On the set-semidefinite representation of nonconvex quadratic programs over arbitrary feasible sets

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Abstract

In the paper we prove that any nonconvex quadratic problem over some set $K \subseteq \mathbb{R}^n$ with additional linear and binary constraints can be rewritten as linear problem over the cone, dual to the cone of K-semidefinite matrices.

We show that when K is defined by one quadratic constraint or by one concave quadratic constraint and one linear inequality, then the resulting K-semidefinite problem is actually a semidefinite programming problem. This generalizes a results obtained by Sturm and Zhang ([J.F. Sturm and S. Zhang, On cones of nonnegative quadratic functions. Math. Oper. Res. 28 (2003)]), since we can handle problems with many linear and binary constraints.

Our result also generalizes the well-known completely positive representation result from Burer ([S. Burer, On the copositive representation of binary and continuous nonconvex quadratic programs. Math. Program. 120 (Ser. A) (2009), pp. 479-495]), which is actually a special instance of our result with $K = \mathbb{R}^n_+$.

Key Words: set-positivity, semidefinite programming, copositive programming, mixed integer programming.

Mathematics subject classifications (MSC 2000): 90C11, 90C20, 90C22.

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1 Introduction

In [2, 14, 16, 17] several hard problems from combinatorial optimization have been reformulated as linear programs over the cone of copositive or completely positive matrices. In [5], Burer generalized these results as follows: under rather weak assumptions any nonconvex quadratic problem over the nonnegative orthant with some additional linear and binary constraints can be rewritten as linear problem over the cone of completely positive matrices.

The main contribution of this paper consists of the following result: optimization problems with a nonconvex quadratic objective function where the feasible set consists of all vectors from a given set K which satisfy given linear and binary constraints have a conic linear programming formulation, i.e. the optimal value of such problems is equal to the optimal value of a liner function over the domain of symmetric matrices which satisfy a bunch of linear constraints and are contained in the cone, dual to the K-semidefinite cone. We also explain the relations between the original feasible set and the feasible set of the conic linear problem.

If K is the non-negative orthant $\mathbb{R}^n_+$, than our result states essentially the same as the well-known Burer's completely positive representation result [5].

Our result also captures and generalizes the quadratic cases from Sturm and Zhang [18]: if K is defined by a single quadratic constraint or by one concave quadratic constraint and by one linear inequality, then the resulting conic linear program is actually a semidefinite programming problem, a result that cannot be obtained straightforwardly from the approach in [18] because we allow in the original problem linear constraints and binary constraints.

Since the set K can be arbitrary set we can handle also sets K defined by more than one quadratic constraint. Unfortunately, in these cases the resulting conic linear problem is usually no longer a semidefinite programming problem.

1.1 Notation

In [5] the reformulation is done over the cone of *completely positive* matrices

$$C^*_{\mathbb{R}^n_+} := \left\{ \sum_i x^i (x^i)^\top : \ x^i \in \mathbb{R}^n_+ \right\} \tag{1}$$

which is the dual cone of the cone of *copositive* matrices defined by

$$C_{\mathbb{R}^n_+} := \left\{ A \in \mathscr{S}^n : x^\top A x \geq 0 \ \text{ for all } \ x \in \mathbb{R}^n_+ \right\}. \tag{2}$$

Here, $\mathscr{S}^n$ denotes the space of real symmetric $n \times n$ matrices equipped with the inner product defined by $\langle A, B \rangle := \text{trace}(AB)$ for all $A, B \in \mathscr{S}^n$. Recall that the dual cone of a cone C in a topological space X is in general defined by

$$C^* := \{x^* \in X^* : x^*(x) \geq 0 \text{ for all } x \in C\}$$

with X^* denoting the topological dual space, i.e. the space of all continuous linear maps from X to $\mathbb{R}$.

Replacing $\mathbb{R}^n_+$ in (1) and (2) by an arbitrary nonempty set $K \subseteq \mathbb{R}^n$ we obtain

$$C_K := \{A \in \mathscr{S}^n : x^\top A x \geq 0 \text{ for all } x \in K\}$$

which is called *K-semidefinite* (or *set-semidefinite*) cone. In opposition to [10, 11] we define here the K-semidefinite cone in the subspace of symmetric matrices instead of in the whole space of linear maps mapping from the Euclidean space $\mathbb{R}^n$ to $\mathbb{R}^n$. The K-semidefinite cone is a convex cone and hence defines a partial ordering in the space of symmetric matrices.

Remark 1.1. *If $K = \mathbb{R}^n$ then C_K and C_K^* are exactly the cone of positive semidefinite matrices denoted by $\mathscr{S}_n^+$.*

In this paper, $\operatorname{cone}(\Omega)$ for some set Ω denotes the cone generated by the set, $\operatorname{conv}(\Omega)$ is the convex hull and $\operatorname{cone}\operatorname{conv}(\Omega)$ denotes the convex cone generated by the set Ω, i.e. $\operatorname{cone}\operatorname{conv}(\Omega) = \{\sum_i \alpha_i x^i : \alpha_i \geq 0,\ x^i \in \Omega\}$, and $\operatorname{cl}(\Omega)$ is the closure of the set Ω. Further, we assume K to be a nonempty subset of $\mathbb{R}^n$.

