Strong Dual for Conic Mixed-Integer Programs*

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July 14, 2011

Abstract

Mixed-integer conic programming is a generalization of mixed-integer linear programming. In this paper, we present an extension of the duality theory for mixed-integer linear programming (see [4], [11]) to the case of mixed-integer conic programming. In particular, we construct a subadditive dual for mixed-integer conic programming problems. Under a simple condition on the primal problem, we are able to prove strong duality.

1 Introduction

One of the fundamental goals of optimization theory is the study of structured techniques to obtain bounds on the optimal objective function value for a given class of optimization problems. For a minimization (resp. maximization) problem, upper (resp. lower) bounds are provided by points belonging to the feasible region. Dual bounds, i.e., lower (resp. upper) bounds for a minimization (resp. maximization) problem are typically obtained by constructing various types of dual optimizations problems whose feasible solutions provide these bounds. We will say that an optimization problem is finite it its feasible region is non-empty and the objective function is bounded. A strong dual is typically characterized by two properties:

1. The primal program is finite if and only if the dual program is finite.
2. If the primal and the dual are finite, then the optimal objective function values of the primal and dual are equal.

In the case of linear programming problems and more generally for conic (convex) optimization problems the dual optimization problem is well understood and plays a key role in various algorithmic devices [2]. The subadditive dual for mixed-integer linear programs is also well understood [7, 9, 5, 6, 13]. In this paper, we evaluate the possibility of extending the subadditive dual for mixed-integer conic programs.

The rest of paper is organized as follows. In Section 2, we present the necessary notation, definitions and the statement of our main result. In Section 3, we verify the basic weak duality, i.e., the fact that the dual feasible solutions produce the correct bounds. Apart from weak duality, the proof of strong duality relies on the following additional three results: (i) The finiteness of the primal being equivalent to the finiteness of its continuous relaxation. (ii) Conic duality. (iii) The possibility of constructing a subadditive function defined over $\mathbb{R}^m$ such that it is dual feasible and matches the value function of the primal in a relevant subset of $\mathbb{R}^m$. In Section 4, we develop and present (in case of conic duality) these preliminary results. In Section 5, we present the proof of the strong duality. In particular, in Section 5.1, we present

*This research was partially supported by NSF under grants CMMI-1030422 and CMMI-1030662.

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a sufficient condition for the finiteness of the primal program being equivalent to the finiteness of the dual program. In Section 5.2, we prove that under this sufficient condition, if the primal and dual are finite, then their optimal values must be equal. We make concluding remarks in Section 6.

2 Notation, definitions and main result

Let $A \in \mathbb{R}^{m\times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$. Let $K \subseteq \mathbb{R}^m$ be a full-dimensional, closed and pointed cone. A conic vector inequality is defined as follows ([2]):

Definition 1 (Conic vector inequality). *For $a, b \in \mathbb{R}^m$, $a \succeq_K b$ if and only if $a - b \in K$. In addition, we write $a \succ_K b$ whenever $a - b \in \text{int}(K)$.*

A mixed-integer conic programming problem (the primal optimization problem) is an optimization problem of the following form:

$$(\mathcal{P}) \begin{cases} z^* = \inf & c^t x \\ \quad s.t. & Ax \succeq_K b \\ & x_i \in \mathbb{Z}, \forall i \in \mathcal{I}. \end{cases}$$

where $\mathcal{I} = \{1, \dots, n_1\} \subseteq \{1, \dots, n\}$ is the set of indexes of integer variables.

Notice that $(\mathcal{P})$ is a generalization of a mixed-integer linear programming problem, by setting $K = \mathbb{R}^m_+$. Hence, a natural way of defining a dual optimization problem for mixed-integer conic programming is to generalize the well-known subadditive dual of mixed-integer linear programming (see, for example, [4] and [11]). Consequently, to define the dual of $(\mathcal{P})$, we first present some notation and definitions that are slight variations of those necessary to define the subadditive dual problem for the mixed-integer linear programming case.

Definition 2 (Subadditive function). *Let $\mathcal{S} \subseteq \mathbb{R}^m$. A function $g : \mathcal{S} \mapsto \mathbb{R} \cup \{-\infty\}$ is said to be subadditive if for all $u, v \in \mathcal{S}$ such that $u + v \in \mathcal{S}$, the inequality $g(u + v) \leq g(u) + g(v)$ holds.*

Definition 3 (Nondecreasing w.r.t K). *Let $\mathcal{S} \subseteq \mathbb{R}^m$. A function $g : \mathcal{S} \mapsto \mathbb{R} \cup \{-\infty\}$ is said to be nondecreasing w.r.t K if for $u, v \in \mathcal{S}$, $u \succeq_K v$ implies $g(u) \geq g(v)$.*

We define the subadditive dual problem for $(\mathcal{P})$ as follows:

$$(\mathcal{D}) \begin{cases} \rho^* = \sup & g(b) \\ \quad s.t. & g\left(A^i\right) = c_i, \quad \forall i \in \mathcal{I} \\ & g\left(-A^i\right) = -c_i, \quad \forall i \in \mathcal{I} \\ & \bar{g}\left(A^i\right) = c_i, \quad \forall i \in \mathcal{C} \\ & \bar{g}\left(-A^i\right) = -c_i, \quad \forall i \in \mathcal{C} \\ & g(0) = 0 \\ & g \in \mathcal{F}. \end{cases}$$

where $\mathcal{C} = \{n_1 + 1, \dots, n\}$ is the set of indexes of continuous variables, A^i denotes the ith column of A, for a function $g : \mathbb{R}^m \mapsto \mathbb{R}$ we write $\overline{g}(d) = \limsup_{\delta \to 0^+} \frac{g(\delta d)}{\delta}$ and $\mathcal{F} = \{g : \mathbb{R}^n \mapsto \mathbb{R} \ : g \text{ is subadditive and nondecreasing w.r.t } K\}$.

Notice that when $K = \mathbb{R}^m_+$ we retrieve the subadditive dual for a mixed-integer linear programming problem.

The main result of this paper is to show that strong duality for mixed-integer conic programming holds under the technical condition that

$$\text{there exists } \hat{x} \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} \text{ such that } A\hat{x} \succ_K b. \quad (*)$$

We state this result formally next. Denote $n_2 = n - n_1$.

Theorem 1 (Strong duality)**.** *If there exists $\hat{x} \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ such that $A\hat{x} \succ_K b$, then*

1. *$(\mathcal{P})$ has a finite optimal value if and only if $(\mathcal{D})$ has a finite optimal value.*

2. *If $(\mathcal{P})$ has a finite optimal value, then there exists a function g^* feasible for $(\mathcal{D})$ such that $g^*(b) = z^*$ and, consequently, $z^* = \rho^*$.*

2.1 Form of the primal problem

Before proceeding, in this section, we comment on the generality of form $(\mathcal{P})$. Consider the seemingly more general problem given by

$$(\mathcal{P}') \begin{cases} z^* = \inf & c^t x \\ \quad s.t. & Ex \succeq_{K_1} f \\ & Dx = g \\ & x \in K_2 \\ & x_i \in \mathbb{Z}, \forall i \in \mathcal{I}. \end{cases}$$

where both K_1 and K_2 are full-dimensional, closed and pointed cones. (Here $K_2 = \mathbb{R}^n_+$ is perhaps of particular interest.) Let r be the dimension of vector g. It is well known that $(\mathcal{P}')$ can be written in form $(\mathcal{P})$ by letting $K = K_1 \times \mathbb{R}^{2r}_+ \times K_2$,

$$A = \begin{pmatrix} E \\ D \\ -D \\ I \end{pmatrix} \text{ and } b = \begin{pmatrix} f \\ g \\ -g \\ 0 \end{pmatrix}.$$

