Representing quadratically constrained quadratic programs as generalized copositive programs*

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Abstract

We show that any nonconvex quadratically constrained quadratic program (QCQP) can be represented as a generalized copositive program. In fact, we provide two representations. The first is based on the concept of completely positive (CP) matrices over second order cones, while the second is based on CP matrices over the positive semidefinte cone. Our analysis assumes that the feasible region of the QCQP is nonempty and bounded.

Keywords: Copositive programming, quadratically constrained quadratic programs, convex representation.

Mathematics Subject Classification: 90C20, 90C25, 90C26

1 Introduction

Consider the nonconvex quadratically constrained quadratic program (QCQP)

$$\min_{x \in \Re^n} \left\{ \langle x, Qx \rangle + 2\langle q, x \rangle \;\middle|\; \langle x, Q^j x \rangle + 2\langle q^j, x \rangle \le \chi^j \ (j = 1, ..., r) \right\}, \tag{1}$$

where, in particular, Q and Q^j are general $n \times n$ symmetric matrices. Let F denote the feasible set of (1).

Problem (1) is quite general, encompassing many classes of difficult optimization problems, and thus is NP-hard. For example, (1) models quadratic programs (when all constraints are linear) and binary integer programs (when all constraints are linear and the conditions $x_i^2 = x_i \Leftrightarrow x_i \in \{0, 1\}$ are present). Also, polynomial optimization problems may be reduced

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to (1) by introducing extra variables and constraints. For example, a cubic term $x_i x_j x_k$ can be made quadratic with an extra variable and constraint: $x_i y$, where $y = x_j x_k$. For more background on QCQP, we refer the reader to [1, 2].

One method for globally solving (1) is first to linearize all quadratics by introducing a new symmetric matrix variable $X = xx^T$, e.g., $\langle x, Qx \rangle + 2\langle q, x \rangle = \langle Q, X \rangle + 2\langle q, x \rangle$, where $\langle Q, X \rangle := \text{trace}(QX)$. Then the feasible region is convexified via $\mathcal{C}'(F) := \text{cl conv}\{(x, X) \mid x \in F,\ X = xx^T\}$, which allows (1) to be cast as the equivalent problem of minimizing $\langle Q, X \rangle + 2\langle q, x \rangle$ over $(x, X) \in \mathcal{C}'(F)$. So, in a certain sense, solving (1) is equivalent to characterizing $\mathcal{C}'(F)$. In fact, many existing techniques for QCQP can be interpreted as providing tractable relaxations of $\mathcal{C}'(F)$; see [3, 4].

Characterizing $\mathcal{C}'(F)$ in a tractable manner is difficult; if it were easy, then we could solve (1) easily. For the case of nonconvex quadratic programming when all $Q^j = 0$, however, progress has been made using the dual concepts of *copositive* and *completely positive* (CP) matrices (see Section 1.1 below for definitions). In particular, Burer [5] has shown that every nonconvex QP is equivalent to an explicit *copositive program*, which is a linear conic program over the convex cone of CP matrices. This approach focuses the difficulty of nonconvex QPs completely on the CP matrices. Said differently, any knowledge concerning CP matrices can be uniformly applied to help solve all nonconvex QPs. Fortunately, a fair amount is known about how to approximate CP matrices [6, 7, 8, 9]. Burer's result also holds for specific types of quadratic constraints such as the binary condition $x_i^2 = x_i$.

Burer [10] and Eichfelder and Povh [11] extend the results of [5] to the case of nonconvex QPs with an additional convex cone constraint $x \in \mathcal{K}$. Here again, certain types of quadratic constraints are allowed [10, Section 2.4]. In this case, instead of the CP matrices, the cone of interest is the generalized CP matrices over $\mathcal{K}$: $\mathcal{C}(\mathcal{K}) := \text{cl conv}\{xx^T \mid x \in \mathcal{K}\}$. In particular, Burer suggests that standard cones $\mathcal{K}$ such as the nonnegative orthant, the second-order cone, and the semidefinite cone (or direct products of these) could be of particular importance.

In this paper, we use the results of [10] to show how the nonconvex QCQP (1)—under the assumption of a bounded, nonempty feasible region—can be expressed as a linear conic program over a cone of the form $\mathcal{C}(\mathcal{K})$. In fact, we provide two approaches: one with $\mathcal{K}$ being, in essence, the Cartesian product of second order cones, and the other with $\mathcal{K}$ the Cartesian product of a positive semidefinite cone and a nonnegative orthant.