1.2 Technical preliminaries

Under the assumptions here, i.e. $K \subseteq \mathbb{R}^n$, the dual cone of the K-semidefinite cone was given in [18, Prop. 1, Lemma 1]:

Lemma 1.2. *Let $K \subseteq \mathbb{R}^n$ be a nonempty set, then*

$$C_K^* = \operatorname{cl}\operatorname{cone}\operatorname{conv}\ \{xx^\top : x \in K\}.$$

Since C_K^* is the closure of a convex cone generated by $\{xx^\top : x \in K\}$, using Carathéodory theorem we can represent the dual cone by

$$C_K^* = \operatorname{cl}\left(\left\{\sum_{i=1}^{n(n+1)/2} \alpha_i x^i (x^i)^\top \ :\ \alpha_i \geq 0,\ x^i \in K,\ \forall i = 1,\dots,\frac{n(n+1)}{2}\right\}\right).$$

For shortness of the representation we omit the upper limit $p := n(n+1)/2$ in the sum above and write instead in the following $C_K^* = \operatorname{cl}(\{\sum_i \alpha_i x^i (x^i)^\top \ :\ \alpha_i \geq 0,\ x^i \in K\})$. We would like to add, that in [18, Lemma 1] it was shown that

$$\operatorname{cl}\operatorname{cone}\operatorname{conv}\ \{xx^\top : x \in K\} = \operatorname{cone}\operatorname{conv}\ \{xx^\top : x \in \operatorname{cl}(K)\}, \tag{3}$$

for an arbitrary set $K \subseteq \mathbb{R}^n$. We have realized that there is a mistake in the proof of this lemma which is based on the fact that $\operatorname{cl}\operatorname{cone}(K) \neq \operatorname{cone}\operatorname{cl}(K)$ for some set K (e.g. for $K = \{1\} \times [0,\infty)$), see the next example.

Example 1.3. *Consider the closed set $K = \{1\} \times [0,\infty)$ and let $x > 0$ be given. Then*

$$\begin{aligned} Y^* := \begin{pmatrix} 0 & 0 \\ 0 & x^2 \end{pmatrix} \ &\in\ \operatorname{cl}\left(\left\{\sum_i \alpha_i \begin{pmatrix} 1 \\ x^i \end{pmatrix}\begin{pmatrix} 1 \\ x^i \end{pmatrix}^\top \ :\ \alpha_i \geq 0,\ x^i \geq 0\right\}\right) \\ &=\ \operatorname{cl}\operatorname{cone}\operatorname{conv}\ \{yy^\top : y \in K\} \end{aligned}$$

as for all $k \in \mathbb{N}$,

$$Y_k := \frac{1}{k^2} \begin{pmatrix} 1 \\ kx \end{pmatrix} \begin{pmatrix} 1 \\ kx \end{pmatrix}^\top \in \text{cone conv } \{yy^\top : y \in K\}$$

and $\lim_{k\to\infty} Y_k = Y^*$, *but*

$$\begin{aligned} Y^* &\notin \left\{ \sum_i \alpha_i \begin{pmatrix} 1 \\ x^i \end{pmatrix} \begin{pmatrix} 1 \\ x^i \end{pmatrix}^\top : \alpha_i \geq 0,\, x^i \geq 0 \right\} \\ &= \text{cone conv } \{yy^\top : y \in \text{cl}\, K\} = \text{cone conv } \{yy^\top : y \in K\}. \end{aligned}$$

So when $K \subseteq \mathbb{R}^n$ is an arbitrary set then result (3) is not necessarily true. However, if K is a cone, we retrieve the result gained in [18, Lemma 1].

Lemma 1.4. *Let* $K \subseteq \mathbb{R}^n$ *be a nonempty set, then*

$$C_K^* = \text{cl cone conv } \{xx^\top : x \in K\} = \text{conv } \{xx^\top : x \in \text{cl}\,\text{cone}\, K\}.$$

Proof. First, suppose $Z \in \text{cl cone conv } \{xx^\top : x \in K\}$. Then $Z = \lim_{k\to\infty} Z_k$ with

$$Z_k \in \text{cone conv } \{xx^\top : x \in K\}$$

and

$$Z_k = \sum_i \alpha_{k,i}^2 x^{k,i} (x^{k,i})^\top$$

for nonnegative scalars $\alpha_{k,i}$ and vectors $x^{k,i} \in K$ for all $i \leq n(n+1)/2$. By defining Y_k to be the matrix with columns $\alpha_{k,i} x^{k,i}$, we can write $Z_k = Y_k Y_k^\top$. We have

$$\lim_{k\to\infty} \|Y_k\|^2 = \lim_{k\to\infty} \text{trace}(Y_k Y_k^\top) = \lim_{k\to\infty} \text{trace}(Z_k) = \text{trace}(Z)$$

(with the norm denoting the Frobenius norm). Thus, the sequence Y_k is bounded and has a cluster point Y^*. Each column of Y^* is thus a limit for $k \to \infty$ of a sequence of elements $\alpha_{k,i} x^{k,i}$ of the cone generated by K. Hence, the columns of Y^* belong to the closure of the cone generated by K. So, $Z = Y^*(Y^*)^\top \in \text{conv } \{xx^\top : x \in \text{cl}\,\text{cone}\, K\}$.

