pseudo $\nabla$-monotonicity property any stationary point of a regularized gap function is a solution of the penalized equilibrium problem. As an application, we discuss a special case that arises from the Tikhonov regularization method for pseudo monotone equilibrium problems.

**Keywords.** Bilevel Equilibrium Problems, Auxiliary Problem Principle, Pseudo $\nabla$-Monotone, Gap Function; Descent Method.

## 1 Introduction

Let $C$ be a nonempty closed convex subset in $\mathbb{R}^n$ and $f, g : C \times C \to \mathbb{R}$ be two bifunctions satisfying $f(x,x) = g(x,x) = 0$ for every $x \in C$. Such a bifunction is called an equilibrium bifunction. We consider the following bilevel equilibrium problem (BEP for short):

$$\text{Find } \bar{x} \in S_g \text{ such that } f(\bar{x}, y) \geq 0, \forall y \in S_g, \tag{1.1}$$

where $S_g = \{u \in C : g(u,y) \geq 0, \forall y \in C\}$ i.e.. $S_g$ is the solution set of the equilibrium problem

$$\text{Find } u \in C \text{ such that } g(u,y) \geq 0, \forall y \in C. \tag{1.2}$$

As usual, we call problem ( 1.1) the upper problem and ( 1.2) the lower one. BEPs are special cases of mathematical programs with equilibrium constraints. Sources for such problems can be found in [13, 14, 20]. Bilevel monotone variational inequality, which is a special case of problem ( 1.1) was considered in [1, 11]. Moudafi in [18] suggested the use of the proximal point method for monotone BEPs. Recently, Ding in [9] used the auxiliary problem principle to BEPs. In both papers, the bifunctions $f$ and $g$ are required to be monotone on $C$. It should be noticed that under the

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pseudomonotonicity assumption on $g$ the solution-set $S_g$ of the lower problem ( 1.2) is a closed convex set whenever $g(x,.)$ is lower seminoutinuous and convex on $C$ for each $x$. However, the main difficulty is that, even the constrained set $S_g$ is convex, it is not given explicitly as in a standard mathematical programming problem, and therefore the available methods (see e.g. [4, 7, 15, 16, 21, 22, 24] and the references therein) cannot be applied directly.

In this paper, first, we propose a penalty function method for Problem ( 1.1). Next, we use a regularized gap function for solving the penalized problems. Under certain pseudo $\nabla$-monotonicity properties of the regularized bifunction we show that any stationary point of the gap function on the convex set $C$ is a solution to the penalized subproblem. Finally, we apply the proposed method to the Tikhonov regularization method for pseudomonotone equilibrium problems.

# 2 A Penalty Function Method

Penalty function method is a fundamental tool widely used in optimization to convert a constrained problem into unconstraint (or easier constrained) ones. This method was used to monotone variational inequalities in [11] and equilibrium problems in [19]. In this section we use the penalty function method to the bilevel problem ( 1.1). First, let us recall some well-known concepts on monotonicity and continuity (see e.g. [5]) that will be used in the sequel.

**Definition 2.1** *The bifunction $\phi : C \times C \to \mathbb{R}$ is said to be:*
*a) strongly monotone on $C$ with modulus $\beta > 0$ if*

$$\phi(x,y) + \phi(y,x) \le -\beta||x-y||^2 \quad \forall x,y \in C;$$

*b) monotone on $C$, if*

$$\phi(x,y) + \phi(y,x) \le 0 \quad \forall x,y \in C;$$

*c) pseudomonotone on $C$ if*

$$\forall x,y \in C : \ \phi(x,y) \ge 0 \implies \phi(y,x) \le 0;$$

*d) upper-semicontiniuos at $x$ with respect to the first argument on $C$ if*

$$\overline{\lim}_{z\to x}\phi(z,y) \le \phi(x,y) \quad \forall y \in C;$$

*e) lower-semicontiniuos at $y$ with respect to the second argument on $C$ if*

$$\underline{\lim}_{w\to y}\phi(x,w) \ge \phi(x,y) \quad \forall x \in C;$$

Clearly, $a) \implies b) \implies c)$.

**Definition 2.2** *( [6]) The bifunction $\phi : C \times C \to \mathbb{R}$ is said to be coercive on $C$ if there exists a compact subset $B \subset I\!R^n$ and a vector $y_0 \in B \cap C$ such that*

$$\phi(x, y_0) < 0, \quad \forall x \in C \setminus B.$$

**Theorem 2.1** *([12] Proposition 2.1.14 ) Let $\phi : C \times C \to \mathbb{R}$ be a equilibrium bifunction such that $\phi(., y)$ is upper semicontinuous on $C$ for each $y \in C$ and $\phi(x, .)$ is convex on $C$ for each $x \in C$. Suppose that $C$ is compact or $\phi$ is coercive on $C$, then there exists at least one $x^* \in C$ such that $\phi(x^*, y) \geq 0$ for every $y \in C$.*

The following theorem tell us a relationship between the coercivity and the strong monotonicity.