We remark that our results complement a recent procedure by Peña et al. [12], which recasts polynomial optimization problems (even with unbounded feasible sets) as linear conic programs over "higher order" CP matrices. In particular, their approach involves a different generalization of CP matrices than the one we employ here.

1.1 Notation and terminology

We use $\Re^n$ to denote n-dimensional Euclidean space, and $\Re^n_+$ is the nonnegative orthant. $\mathcal{S}^n$ is the space of $n \times n$ symmetric matrices. For $X \in \mathcal{S}^n$, we write $X \succeq 0$ or $X \in \mathcal{S}^n_+$ if X is positive semidefinite. The second order cone in $\Re^n$ is defined as $\mathrm{SOC}(n) := \{x \in \Re^n \mid \sqrt{x_2^2 + ... + x_n^2} \leq x_1\}$. The notation $\circ$ between vectors indicates the Hadamard product, and $\mathrm{diag}(X)$ extracts the diagonal of X as a vector.

For any set $D \subseteq \Re^n$, we define $\mathcal{C}(D) := \mathrm{cl\,conv}\{X \,|\, x \in D,\, X = xx^T\}$ and $\mathcal{C}'(D) := \mathrm{cl\,conv}\{(x, X) \,|\, x \in D,\ X = xx^T\}$. Regarding $\mathcal{C}(D)$, we will be particularly interested in the case when D is a direct product of nonnegative orthants, second order cones, and positive semidefinite cones. In this case, $\mathrm{conv}\{X \,|\, x \in D,\, X = xx^T\}$ is already closed [13, Lemma 1], and $\mathcal{C}(D)$ is a convex cone. The closure operation is also unnecessary for $\mathcal{C}'(D)$ when D is compact, which will be the case whenever we use $\mathcal{C}'(\cdot)$. When $D = \Re^n_+$, the closed convex cone $\mathcal{C}(\Re^n_+)$ is called the *completely positive cone* and consists of *completely positive matrices*. The dual cone $\mathcal{C}(\Re^n_+)^* \subseteq \mathcal{S}^n$ is called the *copositive cone* with *copositive matrices*. More generally, we call $\mathcal{C}(D)$ the *generalized completely positive cone over* D and its dual cone in $\mathcal{S}^n$ the *generalized copositive cone*. Eichfelder and Jahn [14, 15] study these cones under the name *set-semidefinite cones*. (A slight difference is, in [14, 15], set-semidefinite cones are defined in the space of square matrices, with no assumption of symmetry.) We also use the abbreviation "CP" for "completely positive" and "GCP" for "generalized CP."

Note that one must consider semidefinite matrices in vector form to define $\mathcal{C}(\mathcal{S}^n_+)$ properly. For example, one could express $\mathcal{S}^n_+$ as a subset of $\Re^{n^2}$ with columns of the semidefinite matrix stacked in order. In this way, $\mathcal{C}(\mathcal{S}^n_+)$ would consist of matrices of size $n^2 \times n^2$. Other representations of $\mathcal{S}^n_+$ in vector form are possible. We will not specify a particular encoding and will mainly use matrix notation in $\mathcal{S}^n$. However, whenever $\mathcal{C}(\mathcal{S}^n_+)$ is presented, it is understood to be based on a chosen vector encoding.

2 Main Technical Result

The main tool in our analysis will be the following theorem of GCP formulations for sets $\mathcal{C}'(D)$, where D has a certain form.

Theorem 1. *[10, Theorem 3] Consider a nonempty set of the form $\mathcal{L} \cap \mathcal{Q} \subseteq \Re^d$, where $\mathcal{L} := \{z \in \mathcal{K} : Az = b\}$ is a bounded, affine slice of a closed, convex cone $\mathcal{K}$ and*

$$\mathcal{Q} := \left\{ z \,|\, \langle z, G^k z\rangle + 2\langle g^k, z\rangle = \gamma^k \ (k = 1, ..., \ell) \right\}$$

is the intersection of level sets of several quadratic functions, where γ^k is the maximum value of $\langle z, G^k z\rangle + 2\langle g^k, z\rangle$ over $z \in \mathcal{L}$. Then $\mathcal{C}'(\mathcal{L} \cap \mathcal{Q})$ equals

$$\left\{ (z, Z) \;\middle|\; \begin{pmatrix} 1 & z^T \\ z & Z \end{pmatrix} \in \mathcal{C}(\Re_+ \times \mathcal{K}), \quad \begin{array}{c} Az = b, \ \mathrm{diag}(AZA^T) = b \circ b \\ \langle G^k, Z\rangle + 2\langle g^k, z\rangle = \gamma^k \ (k = 1, \ldots, \ell) \end{array} \right\}.$$

3 Representation With Second Order Cones

In this section, we show how to represent (1) as a linear conic program over the GCP cone $\mathcal{C}(\Re_+ \times \mathcal{K})$, where $\mathcal{K}$ is a direct product of second order cones. We make the assumption that F, the feasible region of (1), is nonempty and bounded and, in particular, is contained in the unit ball $\{x \mid \langle x, x\rangle \le 1\}$ (perhaps after a simple variable scaling).

