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How to generate weakly infeasible semidefinite programs via Lasserre's relaxations for polynomial optimization

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Abstract

Examples of weakly infeasible semidefinite programs are useful to test whether semidefinite solvers can detect infeasibility. However, finding non trivial such examples is notoriously difficult. This note shows how to use Lasserre's semidefinite programming relaxations for polynomial optimization in order to generate examples of weakly infeasible semidefinite programs. Such examples could be used to test whether a semidefinite solver can detect weak infeasibility. In addition, in this note, we generate weakly infeasible semidefinite programs from an instance of polynomial optimization with nonempty feasible region and solve them by semidefinite solvers. Although all semidefinite programming relaxation problems are infeasible, we observe that semidefinite solvers do not detect the infeasibility and that values returned by semidefinite solvers are equal to the optimal value of the instance due to numerical round-off errors.

Key words: Semidefinite programming, weakly infeasible, semidefinite programming relaxation.

1. Introduction

Semidefinite program (SDP) is the problem of minimizing a linear function over the intersection of the n-dimensional positive semidefinite cone and an affine space. Unlike linear program, SDP is one of nonlinear optimization problems and it has two possible infeasibility, *i.e.*, strongly infeasible and weakly infeasible. The strong infeasibility is the same property as the infeasibility in linear program and there exists a certificate for the infeasibility. For strongly infeasible SDP, some efficient algorithms for detecting the infeasibility are proposed in [14, 7]. In contrast, the weak infeasibility is an asymptotic property peculiar to nonlinear optimization. By adding a small perturbation in weakly infeasible SDP, the perturbed SDP may become feasible. In addition, weakly infeasible SDPs do not have certificate of the infeasibility. Consequently, it is very difficult to decide if the SDP is either feasible or infeasible numerically because numerical errors, such as round-off errors, always occur in the computation of interior-point methods.

The contribution of this note is to show that Lasserre's SDP relaxation [3] cam be used to generate examples of weakly infeasible semidefinite programs. It is interesting that weakly infeasible SDP problems can be obtained by applying Lasserre's SDP relaxation into polynomial optimization problems (POPs) because we seldom encounter weakly infeasible SDP problems in applications except for artificial examples. We can use these SDP problems as one of sample problems when we measure performance of algorithms for checking whether a given SDP is feasible or infeasible.

In this note, we solve these SDP problems obtained from this POP by the existing SDP solvers, SeDuMi [12] and SDPA [1]. Interestingly, we observe that the optimal values of SDP relaxation with higher relaxation order obtained by SeDuMi and SDPA coincide with the optimal value of the original POP. This phenomenon is also presented in [2, 15]. In fact, the authors applied Lasserre's SDP relaxation into specific POPs, and then confirmed that the returned values are the exact optimal value of the POP although SDP solvers cannot solve the resulting SDP relaxation problems correctly. In addition, by choosing an appropriate parameter set of SDPA, we observe that SDPA can detect that at least primal or dual is infeasible, while SeDuMi cannot.

The organization of this note is as follows: in Section 2, we give some facts on SDP, and Lasserre's SDP relaxation for POPs. We show in Section 3 that all SDP relaxation problems of POP (6) are weakly infeasible. The numerical result for the SDP relaxation problems is given in Section 4. Discussion on these SDP relaxation problems and the numerical result are given in Section 5.

1.1. Notation and symbols.

For every finite set A, $\#(A)$ denotes the number of elements in A. Let $\mathbb{N}^n$ be the set of n-dimensional nonnegative integer vectors. We define $\mathbb{N}^n_r := \{\alpha \in \mathbb{N}^n \mid \sum_{i=1}^n \alpha_i \leq r\}$. Let $\mathbb{R}[x]$ be the set of polynomials with n-dimensional real vector x. For every $\alpha \in \mathbb{N}^n$, x^α denotes the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. For $f \in \mathbb{R}[x]$, let F be the set of exponentials α of monomials x^α whose coefficients f_α are nonzero. Then we can write $f(x) = \sum_{\alpha \in F} f_\alpha x^\alpha$. We call F the *support* of f. The degree $\deg(f)$ of f is the maximum value of $|\alpha| := \sum_{i=1}^n \alpha_i$ over the support F. $\mathbb{R}_r[x]$ is the set of polynomials with the degree up to r. $\mathbb{S}^n$ and $\mathbb{S}^n_+$ denote the sets of $n \times n$ symmetric matrices and $n \times n$ symmetric positive semidefinite matrices, respectively. For $A, B \in \mathbb{S}^n$, we define $A \bullet B = \sum_{i,j=1}^n A_{ij} B_{ij}$.

