Algorithmic and Complexity Results for Cutting Planes Derived from Maximal Lattice-Free Convex Sets

Amitabh Basu* Robert Hildebrand† Matthias Köppe‡

July 25, 2011§

Abstract

We study a mixed integer linear program with m integer variables and k non-negative continuous variables in the form of the relaxation of the corner polyhedron that was introduced by Andersen, Louveaux, Weismantel and Wolsey [*Inequalities from two rows of a simplex tableau*, Proc. IPCO 2007, LNCS, vol. 4513, Springer, pp. 1–15]. We describe the facets of this mixed integer linear program via the extreme points of a well-defined polyhedron. We then utilize this description to give polynomial time algorithms to derive valid inequalities with optimal l_p norm for arbitrary, but fixed m. For the case of $m = 2$, we give a refinement and a new proof of a characterization of the facets by Cornuéjols and Margot [*On the facets of mixed integer programs with two integer variables and two constraints*, Math. Programming **120** (2009), 429–456]. The key point of our approach is that the conditions are much more explicit and can be tested in a more direct manner, removing the need for a reduction algorithm. These results allow us to show that the relaxed corner polyhedron has only polynomially many facets.

1 Introduction

The integer programming community has recently focused on developing a unifying theory for cutting planes. This has involved applying tools from convex analysis and the geometry of numbers to combine the ideas behind Gomory's corner polyhedron [13] and Balas' intersection cuts [2] into one uniform framework. It is fair to say that this recent line of research was started by the seminal paper by Andersen, Louveaux, Weismantel and Wolsey [1], which took a fresh look at the work done by Gomory and Johnson in the 1960's. We refer the reader to [8] for a survey of these results.

It can be argued that the theoretical research has tended to emphasize the structural aspects of these cutting planes and the algorithmic aspects have not been developed as

*Dept. of Mathematics, University of California, Davis, abasu@math.ucdavis.edu
†Dept. of Mathematics, University of California, Davis, rhildebrand@math.ucdavis.edu
‡Dept. of Mathematics, University of California, Davis, mkoeppe@math.ucdavis.edu
§Revision: 201 − Date: 2011-07-25 12:55:40 -0700 (Mon, 25 Jul 2011)

intensively. Our goal in this paper is to derive structural results which, we hope, will be useful from an algorithm design perspective. Hence, our emphasis is on deriving polynomiality results about the structure of these cutting planes. We also provide concrete polynomial time algorithms for generating the "best" or "deepest" cuts, according to some standard criteria.

To this end, we study the following system, introduced by Andersen et al. [1] and Borozan and Cornuéjols [7].

$$\begin{aligned} &x = f + \sum_{j=1}^{k} r^j s_j, \\ &x \in \mathbb{Z}^m, \quad s_j \geq 0 \quad \text{for all } j = 1, \ldots, k. \end{aligned} \tag{1}$$

We will assume that the data is rational, i.e., $f \in \mathbb{Q}^m$ and $r^j \in \mathbb{Q}^m$ for all $j \in \{1, \ldots, k\}$. This model appears as a natural relaxation of Gomory's corner polyhedron [13]. As mentioned above, this model has received significant attention in recent years for developing the theory behind cutting planes derived from multiple rows of the optimal simplex tableaux. Note that to describe the solutions of (1), one only needs to record the values of the s_j variables. We use $R_f = R_f(r^1, \ldots, r^k)$ to denote the set of all points s such that (1) is satisfied. It is well-known that all valid inequalities for $\operatorname{conv}(R_f)$, where conv denotes the convex hull, can be derived using the Minkowski functional of maximal lattice-free convex sets. We state this formally in Theorem 2.2 below. In this paper we give algorithms and theorems about the facet structure of $\operatorname{conv}(R_f)$, which are expected to be useful for generating strong cutting planes for general mixed integer linear programs.

Motivation and Results. It is well-known that the integer hull $\operatorname{conv}(R_f)$ is a polyhedron of the *blocking type*. In Section 3, we first describe the so-called *blocking polyhedron* for $\operatorname{conv}(R_f)$. This is the convex set of all valid inequalities for $\operatorname{conv}(R_f)$. For a detailed account of blocking polyhedra and the "polar" set of the valid inequalities for such polyhedra, see Chapter 9 in [19]. The main result of Section 3 gives an explicit description of the blocking polyhedron of $\operatorname{conv}(R_f)$ using a polynomial number of inequalities (Theorem 3.3). This implies that all facets of $\operatorname{conv}(R_f)$ can be obtained by enumerating the extreme points of a polyhedron with a polynomial number of facets in the dual space. This result has the same flavor as Gomory's result for describing all facets of the corner polyhedron implicitly via the extreme points of a well-defined polyhedron (see Theorem 18 in [13]).