We first argue that, by lifting up one dimension, we can represent F as the intersection of a unit sphere and a quadratically constrained convex region. We require the following proposition:

Proposition 1. *Choose $\lambda^j \ge 0$ $(j = 1, \ldots, r)$ arbitrarily and define $\tilde{\mathcal{L}} \cap \tilde{\mathcal{Q}}$, where*

$$\begin{aligned} \tilde{\mathcal{L}} &:= \left\{ w \in \Re^{n+1} \mid \langle w, w\rangle \le 1, \langle w, P^j w\rangle + 2\langle p^j, w\rangle \le \rho^j \ (j = 1, \ldots, r) \right\}, \\ \tilde{\mathcal{Q}} &:= \left\{ w \in \Re^{n+1} \mid \langle w, w\rangle = 1 \right\}, \end{aligned}$$

and

$$P^j = \begin{pmatrix} Q^j + \lambda^j I & 0 \\ 0 & \lambda^j \end{pmatrix}, \qquad p^j = \begin{pmatrix} q^j \\ 0 \end{pmatrix}, \qquad \rho^j = \chi^j + \lambda^j.$$

Then $F = \pi(\tilde{\mathcal{L}} \cap \tilde{\mathcal{Q}})$, where $\pi : \Re^{n+1} \to \Re^n$ is projection onto the first n coordinates.

Proof. Straightforward by identifying x with $\pi(w)$ and using the assumption that $F \subseteq \{x \mid \langle x, x\rangle \le 1\}$. In particular, the squared variable w_{n+1}^2 can be viewed as a slack for the inequality $\langle x, x\rangle \le 1$. □

If all λ^j are sufficiently large such that $P^j \succeq 0$ (which we assume from now on), then $\tilde{\mathcal{L}} \cap \tilde{\mathcal{Q}}$ is the intersection of a sphere and a convex region defined by convex quadratic inequalities. Furthermore, problem (1) may be recast as the following optimization:

$$\min_{w \in \Re^{n+1}} \left\{ \langle w, Pw\rangle + 2\langle p, w\rangle \;\middle|\; w \in \tilde{\mathcal{L}} \cap \tilde{\mathcal{Q}} \right\}, \qquad \text{where } P = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}, p = \begin{pmatrix} q \\ 0 \end{pmatrix}. \tag{2}$$

In order to apply Theorem 1, we next show that, by introducing auxiliary variables and constraints, we can represent $\tilde{\mathcal{L}} \cap \tilde{\mathcal{Q}}$ in the form $\mathcal{L} \cap \mathcal{Q}$, where $\mathcal{L}$ and $\mathcal{Q}$ satisfy the conditions of Theorem 1. We use the following proposition due to Alizadeh and Goldfarb [16]:

Proposition 2. *For each $j = 1, \ldots, r$, let $P^j = (R^j)^T R^j$ for some matrix $R^j \in \Re^{\text{rank}(P^j)\times(n+1)}$. Then a point $w \in \mathbb{R}^{n+1}$ satisfies $\langle w, P^j w\rangle + 2\langle p^j, w\rangle \le \rho^j$ if and only if*

$$\begin{pmatrix} \frac{1}{2}(1+\rho^j) - \langle p^j, w\rangle \\ \frac{1}{2}(1-\rho^j) + \langle p^j, w\rangle \\ R^j w \end{pmatrix} \in \text{SOC}(\text{rank}(P^j)+2).$$

Now, define $d := (n+2) + \sum_{j=1}^r (\text{rank}(P^j)+2)$ and introduce a long vector of variables $z = (w_0; w; w^1; \ldots; w^r) \in \Re^d$, where $w_0 \in \Re$, $w \in \Re^{n+1}$, and $w^j \in \Re^{\text{rank}(P^j)+2}$. Also define the following subsets of $\Re^d$: $\mathcal{Q} := \{z \,|\, \langle w, w\rangle = 1\}$, and

$$\mathcal{L} := \left\{ z \in \mathcal{K} \;\middle|\; w_0 = 1,\ w^j = \begin{pmatrix} \frac{1}{2}(1+\rho^j) - \langle p^j, w\rangle \\ \frac{1}{2}(1-\rho^j) + \langle p^j, w\rangle \\ R^j w \end{pmatrix} \quad (j = 1, \ldots, r) \right\},$$

where

$$\mathcal{K} := \text{SOC}(n+2) \times \text{SOC}(\text{rank}(P^1)+2) \times \cdots \times \text{SOC}(\text{rank}(P^r)+2).$$