Using this representation we can use the results in this paper to construct a subadditive dual of $(\mathcal{P}')$. However, some of the relations between the primal and dual problems can be lost in this translation. For instance, one would like to restrict b to be in an appropriate subspace to account for some of its components being equal to g and $-g$ respectively. For this reason we now consider a special case of $(\mathcal{P}')$ for which we can explicitly write a subadditive dual without using the equivalence. This problem is given by

$$(\mathcal{P}'') \begin{cases} z^{''} = \inf & c^t x \\ \quad s.t. & Ax \succeq_K b \\ & x \in \mathbb{R}^n_+ \\ & x_i \in \mathbb{Z}, \forall i \in \mathcal{I}. \end{cases}$$

For this problem it is fairly straightforward from the rest of this paper to see that a subadditive dual is given by

$$(\mathcal{D}'') \begin{cases} \rho^{''} = \sup & g(b) \\ \quad s.t. & g\left(A^i\right) \le c_i, \quad \forall i \in \mathcal{I} \\ & \bar{g}\left(A^i\right) \le c_i, \quad \forall i \in \mathcal{C} \\ & g(0) = 0 \\ & g \in \mathcal{F}. \end{cases}$$

We formally state this result in the next theorem.

Theorem 2. *1. (Weak duality) For all $x \in \mathbb{R}^n$ feasible for $(\mathcal{P}'')$ and for all $g : \mathbb{R}^m \mapsto \mathbb{R}$ feasible for $(\mathcal{D}'')$, we have $g(b) \leq c^t x$.*

2. (Strong duality) If there exists $\hat{x} \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ such that $A\hat{x} \succ_K b, \hat{x} > 0$, then

(a) $(\mathcal{P}'')$ has a finite optimal value if and only if $(\mathcal{D}'')$ has a finite optimal value.

(b) If $(\mathcal{P}'')$ has a finite optimal value, then there exists a function g^* feasible for $(\mathcal{D}'')$ such that $g^*(b) = z^{''}$ and, consequently, $z^{''} = \rho^{''}$.

The proof of Theorem 2 is analogous to the proofs of Proposition 1 and Theorem 1. For completeness, an alternative proof of Theorem 2, that uses Theorem 1, is presented in Appendix (Section 7).

3 Weak duality

As in the case of mixed-integer linear programming, weak duality is a straightforward consequence of the definition of the subadditive dual. We first require a well-known result relating g and $\bar{g}$ when g is a subadditive function.

Theorem 3 ([8], [5], and [11]). *If $g : \mathbb{R}^m \to \mathbb{R}$ is a subadditive function, then $\forall\ d \in \mathbb{R}^m$ with $\overline{g}(d) < \infty$ and $\forall\ \lambda \geq 0$, $g(\lambda d) \leq \lambda \overline{g}(d)$.*

Proposition 1 (Weak duality). *For all $x \in \mathbb{R}^n$ feasible for $(\mathcal{P})$ and for all $g : \mathbb{R}^m \mapsto \mathbb{R}$ feasible for $(\mathcal{D})$, we have*

$$g(b) \leq c^t x.$$

Proof. Let $u, v \geq 0$ such that $x = u - v$. We have

$$\begin{aligned}
g(b) &\leq g(Ax) \\
&= g(Au - Av) \\
&= g\left(\sum_{i=1}^{n} A^i u_i + \sum_{i=1}^{n} (-A^i) v_i\right) \\
&= g\left(\sum_{i\in\mathcal{I}} A^i u_i + \sum_{i\in\mathcal{I}} (-A^i) v_i + \sum_{i\in\mathcal{C}} A^i u_i + \sum_{i\in\mathcal{C}} (-A^i) v_i\right) \\
&\leq \sum_{i\in\mathcal{I}} g(A^i u_i) + \sum_{i\in\mathcal{I}} g(-A^i v_i) + \sum_{i\in\mathcal{C}} g(A^i u_i) + \sum_{i\in\mathcal{C}} g(-A^i v_i) \\
&\leq \sum_{i\in\mathcal{I}} g(A^i) u_i + \sum_{i\in\mathcal{I}} g(-A^i) v_i + \sum_{i\in\mathcal{C}} \overline{g}(A^i) u_i + \sum_{i\in\mathcal{C}} \overline{g}(-A^i) v_i \\
&= \sum_{i=1}^{n} c_i u_i + \sum_{i=1}^{n} (-c_i) v_i \\
&= c^t x.
\end{aligned}$$

The first inequality relies on the fact that x satisfies $Ax \succeq_K b$ and g is nondecreasing w.r.t K and the second inequality relies on the fact that g is subadditive. The third inequality is based on the subadditivity of g, the fact that $g(0) = 0$ and Theorem 3. □

We obtain the following corollary of Proposition 1.

Corollary 1.

1. *If the primal problem $(\mathcal{P})$ is unbounded, then the dual problem $(\mathcal{D})$ is infeasible.*
2. *If the dual problem $(\mathcal{D})$ is unbounded, then the primal problem $(\mathcal{P})$ is infeasible.*

4 Preliminary results for proving strong duality

4.1 Finiteness of a convex mixed-integer problem being equivalent to the finiteness of its continuous relaxation

In this section, we study a sufficient condition for the finiteness of the primal $(\mathcal{P})$ being equivalent to the finiteness of its continuous relaxation. This condition is required to show that the primal program is finite if and only if the dual program is finite.

The main result of this section is a sufficient condition to have this property in the context of general convex mixed-integer optimization, that is, the feasible region of the primal is of the form $B \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$, where $B \subseteq \mathbb{R}^n$ is a closed convex set and $n = n_1 + n_2$.

The next example shows that the property of finiteness of a convex mixed-integer problem being equivalent to finiteness of its relaxation is not always satisfied, not even when the feasible set is a polyhedron.

Example 1. *Consider the polyhedral set* $B_1 = \{x \in \mathbb{R}^2 \,:\, x_2 - \sqrt{2}x_1 = 0\}$ *and let the objective function be given by* $c = (1, \sqrt{2})$*. In this case* $B_1 \cap \mathbb{Z}^2 = \{0, 0\}$*, so* $\inf\{c^t x : \; x \in B_1 \cap \mathbb{Z}^n\} = 0$*. On the other hand,* $\inf\{c^t x : \; x \in B_1\} = -\infty$*. Therefore, the integer programming problem has finite optimal value, but its relaxation has unbounded objective value.*

When the feasible set is a polyhedron, a well-known sufficient condition for this property to be true is that of the polyhedron to be defined by rational data. However, the following example shows that the property is not necessarily true when the convex set B is full dimensional, $\text{conv}(B \cap \mathbb{Z}^n)$ is a polyhedron and B is conic quadratic representable using rational data.