2. Preliminaries

For given $C, A_1, \ldots, A_m \in \mathbb{S}^n$ and $b \in \mathbb{R}^m$, SDP and its dual can be formulated as follows:

$$\text{(Primal) } \theta_P := \sup_{X \in \mathbb{S}^n_+} \{C \bullet X \mid A_j \bullet X = b_j \ (j = 1, \ldots, m)\}, \tag{1}$$

$$\text{(Dual) } \theta_D := \inf_{y \in \mathbb{R}^m} \left\{ b^T y \,\middle|\, \sum_{j=1}^m y_j A_j - C \in \mathbb{S}^n_+ \right\}. \tag{2}$$

It is well-known that the strong duality holds for SDPs (1) and (2).

Theorem 2.1. *(Renegar* [10, Theorem 3.2.6.]*) $\theta_P = \theta_D$ if either (1) or (2) has an interior feasible solution.*

SDP has two types of the infeasibility, *i.e.*, *strong infeasibility* and *weak infeasibility*. We call $\bar{y}$ a *dual improving ray* if and only if $\bar{y}$ satisfies $b^T \bar{y} < 0$ and $\sum_{j=1}^m \bar{y}_j A_j \in \mathbb{S}^n_+$. Clearly, SDP (1) is infeasible if there exists a dual improving ray $\bar{y}$. Indeed, if SDP (1) has a feasible solution $\tilde{X}$, then we have

$$0 > b^T\bar{y} = \sum_{j=1}^m \left(A_j \bullet \tilde{X}\right)\bar{y}_j = \left(\sum_{j=1}^m \bar{y}_j A_j\right) \bullet \tilde{X} \geq 0,$$

which is a contradiction. We say that SDP (1) is *strongly infeasible* if there exists a dual improving ray. On the other hand, we say that SDP (1) is *weakly infeasible* if the feasible region of SDP (1) is empty and for any $\epsilon > 0$, there exists an $\bar{X} \in \mathbb{S}^n_+$ such that

$$|A_j \bullet \bar{X} - b_j| < \epsilon \text{ for all } j = 1\ldots, m.$$

We remark that from Theorem 3.2.2 in [10], an infeasible SDP is either strongly infeasible or weakly infeasible.

$\sigma \in \mathbb{R}[x]$ is a *sum of squares* if and only if $\sigma = \sum_{j=1}^\ell g_j(x)^2$ for some $g_1, \ldots, g_\ell \in \mathbb{R}[x]$. Σ and Σ_r denote the sets of sum of squares and sum of squares with the degree up to $2r$, respectively. It is well-known that Σ is not equivalent to the set of nonnegative polynomials over $\mathbb{R}^n$ in general. See [5, 11] for more details.

Let $f_1, \ldots, f_m \in \mathbb{R}[x]$ and $\bar{r} := \max\{\lceil \deg(f)/2\rceil, \lceil\deg(f_1)/2\rceil, \ldots, \lceil\deg(f_m)/2\rceil\}$. For an integer r satisfying $r \geq \bar{r}$, we set $r_j := r - \lceil\deg(f_j)/2\rceil$ for all $j = 1, \ldots, m$. We define the sets $M(f_1, \ldots, f_m)$ and $M_r(f_1, \ldots, f_m)$ as follows:

$$\begin{aligned} M(f_1, \ldots, f_m) &:= \left\{ \sigma_0 + \sum_{j=1}^m \sigma_j f_j \,\middle|\, \sigma_0, \ldots, \sigma_m \in \Sigma \right\}, \\ M_r(f_1, \ldots, f_m) &:= \left\{ \sigma_0 + \sum_{j=1}^m \sigma_j f_j \,\middle|\, \sigma_0 \in \Sigma_r, \sigma_j \in \Sigma_{r_j} \text{ for all } j = 1, \ldots, m \right\}, \end{aligned}$$

We remark that $M_r(f_1, \ldots, f_m) \subseteq M(f_1, \ldots, f_m)$ for all $r \geq \bar{r}$. We call $M(f_1, \ldots, f_m)$ the *quadratic module generated by* $f_1, \ldots, f_m$. The quadratic module $M(f_1, \ldots, f_m)$ is said to be *Archimedean* if and only if there exists $N \in \mathbb{N}$ for which $N - \sum_{j=1}^n x_j^2 \in M(f_1, \ldots, f_m)$. For given $f_1, \ldots, f_m \in \mathbb{R}[x]$, let $K := \{x \in \mathbb{R}^n \mid f_1(x) \geq 0, \ldots, f_m(x) \geq 0\}$. The following theorem plays an essential role in asymptotic behaviors of Lasserre's SDP relaxation for POPs.

