An Outcome Space Algorithm for Minimizing the Product of Two Convex Functions over a Convex Set *

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Abstract

This paper presents an outcome-space outer approximation algorithm for solving the problem of minimizing the product of two convex functions over a compact convex set in $\mathbb{R}^n$. The computational experiences are reported. The proposed algrithm is convergent.

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1 Introduction

We consider the convex multiplicative programming problem

$$\min \ f_1(x)f_2(x) \ \text{ s.t. } \ x \in X, \qquad (CP_X)$$

where $X \subset \mathbb{R}^n$ is a nonempty compact convex set, for each $j = 1, 2$, the function $f_j : \mathbb{R}^n \to \mathbb{R}$ is finite, positive and convex on X.

It is well known that problem (CP_X) is a global optimization problem. Furthermore, problem (CP_X) is known to be NP-hard [11], even in special case when X is a polyhedron and f_1, f_2 are a linear function on $\mathbb{R}^n$ for each $i = 1, 2$. Because of its interesting mathematical aspects as well as its wide range of applications, this problem has attracted the attention of many mathematicians as well as engineers and economists. Many algorithm have been proposed for solving this problem; see, for instance, H.P.Benson and G.M. Boger [1, 2], H.P.Benson [3], Gao, Y., Wu, G. and W. Ma [4] N.T.B.Kim [6, 7], H. Konno and T. Kuno [8, 9], L.D.Muu and B.T.Tam [12] and N.V.Thoai[16] ... and references therein.

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For each $x \in \mathbb{R}^n$, let $f(x) = (f_1(x), f_2(x))^T$ and

$$Y = \{y \in \mathbb{R}^2 \mid y = f(x) = (f_1(x), f_2(x)) \text{ for some } x \in X\}.$$

As usual, the set Y is said to be *outcome set* of X under f. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined for each $y \in \mathbb{R}^2$ by

$$g(y) = y_1 y_2.$$

One of the direct reformulations of problem (CP_X) as an outcome-space problem is given by

$$\min\{g(y) = y_1 y_2 \mid y \in Y\}. \qquad (OP_Y)$$

The relationship between two problems (CP_X) and (OP_Y) is described by the following theorem.

Theorem 1.1. (Theorem 2.2 in [3]) *If y^* is a global optimal solution to problem (OP_Y), then any $x^* \in X$ such that $f(x^*) \leq y^*$ is a global optimal solution to problem (CP_X). Furthermore, the global optimal values of two problems (CP_X) and (OP_Y) are equal, i.e.*

$$g(y^*) = y_1^* y_2^* = f_1(x^*) f_2(x^*).$$

According to Theorem 1.1, Problem (CP_X) problem can be solved by two stages:
i) Finding a global optimal solution y^* to problem (OP_Y);
ii) Finding a feasible solution $x^* \in X$ of problem (CP_X) that satisfies $f(x^*) \leq y^*$. Then x^* is an optimal solution to problem (CP_X). It is easy to see that the point x^* can be obtained by solving a convex programming problem with a linear objective function over the nonempty compact convex feasible set $\{x \in X \mid f(x) \leq y^*\}$.

In this paper, we present an outcome-space outer approximation algorithm for globally solving problem (OP_Y). The proposed algorithm is established basing on the link between the global solution to the problem (OP_Y) and the efficient outcome set of the outcome set Y. Since the number of variables n, in practice, often much larger than 2, we expect potentially that considerable computational savings could be obtained.

The paper is organized as follows. In Section 2, theoretical prerequisites for the algorithm are given. The algorithm is presented in Section 3. Computational experiments are reported in Section 4.

2 Theoretical Prerequisites

Let $\mathbb{R}^2_+ = \{y = (y_1, y_2) \in \mathbb{R}^2 | y_1 \geq 0, y_2 \geq 0\}$ and $Q \subseteq \mathbb{R}^2$. We denote the interior of Q by $\text{int}Q$, the boundary of Q by ∂Q, the relative interior of Q by $\text{ri}Q$, the closure of Q by $\overline{Q}$ and the convex hull of B by $\text{conv}Q$.