Example 2. *Consider the set*

$$\begin{aligned} B_2 \quad &= \quad \text{conv}(\{x \in \mathbb{R}^3 \,:\, x_3 = 0, x_1 = 0, x_2 \geq 0\} \\ &\cup \quad \{x \in \mathbb{R}^3 \,:\, x_3 = 0.5, x_2 \geq x_1^2\} \\ &\cup \quad \{x \in \mathbb{R}^3 \,:\, x_3 = 1, x_1 = 0, x_2 \geq 0\}). \end{aligned}$$

Notice B_2 *is full dimensional, closed (the sets defining* B_2 *have the same recession cone) and is defined by rational data. Observe that* $\text{conv}(B_2 \cap \mathbb{Z}^3) = \{x \in \mathbb{R}^3 \,:\, x_1 = 0, x_2 \geq 0, 0 \leq x_3 \leq 1\}$ *is a polyhedron. Since we have* $\inf\{x_1 \,:\, x \in B_2\} = -\infty$ *and* $\inf\{x_1 \,:\, x \in B_2 \cap \mathbb{Z}^3\} = 0 > -\infty$*, the set* B_2 *does not satisfy the property.*

Finally, it can be shown that the set B_2 *is conic quadratic representable using rational data, that is, there exists a rational matrix* A *and a rational vector* b *such that*

$$B_2 = \left\{ x \in \mathbb{R}^3 : \; \exists u, \; A \begin{pmatrix} x \\ u \end{pmatrix} \succeq_L b \right\},$$

where L *is direct product of Lorentz cones.*

Before we state the sufficient condition for the finiteness of a convex mixed-integer problem being equivalent to the finiteness of its continuous relaxation, we give some definitions and preliminary results that will be needed to prove the validity of this condition.

A linear subspace $L \subseteq \mathbb{R}^n$ is said to be rational linear subspace if there exists a basis of L formed by rational vectors. A convex set $B \subseteq \mathbb{R}^n$ is called lattice-free, if $\text{int}(B) \cap \mathbb{Z}^n = \emptyset$. A lattice-free convex set $B \subseteq \mathbb{R}^n$ is called maximal lattice-free convex set if does not exist a lattice-free convex set $B' \subseteq \mathbb{R}^n$ satisfying $B \subsetneq B'$.

The following result is from [10]. See also [1] for a related result.

Theorem 4 ([10])**.** *Every lattice-free convex set is contained in some maximal lattice-free convex set. A full-dimensional lattice-free convex set* $B \subseteq \mathbb{R}^n$ *is a maximal lattice-free convex set if and only if* B *is a polyhedron of the form* $B = P + L$*, where* P *is a polytope,* L *is a rational linear subspace and every facet of* B *contains a point of* $\mathbb{Z}^n$ *in its relative interior.*

We require a Corollary of Theorem 4.

Corollary 2. *Let $B \subseteq \mathbb{R}^n$ be a full dimensional convex set. Let $n_1 + n_2 = n$. If $\mathrm{int}(B) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) = \emptyset$, then there exists a polytope $P \subseteq \mathbb{R}^n$ and a rational subspace $L \subseteq \mathbb{R}^n$ such that $Q = P + L$ satisfies $\mathrm{int}(Q) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) = \emptyset$ and $B \subseteq Q$.*

Proof. Let $p : \mathbb{R}^n \to \mathbb{R}^{n_1}$ denote the projection on to the first n_1 components and $\mathrm{int}_{\mathbb{R}^{n_1}}(p(B))$ denote the interior of $p(B)$ with respect to $\mathbb{R}^{n_1}$. By Theorem 6.6 of [12], $\mathrm{int}_{\mathbb{R}^{n_1}}(p(B)) = \mathrm{rel.int}(p(B)) = p(\mathrm{rel.int}(B)) = p(\mathrm{int}(B))$. Therefore, since B is full-dimensional, we have $\mathrm{int}_{\mathbb{R}^{n_1}}(p(B)) \neq \emptyset$.

We show next that $\mathrm{int}_{\mathbb{R}^{n_1}}(p(B)) \cap \mathbb{Z}^{n_1} = \emptyset$. Since $\mathrm{int}_{\mathbb{R}^{n_1}}(p(B)) = p(\mathrm{int}(B))$, if $x \in \mathrm{int}_{\mathbb{R}^{n_1}}(p(B))$, then there exists $y \in \mathbb{R}^{n_2}$ such that $(x, y) \in \mathrm{int}(B)$. Hence, we obtain that $x \notin \mathbb{Z}^{n_1}$. Thus, $p(B)$ is a full-dimensional lattice-free convex set of $\mathbb{R}^{n_1}$. Therefore, by Theorem 4, there exists a polytope $P_1 \subseteq \mathbb{R}^{n_1}$ and a rational subspace $L_1 \subseteq \mathbb{R}^{n_1}$ such that $Q_1 = P_1 + L_1$ satisfies $\mathrm{int}_{\mathbb{R}^{n_1}}(Q_1) \cap \mathbb{Z}^{n_1} = \emptyset$ and $p(B) \subseteq Q_1$.

Now,

$$\begin{aligned} B &\subseteq p(B) \times \mathbb{R}^{n_2} \\ &\subseteq Q_1 \times \mathbb{R}^{n_2} \\ &= (P_1 + L_1) \times \mathbb{R}^{n_2} \\ &= [P_1 \times \{0\}] + [L_1 \times \mathbb{R}^{n_2}]. \end{aligned}$$

So, by taking, $P = P_1 \times \{0\}$, $L = L_1 \times \mathbb{R}^{n_2}$ and $Q = P + L$, and observing that $\mathrm{int}(Q) = \mathrm{int}_{\mathbb{R}^{n_1}}(Q_1) \times \mathbb{R}^{n_2}$, we arrive at the desired conclusion.

□

The sufficient condition for the finiteness of a convex mixed-integer problem being equivalent to the finiteness of its continuous relaxation is stated in the following result. The proof of this result is modified from a result in [3].

Proposition 2. *Let $n_1 + n_2 = n$ and let $B \subseteq \mathbb{R}^n$ be a convex set such that $\mathrm{int}(B) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) \neq \emptyset$. Then*

$$\inf\{c^t x : \ x \in B\} = -\infty \Leftrightarrow \inf\{c^t x : \ x \in B \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})\} = -\infty$$

Proof.

($\Leftarrow$) Clearly, if $\inf\{c^t x : \ x \in B \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})\} = -\infty$, then we must have $\inf\{c^t x : \ x \in B\} = -\infty$.

($\Rightarrow$) Suppose $\inf\{c^t x : \ x \in B \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})\} = d > -\infty$. We will show that $\inf\{c^t x : x \in B\} > -\infty$. Assume for a contradiction that $\inf\{c^t x : x \in B\} = -\infty$. Consider the set $B^{\leq} = B \cap \{x \in \mathbb{R}^n : c^t x \leq d\}$. Notice that $\mathrm{int}(B) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) \neq \emptyset$, so B must be a full dimensional set. Also, by assumption, we have $B \nsubseteq \{x \in \mathbb{R}^n : c^t x \geq d\}$. Therefore, $\mathrm{int}(B) \cap \{x \in \mathbb{R}^n : c^t x < d\} \neq \emptyset$. This implies that $\mathrm{int}(B^{\leq}) = \mathrm{int}(B) \cap \{x \in \mathbb{R}^n : c^t x < d\} \neq \emptyset$ and thus $B^{\leq}$ is of full dimension.

Moreover, we have $\mathrm{int}(B^{\leq}) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) = (\mathrm{int}(B^{\leq}) \cap B) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) \subseteq \mathrm{int}(B^{\leq}) \cap (B \cap \{x \in \mathbb{R}^n : c^t x \geq d\}) \subseteq \{x \in \mathbb{R}^n : c^t x < d\} \cap \{x \in \mathbb{R}^n : c^t x \geq d\} = \emptyset$, so $\mathrm{int}(B^{\leq}) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) = \emptyset$. Hence, by Corollary 2 there exists a full dimensional polyhedron $Q = \{x \in \mathbb{R}^n : a_j^t x \leq b_j, \ j \in \{1, \ldots, q\}\}$ such that $Q = P + L$, where P is a polytope and L a rational linear subspace; $\mathrm{int}(Q) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) = \emptyset$ and $B^{\leq} \subseteq Q$.