Theorem 2.2. *(Putinar* [9, Lemma 4.1]*; see also Laurent* [5, Theorem 3.20]*) Assume that the quadratic module $M(f_1, \ldots, f_m)$ is Archimedean. For any $p \in \mathbb{R}[x]$, if $p > 0$ over K, then $p \in M(f_1, \ldots, f_m)$.*

For given $f, f_1, \ldots, f_m \in \mathbb{R}[x]$, we consider the POP:

$$f^* := \inf_{x\in\mathbb{R}^n} \{f(x) \mid f_1(x), \ldots, f_m(x) \geq 0\}. \tag{3}$$

Let r be an integer with $r \geq \bar{r}$. POP (3) can obviously be reformulated as

$$f^* = \sup\{\rho \mid f - \rho \geq 0 \text{ on } K\},$$

where $K = \{x \in \mathbb{R}^n \mid f_1(x) \geq 0, \ldots, f_m(x) \geq 0\}$. By replacing the nonnegativity condition by a simpler condition involving $M_r(f_1, \ldots, f_m)$, we can obtain the following problem by applying Lasserre's SDP relaxation into POP (3):

$$\rho_r^* := \sup\left\{\rho \,|\, f - \rho \in M_r(f_1, \ldots, f_m)\right\} \tag{4}$$

We can rewrite (4) into SDP (1), equivalently. See [3, 5] for more details. We call (4) *SDP relaxation problem* of POP (3) with relaxation order r in this note. Clearly, $\rho_r^* \leq \rho_{r+1}^* \leq f^*$ for all $r \geq \bar{r}$ because $\Sigma_r \subseteq \Sigma_{r+1}$. Lasserre showed in [3, Theorem 4.2] that the sequence $\{\rho_r^*\}_{r=\bar{r}}^\infty$ converges to the f^* if the quadratic module $M(f_1, \ldots, f_m)$ is Archimedean.

For every $z \in \mathbb{R}^{\#(\mathbb{N}^n_{2r})}$, we define the $\#(\mathbb{N}^n_r)\times\#(\mathbb{N}^n_r)$ symmetric matrices $L_r(z)$ and the $\#(\mathbb{N}^n_{r_j})\times\#(\mathbb{N}^n_{r_j})$ symmetric matrices $L_{r_j}(f_j z)$ as follows:

$$L_r(z) := \left(z_{\alpha+\beta}\right)_{\alpha,\beta\in\mathbb{N}^n_r}, L_{r_j}(f_j z) := \left(\sum_{\gamma\in F_j} (f_j)_\gamma z_{\alpha+\beta+\gamma}\right)_{\alpha,\beta\in\mathbb{N}^n_{r_j}},$$

where F_j is the support of f_j for all $j = 1, \ldots, m$. $L_r(z)$ and $L_{r_j}(f_j z)$ are called the *moment matrix* and *localizing matrix*, respectively. The following SDP is the dual of (4):

$$\eta_r^* := \inf_{z \in \mathbb{R}^{\#(\mathbb{N}_{2r}^n)}} \left\{ \sum_{\gamma \in F} f_\gamma z_\gamma \,\middle|\, z_0 = 1, L_r(z) \in \mathbb{S}_+^{\#(\mathbb{N}_r^n)}, L_{r_j}(f_j z) \in \mathbb{S}_+^{\#(\mathbb{N}_{r_j}^n)} \text{ for all } j = 1, \ldots, m \right\}, \quad (5)$$

where F is the support of f. From the weak duality of SDP, we have $\rho_r^* \leq \eta_r^*$ for all $r \geq \bar{r}$.

3. The weak infeasibility of the SDP relaxation problems

We consider POP (3) and let $\bar{r}$ be as in Section 2.

Theorem 3.1. *Let $r \geq \bar{r}$. Assume that $\deg(f) < 2r$ and that $f - \rho \notin M_r(f_1, \ldots, f_m)$ for all $\rho \in \mathbb{R}$. Then the resulting SDP relaxation problem (4) with relaxation order r is weakly infeasible.*

To prove this theorem, we use the following lemma on the moment matrix. We give a proof of Lemma 3.2 in Appendix A.

Lemma 3.2. *For $y = (y_\alpha)_{\alpha \in \mathbb{N}_{2r}^n} \in \mathbb{R}^{\#(\mathbb{N}_{2r}^n)}$. We assume that the moment matrix $L_r(y) = (y_{\alpha+\beta})_{\alpha,\beta \in \mathbb{N}_r^n}$ is positive semidefinite and $y_0 = 0$. Then $y_\alpha = 0$ for all $\alpha \in \mathbb{N}_{2r-1}^n$.*

Proof of Theorem 3.1 : The SDP relaxation problem (4) is infeasible because of $f - \rho \notin M(f_1, \ldots, f_m)$ for all $\rho \in \mathbb{R}$. We suppose to the contrary that the SDP is strongly infeasible. Then the SDP has a dual improving ray $\tilde{y}$. From the definition of the improving ray, the $(0, 0)$-th element of the moment matrix $L_r(\tilde{y})$ is 0. Moreover, the moment matrix $L_r(\tilde{y})$ must be positive semidefinite. It follows from Lemma 3.2 that $\tilde{y}_\alpha = 0$ for all $\alpha \in \mathbb{N}_{2r-1}^n$. We have $F \subseteq \mathbb{N}_{2r-1}^n$ because of $\deg(f) < 2r$. Therefore,

$$0 > \sum_{\gamma \in F} f_\gamma \tilde{y}_\gamma = \sum_{\gamma \in \mathbb{N}_{2r-1}^n} f_\gamma \tilde{y}_\gamma = 0,$$

which is the contradiction. Hence, the SDP is weakly infeasible. □

We obtain the following corollary from Theorem 3.1.