Let $a, b \in \mathbb{R}^2$. We use the notation $a \geq b$ to indicate $a \in b + \mathbb{R}^2_+$, i.e. $a_i \geq b_i$ for all $i = 1, 2$. The notation $a \gg b$ indicates $a \in b + \text{int}R^2_+$, i.e. $a_i > b_i$ for all $i = 1, 2$.

For a given nonempty set $Q \subset \mathbb{R}^2$, a point q^* is said to be an *efficient points* of Q if there is no $q \in Q$ satisfying $q^* \geq q$ and $q^* \neq q$, i.e. $Q \cap (q^* - \mathbb{R}^2_+) = \{q^*\}$. Similarly, a point $q^* \in Q$ is a *weakly efficient point* of Q if there is no $q \in Q$ satisfying $q^* \gg q$, i.e. $Q \cap (q^* - \text{int}\mathbb{R}^2_+) = \emptyset$. We denote by Q_E and Q_{WE} the set of all efficient points and the set of all weakly efficient points of Q respectively. By definition, $Q_E \subseteq Q_{WE}$.

Now, define the set G by

$$G := Y + \mathbb{R}^2_+ = \{y \in \mathbb{R}^2 | y \geq f(x) \gg 0 \;\text{ for some }\; x \in X\}.$$

By definition, it is easy to show that $G \subset \text{int}\mathbb{R}^2_+$ is a nonempty, full-dimension closed convex set in the outcome space $\mathbb{R}^2$ of problem (CP_X). Furthermore, from Theorem 3.2 in Yu (pg. 22 in [17]), we know the following fact that is very useful in the sequel.

Proposition 2.1. $Y_E = G_E$.

From the definition of an efficient point and the assumption that the convex functions f_1, f_2 are positive on X, it is easily seen the relationship between the global optimal solution to the problem (OP_Y) and the efficient set of the outcome set Y. Namely, we have

Proposition 2.2. *Any global optimal solution to problem (OP_Y) must belong to the efficient set Y_E.*

We invoke Proposition 2.1 and Proposition 2.2 to deduce that problem (OP_Y) is equivalent to the following problem

$$\min g(y) = y_1 y_2 \qquad \text{s.t.} \quad y \in G_E. \tag{OP_{G_E}}$$

Therefore, to globally solve problem (OP_Y), we instead globally solve problem (OP_{G_E}).

In the next section, basing on the structure of the efficient set G_E of the convex $G \subset \text{int}\mathbb{R}^2_+$, the outer approximation algorithm is developed for solving the problem (OP_{G_E}).

In the remainder of this section, we present some more particular results that will be needed to develop the outer approximation algorithm. Toward this end, for each $i = 1, 2$, let

$$\alpha_i = \min\{y_i \ : \ y \in G\}.$$

It can easily be seen that α_i is also the optimal value of the convex programming problem $\min\{f_i(x) \ : \ x \in X\}$ for $i = 1, 2$ and the problem

$$\min\{y_2 \ : \ y \in G, \ y_1 = \alpha_1\} \tag{P_{12}}$$

has an unique optimal solution $\hat{y}^1$ and the problem

$$\min\{y_1 \ : \ y \in G, \ y_2 = \alpha_2\} \tag{P_{21}}$$

has an unique optimal solution $\hat{y}^2$. Furthermore, we have $\hat{y}^1, \hat{y}^2 \in G_E$.

Since $G \subset \mathbb{R}^2$ is the closed convex set, it is well known [13] that the efficient set G_E is homeomorphic to a nonempty closed interval of $\mathbb{R}^1$. It is clear that if $\hat{y}^1 \equiv \hat{y}^2$ then $G_E = \{\hat{y}^1\}$ and $\hat{y}^1$ is an unique optimal solution to problem (OP_{G_E}). Therefore we assume henceforth that $\hat{y}^1 \neq \hat{y}^2$. Then it is clear that the efficient set $G_E \subset \partial G$ is a curve with starting point $\hat{y}^1$ and end point $\hat{y}^2$ (see Figure 1(a)).