**Proposition 2.1** *Suppose that the equilibrium bifunction $\phi$ is strongly monotone on $C$, and $\phi(x, .)$ is convex, lower-semicontinuous with respect to the second argument, then for each $y \in C$ there exists a compact set $B$ such that $y \in B$ and $\phi(x, y) < 0$ $\forall x \in C \setminus B$.*

*Proof,* Suppose contradiction that the conclusion of the theorem does not hold. Then there exists an element $y_0 \in C$ such that for every closed ball $B_r$ centered at the origin with radius $r > ||y_0||$, there is an element $x^r \in C \setminus B_r$ such that $\phi(x^r, y_0) \geq 0$.

Fix $r_0 > ||y_0||$ and $r_0 > 1$. Take $x^r = y_0 + r(x - y_0)$, where $r > r_0$, $x \in C \cap B_{r_0}$. By the strong monotonicity of $\phi$, we have

$$\phi(y_0, x^r) \leq -\phi(x^r, y_0) - \beta||x^r - y_0||^2 \leq -\phi(x^r, y_0) - \beta r^2||x - y_0||^2.$$

Since $\phi(y_0, .)$ is convex on $C$, it follows that

$$\phi(y_0, x) \leq \frac{1}{r}\phi(y_0, x^r) + \frac{r-1}{r}\phi(y_0, y_0)$$

which implies $\phi(y_0, x) \leq -\beta r||x - y_0||^2$. Thus

$$\phi(y_0, x) \to -\infty \quad \text{as } r \to \infty. \qquad (1).$$

However, since $\phi(y_0, .)$ is lower semicontinuous on $C$, by the well-known Weierstrass Theorem, $\phi(y_0, .)$ attains its minimum on the compact set $B_{r_0} \cap C$. This fact contradicts to (1). □

From this proposition we can derive the following corallaries.

**Corollary 2.1** *([12]) If bifunction $\phi$ is strongly monotone on $C$, and $\phi(x,.)$ is convex, lower-semicontinuous with respect to the second argument, then $\phi$ is coercive on $C$.*

**Corollary 2.2** *Suppose that bifunction $f$ is strongly monotone on $C$, and $f(x,.)$ is convex, lower-semicontinuous with respect to the second argument. If the bifunction $g$ is coercive on $C$ then, for every $\epsilon > 0$, the bifunction $g + \epsilon f$ is uniformly coercive on $C$ e.g.., there exists a point $y_0 \in C$ and a compact set $B$ both independent of $\epsilon$ such that*

$$g(x, y_0) + \epsilon f(x, y_0) < 0 \ \forall x \in C \setminus B.$$

*Proof.* From the coercivity of $g$ we conclude that there exists a compact $B_1$ and $y_0 \in C$ such that $g(x, y_0) < 0 \ \forall x \in C \setminus B_1$. Since $f$ is strongly monotone, convex, lower semicontinuous on $C$, by choosing $y = y_0$, from Proposition 2.1, there exists a compact $B_2$ such that $f(x, y_0) < 0 \ \forall x \in C \setminus B_2$. Set $B = B_1 \cup B_2$. Then $B$ is compact and $g(x, y_0) + \epsilon f(x, y_0) < 0 \ \forall x \in C \setminus B$. □

**Remark 2.1** *It is worth to note that, if both $f$, $g$ are coercive and pseudomonotone on $C$, then the function $f + g$ are not necessary coercive or pseudomonotone on $C$*

To see this, let us consider the following bifunctions

**Example 2.1** *Let $f(x, y) := (x_1y_2 - x_2y_1)e^{x_1}$, $g(x, y) := (x_2y_1 - x_1y_2)e^{x_2}$ and $C = \{(x_1, x_2) : x_1 \geq -1, \frac{1}{10}(x_1 - 9) \leq x_2 \leq 10x_1 + 9\}$. Then we have*

*i) $f(x, y), g(x, y)$ are pseudomonotone and coercive on $C$;*

*ii) $\forall \epsilon > 0$ the bifunctions $f_\epsilon(x, y) = g(x, y) + \epsilon f(x, y)$ are neither pseudomonotone nor coercive on $C$.*

Indeed,

i) If $f(x, y) \leq 0$ then $f(y, x) \geq 0$, thus $f$ is pseudomonotone on $C$. By choosing $y^0 = (y_1^0, 0), (0 < y_1^0 \leq 1)$ and B $= \{(x_1, x_2) : x_1^2 + x_2^2 \leq r\} (r > 1)$ we have $f(x, y^0) = -x_2y_1^0e^{x_1} < 0 \ \forall y \in C \setminus B$, which means that $f$ is coercive on $C$. Similarly, we can see that $g$ is coercive on $C$

ii) By definition of $f$ we have

$$f_\epsilon(x, y) = (x_2y_1 - x_1y_2)(e^{x_2} - \epsilon e^{x_1}), \forall \epsilon > 0.$$