Letting $\tilde{\pi} : \Re^d \longrightarrow \Re^{n+1}$ denote projection onto coordinates 2 through $n+2$, i.e., $\tilde{\pi}(z) = w$, then it is easy to see by Proposition 2 that $\tilde{\mathcal{L}} \cap \tilde{\mathcal{Q}} = \tilde{\pi}(\mathcal{L} \cap \mathcal{Q})$. Obviously we may define A and b to write $\mathcal{L} = \{z \in \mathcal{K} \,|\, Az = b\}$, and $\mathcal{L}$ is bounded because $\langle w, w\rangle \le w_0 = 1$ and each w^j depends affinely on w. Then since $F \ne \emptyset$ ensures $1 = \max_{z\in\mathcal{L}} \langle w, w\rangle$, $\mathcal{L}$ and $\mathcal{Q}$ satisfy all conditions of Theorem 1. We thus obtain a generalized CP representation of $\mathcal{C}'(\mathcal{L} \cap \mathcal{Q})$.

Now properly defining $H \in \mathcal{S}^d$ and $h \in \Re^d$, (2) can be recast as:

$$\min_{z\in\Re^d} \left\{ \langle z, Hz\rangle + 2\langle h, z\rangle \;\middle|\; z \in \mathcal{L} \cap \mathcal{Q} \right\}.$$

This, together with the CP representation of $\mathcal{C}'(\mathcal{L} \cap \mathcal{Q})$ and the linearization procedure described in the Introduction, we obtain a generalized CP formulation of (1):

Corollary 1. *Problem (1) is equivalent to a linear conic program over $\mathcal{C}(\Re_+ \times \mathcal{K})$, where $\mathcal{K}$ is the product of $r+1$ second-order cones.*

4 Representation With a Semidefinite Cone

In this section, we describe another representation of (1) as a generalized CP program. We assume again that $F \subseteq \{x | \langle x, x\rangle \le 1\}$.

First, note that (1) can be reformulated as follows:

$$\min_{x,X} \left\{ \langle Q, X\rangle + 2\langle q, x\rangle \;\middle|\; \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0, \quad \begin{array}{c} \langle Q^j, X\rangle + \langle q^j, x\rangle \le \chi^j \ (j = 1, \ldots, r) \\ \mathrm{trace}(X) \le 1, \ x_i^2 - X_{ii} = 0 \ (i = 1, \ldots, n) \end{array} \right\}. \quad (3)$$

The reformulation is valid because $X - xx^T \succeq 0$ with zero diagonal ensures $X = xx^T$. Also, $\mathrm{trace}(X) \le 1$ is equivalent to $\langle x, x\rangle \le 1$. Next, we introduce a slack vector $s \in \Re^r_+$, and define the sets $\mathcal{L}$ and $\mathcal{Q}$ as

$$\mathcal{L} := \left\{ (x, X, s) \;\middle|\; \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0, \ s \ge 0, \quad \begin{array}{c} \mathrm{trace}(X) \le 1 \\ \langle Q^j, X\rangle + 2\langle q^j, x\rangle + s_j = \chi^j \ (j = 1, \ldots, r) \end{array} \right\},$$

$$\mathcal{Q} := \left\{ (x, X, s) \,|\, x_i^2 - X_{ii} = 0 \ (i = 1, ..., n) \right\}.$$

Then (3) is equivalent to optimizing $\langle Q, X\rangle + 2\langle q, x\rangle$ over $\mathcal{L} \cap \mathcal{Q}$.

It is easy to see that $\mathcal{L}$ is bounded and can be written as $\{z \in \mathcal{K} \,|\, Az = b\}$ with $\mathcal{K} = \mathcal{S}^{n+1}_+ \times \Re^r_+$ for some properly defined system $Az = b$. Furthermore, because the inequalities $x_i^2 - X_{ii} \le 0$ are implied by semidefiniteness in $\mathcal{L}$, and $F \ne \emptyset$, one sees that $0 = \max_{(x,X,s)\in\mathcal{L}} (x_i^2 - X_{ii})$. Therefore $\mathcal{L}$ and $\mathcal{Q}$ satisfy all conditions in Theorem 1.