Let $y^0 = (y^0_1, y^0_2) = (\alpha_1, \alpha_2)$. Then, the set

$$S^0 = \text{conv}\{y^0, \hat{y}^1, \hat{y}^2\} \tag{1}$$

is a $2-$simplex and S^0 contains G_E (see Figure 1(b)). The simplex S^0 is an initial simplex in the our approximation algorithm for problem (OP_{G_E}).

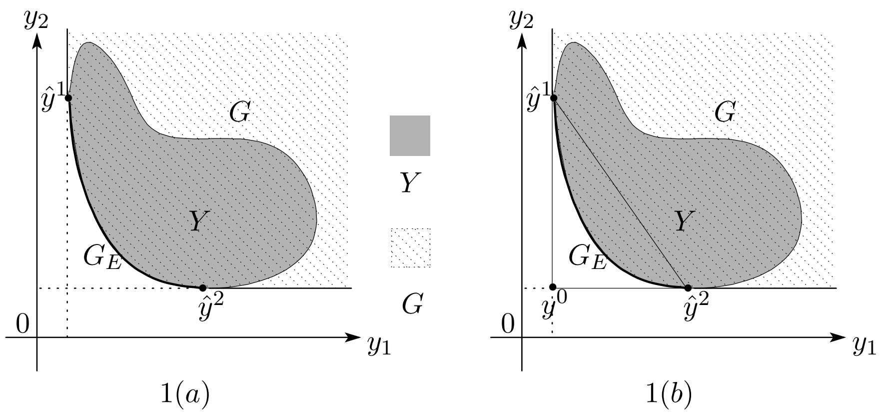


Figure 1.

Let y^k and y^ℓ be any points in G_E that satisfy

$$y_1^k < y_1^\ell \quad \text{and} \quad y_2^\ell < y_2^k. \tag{2}$$

Direct computation shows that the equation of the line through y^k and y^ℓ is $\langle d^{k\ell}, y\rangle = \alpha_{k\ell}$, where

$$\alpha_{k\ell} = \frac{y_1^k}{y_1^\ell - y_1^k} + \frac{y_2^k}{y_2^k - y_2^\ell}.$$

and the normal vector $d^{k\ell}$ defined by

$$d^{k\ell} = \Big(\frac{1}{y_1^\ell - y_1^k}, \frac{1}{y_2^k - y_2^\ell}\Big). \tag{3}$$

See an illustration in Figure 2.

Remark 2.1. Combining (2) and (3) yields the normal vector $d^{k\ell}$ is strictly positive, i.e. $d^{k\ell} \gg 0$.

Consider the convex programming problem with linear objective function

$$\min \ \langle d^{k\ell}, y\rangle \ \text{ s.t. } \ y \in G. \tag{$CP_{k,\ell}$}$$

Let $F^{opt} = \text{Argmin}\{\langle d^{k\ell}, y\rangle \mid y \in G\}$ be the set of all optimal solutions to problem $(CP_{k,\ell})$. It is plain that $F^{opt} \subset G$ is a compact convex set.

The following theorem will play a crucial role in the algorithm for solving the problem (OP_{G_E}).

Theorem 2.1. *Let y^k and y^ℓ be any points in G_E that satisfy (2). Let y^* be an optimal solution to problem $(CP_{k,\ell})$. If $\langle d^{k\ell}, y^*\rangle = \langle d^{k\ell}, y^k\rangle$ then $[y^k, y^\ell]$ is an efficient line segment of G_E, otherwise $y^* \in G_E$ satisfying*

$$y_1^k < y_1^* < y_1^\ell \ \ \textit{and} \ y_2^\ell < y_2^* < y_2^k. \tag{4}$$

Proof. According to Theorem 2.10 (Chapter 4 in [10]), a point $y^* \in G$ is a efficient point of G if and only if there is a vector $v \in \mathbb{R}^p$ and $v \gg 0$ such that y^* is an optimal solution to the convex programming problem

$$\min\{\langle v, y\rangle \mid y \in G\}.$$

Combining this fact, the theory of linear programming and Remark 2.1 gives F^{opt} is an efficient face of G. That means F^{opt} is a face of G and $F^{opt} \subset G_E$.