Take $x(t) = (t, 2t), y(t) = (2t, t)$ then $f_\epsilon(x(t), y(t)) = 3t^2(e^{2t} - \epsilon e^t) > 0$, whereas $f_\epsilon(y(t), x(t)) = -3t^2(e^t - \epsilon e^{2t}) > 0$ for $t$ is sufficiently large. So $f_\epsilon$ is not pseudomonotone on $C$.

Now we show that the bifunction $f_\epsilon(x,y) = (x_2y_1 - x_1y_2)(e^{x_2} - \epsilon e^{x_1})$ is not coercive on $C$. Suppose, by contradiction, that there exist a compact set $B$ and $y^0 = (y_1^0, y_2^0) \in B \cap C$ such that $f_\epsilon(x, y^0) < 0\ \forall x \in C \setminus B$. Then, by coercivity of $f_\epsilon$, it follows $y_1^0, y_2^0 > 0$ and $y_1^0 \neq y_0^2$ . With $x(t) = (t, kt), (t > 0)$ we have $f_\epsilon(x(t), y^0) = t(ky_1^0 - y_2^0)(e^{kt} - \epsilon e^t)$. However:

- If $y_1^0 > y_2^0$, then, from $1 < k < 10$ follows $x(t) \in C$ and $f_\epsilon(x(t), y^0) > 0$ for $t$ is sufficiently large, which contradicts with coercivity.

- If $y_1^0 < y_2^0$ , then, by choosing $\frac{1}{10} < k < 1$ we obtain $x(t) \in C$ and $f_\epsilon(x(t), y^0) > 0$ for $t$ is large enough. But this can not be happened because of the coercivity of $f_\epsilon$.

Now, for each fixed $\epsilon > 0$, we consider the penalized equilibrium problem $PEP(C, f_\epsilon)$ defined as

$$\text{Find } \bar{x}_\epsilon \in C \text{ such that } f_\epsilon(\bar{x}_\epsilon, y) := g(\bar{x}_\epsilon, y) + \epsilon f(\bar{x}_\epsilon, y) \geq 0\ \forall y \in C. \tag{2.1}$$

By SOL$(C, f_\epsilon)$ we denote the solution-set of $PEP(C, f_\epsilon)$.

**Theorem 2.2** *Suppose that the equilibrium bifunctions $f, g$ are pseudomonotone, upper semicontinuous with respect to the first argument and lower semicontinuous, convex with respect to the second argument on $C$. Then any cluster point of the sequence $\{x_k\}$ with $x_k \in SOL(C, f_{\epsilon_k})$, $\epsilon_k \to 0$ is a solution to the original bilevel problem. In addition, if $f$ is strongly monotone and $g$ is coercive on $C$, then for each $\epsilon_k > 0$ the penalized problem $PEP(C, f_{\epsilon_k})$ is solvable and any sequence $\{x_k\}$ with $x_k \in SOL(C, f_{\epsilon_k})$ converges to the unique solution of the bilevel problem ( 1.1) as $k \to \infty$.*

*Proof.* By the assumption, the equilibrium bifunction $f_{\epsilon_k}$ is upper - semicontiniuos with respect to the first argument and lower semicontinuous, convex with respect to the second argument on $C$. Then, by Corollary 2.2, $f_{\epsilon_k}$ is uniformly coercive on $C$. Thus Problem $PEP(C, f_{\epsilon_k})$ is solvable and, for all $\epsilon_k > 0$, the solution-sets of these problems are contained in a compact set $B$. So any infinite sequence $\{x_k\}$ of the solutions has a cluster point, say, $\bar{x}$. Without lost of generality, we may assume that $x_k \to x$ as k $\to \infty$. Since $x_k \in SOL(C, f_{\epsilon_k})$, one has

$$g(x_k, y) + \epsilon_k f(x_k, y) \geq 0\ \forall\ y \in C. \tag{1}$$

For any $z \in S_g$, we have $g(z, y) \geq 0\ \forall y \in C$, particularly, $g(z, x_k) \geq 0$. Then, by the pseudomonotonicity of $g$, we have $g(x_k, z) \leq 0$. Replacing $y$ by $z$ in (1) we obtain

$$g(x_k, z) + \epsilon_k f(x_k, z) \geq 0,$$

which implies

$$\epsilon_k f(x_k, z) \geq -g(x_k, z) \geq 0 \Rightarrow f(x_k, z) \geq 0.$$

Let $\epsilon_k \to 0$, by upper semicontiniuity of $f$, we have $f(\bar{x}, z) \geq 0 \ \forall z \in S_g$.