To prove the converse, we assume $Z \in \text{conv } \{xx^\top : x \in \text{cl}\,\text{cone}\, K\}$. Then

$$Z = \sum_i \alpha_i \bar{x}^i (\bar{x}^i)^\top$$

with nonnegative scalars α_i, $\bar{x}^i \in \text{cl}\,\text{cone}\, K$ for all $i \leq n(n+1)/2$. Thus, there exist sequences $\lambda_{i,k} \geq 0$ and $x^{i,k} \in K$ with $\bar{x}^i = \lim_{k\to\infty} \lambda_{i,k} x^{i,k}$. Then

$$Z = \sum_i \alpha_i \left(\lim_{k\to\infty} \lambda_{i,k} x^{i,k} \right) \left(\lim_{k\to\infty} \lambda_{i,k} x^{i,k} \right)^\top = \lim_{k\to\infty} \sum_i \left(\alpha_i \lambda_{i,k}^2 \right) x^{i,k} (x^{i,k})^\top$$

and $Z \in \text{cl cone conv } \{xx^\top : x \in K\}$. □

Corollary 1.5. *If $K \subseteq \mathbb{R}^n$ is a nonempty cone, then*

$$C_K^* = \mathrm{cl\,conv}\ \{xx^\top : x \in K\} = \mathrm{conv}\ \{xx^\top : x \in \mathrm{cl}(K)\}.$$

If K is a nonempty closed cone, then the dual cone reduces to

$$C_K^* = \mathrm{conv}\{xx^\top : x \in K\} = \left\{ \sum_i x^i (x^i)^\top : x^i \in K \right\}$$

and C_K^* is closed.

We call constraints $X \in C_K$ (or $X \in C_K^*$) *set-semidefinite* constraints. Anstreicher and Burer give in [1] for low dimensions computable representations of C_K^* in terms of matrices that are positive semidefinite and componentwise nonnegative. For $n = 5$ and $K = \mathbb{R}^5_+$ examinations of the cone of completely positive matrices $C^*_{\mathbb{R}^5_+}$ are done by Burer, Anstreicher and Dür in [7]. Jarre and Schmallowsky present in [13] a numerical test for checking whether some matrix is an element of the cone of completely positive matrices $C^*_{\mathbb{R}^n_+}$. Recently, a numerical test for detecting copositivity based on simplical partitions and several sufficient conditions has been developed by Bundfuss and Dür, see [8, 9], and Bomze and Eichfelder [3]. We are not aware of results about separation problems for C_K or C_K^* for general K.

The following lemma is the base for our main result and states that the optimal value of a quadratic function over an arbitrary set S is equal to the optimal value of the corresponding linear function over the convex set generated by dyadic products of elements from this set S.

Lemma 1.6. *Let a matrix $Q \in \mathscr{S}^n$, a vector $c \in \mathbb{R}^n$ and a nonempty set $S \subseteq \mathbb{R}^n$ be given. Then the following is true*

$$\inf \quad \{x^\top Q x + 2c^\top x : x \in S\} \tag{4}$$

$$=$$

$$\inf \quad \left\{ \langle \tilde{Q}, Y \rangle : Y \in \mathrm{conv}\left\{ \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^\top : x \in S \right\} \right\}, \tag{5}$$

where $\tilde{Q} = \begin{pmatrix} 0 & c^\top \\ c & Q \end{pmatrix}$. Moreover, if the optimal value of (5) is attained then there exists a rank one optimal solution.

Proof. The "$\geq$" part is easy. For any $x \in S$ the matrix $Y = (1\ x^\top)^\top (1\ x^\top)$ is feasible for (5) and gives the same objective value. To prove the converse let us consider $Y = \sum_i \lambda_i \begin{pmatrix} 1 \\ x^i \end{pmatrix} \begin{pmatrix} 1 \\ x^i \end{pmatrix}^\top$, where $\lambda_i \geq 0$, $\sum_i \lambda_i = 1$ and $x^i \in S$, for all i. Let $\bar{x} \in S$ such that

$$\left\langle \tilde{Q},\ \begin{pmatrix} 1 \\ \bar{x} \end{pmatrix} \begin{pmatrix} 1 \\ \bar{x} \end{pmatrix}^\top \right\rangle = \min_i \left\{ \left\langle \tilde{Q},\ \begin{pmatrix} 1 \\ x^i \end{pmatrix} \begin{pmatrix} 1 \\ x^i \end{pmatrix}^\top \right\rangle \right\}.$$

Then

$$\begin{aligned}\langle \tilde{Q}, Y\rangle &= \sum_i \lambda_i(\langle Q, x^i(x^i)^\top\rangle + 2c^\top x^i) \;\geq\; \sum_i \lambda_i(\langle Q, \bar{x}\bar{x}^\top\rangle + 2c^\top \bar{x}) \\ &= \langle Q, \bar{x}\bar{x}^\top\rangle + 2c^\top \bar{x} \;=\; \bar{x}^\top Q\bar{x} + 2c^\top \bar{x}.\end{aligned}$$

It follows that the optimal value of (4) is less or equal to the optimal value of (5). Together with the first part we have the equality. The last assertion is trivial. □