If $\langle d^{k\ell}, y^* \rangle = \langle d^{k\ell}, y^k \rangle$ then we have y^k and y^ℓ belong to the convex set $F^{opt} \subset G_E$. Hence $[y^k, y^\ell]$ is an efficient line segment of G_E. Now, suppose that $\langle d^{k\ell}, y^* \rangle < \langle d^{k\ell}, y^k \rangle$. Then, by structure of the efficient set G_E, the point y^* has to belong to the curve $\Gamma \subseteq G_E$ and the two points y^k and y^ℓ are connected by Γ (see Figure 2). By the definition of the efficient point, we obtain the assertion (4). ■

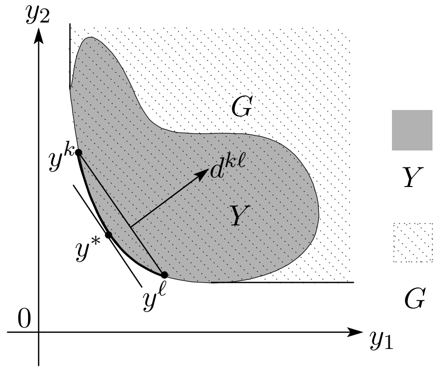


Figure 2.

Remark 2.2. Let y^* be an optimal solution of problem $(CP_{k,\ell})$. Then

$$H = \{y \in \mathbb{R}^2 : \langle d^{k\ell}, y \rangle = \langle d^{k\ell}, y^* \rangle\},$$

is a supporting hyperplane of G at y^*, i.e.

$$y^* \in H \text{ and } \langle d^{k\ell}, y \rangle \geq \langle d^{k\ell}, y^* \rangle \ \forall y \in G.$$

The outer approximation algorithm for problem (OP_{G_E}) will also make use of the alternate representation of the simplex S^0 given in the following corollary.

Corollary 2.1. *The simplex* S^0 *defined in* (1) *may also be written*

$$S^0 = \{y \in \mathbb{R}^2 \mid \langle d^{12}, y \rangle \leq \alpha_{12}, y \geq y^0\}. \tag{5}$$

Proof. It is clear that the points $y^1 = \hat{y}^1$ and $y^2 = \hat{y}^2$ satisfy (1), where $k = 1$ and $\ell = 2$. The corollary is immediate from Theorem 2.1 and the definition of the simplex S^0. ■

3 Outer Approximation Algorithm

Starting with simplex $S^0 = \text{conv}\{y^0, \hat{y}^1, \hat{y}^2\}$, where $\hat{y}^1, \hat{y}^2 \in G_E$ and the inequality representation of S^0 given in Corollary 2.1, the algorithm will iteratively generate of polytope S^h, $h = 0, 1, 2, \ldots$, such that

$$S^0 \supset S^1 \supset S^2 \supset \cdots \supset G_E.$$

We denote the set of known efficient extreme points by V_{eff}, the number of the set V_{eff} by $|V_{eff}|$ and the vertex set of S^h by $V(S^h)$.

At a typical iteration h, we have a polytope S^h with its vertex set $V(S^h)$, the inequality representation of S^h, the set $V_{eff} = \{y^1, y^2, \ldots, y^{N_e}\}$ with $N_e = |V_{eff}| = 2 + h$, where $y^1 = \hat{y}^1$ and $y^{N_e} = \hat{y}^2$. Find an optimal solution v^h for the problem

$$\min \ g(y) \text{ s.t. } \ y \in S^h. \tag{$P_S(h)$}$$

If a stopping condition is met then stop, else find the index $i \in \{1, 2, \ldots, N_e\}$ such that