To complete the proof, we need only to show that $\bar{x} \in S_g$. Indeed, for any $y \in C$ we have

$$g(x_k, y) + \epsilon_k f(x_k, y) \geq 0 \ \forall \ y \in C. \tag{2}$$

Again, by upper semicontiniuity of $f$ and $g$ we obtain in the limit, as $\epsilon_k \to 0$, that $g(\bar{x}, y) \geq 0 \ \forall y \ \in C$. Hence $\bar{x} \in S_g$.

On the other hand, from the assumption on $g$ the solution-set $S_g$ of the lower equilibrium $EP(C, g)$ is a closed, convex, compact set. Since $f$ is lower semicontinuous and convex with respect to the second argument and is strongly monotone on $C$, the upper equilibrium problem $EP(S_g, f)$ has a unique soultion. By the first part of this theorem, this unique solution much be the limit point of any sequence $\{x_k\}$ with $x_k$ being a solution to the penalized problem $PEP(C, f_{\epsilon_k})$. □

**Remark 2.2** *In a special case considered in [18], where both f and g are monotone, the penalized problem (PEP) is monotone too. In this case (PEP) can be solved by some existing methods (see. e.g. [16, 17, 18, 21, 22, 24]) and the references therein. However, when one of these two bifunctions is pseudomonotone, the penalized problem (PEP). in general, does not inherit any monotonicity property from f and g. In this case, Problem (PEP) cannot be solved by the aboe mentioned existing methods.*

# 3 Gap Function and Descent Direction

A well-known tool for solving equilibrium problem is the gap function. The regularized gap function has been introduced by Fukushima and Taji in [23] for variational inequalities, and extended by Mastroeni in [16] to equilibrium problems. In this section we use the regularized gap function for the penalized equilibrium problem (PEP). As we have mentioned above, this problem, even when $g$ is pseudomonotone and $f$ is strongly monotone is still difficult to solve.

Throughout this section we suppose that both $f$ and $g$ are lower semicontinuous, convex on $C$ with respect to the second argument. First we recall (see e.g. [16]) the definition of a gap function for the equilibrium problem.

**Definition 3.1** *A function $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ is said to be a gap function for (PEP) if*

*i)* $\varphi(x) \geq 0 \ \forall x \ \in C$

*ii)* $\varphi(\bar{x}) = 0$ *iff* $\bar{x}$ *is a solution for (PEP).*

A gap function for (PEP) is $\varphi(x) = -\min_{y \in C} f_\epsilon(x, y)$. This gap function may not be finite and, in general, is not differentiable. To obtain a finite, differentiable gap function, we use the regularized gap function introduced in [23] and recently is used by Matroeni in [16] to equilibrium problems. From Proposition 2.2 and Theorem 2.1 in [16] the following proposition is immediate.

**Proposition 3.1** *Suppose that $l : C \times C \to \mathbb{R}$ is a nonnegative differentiable, strongly convex bifunction on $C$ with respect to the second argument and satisfies*

*a)* $l(x,x) = 0 \ \forall x \in C$

*b)* $\nabla_y l(x,x) = 0 \ \forall x \in \ C$.

*Then the function*

$$\varphi_\epsilon(x) = -\min_{y \in C} [g(x,y) + \epsilon[f(x,y) + l(x,y)]]$$

*is a finite gap function for (PEP). In addition, if $f$ and $g$ are differentiable with respect to the first argument and $\nabla_x f(x,y), \nabla_x g(x,y)$ are continuous on $C$, then $\varphi_\epsilon(x)$ is continuously differentiable on $C$ and*

$$\nabla \varphi_\epsilon(x) = -\nabla_x g(x, y_\epsilon(x)) - \epsilon \nabla_x [f(x, y_\epsilon(x)) + l(x, y_\epsilon(x))] = -\nabla_x g_\epsilon(x, y_\epsilon(x))$$

*where*

$$g_\epsilon(x,y) = g(x,y) + \epsilon[f(x,y) + l(x,y)]$$

*and*

$$y_\epsilon(x) = arg \min_{y \in C} \{g_\epsilon(x,y)\}.$$

Note that, the function $l(x,y) := \frac{1}{2}\langle M(y-x), y-x\rangle$, where $M$ is a symmetric positive definite matrix of order $n$ satisfies the assumptions on $l$.

We need some definitions on $\nabla$-monotonicity.