2 Set-semidefinite reformulation of quadratic programs

In this section we examine the equivalence between a quadratic optimization problem with linear constraints, a *set* constraint and binary variables, and the relaxed problem over the dual cone of set-semidefinite matrices. Let $Q \in \mathscr{S}^n$ be a symmetric matrix, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, $K \subseteq \mathbb{R}^n$ a nonempty set and $B \subseteq \{1,\dots,n\}$ an index set. We study the following quadratic optimization problem

$$\begin{aligned} \mathrm{OPT}_{QP} \;:=\; &\inf x^\top Qx + 2c^\top x \\ &\text{such that} \\ &Ax = b, \\ &x_j \in \{0,1\} \text{ for all } j \in B, \\ &x \in K. \end{aligned} \tag{QP}$$

We will refer to the following notation

$$\begin{aligned}\mathrm{Feas}(QP) &:= \{x\colon x \text{ feasible for } (QP)\}, \\ \mathrm{Feas}^+(QP) &:= \mathrm{conv}\left\{\begin{pmatrix}1\\x\end{pmatrix}\begin{pmatrix}1\\x\end{pmatrix}^\top : x \in \mathrm{Feas}(QP)\right\}.\end{aligned}$$

We follow the line of [5] and assume in the following:

Assumption 1. *If $Ax = b$ and $x \in K$, then $x_j \in [0,1]$ for all $j \in B$.*

Remark 2.1. *Assumption 1 is not very restrictive, if we are allowed to change the set K. Suppose that it does not hold for some $j \in B$, e.g. we have $Ax = b$ but this does not imply $x_j \in [0,1]$. Then we can add two more equations $x_j + y_j = 1$, $x_j - z_j = 0$ and two sign constraints: $y_j, z_j \geq 0$. Hence, by using $K' := K \times \mathbb{R}^2_+$ with constraints $\{Ax = b,\ x_j + y_j = 1,\ x_j - z_j = 0\}$, we fulfill the assumption.*

The described method is especially of interest for K a cone, as $K' = K \times \mathbb{R}^2_+$ remains to be a cone and the special structure is not destroyed. Otherwise, we can of course simply replace K by $K' = K \cap \{x \in \mathbb{R}^n :\ x_j \in [0,1] \text{ for all } j \in B\}$.

However, if we can not change the set K, as is the case in (12), (13) and (14) then Assumption 1 is very restrictive.

If the set B is empty this assumption is trivial.

If $\mathrm{Feas}(QP)$ is unbounded then $\mathrm{Feas}^+(QP)$ might not be closed. Using the following definitions,

$$L_\infty := \{d \in \mathbb{R}^n_+ : Ad = 0\}, \qquad L^+_\infty := \mathrm{conv}\left\{ \begin{pmatrix} 0 \\ d \end{pmatrix} \begin{pmatrix} 0 \\ d \end{pmatrix}^\top : d \in L_\infty \right\},$$

Bomze and Jarre [4] proved for the case $K = \mathbb{R}^n_+$ that $\mathrm{cl}(\mathrm{Feas}^+(QP)) = \mathrm{Feas}^+(QP) + L^+_\infty$, under the assumption that the constraint $Ax = b$ together with $x \in \mathbb{R}^n_+ = K$ implies that x_j is bounded for all $j \in B$ (this assumption is implied by Assumption 1).

We cannot extend this results to a general K since $\mathrm{cl}(\mathrm{Feas}^+(QP))$ is no longer a polyhedron. However, we can prove the following result.

Lemma 2.2. *Let us consider*

$$OPT_{P1} = \inf \left\{ \left\langle \begin{pmatrix} 0 & c^\top \\ c & Q \end{pmatrix}, Y \right\rangle : Y \in \mathrm{cl}(Feas^+(QP)) \right\}.$$

Then $OPT_{QP} = OPT_{P1}$.

Proof. Lemma 1.6 implies that $\mathrm{OPT}_{QP} \geq \mathrm{OPT}_{P1}$. Otherwise, choose $Y^k \in \mathrm{Feas}^+(QP)$ such that $\left\langle \begin{pmatrix} 0 & c^\top \\ c & Q \end{pmatrix}, Y^k \right\rangle \to \mathrm{OPT}_{P1}$. As we show in Lemma 1.6 we can find for every Y^k an $x^k \in \mathrm{Feas}(QP)$ such that

$$\left\langle \begin{pmatrix} 0 & c^\top \\ c & Q \end{pmatrix}, Y^k \right\rangle \geq (x^k)^\top Q x^k + 2c^\top x^k.$$

By taking the limit on both sides we obtain $\mathrm{OPT}_{P1} \geq \mathrm{OPT}_{QP}$, hence we have the equality. □

We can further rewrite the feasible set $\mathrm{cl}(\mathrm{Feas}^+(QP))$ as an intersection of the cone $C^*_{1\times K}$ with an affine space defined by the other constraints from (QP). Here, by $1 \times K$ we shortcut the set $\{1\} \times K$. First, we point out that any matrix $Y = \begin{pmatrix} Y_{11} & x^\top \\ x & X \end{pmatrix} \in \mathrm{Feas}^+(QP)$ is feasible for:

$$\begin{aligned} Y_{11} &= 1, && (6)\\ Ax &= b, && (7)\\ \mathrm{Diag}(AXA^\top) &= b \circ b := (b_1^2, b_2^2, \ldots, b_m^2)^\top, && (8)\\ x_j &= X_{jj} \text{ for all } j \in B. && (9) \end{aligned}$$

We consider the dual cone of the $\{1\} \times K$-semidefinite cone:

$$C^*_{1\times K} = \mathrm{cl}\left(\left\{ \sum_i \lambda_i \begin{pmatrix} 1 \\ x^i \end{pmatrix} \begin{pmatrix} 1 \\ x^i \end{pmatrix}^\top : \lambda_i \geq 0,\, x^i \in K \right\} \right).$$

Note that the cone $C^*_{1\times K}$ is a closed convex cone. We have the following equality.