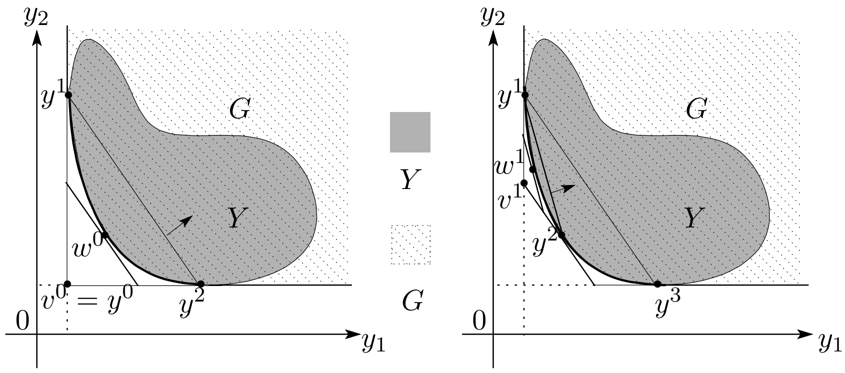
$$y_1^i \leq v_1^h < y_1^{i+1}. \tag{6}$$

Set $k = i$ and $\ell = i + 1$. Find an optimal solution y^* for problem $(CP_{k,\ell})$ and construct the polytope

$$S^{h+1} = \{y \in S^h \ : \ell_h(y) = \langle d^{k\ell}, y\rangle - \langle d^{k\ell}, y^*\rangle \geq 0\}.$$

Determine the vertex set $V(S^{h+1})$ and renumber the element of V_{eff}. The set $V(S^{h+1})$ can be calculated by using some existing method (see, for example [5], [15]).

It is clear that $y^* \in G_E$ and y^*, y^k, y^ℓ satisfy (4). Figure 3 illustrates Iteration 0 and Iteration 1 of the algorithm.



3(a) 3(b)

Figure 3.

Note that the objective function $g(y) = y_1y_2$ of problem $(P_S(h))$ is quasiconcave on S^h [1] and attains its minimum at an extreme point of S^h . Therefore, instead of solving problem $(P_S(h))$, we solve the simpler problem

$$\min \ g(y) \text{ s.t. } \ y \in V(S^h). \tag{$P_{VS}(h)$}$$

Let β_h be an optimal value of problem $(P_S(h))$ and we have β_h is a lower bound of problem (OP_{G_E}), i.e $\beta_h \leq g_{opt}$, where g_{opt} is the optimal value of problem (OP_{G_E}). By constructing, $\{\beta_h\}$ is a nondecreasing sequence i.e, $\beta_{h+1} \geq \beta_h$ for all $h = 0, 1, 2, ...$

Let ε be a given sufficient small real number. Let $y^e \in G_E$. Then $g(y^e)$ is an upper bound of problem (OP_{G_E}). The point y^e is said to be *an ε-optimal solution for the problem* (OP_{G_E}) if there is a lower bound $\bar{\beta}$ such that $g(y^e) - \bar{\beta} < \varepsilon$.

Below, we will present an algorithm for finding ε-optimal solution to problem (OP_{G_E}).

Outer Approximation Algorithm

Initialization Step. See Steps 1 through 5 below

Step 1. Determine two efficient points $\hat{y}^1$ and $\hat{y}^2$ by solving problem (P_{12}) and (P_{21}), respectively.

Step 2. **If** $\hat{y}^1 \equiv \hat{y}^2$ **Then** STOP
($\hat{y}^1$ *is an unique optimal solution to Problem* (OP_{G_E}));

Step 3. Start with the simplex S^0 defined by (5) and S^0 has the vertex set $V(S^0) = \{y^0, \hat{y}^1, \hat{y}^2\}$ (see (1)). Let $y^1 = \hat{y}^1$ and $y^2 = \hat{y}^2$

Step 4. Let $V_{eff} = \{y^1, y^2\}$; (*The set of known efficient extreme points*)
$N_e = 2$; (*The element number of the set* V_{eff})
$h = 0$;

Step 5. **If** $g(\hat{y}^1) < g(\hat{y}^2)$ **Then** $\theta_0 = g(\hat{y}^1)$ and $y^{best} = y^1$
Else $\theta_0 = g(\hat{y}^2)$ and $y^{best} = y^2$
(θ_0 - *current best upper bound and* y^{best} - *current best feasible point*)

Iteration h, $h = 0, 1, 2\dots$ See Steps $h1$ through $h6$ below

Step h1. Find the optimal solution v^h for the problem

$$\min\{g(y) \;:\; y \in S^h\} = \min\{g(y) \;:\; y \in V(S^h)\}$$

and set $\beta_h = g(v^h)$ (*the current lower bound*);

If $v^h \in G$ **Then** STOP
($y^{opt} = v^h$ *is an optimal solution to problem* (OP_{G_E})).