**Definition 3.2** *A differentiable bifunction $h : C \times C \to \mathbb{R}$ is called:*

*a) strongly $\nabla$- monotone on $C$ if there exists a constant $\tau > 0$ such that:*

$$\langle \nabla_x h(x,y) + \nabla_y h(x,y), y - x\rangle \geq \tau ||y - x||^2 \ \forall x, y \ \in C;$$

*b) strictly $\nabla$-monotone on $C$ if*

$$\langle \nabla_x h(x,y) + \nabla_y h(x,y), y - x\rangle > 0 \ \forall x, y \ \in C \text{ and } x \neq y;$$

*c) $\nabla$-monotone on $C$ if*

$$\langle \nabla_x h(x,y) + \nabla_y h(x,y), y - x\rangle \geq 0 \ \forall x, y \ \in C;$$

*d)strictly pseudo $\nabla$-monotone on $C$ if*

$$\langle \nabla_x h(x,y), y - x\rangle \leq 0 \Longrightarrow \ \langle \nabla_y h(x,y), y - x\rangle > 0 \ \forall x, y \in C \text{ and } x \neq y;$$

*e) pseudo $\nabla$-monotone on $C$ if*

$$\langle \nabla_x h(x,y), y - x\rangle \leq 0 \Longrightarrow \ \langle \nabla_y h(x,y), y - x\rangle \geq 0 \ \forall x, y \in C.$$

**Remark 3.1** *The definitions a), b), c) can be found, for example, in [4, 16]. The definitions d) and e), to our best knowledges, are not used before. From the definitions we have*

$$a) \Rightarrow b) \Rightarrow c) \Rightarrow e) \text{ and } a) \Rightarrow b) \Rightarrow d) \Rightarrow e).$$

*However, c) may not imply d) and vice versa as shown by the following simple examples.*

**Example 3.1** *Consider the bifunction $h(x,y) = e^{x^2}(y^2 - x^2)$ defined on $C \times C$ with $C = \mathbb{R}$. This bifunction is not $\nabla$-monotone on $C$, because*

$$\langle \nabla_x h(x,y) + \nabla_y h(x,y), y - x \rangle = 2e^{x^2}(y-x)^2(x^2 + xy + 1)$$

*is negative for $x = -1, y = 3$. However, $h(x,y)$ is strictly pseudo $\nabla$-monotone. Indeed, we have*

$$\langle \nabla_x h(x,y), y - x \rangle = 2xe^{x^2}(y^2 - x^2 - 1)(y - x) \leq 0$$

$$\Leftrightarrow x(y^2 - x^2 - 1)(y - x) \leq 0,$$

$$\langle \nabla_y h(x,y), y - x \rangle = 2ye^{x^2}(y - x) > 0 \Leftrightarrow y(y - x) > 0.$$

*It is not difficult to verify that*

$$x(y^2 - x^2 - 1)(y - x) \leq 0 \Rightarrow y(y - x) > 0 \text{ as } x \neq y.$$

*Hence this function is strictly pseudo $\nabla$-monotone, but it is not $\nabla$-monotone*

*Vice versa, consider the bifunction $h(x,y) = (y-x)^T M(y-x)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$, where $M$ is a matrix of order $n \times n$. We have:*

*i) $h$ is $\nabla$-monotone, because*

$$\langle \nabla_x h(x,y) + \nabla_y h(x,y), y - x \rangle$$

$$= \langle -(y-x)^T(M + M^T) + (y-x)^T(M + M^T), y - x \rangle = 0 \ \forall x, y.$$

*Clearly, $h$ is not strictly $\nabla$-monotone.*

*ii) $h$ is strictly pseudo $\nabla$-monotone. iff*

$$\langle \nabla_x h(x,y), y - x \rangle = -\langle (y-x)^T(M + M^T), y - x \rangle \leq 0$$

*implies*

$$\langle \nabla_y h(x,y), y - x \rangle = (y-x)^T(M + M^T), y - x \rangle > 0 \ \forall x, y, \ x \neq y.$$

*The latter inequality is equivalent to $M + M^T$ is a positive definite matrix of order $n \times n$.*

**Remark 3.2** *As shown in [4] that when $h(x,y) = \langle T(x), y-x\rangle$ with $T$ differentiable monotone operator on $C$, then $h$ is monotone on $C$ if and only if $T$ is monotone on $C$, and in this case monotonicity of $h$ on $C$ coincides with $\nabla$-monotonicity of $h$ on $C$.*

The following example shows that pseudomonotonicity may not imply pseudo $\nabla$-monotonicity.