Lemma 2.3. *Under Assumption 1 we have*

$$\mathrm{cl}\, Feas^{+}(QP) = C^{*}_{1\times K} \cap \{Y \in \mathscr{S}^{n+1} \colon Y \ \text{feasible for} \ (6)-(9)\}$$

Proof. The inclusion "⊆" follows from above since the set on the right hand side is closed. To prove the converse inclusion let us consider

$$Y = \sum_i \lambda_i \begin{pmatrix} 1 \\ x^i \end{pmatrix} \begin{pmatrix} 1 \\ x^i \end{pmatrix}^{\top} \in C^{*}_{1\times K} \cap \{Y \in \mathscr{S}^{n+1} \colon Y \ \text{feasible for} \ (6)-(9)\}, \tag{10}$$

where $\lambda_i > 0$ and $x^i \in K$. Constraints (6)–(8) imply that $\sum_i \lambda_i = 1$ and for every row a_j of matrix A we have

$$\sum_i \lambda_i a_j^{\top} x^i = b_j \ \text{ and } \ \sum_i \lambda_i (a_j^{\top} x^i)^2 = b_j^2$$

It follows that

$$0 = \sum_i \lambda_i (a_j^{\top} x^i)^2 - \left(\sum_i \lambda_i a_j^{\top} x^i \right)^2 = \sum_i \lambda_i \left(a_j^{\top} x^i - \sum_k \lambda_k a_j^{\top} x^k \right)^2 \geq 0,$$

hence the equality is throughout. This is possible only if $a_j^{\top} x^i - \sum_k \lambda_k a_j^{\top} x^k = 0$ for all i, hence $a_j^{\top} x^i = a_j^{\top} x^k$ for all i,k and finally $a_j^{\top} x^i = b_j$, for all i. Constraint (9) is equivalent to

$$\sum_i \lambda_i x^i_j - \sum_i \lambda_i (x^i_j)^2 = \sum_i \lambda_i x^i_j (1 - x^i_j) = 0. \tag{11}$$

Assumption 1 implies that $x^i_j \in [0,1]$ for all $j \in B$. Then (11) is possible if and only if $x^i_j \in \{0, 1\}$, for all i and for all $j \in B$. Therefore $x^i \in \mathrm{Feas}(QP)$ and $Y \in \mathrm{Feas}^{+}(QP)$.

The set $C^{*}_{1\times K} \cap \{Y \in \mathscr{S}^{n+1} \colon Y \ \text{feasible for} \ (6)-(9)\}$ is a closure of matrices which can be decomposed as (10). Since $\mathrm{cl}\,\mathrm{Feas}^{+}(QP)$ is closed, the inclusion "⊇" follows.

□

Lemmas 2.2 and 2.3 directly imply that

Theorem 2.4. *Let Assumption 1 be satisfied. The optimal value OPT_{QP} is equal to the optimal value of*

$$\begin{array}{rrl} OPT_C := & \inf & \left\langle \begin{pmatrix} 0 & c^{\top} \\ c & Q \end{pmatrix}, Y \right\rangle \\ & & \textit{such that} \\ & Y \in & C^{*}_{1\times K}, \\ & Y & \textit{feasible for} \ (6)-(9). \end{array} \tag{QP$_C$}$$

Note that (QP$_C$) is a linear program over the dual cone $C^{*}_{1\times K}$. We transformed all nonlinearity and nonconvexity into the structure of the closed convex cone $C^{*}_{1\times K}$.

Theorem 2.4 is a generalization of the completely positive representation results by Burer [5, 6] and Eichfelder and Povh [12]. Burer represented a completely positive reformulation of (QP) for the case $K = \mathbb{R}^n_+$. This result was independently generalized further by Burer [6] and Eichfelder and Povh [12] to K an arbitrary closed convex cone. Theorem 2.4 is therefore the most general representation result since we assumed only that K is an arbitrary set.

3 Some practical applications

In the following we provide some practical applications of Theorem 2.4.

3.1 Optimization over the non-negative orthant

Burer considered in [5] problem (QP) when K is the nonnegative orthant $\mathbb{R}^n_+$ and obtained the following result

$$\begin{array}{rcl} \mathrm{OPT}_{QP} = & & \inf \left\langle \begin{pmatrix} 0 & c^\top \\ c & Q \end{pmatrix}, Y \right\rangle \\ & & \text{such that} \\ Y & \in & C^*_{\mathbb{R}^{n+1}}, \\ Y & & \text{feasible for } (6)-(9). \end{array}$$

where $C^*_{\mathbb{R}^{n+1}}$ is the cone of completely positive matrices. Note that the only difference between this formulation and our formulation from Theorem 2.4 is the constraint $C^*_{\mathbb{R}^{n+1}}$. We can prove

Lemma 3.1. $C^*_{\mathbb{R}^{n+1}} = C^*_{1\times\mathbb{R}^+_n}$.