Corollary 3.3. *Assume that $\deg(f) < 2\bar{r}$ and that $f - \rho \notin M(f_1, \ldots, f_m)$ for all $\rho \in \mathbb{R}$. Then all the resulting SDP relaxation problems (4) with relaxation order $r \geq \bar{r}$ are weakly infeasible.*

Now, we consider a POP whose optimal value f^* is $-\infty$. Then the POP satisfies $f - \rho \notin M(f_1, \ldots, f_m)$ for all $\rho \in \mathbb{R}$. Indeed, if there exists $\hat{\rho} \in \mathbb{R}$ such that $f - \hat{\rho} \in M(f_1, \ldots, f_m)$, then $f^* \geq \hat{\rho}$, which contradicts to $f^* = -\infty$. Moreover, from the definition of $\bar{r}$, if $r > \bar{r}$, then $r > \deg(f)$. Therefore, it follows from Theorem 3.1 that the resulting SDP relaxation problem with $r > \bar{r}$ is weakly infeasible.

We give another example satisfying the assumptions in Theorem 3.1 and Corollary 3.3. This is the POP which we mentioned in Section 1. It should be noted that POP (6) is almost the same as a POP in Example 6.3.1 in [8]. It is proved in Example 6.3.1 in [8] that the quadratic module associated with the POP given in Example 6.3.1 is not Archimedean. We will show in Example 3.4 that POP (6) satisfies the assumptions in Theorem 3.1 and Corollary 3.3 by using the same way as Example 6.3.1 in [8].

Example 3.4.

$$\inf_{x,y \in \mathbb{R}} \left\{ f(x,y) := -x - y \,\middle|\, \begin{array}{l} f_1(x,y) := x - 0.5 \geq 0, \\ f_2(x,y) := y - 0.5 \geq 0, \\ f_3(x,y) := 0.5 - xy \geq 0 \end{array} \right\}. \quad (6)$$

The optimal value is -1.5 and the optimal solution is $(x, y) = (1, 0.5), (0.5, 1)$. Clearly, the feasible region of POP (6) has nonempty, compact and full-dimensional.

We show that all the SDP relaxation problems (4) for POP (6) are weakly infeasible by using Corollary 3.3. For POP (6), we have $\bar{r} = 1$ and thus $\deg(f) < 2\bar{r}$. We show that $f - \rho \notin M(f_1, f_2, f_3)$. To this end, we introduce some notation and symbols used in [8, Example 6.3.1]. *We define a subset S of* $\mathbb{R}[x_1, \ldots, x_n]$ *as follows: $0 \in S$ and for all $p \in \mathbb{R}[x_1, \ldots, x_n]$,*

$$p \in S \Leftrightarrow \left\{ \begin{array}{l} p_\alpha > 0 \text{ and } \alpha \not\equiv (1, \ldots, 1) \text{ mod } 2, \text{ or} \\ p_\alpha < 0 \text{ and } \alpha \equiv (1, \ldots, 1) \text{ mod } 2, \end{array} \right.$$

where p_α is the coefficient of the largest monomial x^α with the lexicographic order in p. It is not difficult to prove the following properties:

$$1 \in S, S+S \subseteq S, \Sigma S := \{\sigma f \mid \sigma \in \Sigma, f \in S\} \subseteq S, -1 \notin S, S \cup (-S) = \mathbb{R}[x_1, \dots, x_n], S \cap (-S) = \{0\}.$$

Because $f_1, f_2, f_3 \in S$, it follows from these properties that $M(f_1, f_2, f_3) \subseteq S$. On the other hand, we have $f - \rho \in -S$ for all $\rho \in \mathbb{R}$, which implies $f - \rho \notin M(f_1, f_2, f_3)$ for all $\rho \in \mathbb{R}$. Therefore from Corollary 3.3, all SDP relaxation problems (4) with relaxation order $r \geq 1$ obtained from POP (6) are weakly infeasible. In addition, this example means that one can generate weakly infeasible SDP relaxation problems by applying Lasserre's SDP relaxation into POPs which satisfy $f \in -S$ and $f_1, \dots, f_m \in S$.