Since $\mathrm{int}(B) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) \neq \emptyset$, we obtain that $B \nsubseteq Q$. Therefore, there exists $x_0 \in B \setminus Q$, that is, $x \in B$ and $a_j^t x_0 > b_j$, for some $j \in \{1, \ldots, q\}$. Notice that, since $B^{\leq} \subseteq Q$, we have $x_0 \notin B^{\leq}$, and so, $c^t x_0 > d$. On the other hand, since $Q \subseteq \{x \in \mathbb{R}^n : \ a_j^t x \leq b_j\}$, we have $\sup\{a_j^t x : \ x \in Q\} < \infty$. Therefore, since $\mathrm{rec.cone}(Q) = L$, we must have $a_j^t r = 0$, for all $r \in \mathrm{rec.cone}(Q)$. Hence, $\inf\{a_j^t x : \ x \in Q\} > -\infty$, so there exists $M > 0$ such that $Q \subseteq \{x \in \mathbb{R}^n : \ a_j^t x \geq b_j - M\}$.

Let $\{x_n\}_{n \geq 1} \subseteq B^{\leq}$ such that $\lim_{n\to\infty} c^t x_n = -\infty$ and $\lambda_n \in (0, 1)$ such that the point $y_n = (1 - \lambda_n) x_0 + \lambda_n x_n$ satisfies $c^t y_n = d$. Since $x_0, x_n \in B$, by convexity of B, we have $y_n \in B$. Therefore we obtain that $y_n \in B^{\leq}$.

On the other hand,

$$\begin{aligned} a_j^t y_n - b_j &= (1-\lambda_n) a_j^t x_0 + \lambda_n a_j^t x_n - b_j \\ &\geq (1-\lambda_n)(a_j^t x_0 - b_j) - \lambda_n M \\ &= (a_j^t x_0 - b_j) - \lambda_n [(a_j^t x_0 - b_j) + M], \end{aligned} \tag{1}$$

where the inequality follows from the fact that $\{x_n\}_{n\geq 1} \subseteq B^{\leq} \subseteq Q \subseteq \{x \in \mathbb{R}^n : \; a_j^t x \geq b_j - M\}$.

Notice that, by definition, $\lambda_n = \frac{d - c^t x_0}{c^t x_n - c^t x_0}$ and thus $\lim_{n\to\infty} \lambda_n = 0$. Hence, by (1), for sufficiently large n, we have $a_j^t y_n > b_j$, a contradiction with the fact $y_n \in B^{\leq} \subseteq Q$. Therefore, we must have $\inf\{c^t x \, : \, x \in B\} > -\infty$. □

The condition that there exists a mixed-integer feasible solution in the interior of the continuous relaxation is crucial for Proposition 2. This is illustrated in Example 1 and Example 2, where B_1 and B_2, respectively, do not satisfy the property of finiteness of a convex mixed-integer problem being equivalent to the finiteness of its continuous relaxation. Finally, observe that the converse of Proposition 2 is not true; consider any lattice-free rational unbounded polyhedron.

4.2 Strong duality for conic programming

In mixed-integer linear programming, the proof of strong duality for the corresponding subadditive dual relies on the existence of a strong duality result for linear programming. Unfortunately, unlike the case of linear programming, strong duality for conic programming requires some additional assumptions. Therefore, it is not surprising that we require the extra condition $(*)$ to prove strong duality for mixed-integer conic programming. We recall the duality theorem for conic programming ([2]).

Theorem 5 (Duality for conic programming)**.** *Let $A \in \mathbb{R}^{m\times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$. Let $K \subseteq \mathbb{R}^m$ be a full-dimensional, closed and pointed cone. Denote $K_* = \{y \in \mathbb{R}^m : \; y^t x \geq 0, \; \forall x \in K\}$, the dual cone of K. Then:*

1. *(*Weak duality*) For all $x \in \{x \in \mathbb{R}^n : \; Ax \succeq_K b\}$, $y \in \{y \in \mathbb{R}^m : \; A^t y = c, \; y \succeq_{K_*} 0\}$ we have*

$$b^t y \leq c^t x.$$

2. *(*Strong duality*) If there exists $\hat{x} \in \mathbb{R}^n$ such that $A\hat{x} \succ_K b$ and $\inf\{c^t x : Ax \succeq_K b\} > -\infty$, then*

$$\inf\{c^t x : Ax \succeq_K b\} = \max\{b^t y : A^t y = c, \; y \succeq_{K_*} 0\}.$$

4.3 Value function of $(\mathcal{P})$

In this section we study the value function of $(\mathcal{P})$ and some of its properties. The motivation to study the value function is the following: we will verify that the value function satisfies all the properties of the feasible functions of the dual, except that it might not be defined over all of $\mathbb{R}^m$.

We begin with some notation. For $u \in \mathbb{R}^m$, let $S(u) = \{x \in \mathbb{R}^n \; : \; Ax \succeq_K u, \; x_i \in \mathbb{Z}, \; \forall \, i \in \mathcal{I}\}$ be the feasible region with right-hand side u and let $P(u) = \{x \in \mathbb{R}^n : \; Ax \succeq_K u\}$ be its continuous relaxation. Let $\Omega = \{u \in \mathbb{R}^m : \; S(u) \neq \emptyset\}$. Notice that $0 \in S(0)$, so $\Omega \neq \emptyset$.

Definition 4 (Value function of $(\mathcal{P})$)**.** *The value function of a mixed-integer conic programming is the function $f : \Omega \mapsto \mathbb{R} \cup \{-\infty\}$, defined as*

$$f(u) = \inf\{c^t x \;\; : \;\; x \in S(u)\}.$$

We show next some basic properties of the value function.

Proposition 3. *Let $f : \Omega \mapsto \mathbb{R} \cup \{-\infty\}$ be the value function of $(\mathcal{P})$, then*

1. *f is subadditive on Ω.*

2. *f is nondecreasing w.r.t K on Ω.*

3. *If $f(0) = 0$, then $f(\alpha A^i) = \alpha c_i$, $\forall\ i \in \mathcal{I}$ and $\alpha \in \{-1, 1\}$.*

4. *If $f(0) = 0$, then $\overline{f}(\alpha A^i) = \alpha c_i$, $\forall\ i \in \mathcal{C}$ and $\alpha \in \{-1, 1\}$.*

5. *Let $u \in \Omega$. If $f(u) > -\infty$, then $f(0) = 0$.*

Proof.

1. Let $u_1, u_2 \in \Omega$, $x_1 \in S(u_1)$ and $x_2 \in S(u_2)$. By additivity of $\succeq_K$, $x_1 + x_2 \in S(u_1 + u_2)$. This implies that $c^t x_1 + c^t x_2 \geq f(u_1 + u_2)$. By taking the infimum on $x_i \in S(u_i)$, $i = 1, 2$ we conclude that $f(u_1) + f(u_2) \geq f(u_1 + u_2)$.

2. Let $u_1, u_2 \in \Omega$, $u_1 \succeq_K u_2$. Let $x \in S(u_1)$. By transitivity of $\succeq_K$, we have $x \in S(u_2)$. Therefore, $S(u_1) \subseteq S(u_2)$, and so $f(u_1) \geq f(u_2)$.