Hence, an application of Theorem 1 and the linearization procedure described in the Introduction section yields a generalized copositive formulation for (1):

Corollary 2. *Problem (1) is equivalent to a linear conic program over $\mathcal{C}(\Re_+ \times \mathcal{K})$, where $\mathcal{K} = \mathcal{S}^{n+1} \times \Re^r_+$.*

5 Additional Remarks

As mentioned in the Introduction, a fair amount is known about approximating the cone $\mathcal{C}(\Re^n_+)$ of CP matrices. While one can, in principle, approximate $\mathcal{C}(\Re^n_+)$ to any accuracy if one is willing to spend the computational effort, the following relaxation is often used in practice: $\mathcal{D}(\Re^n_+) := \{X \succeq 0 \mid X \ge 0\}$. This is the so-called *doubly nonnegative matrices*, and it is known that $\mathcal{C}(\Re^n_+) \subseteq \mathcal{D}(\Re^n_+)$ with equality if and only if $n \le 4$ [17].

Relatively little is known about approximating the GCP cones $\mathcal{C}(\mathcal{K})$ or $\mathcal{C}(\Re_+ \times \mathcal{K})$ as studied in this paper, even in small dimension. In analogy with approximation hierarchies for the CP matrices, Zuluaga et al. [18] introduce generalized hierarchies that could be applied in this case.

As an alternative to their approach, however, we end the paper by proposing a direct generalization of $\mathcal{D}(\Re^n_+)$ that could potentially be of computational interest. Let $\mathcal{K}$ be a closed, convex cone, and let $\mathcal{C}(\mathcal{K})$ be the GCP cone over $\mathcal{K}$. We propose

$$\mathcal{D}(\mathcal{K}) := \{X \succeq 0 \mid Xs \in \mathcal{K}, \ \forall \ s \in \text{Ext}(\mathcal{K}^*)\},$$

where $\text{Ext}(\mathcal{K}^*)$ is the set of extreme rays of the dual cone $\mathcal{K}^*$ of $\mathcal{K}$. Note that, in our cases of interest, $\mathcal{K}$ is self-dual, i.e., $\mathcal{K}^* = \mathcal{K}$. When $\mathcal{K} = \Re^n_+$, this reads $\mathcal{D}(\Re^n_+) = \{X \succeq 0 \mid X_{\cdot i} \in \Re^n_+ \ (i = 1, \dots, n)\}$, which matches. We also have the following straightforward proposition:

Proposition 3. *$\mathcal{C}(\mathcal{K}) \subseteq \mathcal{D}(\mathcal{K})$.*

Proof. It suffices to show that the extreme rays xx^T of $\mathcal{C}(\mathcal{K})$ are in $\mathcal{D}(\mathcal{K})$, where $x \in \mathcal{K}$. Let $s \in \text{Ext}(\mathcal{K}^*)$ be arbitrary. Then $(xx^T)s = \langle x, s\rangle x \in \mathcal{K}$, as desired, because $\langle x, s\rangle \geq 0$. □

In a related, yet different, context, Burer and Anstreicher [19] have proposed something very similar to $\mathcal{D}(\mathcal{K})$, when $\mathcal{K}$ is the product of two second order cones. Borrowing some ideas from their approach, we have been able to show that, when $\mathcal{K}$ is the direct product of nonnegative orthants and second order cones, $\mathcal{D}(\mathcal{K})$ is tractable despite its semi-infinite presentation. In addition, when $\mathcal{K} = \Re^{n_1}_+ \times \text{SOC}(n_2)$, we have $\mathcal{C}(\mathcal{K}) = \mathcal{D}(\mathcal{K})$ if and only if $n_1 = 1$ or $n_1 = n_2 = 2$. A similar result is that, when $\mathcal{K} = \text{SOC}(n_1) \times \text{SOC}(n_2)$ with $n_1 \leq n_2$ without loss of generality, then $\mathcal{C}(\mathcal{K}) = \mathcal{D}(\mathcal{K})$ if and only if the same conditions on n_1 and n_2 hold.

When $\mathcal{K}$ involves direct products with the semidefinite cone, we do not know if $\mathcal{D}(\mathcal{K})$ is tractable or if $\mathcal{C}(\mathcal{K})$ equals $\mathcal{D}(\mathcal{K})$ in some cases.

Overall, we believe the results in this paper motivate further study of the GCP cones $\mathcal{C}(\mathcal{K})$, where $\mathcal{K}$ is the direct product of nonnegative orthants, second-order cones, and semidefinite cones.

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