Remark 3.5. *The strong duality holds for all SDP relaxation problems in Example 3.4. Indeed, the original POP (6) has an interior feasible solution. It follows from Theorem 4.2 in* [3] *that the dual of the SDP relaxation problem with relaxation order r has an interior feasible solution for all $r \geq 1$, and thus the strong duality holds due to Theorem 2.1. This implies that the optimal values of duals of all SDP relaxation problems are also $-\infty$.*

In addition, an assumption of Lasserre's theorem [3, Theorem 4.2] *on the asymptotic convergence of the optimal values of SDP relaxation problems does not hold. Indeed, the quadratic module $M(f_1, f_2, f_3)$ generated by f_1, f_2, f_3 is not Archimedean because $N - x^2 - y^2 \in -S$ for any $N \in \mathbb{N}$. Therefore, we cannot ensure theoretically that one can obtain the optimal value of POP (6) by applying Lasserre's SDP relaxation. However, as we will see in Section 4, we can obtain the optimal value by solving the SDP relaxation problems with higher relaxation order by SeDuMi and SDPA.*

It should be noted that if POP does not satisfy $\deg(f) < 2r$ in Theorem 3.1, then the resulting SDP relaxation problem with relaxation order r may be strongly infeasible even if POP satisfies $f - \rho \notin M(f_1, \dots, f_m)$ for all $\rho \in \mathbb{R}$. We give such an example.

Example 3.6. *For POP (6) in Example 3.4, we replace $f(x, y) = -x - y$ by $f(x, y) = -x^2 - y^2$ and apply Lasserre's SDP relaxation with relaxation order $r = 1$ into the POP. Clearly, this POP does not satisfy $\deg(f) < 2r$ although it satisfies $f - \rho \notin M(f_1, f_2, f_3)$ for all $\rho \in \mathbb{R}$. Any dual improving ray $\tilde{z} \in \mathbb{R}^6$ for the resulting SDP relaxation problem (4) must satisfy*

$$-\tilde{z}_{(2,0)} - \tilde{z}_{(0,2)} < 0, \tilde{z}_{(0,0)} = 0, \begin{pmatrix} \tilde{z}_{(0,0)} & \tilde{z}_{(1,0)} & \tilde{z}_{(0,1)} \\ \tilde{z}_{(1,0)} & \tilde{z}_{(2,0)} & \tilde{z}_{(1,1)} \\ \tilde{z}_{(0,1)} & \tilde{z}_{(1,1)} & \tilde{z}_{(0,2)} \end{pmatrix} \in \mathbb{S}^3_+, -\tilde{z}_{(1,0)}, -\tilde{z}_{(0,1)}, \tilde{z}_{(1,1)} \geq 0,$$

where $\tilde{z} := (\tilde{z}_{(0,0)}, \tilde{z}_{(1,0)}, \tilde{z}_{(0,1)}, \tilde{z}_{(2,0)}, \tilde{z}_{(1,1)}, \tilde{z}_{(0,2)})^T$. It is easy to find such a vector. Indeed, the vector $\tilde{z} = (0, 0, 1, 0, 1)^T$ is an improving ray and thus the SDP relaxation problem (4) with relaxation order $r = 1$ is strongly infeasible. In contrast, it follows from Corollary 3.3 that SDP relaxation problem (4) with relaxation order $r \geq 2$ are weakly infeasible.

Remark 3.7. *For a given POP, if the quadratic module $M(f_1, \dots, f_m)$ is Archimedean, then there exists $\hat{\rho} \in \mathbb{R}$ such that $f - \hat{\rho} \in M(f_1, \dots, f_m)$. Indeed, because the quadratic module is Archimedean, there exists $N \in \mathbb{N}$ for which $N - \sum_{i=1}^n x_i^2 \in M(f_1, \dots, f_m)$, which implies that the feasible region of the POP is compact. For sufficiently small $\hat{\rho}$, we have $f - \hat{\rho} > 0$ over the feasible region. Therefore it follows from Theorem 2.2 that $f - \hat{\rho} \in M(f_1, \dots, f_m)$.*

On the other hand, even if $M(f_1, \dots, f_m)$ is not Archimedean, there may exist $\rho \in \mathbb{R}$ such that $f - \rho \in M(f_1, \dots, f_m)$. Indeed, in Example 3.4, we replace $f(x, y) = -x - y$ by $f(x, y) = x + y$. Then the quadratic module is not Archimedean. But, we have $f - 1 = 1 \cdot (x - 0.5) + 1 \cdot (y - 0.5) + 0 \cdot (0.5 - xy) \in M(f_1, \dots, f_m)$.

4. The computation result

In this section, we solve SDP relaxation problems of POP (6) in Example 3.4 by SDP solvers, SeDuMi [12] and SDPA [1]. We observe that (i) SeDuMi and SDPA cannot detect the infeasibility, and some values returned by SeDuMi and SDPA are the almost same as the optimal value -1.5 of POP (6), (ii) by choosing an appropriate parameter for SDPA, SDPA can detect that at least primal or dual SDPs is infeasible. One of the reasons for (i) is numerical errors, such as round-off errors, in the practical computation of primal-dual interior-point methods. In Section 5.2, we discuss the reason more in detail.