Proof. The direction "$\subseteq$" is obvious. For the other direction let us consider $Y \in C^*_{\mathbb{R}^{n+1}}$. Then there are $\alpha_i \geq 0$ and $x^i \in \mathbb{R}^n_+$ with

$$Y = \sum_i \begin{pmatrix} \alpha_i \\ x^i \end{pmatrix} \begin{pmatrix} \alpha_i \\ x^i \end{pmatrix}^\top = \underbrace{\sum_{i:\ \alpha_i \neq 0} \alpha_i^2 \begin{pmatrix} 1 \\ \frac{1}{\alpha_i} x^i \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{\alpha_i} x^i \end{pmatrix}^\top}_{Y_1} + \underbrace{\sum_{i:\ \alpha_i = 0} \begin{pmatrix} 0 \\ x^i \end{pmatrix} \begin{pmatrix} 0 \\ x^i \end{pmatrix}^\top}_{Y_2} \in C^*_{\mathbb{R}^{n+1}}.$$

Obviously $Y_1 \in C_{1\times\mathbb{R}^n_+}$. Since for all i with $\alpha_i = 0$

$$\begin{pmatrix} 0 \\ x^i \end{pmatrix} \begin{pmatrix} 0 \\ x^i \end{pmatrix}^\top = \lim_{n\to\infty} \underbrace{\frac{1}{n^2} \begin{pmatrix} 1 \\ nx^i \end{pmatrix} \begin{pmatrix} 1 \\ nx^i \end{pmatrix}^\top}_{\in C_{1\times\mathbb{R}^n_+}},$$

we have $Y_2 \in C^*_{1\times\mathbb{R}^n_+}$, too. As $C_{1\times\mathbb{R}^n_+}$ is a convex cone this implies $Y \in C_{1\times\mathbb{R}^n_+}$. □

Corollary 3.2. *For the case $K = \mathbb{R}^n_+$ the set-semidefinite representation of (QP) from Theorem 2.4 coincides with the completely positive representation from [5].*

3.2 Optimization problems with one quadratic constraint

Let us consider the case when K is a (nonconvex) nonempty set defined by one quadratic constraint:

$$K = \{x \in \mathbb{R}^n :\ x^\top P x + 2p^\top x + p_0 \leq 0\} \tag{12}$$

where $p, p_o \in \mathbb{R}^n$ and $P \in \mathscr{S}^n$. The dual of the $1 \times K$-semidefinite cone is

$$C^*_{1\times K} = \mathrm{cl}\left(\left\{\sum_i \lambda_i \begin{pmatrix} 1 \\ x^i \end{pmatrix}\begin{pmatrix} 1 \\ x^i \end{pmatrix}^\top : \lambda_i \geq 0,\ \left\langle \begin{pmatrix} p_0 & p^\top \\ p & P \end{pmatrix},\ \begin{pmatrix} 1 \\ x^i \end{pmatrix}\begin{pmatrix} 1 \\ x^i \end{pmatrix}^\top \right\rangle \leq 0\right\}\right).$$

We have the following representation for $C^*_{1\times K}$:

Lemma 3.3.

$$C^*_{1\times K} = \left\{ \begin{pmatrix} Y_{11} & x^\top \\ x & X \end{pmatrix} \in \mathscr{S}^+_{n+1} : \left\langle \begin{pmatrix} p_0 & p^\top \\ p & P \end{pmatrix},\ \begin{pmatrix} Y_{11} & x^\top \\ x & X \end{pmatrix} \right\rangle \leq 0 \right\}.$$

Proof. The direction "$\subseteq$" is obvious. For the converse let us consider $Y \in \mathscr{S}^+_{n+1}$ with

$$\left\langle \begin{pmatrix} p_0 & p^\top \\ p & P \end{pmatrix},\ \begin{pmatrix} Y_{11} & x^\top \\ x & X \end{pmatrix} \right\rangle \leq 0.$$

Lemma 2.4 from [15] (see also Proposition 3 from [18]), which is also the crucial result for an alterative proof of the famous S-lemma, implies that there exist $\begin{pmatrix} \alpha_i \\ x^i \end{pmatrix} \in \mathbb{R}^{n+1}$ such that $Y = \sum_i \begin{pmatrix} \alpha_i \\ x^i \end{pmatrix}\begin{pmatrix} \alpha_i \\ x^i \end{pmatrix}^\top$ and

$$\left\langle \begin{pmatrix} p_0 & p^\top \\ p & P \end{pmatrix},\ \begin{pmatrix} \alpha_i \\ x^i \end{pmatrix}\begin{pmatrix} \alpha_i \\ x^i \end{pmatrix}^\top \right\rangle = (x^i)^\top P x^i + 2\alpha_i p^\top x^i + \alpha_i^2 p_0 \leq 0, \text{ for all } i.$$

Without loss of generality we may assume that $\alpha_i \geq 0$. If $\alpha_i > 0$ then $\begin{pmatrix} \alpha_i \\ x^i \end{pmatrix}\begin{pmatrix} \alpha_i \\ x^i \end{pmatrix}^\top = \alpha_i^2 \begin{pmatrix} 1 \\ \frac{1}{\alpha_i}x^i \end{pmatrix}\begin{pmatrix} 1 \\ \frac{1}{\alpha_i}x^i \end{pmatrix}^\top \in C^*_{1\times K}$. The remaining of the proof deals with the case $\alpha_i = 0$. Then $(x^i)^\top P x^i \leq 0$.