3. For $i \in \mathcal{I}$ and $\alpha \in \{-1, 1\}$, since the vector $x_i = \alpha,\ x_j = 0, j \neq i$ is feasible for $(\mathcal{P})$ with r.h.s. $b = \alpha A^i$, we have $f(\alpha A^i) \leq \alpha c_i$. Since f is subadditive, we have $0 = f(0) \leq f(\alpha A^i) + f(-\alpha A^i)$. Therefore, $\alpha c_i \leq -f(-\alpha A^i) \leq f(\alpha A^i) \leq \alpha c_i$, so, $f(\alpha A^i) = \alpha c_i$. Thus, for $i \in \mathcal{I}$, $f(A^i) = c_i$ and $f(-A^i) = -c_i$.

4. For $i \in \mathcal{C}$ and $\delta \in \mathbb{R}$, since the vector $x_i = \delta,\ x_j = 0, j \neq i$ is feasible for $(\mathcal{P})$ with r.h.s. $b = \delta A^i$, we have $f(\delta A^i) \leq \delta c_i$. Since f is subadditive, we have $0 = f(0) \leq f(\delta A^i) + f(-\delta A^i)$. Therefore, $\delta c_i \leq -f(-\delta A^i) \leq f(\delta A^i) \leq \delta c_i$, so, $f(\delta A^i) = \delta c_i$. This implies $\overline{f}(A^i) = \limsup_{\delta \to 0^+} \frac{f(\delta A^i)}{\delta} = c_i$ and $\overline{f}(-A^i) = \limsup_{\delta \to 0^+} \frac{f(\delta - A^i)}{\delta} = -c_i$. Therefore, for $i \in \mathcal{C}$, $\overline{f}(A^i) = c_i$ and $\overline{f}(-A^i) = -c_i$.

5. By contrapositive. Assume $f(0) < 0$. Then there exists $\bar{x} \in S(0)$ such that $c^t \bar{x} < 0$. For all $\lambda \in \mathbb{Z}_+$, we have $\lambda \bar{x} \in S(0)$ and $c^t(\lambda \bar{x}) = \lambda c^t \bar{x} < 0$. Let $x \in S(u)$. By additivity of $\succeq_K$, $x + \lambda \bar{x} \in S(u)$, for all $\lambda \in \mathbb{Z}_+$ and $c^t(x + \lambda \bar{x}) = c^t x + \lambda c^t \bar{x}$. This implies that $f(u) = -\infty$.

□

Since the value function f might not be defined over $\mathbb{R}^m$, it is not necessarily a feasible solution to the dual. In the next section, we will construct a new function that is equal to f on b, is finite-valued over $\mathbb{R}^m$, and continues to satisfy all the conditions of the dual. The following corollary of Proposition 3 and the subsequent two propositions are crucial in this construction.

Corollary 3. *Let $K_1 \subseteq \mathbb{R}^p$, $K_2 \subseteq \mathbb{R}^q$ be closed and pointed convex cones. Let $S(u, v) = \{(x, y) \in \mathbb{R}^{(p+q)} : Ax + Cy \succeq_{K_1} u,\ Fy \succeq_{K_2} v,\ x_i \in \mathbb{Z}, \forall\ i \in \mathcal{I}_p,\ y_i \in \mathbb{Z}, \forall\ i \in \mathcal{I}_q\}$, where $\mathcal{I}_p \subseteq \{1, \ldots, p\}$, $\mathcal{I}_q \subseteq \{1, \ldots, q\}$, and let $\Omega_p = \{u \in \mathbb{R}^p :\ S(u, 0) \neq \emptyset\}$. Let*

$$G(u, v) = \inf\{c^t x + d^t y :\ (x, y) \in S(u, v)\}$$

Consider $g : \Omega_p \to \mathbb{R}$ defined as $g(u) = G(u, 0)$. Then

1. *g is subadditive on Ω_p.*

2. *g is nondecreasing w.r.t K_1 on Ω_p.*

3. *If $G(0, 0) = 0$, then, for $\alpha \in \{-1, 1\}$, $g(\alpha A^i) = \alpha c_i$, $\forall\ i \in \mathcal{I}_p$ and $\overline{g}(\alpha A^i) = \alpha c_i$, $\forall\ i \in \{1, \ldots, p\} \setminus \mathcal{I}_p$.*

Proof.

1. Observe that, by Proposition 3, G is subadditive on its domain. Let $u_1, u_2 \in \Omega_p$. Then

$$g(u_1 + u_2) = G[(u_1, 0) + (u_2, 0)] \leq G(u_1, 0) + G(u_2, 0) = g(u_1) + g(u_2),$$

   where the inequality is a consequence of G being subadditive.

2. Observe that, by Proposition 3, G is nondecreasing w.r.t $K_1 \times K_2$ on its domain. Also, $u_1 \succeq_{K_1} u_2$ if and only if $(u_1, 0) \succeq_{K_1 \times K_2} (u_2, 0)$. Therefore, if $u_1 \succeq_{K_1} u_2$, then $g(u_1) = G(u_1, 0) \geq G(u_2, 0) = g(u_2)$, as desired.

3. If $G(0, 0) = 0$, then, by Proposition 3, for $\alpha \in \{-1, 1\}$, $g(\alpha A^i) = G(\alpha A^i, 0) = \alpha c_i$, $\forall\ i \in \mathcal{I}_p$ and $\overline{g}(\alpha A^i) = \overline{G}(\alpha A^i, 0) = \alpha c_i$, $\forall\ i \in \{1, \ldots, p\} \setminus \mathcal{I}_p$.

□

The next proposition states a sufficient condition to have $\Omega = \mathbb{R}^m$, that is, $\forall\ b \in \mathbb{R}^m$, $S(b) \neq \emptyset$.

Proposition 4. *If there exists $\hat{x} \in \mathbb{R}^n$ such that $A\hat{x} \succ_K 0$, then $\forall\ b \in \mathbb{R}^m$, there exists $x \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ such that $Ax \succ_K b$.*

Proof. Let $b \in \mathbb{R}^m$. Since $A\hat{x} \succ_K 0$, then there exists $\varepsilon > 0$ such that $B(A\hat{x}, \varepsilon)$, i.e., the open ball of radius ϵ around $A\hat{x}$, is contained in K. Therefore, by continuity of Ax and density of $\mathbb{Q}^n$ in $\mathbb{R}^n$, there exists $q \in \mathbb{Q}^n$ such that $Aq \in B(A\hat{x}, \varepsilon)$. This implies, by a suitable positive scaling of q, that there exists $z \in \mathbb{Z}^n$ such that $Az \in \text{int}(K)$. Hence, there exists $\delta > 0$ such that $B(Az, \delta) \subseteq K$. For $M \in \mathbb{N}$ sufficiently large, we obtain that $Az - \frac{b}{M} \in B(Az, \delta) \subseteq K$. Thus, scaling by $M > 0$, we obtain that $A(Mz) - b \in \text{int}(K)$, that is, $A(Mz) \succ_K b$, as desired. □

The following result gives a condition such that if the primal is finite for some r.h.s. b, then it is also is finite for every r.h.s. $u \in \Omega$.

Proposition 5. *If there exists $\hat{x} \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ such that $A\hat{x} \succ_K b$ and $f(b) > -\infty$, then $\forall\ u \in \Omega$, we have $\inf\{c^t x\ :\ x \in P(u)\} > -\infty$. In particular, $\forall\ u \in \Omega$, $f(u) > -\infty$.*

Proof. Since $\{x \in \mathbb{R}^n :\ Ax \succ_K b\} = \text{int}(P(b))$, we have $\text{int}(P(b)) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) \neq \emptyset$. Therefore, since $f(b) > -\infty$, by Proposition 2, we have $\inf\{c^t x\ :\ x \in P(b)\} > -\infty$. This implies, by (2) of Theorem 5, that the set $\{y :\ A^t y = c,\ y \succeq_{K_*} 0\}$ is nonempty.