To solve the SDP relaxation problems of POP (6), we use a computer with 4 Intel Xeon 2.66 GHz cpus and 8GB memory. Table 1 shows the numerical result by SeDuMi 1.21. Asterisks in the first column of

Table 1 indicate that SeDuMi retunes the message "Run into numerical problems". This implies that SeDuMi terminates before it finds an accurate optimal solution for the SDP relaxation problems (4). In Table 1, the second and the third columns show the optimal values of the SDP relaxation problems (4) and its dual. The fourth to the ninth columns show DIMACS errors printed by SeDuMi. DIMCAS errors are measures for the optimality on the obtained value and solution, and are defined in [6]. We see that SeDuMi can not detect the infeasiblity and return the optimal value of POP (6) at $r \geq 3$. Moreover, we observe that the obtained solutions are very accurate because DIMACS errors for all SDP relaxation problems are sufficiently small.

TABLE 1. The numerical results and DIMACS errors by SeDuMi.

r	θ_P	θ_D	error 1	error 2	error 3	error 4	error 5	error 6
1*	-5.0832606527e+07	-7.7672643211e+07	7.8e-09	0.0e+00	0.0e+00	0.0e+00	-2.1e-01	8.6e-01
2	-7.1309019292e+02	-7.6471569258e+02	5.7e-10	0.0e+00	0.0e+00	0.0e+00	-3.5e-02	2.9e-01
3*	-1.4999999877e+00	-1.4999999933e+00	7.3e-09	0.0e+00	0.0e+00	3.0e-10	-1.4e-09	2.2e-08
4*	-1.4999999932e+00	-1.4999999947e+00	2.9e-09	0.0e+00	0.0e+00	2.0e-10	-3.5e-10	8.8e-09
5	-1.4999999978e+00	-1.4999999984e+00	8.3e-10	0.0e+00	0.0e+00	5.4e-11	-1.3e-10	2.7e-09
6	-1.4999999967e+00	-1.4999999976e+00	1.2e-09	0.0e+00	0.0e+00	6.4e-11	-2.3e-10	3.7e-09
7	-1.4999999962e+00	-1.4999999967e+00	1.3e-09	0.0e+00	0.0e+00	7.7e-11	-1.2e-10	4.4e-09
8*	-1.4999999946e+00	-1.4999999952e+00	1.7e-09	0.0e+00	0.0e+00	9.5e-11	-1.5e-10	6.1e-09
9	-1.4999999949e+00	-1.4999999955e+00	1.6e-09	0.0e+00	0.0e+00	7.8e-11	-1.3e-10	5.6e-09
10*	-1.4999999509e+00	-1.4999999556e+00	1.5e-08	0.0e+00	0.0e+00	6.9e-10	-1.2e-09	5.1e-08

TABLE 2. The numerical results and DIMACS errors by SDPA.

r	θ_P	θ_D	Status	error 1	error 2	error 3	error 4	error 5	error 6
1	-2.4383121069e+04	-4.8766242118e+04	pFEAS	7.2e-06	0.0e+00	3.7e-17	1.1e-16	-3.3e-01	5.3e-16
2	-6.0086464934e+01	-6.3081102925e+01	pFEAS	4.3e-07	0.0e+00	3.3e-10	0.0e+00	-2.4e-02	1.5e-01
3	-1.5001158225e+00	-1.5001158225e+00	pdOPT	2.0e-09	0.0e+00	2.6e-12	0.0e+00	-5.6e-17	1.1e-04
4	-1.5000010834e+00	-1.4999981434e+00	pdFEAS	2.3e-10	0.0e+00	2.8e-12	0.0e+00	7.3e-07	1.6e-06
5	-1.5000003346e+00	-1.4999999009e+00	pdFEAS	2.1e-10	0.0e+00	2.0e-07	0.0e+00	1.1e-07	1.1e-06
6	-5.0000000000e+01	-0.0000000000e+00	noINFO	1.3e+02	2.1e-01	1.4e+02	0.0e+00	9.8e-01	6.7e+03
7	-1.5000003078e+00	-1.4999997780e+00	pdFEAS	1.4e-09	0.0e+00	2.4e-07	0.0e+00	1.3e-07	2.0e-06
8	-1.5000000904e+00	-1.4999999929e+00	pdOPT	6.8e-10	0.0e+00	3.3e-07	0.0e+00	2.4e-08	8.0e-07
9	-1.5000003952e+00	-1.4999995706e+00	pdFEAS	4.5e-08	0.0e+00	3.4e-07	0.0e+00	2.1e-07	4.4e-06
10	-1.6090094249e+00	-1.4203975957e+00	noINFO	2.0e-03	0.0e+00	1.2e-06	0.0e+00	4.7e-02	1.6e-01