**Example 3.2** *Let $h(x,y) = -ax(y-x)$, defined on $\mathbb{R}_+ \times \mathbb{R}_+$, $(a > 0)$. It is easy to see that*

$$h(x,y) \geq 0 \implies h(y,x) \leq 0 \ \forall x, y \geq 0.$$

*Thus $h$ is pseudomonotone on $\mathbb{R}_+$*

*We have*

$$\langle \nabla_x h(x,y), y-x\rangle = -a(y-x)(y-2x) < 0 \ \forall y > 2x > 0.$$

*But*

$$\langle \nabla_y h(x,y), y-x\rangle = -ax(y-x) < 0 \ \forall y > 2x > 0.$$

*So $h$ is not pseudo $\nabla$-monotone on $\mathbb{R}_+$.*

From the definition of the gap function $\varphi_\epsilon$, a global minimal point of this function over $C$ is a solution to Problem (PEP). Since $\varphi_\epsilon$ is not convex, its a global minimum is extremely difficult to compute. In [4] the authors shown that under the strict $\nabla$-monotonicity a stationary point is also a global minimum of gap function. By an counter-example, the authors in [4] also pointed out that the strict $\nabla$-monotonicity assumption can not be relaxed to $\nabla$-monotonicity. The following theorem shows that the stationary property is still guaranteed under the strict pseudo $\nabla$-monotonicity.

**Theorem 3.1** *Suppose that $g_\epsilon$ is strictly pseudo $\nabla$-monotone on $C$. If $\bar{x}$ is a stationary point of $\varphi_\epsilon$ over $C$ i.e.*

$$\langle \nabla\varphi_\epsilon(\bar{x}), y-\bar{x}\rangle \ \geq 0 \quad \forall y \ \in \ C.$$

*Then $\bar{x}$ solves (PEP).*

*Proof.* Suppose that $\bar{x}$ does not solve (PEP). Then $y_\epsilon(\bar{x}) \neq \bar{x}$.

Since $\bar{x}$ is a stationary point of $\varphi_\epsilon$ on $C$, from the defition of $\varphi_\epsilon$, we have

$$\langle \nabla\varphi_\epsilon(\bar{x}), y-\bar{x}\rangle = -\langle \nabla_x g_\epsilon(x, y_\epsilon(x)), y_\epsilon(x) - x\rangle \geq 0$$

By strict pseudo $\nabla$-monotonicity of $g_\epsilon$, it follows that

$$\langle \nabla_y g_\epsilon(\bar{x}, y_\epsilon(\bar{x})), y_\epsilon(\bar{x}) - \bar{x}\rangle \ > 0. \tag{1}$$

On the other hand, since $y_\epsilon(\bar{x})$ minimizes $g_\epsilon(x, .)$ over $C$, we have

$$\langle \nabla_y g_\epsilon(\bar{x}, y_\epsilon(\bar{x})), y_\epsilon(\bar{x}) - \bar{x}\rangle \leq 0$$

which is conflicts with (1). □

To computing a stationary point of a differentiable function over a closed convex set, we can use the existing descent direction algorithms in mathematical programming (see, e.g.. [3], [4]). The next proposition shows that if $y(x)$ is a solution of the problem $\min_{y \in C} g_\epsilon(x, y)$, then $y(x) - x$ is a descent direction on $C$ of $\varphi_\epsilon$ at $x$. Namely, we have the following proposition.

**Proposition 3.2** *Suppose that $g_\epsilon$ is strictly pseudo $\nabla$-monotone on $C$ and $x$ is not a solution to Problem (PEP), then*

$$\langle \nabla \varphi_\epsilon(x), y_\epsilon(x) - x\rangle \ < 0.$$

*Proof,* Let $d_\epsilon(x) = y_\epsilon(x) - x$. Since $x$ is not a solution to (PEP) implies $d_\epsilon(x) \neq 0$. Suppose contradiction that $d_\epsilon(x)$ is not a descent direction on $C$ of $\varphi_\epsilon$ at $x$. Then

$$\langle \nabla \varphi_\epsilon(x), y_\epsilon(x) - x\rangle \geq 0 \ \Leftrightarrow -\langle \nabla_x g_\epsilon(x, y_\epsilon(x)), y_\epsilon(x) - x\rangle \geq 0,$$

which, by strict pseudo$\nabla$-monotonicity of $g_\epsilon$, implies

$$\langle \nabla_y g_\epsilon(x, y_\epsilon(x)), y_\epsilon(x) - x\rangle > 0. \tag{1}$$

On the other hand, since $y_\epsilon(x)$ minimizes $g_\epsilon(x, .)$ over $C$, by the well-known optimality condition, we have

$$\langle \nabla_y g_\epsilon(x, y_\epsilon(x)), y_\epsilon(x) - x\rangle \leq 0$$

which contradicts to (1). □

**Proposition 3.3** *Suppose that $g(x, .)$ is strictly convex on $C$ for every $x \in C$ and $g_\epsilon$ is strictly pseudo $\nabla$- monotone on $C$. If $x \in C$ is not a solution of (PEP) then there exists $\bar{\epsilon} > 0$ such that $y_\epsilon(x) - x$ is a descent direction of $\varphi_\epsilon$ on $C$ at $x$ for all $0 < \epsilon \leq \bar{\epsilon}$.*