Let $u \in \Omega$ and $\bar{y} \in \{y :\ A^t y = c,\ y \succeq_{K_*} 0\}$. By (1) of Theorem 5 we obtain that $\inf\{c^t x\ :\ x \in P(u)\} \geq u^t \bar{y}$, as required. □

5 Strong duality

For ease of exposition, we recall next the main result of this paper.

Theorem 1 (Strong duality)**.** *If there exists $\hat{x} \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ such that $A\hat{x} \succ_K b$, then*

1. *$(\mathcal{P})$ has a finite optimal value if and only if $(\mathcal{D})$ has a finite optimal value.*

2. *If $(\mathcal{P})$ has a finite optimal value, then there exists a function g^* feasible for $(\mathcal{D})$ such that $g^*(b) = z^*$ and, consequently, $z^* = \rho^*$.*

We will prove Theorem 1 in the following two subsections.

5.1 Finiteness of primal being equivalent to finiteness of the dual

In this section we present a sufficient condition for the following to hold: $(\mathcal{P})$ has a finite optimal value if and only if $(\mathcal{D})$ has a finite optimal value. Observe that essentially we need conditions under which converse of Corollary 1 holds.

We begin by showing that 'a part' of the converse of Corollary 1 holds generally. The proof is modified from a result in [4].

Proposition 6. *If the primal problem is infeasible, then the dual is unbounded or infeasible.*

Proof. If the dual problem is feasible, then we need to verify that it is unbounded. Define $G : \mathbb{R}^m \mapsto \mathbb{R}$ as $G(d) = \min\{x_0 \ : \ Ax + x_0 d \succeq_K d, \ x_i \in \mathbb{Z}, \ i \in \mathcal{I}, \ x_0 \in \mathbb{Z}_+\}$. Notice $G(d) \in \{0, 1\}$ for all $d \in \mathbb{R}^m$, because $x = 0$ and $x_0 = 1$ is always a feasible solution. We have $G(d) = 0$ if and only if $\{x : \ Ax \succeq_K d, \ x_i \in \mathbb{Z}, \ i \in \mathcal{I}\} \neq \emptyset$. Hence, for $d_1, d_2 \in \mathbb{R}^m$, $G(d_1) = G(d_2) = 0$ implies $G(d_1 + d_2) = 0$. Therefore, we obtain that G is subadditive. Also, for $d_1, d_2 \in \mathbb{R}^m$ such that $d_1 \succeq_K d_2$, then $G(d_1) = 0$ implies $G(d_2) = 0$. Hence, we obtain that G is nondecreasing w.r.t K. For $i \in \mathcal{I}$, $\alpha \in \{-1, 1\}$, $G(\alpha A^i) = 0$, because $x_i = \alpha, \ x_j = 0, j \neq i$ is a feasible solution when $d = \alpha A^i$. Similarly, $\overline{G}(A^i) = \overline{G}(A^i) = 0$. Moreover, $G(0) = 0$ and, since the primal is infeasible, $G(b) = 1$.

Let g a feasible solution for the dual, then $g + \lambda G$ is also feasible solution for the dual, for all $\lambda \geq 0$. Since $[g + \lambda G](b) = g(b) + \lambda$, for all $\lambda \geq 0$, we conclude that the dual is unbounded. □

Proposition 7. *If there exists $\hat{x} \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ such that $A\hat{x} \succ_K b$, then $(\mathcal{P})$ has a finite optimal value if and only if $(\mathcal{D})$ has a finite optimal value.*

Proof. $(\Leftarrow)$ Assume that the dual is finite. Then, by Proposition 6, we obtain the primal is feasible. Thus, Corollary 1 implies the primal is finite.

$(\Rightarrow)$ Assume that the primal is finite. Corollary 1 implies that if the dual is feasible then the dual cannot be unbounded. We next verify that dual is indeed feasible.

First observe that, since $\mathrm{int}(P(b)) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) \neq \emptyset$ and $f(b) > -\infty$, then by Proposition 5, we obtain that $\inf\{c^t x \ : \ Ax \succeq_K b\} > -\infty$

Since $\inf\{c^t x \ : \ Ax \succeq_k b\}$ is finite and there exists $\hat{x}$ such that $A\hat{x} \succ_K b$, then by (2) of Theorem 5, we obtain that the set $\{y \in \mathbb{R}^m \ : \ A^t y = c, \ y \succeq_{K_*} 0\}$ is nonempty. Let $\hat{y} \in \{y \in \mathbb{R}^m \ : \ A^t y = c, \ y \succeq_{K_*} 0\}$. Then the function $g(u) = \hat{y}^t u$ is a feasible solution of $(\mathcal{D})$, so the dual problem is feasible. □

Notice that Proposition 7 gives a proof for (1) of Theorem 1. In the next section, we will refine the second half of the proof of Proposition 7, to show that when there exists $\hat{x} \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ such that $A\hat{x} \succ_K b$ and the primal is finite, not only is the dual finite, its optimal value is equal to that of the primal.

5.2 Feasible optimal solution for $(\mathcal{D})$

In this section, we construct a feasible optimal solution for the dual problem $(\mathcal{D})$. The next proposition shows how this can be done.

Proposition 8. *If there exists $\hat{x} \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ such that $A\hat{x} \succ_K b$ and $(\mathcal{P})$ has a finite optimal value, then there exists a function $g^* : \mathbb{R}^m \to \mathbb{R}$, feasible for $(\mathcal{D})$ such that $g^*(b) = z^*$ and, consequently, $z^* = \rho^*$.*

Proof. Let $f : \Omega \to \mathbb{R}$ be the value function of $(\mathcal{P})$. By hypothesis, $f(b) > -\infty$. Notice also that for all $y \in \mathbb{Z}$, $yb \in \Omega$. Hence, by Proposition 5, we have that $f(yb) > -\infty$, for all $y \in \mathbb{Z}$.

For $\lambda \geq 0$ denote $f_R(\lambda b) = \inf\{c^t x : \ x \in P(\lambda b)\}$. Observe that $f_R(\lambda b) = \lambda f_R(b)$. By Proposition 2, we have $f_R(b) > -\infty$. Thus, $f_R(\lambda b) = \lambda f_R(b) > -\infty$, for all $\lambda \geq 0$.

If $\Omega = \mathbb{R}^n$, then by Proposition 3 and Proposition 5, f is feasible for the dual $(\mathcal{D})$. Moreover, by definition, $f(b) = z^*$, so, by taking $g^* = f$, we arrive at the desired conclusion.