If $\theta_h - \beta_h \le \varepsilon$ **Then** STOP (y^{best} *is an* ε*-optimal solution*)
$UB = \theta_h$, $LB = \beta_h$.

Else Let $i := 1$;

Step h2. (*Determine an index* i *satisfying* (6))
Step $h2_1$ **If** $v_1^h < y_1^{i+1}$ **Then** Go to Step $h3$
Step $h2_2$ Let $i := i + 1$; Go to Step $h2_1$;

Step h3. Solve the convex programming Problem $(CP_{k\ell})$ with $k = i$ and $\ell = i + 1$ receiving the optimal solution $w^h \in G_E$;
If $\langle d^{k\ell}, w^h\rangle = \langle d^{k\ell}, y^k\rangle$ **Then** Go to Step $h6$

Step h4.
If $g(w^h) < \theta_h$ **Then** $\theta_{h+1} = g(w^h)$ (*current best upper bound*)
and $y^{best} = w^h$ (*current best feasible point*)

Step h5. (*add* $y^* = w^h$ *to list of known efficient points*)
$u^j = y^j, \quad \forall\, j = 1, \cdots, i;$
$u^{j+1} = y^*;$
$u^{j+2} = y^{j+1}, \quad \forall\, j = i + 1, \dots, N_{eff} - 1;$
$N_{eff} = N_{eff} + 1;$
$V_{eff} = \{y^i = u^i, \quad \forall i = 1, \cdots, N_{eff}\}.$

Step h6.
$S^{h+1} = \{y \in S^h \;: \ell_h(y) = \langle d^{k\ell}, y\rangle - \langle d^{k\ell}, y^*\rangle \ge 0\}.$
$\theta_{h+1} = \theta_h$ (*the upper bound is not improved*)
Determine the vertex set $V(S^{h+1})$, let $h = h + 1$ and go to Iteration h

Theorem 3.1. *The algorithm terminates after finitely many steps and yields an* ε*-optimal solution to problem* (OP_{G_E})*. When* $\varepsilon = 0$*, the sequence* $\{v^h\}$ *has a cluster point that solve problem* (OP_{G_E}) *globally.*

Proof. The algorithm starts from the 2-simplex S^0. The algorithm generates the sequence $\{v^h\}$ belonging to the compact $\overline{(S^0 \setminus G)}$ and the sequence $\{w^t\}$ belonging the compact set G_E. By taking subsequence if necessary we may assume that $v^h \to v^*$ and $w^h \to w^*$. Since the lower bound sequence $\{\beta_h = g(v^h)\}$ and the upper bound sequence $\{\theta_h\}$ are monotone and bounded, we obtain in the limit that

$$\lim_h \beta_h \le g_{opt} \text{ and } \lim_h \theta_h \ge g_{opt}. \tag{6}$$

In every iteration h, the algorithm constructs the polytope $S^{h+1} = \{y \in S^h \ : \ell_h(y) = \langle d^{k\ell}, y\rangle - \langle d^{k\ell}, y^*\rangle \geq 0\}$, where $H_h = \{y \in \mathbb{R}^2 | \ell_h(y) = \langle d^{k\ell}, y\rangle - \langle d^{k\ell}, y^*\rangle = 0\}$ is the supporting hyperplane of G at $y^* \in G_E$. Since $d^{k\ell}$ is strictly positive vector, from [10] we have the set $F = \{x \in S^{h+1} \cap H_h\}$ is the efficient face of set S^{h+1}. Furthermore, by geometric structure of the set G and Theorem 11.5 in [14], we have

$$\lim_h S^h = G,$$

which implies $v^* \in G_E$. Combining this fact and (6) we deduce that v^* is a global optimal solution to problem (OP_{G_E}) and if the algorithm terminates at iteration k the y^{best} is a ε-optimal solution to problem (OP_{G_E}). ■