*Proof,* By contradiction, suppose that the statement of the proposition does not hold. Then there exists $\epsilon_k \searrow 0$ and $x \in C$ such that

$$\langle \nabla \varphi_{\epsilon_k}(x), y_{\epsilon_k}(x) - x \rangle \geq 0.$$

From $y_{\epsilon_k}(x) = \text{argmin}_{y \in C} g_{\epsilon_k}(x, y)$ follows

$$-\langle \nabla_y g_{\epsilon_k}(x, y_{\epsilon_k}(x)), y_{\epsilon_k}(x) - x \rangle \geq 0. \tag{1}$$

Since $g_\epsilon(x, .)$ is strictly convex differentiable on $C$, by Theorem 2.1 in [7], the function $\epsilon \mapsto y_\epsilon(x)$ is continuous with respect to $\epsilon$. Thus $y_{\epsilon_k}(x)$ tends to $y_0(x)$ as $\epsilon_k \to 0$, where $y_0(x) = \text{argmin}_{y \in C} g(x, y)$.
Since $g_{\epsilon_k}(x, y) = g(x, y) + \epsilon_k f(x, y)$ is continuously differentiable, letting $\epsilon_k \to 0$ in (1) we obtain

$$-\langle \nabla_x g(x, y_0(x)), y_0(x) - x \rangle \geq 0.$$

By strict pseudo $\nabla$ -monotonicity of $g_{\epsilon_k}$, it follows

$$\langle \nabla_y g(x, y_0(x)), y_0(x) - x \rangle > 0. \tag{2}$$

On the other hand, since $y_{\epsilon_k}(x)$ minimizes $g_{\epsilon_k}(x, .)$ over $C$, we have

$$\langle \nabla_y g_{\epsilon_k}(x, y_{\epsilon_k}(x)), y_{\epsilon_k}(x) - x \rangle \leq 0.$$

Taking the limit we obtain

$$\langle \nabla_y g(x, y_0(x)), y_0(x) - x \rangle \leq 0,$$

which contradicts to (2).

□

To illustrate Theorem 3.1, let us consider the following examples

**Example 3.3** *Consider the bifunctions* $g(x, y) = e^{x^2}(y^2 - x^2)$ *and* $f(x, y) = 10^{x^2}(y^2 - x^2)$ *defined on* $\mathbb{R} \times \mathbb{R}$. *It is not hard to verify that:*
*i)* $g(x, y), f(x, y)$ *are monotone, strictly pseudo* $\nabla$*-monotone on* $\mathbb{R}$
*ii)* $\forall \epsilon > 0$ *the bifunction* $g(x, y) + \epsilon f(x, y)$ *is monotone and strictly pseudo* $\nabla$*-monotone on* $\mathbb{R}$ *and satisfying all of the assumptions of Theorem 3.1.*

**Example 3.4** *Let* $f(x, y) = -x^2 - xy + 2y^2$ *and* $g(x, y) = -3x^2 y + xy^2 + 2y^3$ *defined on* $\mathbb{R}_+ \times \mathbb{R}_+$ *it is easy to see that:*

*i)* $g, f$ *are pseudomonotone, strictly* $\nabla$*-monotone on* $\mathbb{R}_+$

*ii)* $\forall \epsilon > 0$ *the bifunction* $g(x, y) + \epsilon f(x, y)$ *is pseudomonotone and strictly* $\nabla$*-monotone on* $\mathbb{R}_+$ *and satisfying all of the assumptions of Theorem 3.1*

# 4 Application to the Tikhonov Regularization Method

The Tikhonov method [2] is commonly used for handling ill-posed problems. Recently, in [10] the Tikhonov method has been extended to the pseudomonotone equilibrium problem:

$$\text{Find } x^* \in C \text{ such that } \ g(x^*, y) \geq 0 \ \forall y \in C \qquad EP(C, g)$$

where, as before, $C$ is a closed convex set in $\mathbb{R}^n$ and $g : C \to \mathbb{R}$ is a pseudo monotone bifunction satisfying $g(x, x) = 0$ for every $x \in C$.

In the Tikhonov regularization method considered in [10] , Problem $EP(C, g)$ is regularized by the problems

$$\text{Find } x^* \in C \text{ such that } g_\epsilon(x^*, y) := g(x^*, y) + \epsilon f(x^*, y) \geq 0 \ \forall y \in C, \qquad EP(C, g_\epsilon)$$

where $f$ is an equilibrium bifunction on $C$ and $\epsilon > 0$, which plays as the regularization bifunction and regularization parameter, respectively.

In [10] the following theorem has been proved.