If $\Omega \subsetneq \mathbb{R}^n$, we will show that there exists a feasible solution $g^* : \mathbb{R}^m \to \mathbb{R}$ for the dual $(\mathcal{D})$ such that $g^*(b) = f(b)$. Denote $X(u) = \{(x, y) \in \mathbb{R}^{n+1} : \ Ax - yb \succeq_K u, y \geq 0, \ x_i \in \mathbb{Z}, \forall \ i \in \mathcal{I}, \ y \in \mathbb{Z}\}$. Now we construct g^*, we have 3 cases:

Case 1: $f(b) \geq 0$ and $f_R(b) \geq 0$. Define

$$g^*(u) = \inf\{c^t x + [f(b) - f_R(b)]y : \ (x,y) \in X(u)\}$$

First we prove that $g^*(0) = 0$. Let $(x,y) \in X(0)$. Then we have

$$\begin{aligned} c^t x + [f(b) - f_R(b)]y &\geq f(yb) + f(b)y - f_R(b)y \\ &= [f(yb) - f_R(yb)] + f(b)y \\ &\geq 0. \end{aligned}$$

By taking the feasible solution $(0,0)$, with objective value 0, we conclude $g^*(0) = 0$.
Now we prove that $g^*(b) = f(b)$. Let $(x,y) \in X(b)$ with $y \geq 1$. Then we have

$$\begin{aligned} c^t x + [f(b) - f_R(b)]y &\geq f((y+1)b) + f(b)y - f_R(b)y \\ &= [f((y+1)b) - f_R((y+1)b)] + f_R(b) + f(b)y \\ &\geq f(b)y \\ &\geq f(b). \end{aligned}$$

On the other hand, notice that if $(x,0) \in X(b)$, then $x \in S(b)$ and $c^t x + [f(b) - f_R(b)]y = c^t x$. Therefore, by taking the infimum over $(x,0) \in X(b)$, we conclude $g^*(b) = f(b)$.
Case 2: $f(b) \leq 0$ and $f_R(b) \leq 0$. In this case, define

$$g^*(u) = \inf\{c^t x - 2f_R(b)y : \ (x,y) \in X(u)\}$$

First we prove that $g^*(0) = 0$. Let $(x,y) \in X(0)$. Then we have

$$\begin{aligned} c^t x - 2f_R(b)y &\geq f(yb) - f_R(b)y - f_R(b)y \\ &= [f(yb) - f_R(yb)] - f_R(b)y \\ &\geq 0. \end{aligned}$$

By taking the feasible solution $(0,0)$, with objective value 0, we conclude $g^*(0) = 0$.
Now we prove that $g^*(b) = f(b)$. Let $(x,y) \in X(b)$ with $y \geq 1$. Then we have

$$\begin{aligned} c^t x - 2f_R(b)y &\geq f((y+1)b) - f_R(b)y - f_R(b)y \\ &= [f((y+1)b) - f_R((y+1)b)] - f_R(b)(y-1) \\ &\geq 0 \\ &\geq f(b). \end{aligned}$$

On the other hand, notice that if $(x,0) \in X(b)$, then $x \in S(b)$ and $c^t x - 2f_R(b)y = c^t x$. Therefore, by taking the infimum over $(x,0) \in X(b)$, we conclude $g^*(b) = f(b)$.
Case 3: $f(b) \geq 0$ and $f_R(b) \leq 0$. In this case, define

$$g^*(u) = \inf\{c^t x + [f(b) - 2f_R(b)]y : \ (x,y) \in X(u)\}$$

First we prove that $g^*(0) = 0$. Let $(x,y) \in X(0)$. Then we have

$$\begin{aligned} c^t x + [f(b) - 2f_R(b)]y &\geq f(yb) - f_R(b)y + [f(b) - f_R(b)]y \\ &= [f(yb) - f_R(yb)] + [f(b) - f_R(b)]y \\ &\geq 0. \end{aligned}$$

By taking the feasible solution $(0,0)$, with objective value 0, we conclude $g^*(0)=0$.
Now we prove that $g^*(b)=f(b)$. Let $(x,y)\in X(b)$ with $y\geq 1$. Then we have

$$\begin{aligned} c^t x+[f(b)-2f_R(b)]y &\geq f((y+1)b)-f_R(b)y+[f(b)-f_R(b)]y \\ &= [f((y+1)b)-f_R((y+1)b)]+f_R(b)+[f(b)-f_R(b)]y \\ &\geq f_R(b)(1-y)+yf(b) \\ &\geq yf(b) \\ &\geq f(b). \end{aligned}$$

On the other hand, notice that if $(x,0)\in X(b)$, then $x\in S(b)$ and $c^tx+[f(b)-2f_R(b)]y=c^tx$. Therefore, by taking the infimum over $(x,0)\in X(b)$, we conclude $g^*(b)=f(b)$.

We show next that, in all 3 cases, g^* is feasible for the dual $(\mathcal{D})$. Observe that the point $(\hat{x},1)$ satisfies $A\hat{x}-b\succ_K 0$; $1>0$, so, by Proposition 4, we have $X(u)\neq\emptyset$, for all $u\in\mathbb{R}^m$. Moreover, since $g^*(0)=0$, $g(u)>-\infty$, for all $u\in\mathbb{R}^m$ (Proposition 5). Thus, we have defined a function $g^*:\mathbb{R}^m\to\mathbb{R}$. Finally, by Corollary 3, observe that g^* satisfies all the constraints of the dual $(\mathcal{D})$. In conclusion, g^* is feasible for the dual $(\mathcal{D})$ and $g^*(b)=f(b)=z^*$, so we are done. □

Notice that Proposition 8 gives a proof for (2) of Theorem 1.

6 Remarks

Let $c\in\mathbb{R}^n$ such that the problems $(\mathcal{P})$ and $(\mathcal{D})$ are feasible. Let g feasible for the dual $(\mathcal{D})$, then, by Proposition 1 (Weak duality), the inequality $\sum_{i\in\mathcal{I}} g(A^i)x_i+\sum_{i\in\mathcal{C}}\overline{g}(A^i)x_i\geq g(b)$ is a valid inequality for $(\mathcal{P})$. In other words, we can use subadditive functions to generate valid inequalities for $(\mathcal{P})$. Conversely, if there exists $\hat{x}\in\mathbb{Z}^{n_1}\times\mathbb{R}^{n_2}$ such that $A\hat{x}\succ_K b$, then Theorem 1 yields the following corollary relating valid inequalities and subadditive functions for mixed-integer conic programs.

Corollary 4 (Subadditive valid inequality)**.** *Assume there exists $\hat{x}\in\mathbb{Z}^{n_1}\times\mathbb{R}^{n_2}$ such that $A\hat{x}\succ_K b$. Given a valid inequality $\pi^tx\geq\pi_0$ for $(\mathcal{P})$, then there exists $g\in\mathcal{F}$ satisfying $g(0)=0$, $g(b)\geq\pi_0$, $g(A^i)=\pi_i,\forall i\in\mathcal{I}$, and $\overline{g}(A^i)=\pi_i,\forall i\in\mathcal{C}$ such that $\sum_{i\in\mathcal{I}} g(A^i)x_i+\sum_{i\in\mathcal{C}}\overline{g}(A^i)x_i\geq g(b)$ is also a valid inequality for $(\mathcal{P})$.*

In [13], it is shown that in the case of integer linear programs, given a rational left-hand-side matrix A, there exists a finite set of subadditive functions that yields the subadditive dual for any choice of the right hand side b. Such a result is unlikely in the integer conic setting since, in general, the convex hull of feasible points is not necessarily a polyhedron. The following example illustrates this behavior.

Example 3. *Consider S the epigraph of the parabola $x_2=x_1^2$, that is, $S=\{x\in\mathbb{R}^2:\ x_1^2\leq x_2\}$. It is well-known that S is a conic quadratic representable set ([2]). Indeed, we have*

$$S=\left\{x\in\mathbb{R}^2:\ \begin{bmatrix}1 & 0\\ 0 & \frac{1}{2}\\ 0 & \frac{1}{2}\end{bmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}\succeq_{L^3}\begin{pmatrix}0\\ \frac{1}{2}\\ -\frac{1}{2}\end{pmatrix}\right\},$$

where L^3 is the Lorentz cone in $\mathbb{R}^3$.

On the other hand, we have that $\operatorname{conv}(S\cap\mathbb{Z}^2)$ is a non-polyhedral closed convex set; also see [3]. In fact, we have

$$\operatorname{conv}(S\cap\mathbb{Z}^2)=\bigcap_{n\in\mathbb{Z}}\{x\in\mathbb{R}^2:\ x_2-(2n+1)x_1\geq-(n^2+n)\},$$

where all these inequalities define facets of $\operatorname{conv}(S\cap\mathbb{Z}^2)$.