4 Computational Experiments

We begin with the example given by Benson in [3] to illustrate the outer approximation algorithm. Let $\varepsilon = 0.005$. Consider (CP_X) with $n = 2$, where

$$f_1(x_1, x_2) = (x_1 - 2)^2 + 1,$$
$$f_2(x_1, x_2) = (x_2 - 4)^2 + 1,$$

and X is defined by the constraints

$$g_1(x_1, x_2) = 25x_1^2 + 4x_2^2 - 100 \leq 100,$$
$$g_2(x_1, x_2) = x_1 + 2x_2 - 4 \leq 0$$

and we would like to find an ε-optimal solution.

Initialization Solving two convex problems (P_{12}) and (P_{21}) we obtain two efficient points $y^1 = (1; 16.6028)$ and $y^2 = (5; 5)$

Take $y^0 = (1; 5)$ and the starting simplex $S^0 = \text{conv}\{y^0, y^1, y^2\}$ has the vertex set $V(S^0) = \{y^0, y^1, y^2\}$.

Let $V_{eff} = \{y^1, y^2\}$ and $N_{eff} = 2$.

Since $g(y^1) = 16.6028 < g(y^2) = 25$ we take $\theta_0 = 16.6028$ and $y^{best} = y^1$.

Iteration $h = 0$: Solve $\min\{g(y) : y \in S^0\} = \min\{g(y) : y \in V(S^0)\}$ obtaining an optimal solution $v^0 = y^0$ and the optimal value $\beta_0 = g(v^0) = 5$.

Solve the convex programming Problem (CP_{12}) receiving an optimal solution $= (1.226657; 8.628408) \in G_E$.

Since $g(y^*) = 10.5841 < \theta_0$ then $\theta_1 = 10.5841$ and $y^{best} = \omega^0$.

Update the set of efficient points (sorted by their first coordinate) $V_{eff} = \{y^1 = (1; 16.6028), y^2 = (1.226657; 8.628408), y^3 = (5; 5)\}$; $N_{eff} = 3$.

Update the simplex $S^1 = \{y \in S^0 \mid \langle d^{12}, y\rangle \geq \langle d^{12}, \omega^0\rangle\}$ and its vertex set $V(S^1) = \{\bar{v}^1 = y^1, \bar{v}^2 = y^3, \bar{v}^3, \bar{v}^4\}$ where $\bar{v}^3 = (1; 9.285872), \bar{v}^4 = (2.477533; 5)$.

Iteration $h = 1$: Solve $\min\{g(y) : y \in S^1\} = \min\{g(y) : y \in V(S^1)\}$ obtaining an optimal solution $v^1 = \bar{v}^3$ and the optimal value $\beta_2 = g(v^1) = 9.285872$.

Solve the convex programming Problem (CP_{12}) receiving the optimal solution $\omega^1 = (1.001796; 9.873328) \in G_E$.

Since $g(\omega^1) = 9.891055 < \theta_1$ then $\theta_2 = 9.891055$ and $y^{best} = \omega^1$.

Update the set of efficient points (sorted by their first coordinate) $V_{eff} = \{y^1 = (1; 16.6028), y^2 = (1.001796; 9.873328),$
$y^3 = (1.226657; 8.628408), y^4 = (5; 5)\}$; $N_{eff} = 4$.

Update the simplex $S^2 = \{y \in S^1 \mid \langle d^{12}, y\rangle \geq \langle d^{12}, \omega^1\rangle\}$ and its vertex set $V^2 = \{\bar{v}^1 = y^1, \bar{v}^2 = y^4, \bar{v}^3, \bar{v}^4, \bar{v}^5\}$ where $\bar{v}^3 = (1; 9.936948), \bar{v}^4 = (1.020155; 9.227409)$ and $\bar{v}^5 = (2.477533; 5)$.