Table 2 shows the numerical results by SDPA 7.3.4. In Table 2, the fourth column shows the status of SDPA. Specially, "pdOPT" implies that SDPA solved the problem normally. At $r = 6$, SDPA fails to obtain the accurate value and return "noINFO" because SDPA cannot execute the Cholesky decomposition used in the interior-point method. The fifth to the tenth columns show the DIMACS errors by SDPA. We see from Table 2 that SDPA does not detect the infeasibility and that the obtained optimal values are approximately equal to -1.5. We remark that except for $r = 3, 8$, SDPA does not terminate before it finds accurate solutions. The numerical behavior of SDPA for these SDP relaxation problems are different form SeDuMi.

Moreover, we choose parameters for "Stable_but_Slow" described in the manual of SDPA and solve SDP relaxation problems with relaxation order $r = 1, \ldots, 10$. Table 3 shows the numerical results. We observe that SDPA returns "pdINF" for $r \geq 5$, which implies that SDPA detects that at least one of primal and dual SDPs is infeasible for those SDP relaxation problems. Because those SDP relaxation problems are weakly infeasible and its dual SDPs have an interior feasible point, due to Theorem 4.25 of [13], primal-dual path-follwoing algorithms for self-dual embedding program can detect this infeasibility, theoretically. Despite of the fact that the algorithm of SDPA is not primal-dual path-follwoing algorithm for self-dual embedding program, SDPA detects the infeasibility.

TABLE 3. The numerical results and DIMACS errors by SDPA with parameter "Stable_but_Slow".

r	θ_P	θ_D	Status	error 1	error 2	error 3	error 4	error 5	error 6
1	-1.7064901829e+05	-1.8122348764e+05	pUNBD	9.0e-05	0.0e+00	0.0e+00	0.0e+00	-3.0e-02	2.0e+01
2	-1.8325109245e+02	-2.0240534758e+02	pdFEAS	4.2e-08	0.0e+00	9.6e-12	0.0e+00	-5.0e-02	2.7e-01
3	-1.5269996782e+00	-1.5269996782e+00	pdOPT	2.9e-10	0.0e+00	1.0e-08	0.0e+00	5.5e-17	1.2e-02
4	-1.5000001084e+00	-1.4999996657e+00	pdFEAS	3.7e-11	0.0e+00	8.9e-09	0.0e+00	1.1e-07	2.9e-07
5	-1.3502461494e+03	-1.3251763366e+00	pdINF	6.5e-05	0.0e+00	9.6e-06	0.0e+00	1.0e+00	2.4e+00
6	-1.7789380603e+03	-1.3347554750e+00	pdINF	2.2e-04	0.0e+00	2.2e-05	0.0e+00	1.0e+00	3.2e+00
7	-3.9122087504e+03	-1.3458138318e+00	pdINF	9.2e-04	0.0e+00	1.2e-04	0.0e+00	1.0e+00	4.5e+00
8	-5.0232346525e+03	-1.3514845881e+00	pdINF	3.4e-03	0.0e+00	2.4e-04	0.0e+00	1.0e+00	5.6e+00
9	-6.7273096807e+03	-1.3569571939e+00	pdINF	1.2e-02	0.0e+00	4.5e-04	0.0e+00	1.0e+00	6.6e+00
10	-1.0096490427e+04	-1.3627793996e+00	pdINF	3.1e-02	0.0e+00	9.1e-04	0.0e+00	1.0e+00	7.3e+00

5. Discussions

5.1. **Weak infeasibility.** We seldom see weakly infeasible SDP problems in applications except for artificial examples. In this sense, it is interesting that weakly infeasible SDP problems can be obtained by applying Lasserre's SDP relaxation into POP (6).

As mentioned in [7], it is very difficult to detect the infeasibility of a weakly infeasible SDP problem because the SDP become either feasible or infeasible by perturbing it. The weakly infeasible SDP problems in Example 3.4 will be useful for measuring the performance of algorithms for detecting the infeasibility of a given SDP. Indeed, by tuning parameters of SDPA, SDPA is successful in detecting the infeasibility in Section 4.

5.2. **Similar computational results.** In [2, 15], we can see similar numerical behaviors of SeDuMi and SDPA which we have seen in Section 4. In [2], the authors applied Lasserre's SDP relaxation into the following unconstrained POP:

$$f^* := \inf\left\{f(x,y) := x^4y^2 + x^2y^4 - x^2y^2 \,\middle|\, (x,y) \in \mathbb{R}^2\right\}, \tag{7}$$

where $f^* = -1/27$, and the objective function $f - f^*$ (called the dehomogenized Motzkin polynomial) is nonnegative but not SOS. In the case of unconstrained POP, the condition in Corollary 3.3 is equivalent to the $f - \rho \notin \Sigma$ for all $\rho \in \mathbb{R}$. Consequently, for $r > 3$, all SDP relaxation problems with relaxation order r are weakly infeasible due to Corollary 3.3. Despite of this fact, it is reported in [2] that for those SDP relaxation problems with higher relaxation order, the values returned by SDP solvers are the same as the optimal value of POP (7). In [15], the authors deal with a simple one-dimensional POP. Unlike cases of this note and [2], although all the resulting SDP relaxation problems are feasible, the values returned by SDP solvers are incorrect but the same as the one-dimensional POP.