If $(x^i)^\top P x^i < 0$ then $(x^i)^\top P x^i + 2\varepsilon p^\top x^i + \varepsilon^2 p_0 \leq 0$ for ε sufficiently small, hence

$$Y_\varepsilon = \begin{pmatrix} \varepsilon \\ x^i \end{pmatrix}\begin{pmatrix} \varepsilon \\ x^i \end{pmatrix}^\top = \varepsilon^2 \begin{pmatrix} 1 \\ \frac{1}{\varepsilon}x^i \end{pmatrix}\begin{pmatrix} 1 \\ \frac{1}{\varepsilon}x^i \end{pmatrix}^\top \in C^*_{1\times K}$$

and $\lim_{\varepsilon\to 0} Y_\varepsilon = \begin{pmatrix} 0 \\ x^i \end{pmatrix}\begin{pmatrix} 0 \\ x^i \end{pmatrix}^\top \in C^*_{1\times K}$. By the same line of reasoning we prove the case when $(x^i)^\top P x^i = 0$ and $p^\top x^i \neq 0$. We consider $Y_\varepsilon = \begin{pmatrix} \varepsilon \\ x^i \end{pmatrix}\begin{pmatrix} \varepsilon \\ x^i \end{pmatrix}^\top$ if $p^\top x^i < 0$ and $Y_\varepsilon = \begin{pmatrix} \varepsilon \\ -x^i \end{pmatrix}\begin{pmatrix} \varepsilon \\ -x^i \end{pmatrix}^\top$ if $p^\top x^i > 0$.

If $(x^i)^\top P x^i = 0$ and $p^\top x^i = 0$ then

$$\begin{pmatrix} 0 \\ x^i \end{pmatrix}\begin{pmatrix} 0 \\ x^i \end{pmatrix}^\top = \lim_{\varepsilon\to 0} \varepsilon^2 \begin{pmatrix} 1 \\ z \pm \frac{1}{\varepsilon}x^i \end{pmatrix}\begin{pmatrix} 1 \\ z \pm \frac{1}{\varepsilon}x^i \end{pmatrix}^\top \in C^*_{1\times K}$$

where z is an arbitrary vector from K. Note that we take the sign of $\frac{1}{\varepsilon}x^i$ such that $\pm(x^i)^\top Pz \leq 0$.

As $C^*_{1\times K}$ is a convex cone the assertion is proven. $\square$

We point out that Lemma 3.3 contains essentially the same result as Theorem 1 from [18]. We added a complete proof for the sake of completeness and because it is more straightforward than the proof from [18].

When considering quadratic programs with arbitrary quadratic constraints, Sturm and Zhang show how to rewrite a quadratic problem with one quadratic inequality as a semidefinite program. They extended their result for two more cases: when the feasible set is defined by one strictly concave quadratic equality or by one concave quadratic inequality and by one linear inequality – see Lemmas 3.7, 3.8.

Using Theorem 2.4 we can extend these results from Sturm and Zhang to quadratic programs with one quadratic equality or inequality and arbitrary many linear equations, i.e. we can show that such quadratic problems are semidefinite programming problems and therefore (numerically) tractable. We point out that this result can not be obtained using approach from [18].

Corollary 3.4. *We have the following semidefinite programming representation of (QP) for the case when K is of the form (12):*

$$
\begin{array}{rcl}
OPT_{QP} = & \inf & \left\langle \begin{pmatrix} 0 & c^\top \\ c & Q \end{pmatrix}, Y \right\rangle \\
& \textit{such that} & \\
Y & = & \begin{pmatrix} 1 & x^\top \\ x & X \end{pmatrix} \in \mathscr{S}^+_{n+1}, \\
Ax & = & b, \\
Diag(AXA^\top) & = & b \circ b, \\
\left\langle \begin{pmatrix} p_0 & p^\top \\ p & P \end{pmatrix}, \begin{pmatrix} 1 & x^\top \\ x & X \end{pmatrix} \right\rangle & \leq & 0.
\end{array}
$$

Note that we can handle binary constraints only if Assumption 1 is satisfied without changing the set K (i.e. $Ax = b$ and $x \in K$ for K from (12) must imply $x_j \in [0, 1]$ for all $j \in B$), since changes accordingly to Remark 2.1 change K and Lemma 3.3 is no more true for the new K.

Example 3.5. *Let us consider the following nonconvex quadratic problem:*

$$\inf \{x^2 + xy + y^2 - 2x - 2y : \ y + x/2 = 2,\ x^2 - y^2 - 2xy + 1 \leq 0\}.$$

The feasible set is plotted in Figure 1 as a bold line above the interval $[-\frac{6}{7}, 2]$.