The algorithm terminates in step 5 with global ϵ-optimal solutions given by $y^{bets} = \omega^3 = (1.022741; 9.553280)$ and corresponding $x^* = (1.849198; 1.075401)$. One should notice that, the optimal solution is found after three iterations but we can not stop the algorithm when the LB is not good enough. In addition, the terminal step shows that $9.761987 \leq \alpha \leq 9.770533$ where α is the optimal value of Problem (CP_X) and we can see that the $gap = \dfrac{UB - LB}{UB}$ is smaller than 0.005.

The upper bound obtained by our algorithm, 9.770533 while the one obtained by Benson is 9.7919887 (see [3]).

A set of randomly generated problems was used to test the above algorithm. The test was perform on a laptop MacBook 2Ghz, RAM 2GB , using codes written in C++. Convex Problems $(CP_{k,\ell})$ are solved by using an efficient solver named IPOpt. The test problems are of following two types.

Type I

$$\min \quad \prod_{i=1}^{2}(\alpha^i x + x^T D^i x)$$

subject to:

$$Ax \leq b$$

$$x \geq 0$$

Type II

$$\min \quad \prod_{i=1}^{2}(\alpha^i x + x^T D^i x)$$

subject to:

$$\left(-2 + \sum_{j=1}^{n} \frac{x_j}{j}\right)^2 \leq 100$$

$$Ax \leq b$$

$$x \geq 0$$

The parameters are defined as follows (as in [6-7] and [14]):

- α^1, α^2 are randomly generated vectors with all components belonging to $[0, 1]$.
- $A = (a_{ij}) \in \mathbb{R}^{m \times n}$ is a randomly generated matrix with elements belonging to $[-1, 1]$.

- $b = (b_1, b_2, \dots, b_m)^T$ is randomly generated vector such that

$$b_i = \sum_{j=1}^{n} a_{ij} + 2b_0$$

with b_0 being a randomly generated real in $[0, 1]$

- $D^i \in \mathbb{R}^{n\times n}$ are diagonal matrices with diagonal elements d^i_j randomly generated in $[0, 1]$.

In all problems we find an ϵ-solution where $\epsilon = 0.005$.

In the tables, UB is upper bound, LB is lower bound, gap is defined as $\frac{\text{UB - LB}}{\text{UB}}$, $\#EP$ and $\#VT$ abbreviate for number of efficient points and number of vertices of the working simplex when the algorithm terminates respectively.

n	m	#Iter	UB	LB	Gap	times (in sec.)	#EP	#VT
60	40	4	88.374	88.101	0.0031	0.78	6	7
70	50	5	510.748	510.076	0.0013	1.03	7	8
80	80	4	823.242	821.811	0.0017	2.39	6	7
100	60	4	613.612	612.354	0.0021	1.53	6	7
100	80	4	512.416	511.025	0.0027	2.70	6	7
120	120	2	3518.885	3502.753	0.0046	4.64	4	5
150	100	5	2644.190	2638.212	0.0023	7.19	7	8
150	120	4	2275.959	2270.938	0.0022	7.83	6	7

Table 1: Problems type I

n	m	#Iter	UB	LB	Gap	times (in sec.)	#EP	#VT
60	40	4	298.808	297.368	0.0048	0.94	6	7
70	50	4	344.666	343.887	0.0023	1.24	6	7
80	80	4	1321.174	1319.012	0.0016	1.98	6	7
100	60	5	353.727	353.441	0.0008	2.59	7	8
100	80	5	441.117	440.809	0.0007	4.64	7	8
120	120	4	2404.394	2400.376	0.0015	6.97	6	7
150	100	4	2043.996	2038.233	0.0028	7.73	6	7
150	120	4	1980.837	1977.071	0.0019	10.73	6	7

Table 2: Problems type II

From the tables we can see that in all cases, even in large scale setting, our algorithm works well. The computation times is small since the algorithm terminates after few iterations. Moreover, the quality of final solution obtained is much smaller than 0.005.

The problems type II are somehow more difficult than the ones type I. However we can see that the differences in performance are very little.

5 Conclusion

In the paper we present a global algorithm for solving the convex multiplicative programming problem. The algorithm bases on the outer approximation approach after moving from original space to the outcome space. Expected considerable saving computation was shown by numerical experiments.

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