For the two POPs, the optimal values of the POPs are obtained by Lasserre's SDP relaxation numerically, while one cannot ensure the convergence of Lasserre's SDP relaxation theoretically. Indeed, $f - f^*$ is not SOS in the first POP and the quadratic module associated with the second POP is not Archimedean. However, in [4], it is shown that for a unconstrained POP, by adding an SOS $\tilde{\sigma}$ with small coefficients into the objective function f, the polynomial $f + \tilde{\sigma} - f^*$ can be decomposed into an SOS. This result implies that SDP solvers may return the optimal value of a unconstrained POP due to the numerical error in the numerical computation. For the POP dealt with in [15], a similar result related to this fact is given. Therefore, we obtain the following conjecture on Example 3.4 from these discussion:

Conjecture 5.1. *Let f, f_1, f_2, f_3 be as in Example 3.4. For any $\epsilon > 0$, there exist $r_0 \in \mathbb{N}$ and a polynomial $q \in M_{r_0}(f_1, f_2, f_3)$ such that*

$$f + \epsilon q - f^* \in M(f_1, f_2, f_3).$$

This conjecture ensures that SDP solvers return the optimal value of POP (6) by choosing sufficiently large relaxation order r. Moreover, one may be able to extend this conjecture into a POP with the compact feasible region. The proof of this conjecture or giving a counterexample and the extension of this conjecture into a general case are future works.

Appendix A. A Proof of Lemma 3.2

We prove Lemma 3.2 by induction on r. In the case where $r = 1$, we obtain the following condition from $y_0 = 0$ and $L_1(y)$:

$$L_1(y) = \begin{pmatrix} 0 & y_{e_1} & \cdots & y_{e_n} \\ y_{e_1} & y_{2e_1} & \cdots & y_{e_1+e_n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{e_n} & y_{e_1+e_n} & \cdots & y_{2e_n} \end{pmatrix} \in \mathbb{S}^{n+1}_+,$$

where e_i is the i-th standard vector. Clearly, it follows from this condition that $y_{e_i} = 0$ for all $i = 1, \ldots, n$. This implies $y_\alpha = 0$ for all $\alpha \in \mathbb{N}^n_1$.

We assume that this lemma holds for $r - 1$. Because $L_r(y)$ is positive semidefinite, the principal submatirx $L_{r-1}(y)$ of $L_r(y)$ is also positive semidefinite, and thus $y_\alpha = 0$ for all $\alpha \in \mathbb{N}^n_{2r-3}$. It is sufficient to prove the two cases under the assumption $L_r(y) \in \mathbb{S}^{\#(\mathbb{N}^n_r)}_+$ and $y_0 = 0$:

(i) $\alpha \in \mathbb{N}^n_{2r-2} \setminus \mathbb{N}^n_{2r-3}, y_\alpha = 0$,
(ii) $\alpha \in \mathbb{N}^n_{2r-1} \setminus \mathbb{N}^n_{2r-2}, y_\alpha = 0$.

For α in (i), there exist $\delta_1 \in \mathbb{N}^n_{r-2}$ and $\delta_2 \in \mathbb{N}^n_r$ such that $\alpha = \delta_1 + \delta_2$. Let $\beta = 2\delta_1$ and $\gamma = 2\delta_2$. From the definition of $L_r(y)$ and this decomposition of α, the following matrix is a principal submatrix of $L_r(y)$:

$$\begin{pmatrix} y_\beta & y_\alpha \\ y_\alpha & y_\gamma \end{pmatrix}.$$

Thus, we obtain

$$\begin{pmatrix} y_\beta & y_\alpha \\ y_\alpha & y_\gamma \end{pmatrix} \in \mathbb{S}^2_+.$$

On the other hand, $y_\beta = 0$ because $\beta = 2\delta_1 \in \mathbb{N}^n_{2r-4} \subseteq \mathbb{N}^n_{2r-3}$. From this condition and $y_\beta = 0$, we obtain $y_\alpha^2 \le y_\beta y_\gamma$, and thus $y_\alpha = 0$.

For α in (ii), there exist $\delta_1 \in \mathbb{N}^n_{r-1}$ and $\delta_2 \in \mathbb{N}^n_r$ such that $\alpha = \delta_1 + \delta_2$. By applying the same argument as (i), we can prove (ii). This completes the proof.

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