The optimal value is $-\frac{1}{3}$ *and is attained at* $x = \frac{2}{3}, y = \frac{5}{3}$. *Theorem 2.4 and Lemma 3.3 imply that we can reformulate this optimization problem into*

$$
\begin{array}{c}
\inf \ \langle Q, Y \rangle \\
\textit{such that} \\
Y \in \mathscr{S}^+_3,\ Y_{11} = 1,\ \langle A_1, Y \rangle = 4,\ \langle A_2, Y \rangle = 4,\ \langle A_3, Y \rangle \leq 0
\end{array}
$$

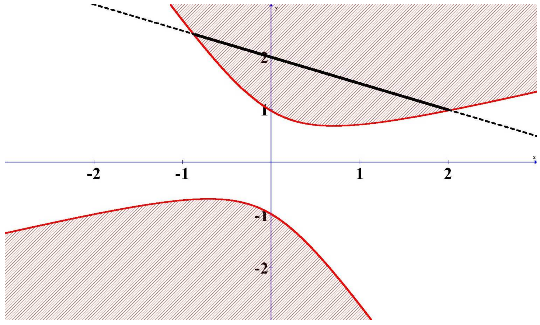

Figure 1: Feasible set of the quadratic problem of Example 3.5.

where

$$Q = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 1 & 0.5 \\ -1 & 0.5 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0.5 & 1 \\ 0.5 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0.25 & 0.5 \\ 0 & 0.5 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & -1 \end{pmatrix}.$$

The optimal solution is a rank one matrix yielding the optimal value $-1/3$*:*

$$Y_{\text{opt}} = \begin{pmatrix} 1 & \frac{2}{3} & \frac{5}{3} \\ \frac{2}{3} & \frac{4}{9} & \frac{10}{9} \\ \frac{5}{3} & \frac{10}{9} & \frac{25}{9} \end{pmatrix}$$

Remark 3.6. *([18, Theorem 1]) For K from (12) we can represent the $1 \times K$-semidefinite cone (the dual to $C^*_{1\times K}$) as*

$$C_{1\times K} = \text{cl}\left\{ Z \in \mathscr{S}_{n+1} \colon Z = X - \lambda \begin{pmatrix} p_0 & p^\top \\ p & P \end{pmatrix} \in \mathscr{S}^+_{n+1}, \;\; \lambda \geq 0 \right\}.$$

The following lemmas contain results from [18]. Although they rely also on Lemma 1 from [18] which was shown to be wrong (see Example 1.3 and comments thereafter), we verified the proofs again and they actually need a weaker version of Lemma 1 which is true (e.g. our Lemma 1.4), hence these results are still holding true.

Lemma 3.7. *[18, Theorem 1] Let us consider a strictly concave function $q(x) = x^\top P x + 2p^\top x + p_0$ and the following set:*

$$K \;\; = \;\; \{x \in \mathbb{R}^n \colon q(x) = 0\}, \tag{13}$$

which we assume to be non-empty. Then the following is true:

$$\begin{aligned} C^*_{1\times K} &= \left\{ \begin{pmatrix} Y_{11} & x^\top \\ x & X \end{pmatrix} \in \mathscr{S}^+_{n+1} \colon \left\langle \begin{pmatrix} p_0 & p^\top \\ p & P \end{pmatrix}, \begin{pmatrix} Y_{11} & x^\top \\ x & X \end{pmatrix} \right\rangle = 0 \right\} \\ C_{1\times K} &= \text{cl}\left\{ Z \in \mathscr{S}_{n+1} \colon Z = X - \lambda \begin{pmatrix} p_0 & p^\top \\ p & P \end{pmatrix} \in \mathscr{S}^+_{n+1}, \;\; \lambda \in \mathbb{R} \right\} \end{aligned}$$

Lemma 3.8. *[18, Theorem 3, Cor. 3] Let $q(x) = x^\top Px + 2p^\top x + p_0$ be a concave function and $a \in \mathbb{R}^{n+1}$. Let*

$$K = \{x \in \mathbb{R}^n \colon q(x) \geq 0,\ (1\ x^\top)\, a \geq 0\} \tag{14}$$

which we assume to be non-empty. Then the following is true:

$$C^*_{1\times K} = \left\{ \begin{pmatrix} Y_{11} & x^\top \\ x & X \end{pmatrix} \in \mathscr{S}^+_{n+1} \colon \left\langle \begin{pmatrix} p_0 & p^\top \\ p & P \end{pmatrix}, \begin{pmatrix} Y_{11} & x^\top \\ x & X \end{pmatrix} \right\rangle \geq 0,\ BXa \in SOC \right\}$$

where SOC denotes the second order (Lorentz) cone and B is defined as

$$B = \begin{pmatrix} p_0 + 1, & 2p^\top \\ p_0 - 1, & 2p^\top \\ 0, & 2R \end{pmatrix},\ R \text{ such that } P = -R^\top R.$$

Corollary 3.9. *We can reformulate any nonconvex quadratic problem (QP) where K is from (13) or (14) as a semidefinite programming problem.*

4 Conclusions

In the paper we present a result that nontrivially generalizes and connects two important results from Burer [5] and Sturm and Zhang [18]. We show that any quadratic problem where the feasible set is defined by linear and binary constraints and is a subset of some arbitrary set K can be rewritten as a linear program over the cone dual to the K-semidefinite cone. When K is the nonnegative orthant then this result coincides with the completely positive representation result from [5]. When K is defined by one quadratic constraint or by one concave quadratic constraint and one linear inequality then our result nontrivially generalizes results from Sturm and Zhang [18] since our approach enables inclusion of linear and binary constraints to the original problem while the approach from [18] does not allow this.

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