**Theorem 4.1** *Suppose that $f(., y)$, $g(., y)$ are upper semicontinuous and lower semicontinuous convex on $C$ for each $x, y \in C$ and that $g$ is pseudomonotone on $C$. Suppose further that $f$ is strongly monotone on $C$ satisfying the condition*

$$\exists \delta > 0. \ |f(x, y)| \leq \delta \|x - x^g\| \|y - x\| \ \forall x, y \in C, \qquad (4.1)$$

*where $x^g \in C$ is given (plays as a guess-solution).*

*Then the following three statements are equivalent:*

*a) The solution-set of $EP(C, g_\varepsilon)$ is nonempty for each $\varepsilon > 0$ and $\lim_{\varepsilon \to 0^+} x(\varepsilon)$ exists, where $x(\varepsilon)$ is arbitrarily chosen in the solution-set of $EP(C, g_\varepsilon)$.*

*b) The solution-set of $EP(C, g_\varepsilon)$ is nonempty for each $\varepsilon > 0$ and $\lim_{\varepsilon \to 0^+} \sup \|x(\varepsilon)\| < \infty$, where $x(\varepsilon)$ is arbitrarily chosen in the solution-set of $EP(C, g_\varepsilon)$.*

*c) The solution-set of $EP(C, g)$ is nonempty.*

*Moreover, if any one of these statements holds, then $\lim_{\varepsilon \to 0^+} x(\varepsilon)$ is equal to the unique solution of the strongly monotone equilibrium problem $EP(S_g, f)$, where $S_g$ denotes the solution-set of the original problem $EP(C, g)$.*

Note that, when $g$ is monotone on $C$, the regularized subproblems are strongly monotone and therefore they can be solved by some existing methods. When $g$ is pseudomonotone, the subproblems, in general, are no longer strongly monotone, monotone, even not pseudomonotone, solving them becomes a difficult task. However, the problem of finding the limit point of the sequences of iterates leads to the unique solution of Problem $EP(S_g, f)$.

In order to apply the penalty and gap function methods described in the preceding sections, let us take, for instant,

$$f(x,y) = \langle x - x^g, y - x\rangle$$

Clearly, $f$ is both strongly monotone and strongly $\nabla$-monotone with the same modulus 1. Moreover, $f$ satisfies the condition ( 4.1). Therefore, the problem of finding the limit point in the above Tikhonov regularization method can be formulated as the bilevel equilibrium problem

$$\text{Find } x \in S_g \text{ such that } f(x^*, y) \geq 0 \ \forall y \in S_g, \tag{4.2}$$

which is of the form ( 1.1). Now, for each fixed $\epsilon_k > 0$, we consider the penalized equilibrium problem $PEP(C, f_{\epsilon_k})$ defined as

$$\text{Find } \bar{x}_k \in C \text{ such that } f_{\epsilon_k}(\bar{x}_k, y) := g(\bar{x}_k, y) + \epsilon_k f(\bar{x}_k, y) \geq 0 \ \forall y \in C. \tag{4.3}$$

As before, by $\mathrm{SOL}(C, f_{\epsilon_k})$ we denote the solution-set of $PEP(C, f_{\epsilon_k})$.

Applying Theorems 2.2 and Theorem 3.1 we obtain the following result.

**Theorem 4.2** *Suppose that bifunction $g$ satisfies the following conditions*

*i) $g(x, .)$ is convex, lower-semicontinuous $\forall x \in C$.*

*ii) $g$ is pseudomonotone and coercive on $C$.*

*Then for any $\epsilon_k > 0$ the penalized problem $PEP(C, f_{\epsilon_k})$ is solvable and any sequence $\{x_k\}$ with $\{x_k\} \in SOL(C, f_{\epsilon_k})$ converges to the unique solution of the problem ( 4.2) as $k \to \infty$.*

*iii) In addition, if $g(x,y) + \epsilon_k f(x,y)$ is strictly pseudo $\nabla$-monotone on $C$ (in particular, $g(x,y)$ is $\nabla$-monotone), and $\bar{x}_k$ is any stationary point of the mathematical program $\min_{x\in C} \varphi_k(x)$ with*

$$\varphi_k(x) := \min_{y\in C}\{g(x,y) + \epsilon_k f(x,y)\}.$$

*then, $\{\bar{x}_k\}$ converges to the unique solution of the problem ( 4.2) as $k \to \infty$.*

**Conclusion**. We have considered a class of bilevel pseudomonotne equilibrium problems. The main difficulty of this problem is that its feasible domain is not given explicitly as in a standard mathematical programming problem. We have proposed a penalty function method to convert the bilevel problem into one-level ones. Then we have applied the regularized gap function method to solve the penalized equilibrium subproblems. We have generalized the pseudo $\nabla$-monotonicity concept from $\nabla$-monotonicity. Under the pseudo $\nabla$-monotonicity property, we have proved that any stationary point of the gap function is a solution to the original bilevel problem. As an application we have shown how to apply the proposed method to the Tikhonov regularization method for pseudomonotone equilibrium problems.

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