By Corollary 4, for all $n \in \mathbb{Z}$, there exists a subadditive function $g_n : \mathbb{R}^3 \to \mathbb{R}$, such that g_n is nondecreasing w.r.t L^3, $g_n(0) = 0$, and

$$\begin{aligned} g_n\left[\begin{pmatrix}0\\ \frac{1}{2}\\ -\frac{1}{2}\end{pmatrix}\right] &= -(n^2+n),\\ g_n\left[\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}\right] &= -(2n+1),\\ g_n\left[\begin{pmatrix}0\\ \frac{1}{2}\\ \frac{1}{2}\end{pmatrix}\right] &= 1. \end{aligned}$$

Therefore, we can write the convex hull of the integer points in S in terms of an infinite number of linear inequalities given by subadditive functions

$$\operatorname{conv}(S \cap \mathbb{Z}^2) = \bigcap_{n\in\mathbb{Z}} \left\{ x \in \mathbb{R}^2 : \; g_n\left[\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}\right] x_1 + g_n\left[\begin{pmatrix}0\\ \frac{1}{2}\\ \frac{1}{2}\end{pmatrix}\right] x_2 \geq g_n\left[\begin{pmatrix}0\\ \frac{1}{2}\\ -\frac{1}{2}\end{pmatrix}\right]\right\}$$

Finally, notice that $\operatorname{conv}(S \cap \mathbb{Z}^2)$ is not a polyhedron, so it cannot be described in terms of a finite number of linear inequalities.

7 Appendix

In this section we present a proof of Theorem 2 that uses Theorem 1. We first recall the problem $(\mathcal{P}'')$ and its dual $(\mathcal{D}'')$

$$(\mathcal{P}'')\begin{cases} z'' = \inf & c^t x \\ \quad s.t. & Ax \succeq_K b \\ & x \in \mathbb{R}^n_+ \\ & x_i \in \mathbb{Z}, \forall i \in \mathcal{I} \end{cases}$$

and

$$(\mathcal{D}'')\begin{cases} \rho'' = \sup & g(b) \\ \quad s.t. & g\left(A^i\right) \leq c_i, \quad \forall i \in \mathcal{I} \\ & \bar{g}\left(A^i\right) \leq c_i, \quad \forall i \in \mathcal{C} \\ & g(0) = 0 \\ & g \in \mathcal{F}. \end{cases}$$

On the other hand, as discussed in section 2.1, we can write $(\mathcal{P}'')$ in form $(\mathcal{P})$ as follows

$$(\mathcal{P})\begin{cases} z^* = \inf & c^t x \\ \quad s.t. & \begin{bmatrix} A \\ I \end{bmatrix} x \succeq_{K\times\mathbb{R}^n_+} \begin{pmatrix} b \\ 0 \end{pmatrix} \\ & x_i \in \mathbb{Z}, \forall i \in \mathcal{I}, \end{cases}$$

where $I \in \mathbb{R}^{n\times n}$ is the identity matrix.

We will also need to consider the subadditive dual of $(\mathcal{P})$

$$
(\mathcal{D})\begin{cases}
\rho^* = \sup & G(b,0) \\
\quad s.t. & G\left(A^i, e_i\right) = c_i, & \forall i \in \mathcal{I} \\
 & G\left(-A^i, e_i\right) = -c_i, & \forall i \in \mathcal{I} \\
 & \bar{G}\left(A^i, e_i\right) = c_i, & \forall i \in \mathcal{C} \\
 & \bar{G}\left(-A^i, e_i\right) = -c_i, & \forall i \in \mathcal{C} \\
 & G(0) = 0 \\
 & G : \mathbb{R}^{m+n} \mapsto \mathbb{R} \ s.t.\ G \text{ is subadditive} \\
 & \text{and nondecreasing w.r.t } K \times \mathbb{R}^n_+,
\end{cases}
$$

where e_i, $i = 1, \ldots, n$, is the ith vector of the canonical basis of $\mathbb{R}^n$.
For ease of exposition, we first recall the statement of Theorem 2 and then present its proof below.

Theorem 2. *1. (Weak duality) For all $x \in \mathbb{R}^n$ feasible for $(\mathcal{P}'')$ and for all $g : \mathbb{R}^m \mapsto \mathbb{R}$ feasible for $(\mathcal{D}'')$, we have $g(b) \leq c^t x$.*

2. (Strong duality) If there exists $\hat{x} \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ such that $A\hat{x} \succ_K b, \hat{x} > 0$, then

(a) $(\mathcal{P}'')$ has a finite optimal value if and only if $(\mathcal{D}'')$ has a finite optimal value.

(b) If $(\mathcal{P}'')$ has a finite optimal value, then there exists a function g^* feasible for $(\mathcal{D}'')$ such that $g^*(b) = z^{''}$ and, consequently, $z^{''} = \rho^{''}$.

Proof.

1. The proof of weak duality is a slight modification of the proof of Proposition 1.

2. Assume $(\mathcal{D}'')$ is finite. First, observe that a similar argument to that in the proof of Proposition 6 shows that the primal is feasible. Also, by weak duality, the primal $(\mathcal{P}'')$ must be bounded. Thus, $(\mathcal{P}'')$ is finite.

   Now assume that $(\mathcal{P}'')$ is finite. Observe that the problem $(\mathcal{P})$ is finite if and only if $(\mathcal{P}'')$ is finite and $z^* = z^{''}$. We also have that there exists $\hat{x} \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ such that $A\hat{x} \succ_K b,\ \hat{x} > 0$. Therefore, by Theorem 1, $(\mathcal{D})$ is a strong dual for $(\mathcal{P})$.

   Now we show that for every G feasible for $(\mathcal{D})$ there exists g feasible for $(\mathcal{D}'')$ such that $g(b) = G(b, 0)$. Let G feasible for $(\mathcal{D})$. For $x \in \mathbb{R}^n$, define $g(x) = G(x, 0)$. Notice that $g(0) = G(0, 0) = 0$. Also, since G is subadditive, g is subadditive. Since $x \succeq_K y$ implies $(x, 0) \succeq_{K\times\mathbb{R}^n_+} (y, 0)$, we have g is nondecreasing w.r.t K. For $i \in \mathcal{I}$, since G is nondecreasing w.r.t $K \times \mathbb{R}^n_+$, we have $g(A^i) = G(A^i, 0) \leq G(A^i, e_i) = c_i$. Similarly, for $i \in \mathcal{C}$ and $\delta \geq 0$, $g(\delta A^i) \leq G(\delta A^i, \delta e_i)$, so by definition of $\bar{g}$ and $\bar{G}$, we obtain $\bar{g}(A^i) \leq \bar{G}(A^i, e_i) = c_i$. Therefore, we have g is feasible for $(\mathcal{D}'')$ and $g(b) = G(b, 0)$, as desired.

   Let $G^* : \mathbb{R}^{m+n} \mapsto \mathbb{R}$ be the function given by Theorem 1, that is, G^* is feasible for $(\mathcal{D})$ and $G^*(b, 0) = z^*$. We have there exists g^* feasible for $(\mathcal{D}'')$ such that $g^*(b) = G^*(b, 0)$. Hence, the dual $(\mathcal{D}'')$ is feasible. By weak duality, the optimal value of $(\mathcal{D}'')$ is bounded above by $z^{''}$. In particular, $(\mathcal{D}'')$ is finite. Finally, we obtain $g^*(b) = G^*(b, 0) = z^* = z^{''}$, as desired.

□

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