We next exploit this to provide efficient algorithms for finding the optimal valid inequality according to certain norms of the coefficient vector. More precisely, let $\|v\|_p = (\sum_{j=1}^k |v_j|^p)^{1/p}$ be the standard l_p norm of a vector $v \in \mathbb{R}^k$. If $\sum_{j=1}^k \gamma_j s_j \geq 1$ is a valid inequality for $\operatorname{conv}(R_f)$, its l_p norm is $\|\gamma\|_p$ where γ is the vector in $\mathbb{R}^m$ with components γ_j. We give polynomial time algorithms to determine cuts with minimum l_p norm for arbitrary, but fixed m. For the special case of the l_1 and l_∞ norms, this reduces to solving a linear program with polynomially many constraints. We also give an alternative approach for the l_∞ norm.

We then investigate the case of $m = 2$ in more detail in Sections 5 and 6. In particular, we show that the number of facets of $\text{conv}(R_f)$ is polynomial in the input. This result is proved in Section 6 (Theorem 6.2). In order to prove this theorem, we first develop some technology in Section 4 to derive necessary conditions for a valid inequality to be a facet. Our hope is that these tools can be utilized to prove useful theorems about facets of $\text{conv}(R_f)$ for $m \geq 3$, in the same vein as the results of Cornuéjols and Margot appearing in [10]. Although we do not derive such results in this paper, we exhibit the promise of this approach by giving alternative proofs of necessary conditions for inequalities to be facets which appear in [10] and providing more refined and new necessary conditions. The necessary conditions in [10] are stated as a particular termination condition of a complicated algorithm. This makes them hard to be used in a practical setting. In contrast, our refined conditions are explicit and can be tested directly. This makes them much more useful from the practical point of view of actually generating facet defining cutting planes. Another advantage of our technique over the Cornuéjols−Margot proof is that when the necessary conditions are violated, we can explicitly express the given valid inequality as a convex combination of other valid inequalities. This is crucial in obtaining a proof of the fact that the so-called *triangle closure* is a polyhedron [6]. This settles an important open problem in this recent line of research. Finally, and perhaps most importantly, we envision that the ideas behind the polynomiality results of Section 6 can be exploited to design algorithms and heuristics for deriving effective cutting planes. We emphasize this by using the constructive nature of the proof for Theorem 6.2 to give a polynomial time algorithm for enumerating all the facets of $\text{conv}(R_f)$ for $m = 2$ (Theorem 6.3).

We mention here that some variations of these ideas have been explored by Louveaux and Poirrier [16], and also by Fukasawa et al. [12].

2 Preliminaries

It is well-known that $\text{conv}(R_f)$ is a full-dimensional polyhedron of blocking type, i.e., $\text{conv}(R_f) \subset \mathbb{R}^k_+$ (where $\mathbb{R}^k_+$ denotes the nonnegative orthant) and if $x \in \text{conv}(R_f)$, then $y \geq x$ implies $y \in \text{conv}(R_f)$. Hence, all nontrivial valid inequalities for $\text{conv}(R_f)$ can be written as $\gamma \cdot s = \sum_{j=1}^k \gamma_j s_j \geq 1$ for some vector $\gamma \in \mathbb{R}^k_+$ (see [19], Chapter 9 for more details on polyhedra of blocking type).

A valid inequality $\sum_{j=1}^k \gamma_j s_j \geq 1$ for $\text{conv}(R_f)$ is called *minimal* if it is not dominated by another inequality, i.e., there does not exist a *different* valid inequality $\sum_{j=1}^k \gamma'_j s_j \geq 1$ such that $\gamma'_j \leq \gamma_j$ for $j = 1, \ldots, k$. A valid inequality $\gamma \cdot s \geq 1$ for $\text{conv}(R_f)$ is called *extreme* if there do not exist valid inequalities $\gamma^1 \cdot s \geq 1$, $\gamma^2 \cdot s \geq 1$ such that $\gamma = \frac{1}{2}\gamma^1 + \frac{1}{2}\gamma^2$. For polyhedra of blocking type, extreme inequalities are always minimal. Moreover, since $\text{conv}(R_f)$ is full-dimensional, facets and extreme inequalities for $\text{conv}(R_f)$ are one and the same thing. We now collect the main results from the recent theory of cutting planes using lattice-free sets. For more details, please see [8].

Definition 2.1. *Let $K \subset \mathbb{R}^m$ be a closed convex set containing the origin in its interior. The* gauge *or the* Minkowski functional *is defined by*

$$\psi_K(x) = \inf\{\, t > 0 \mid t^{-1}x \in K \,\} \quad \textit{for all } x \in \mathbb{R}^m.$$

By definition ψ_K is non-negative.

Theorem 2.2 (Intersection cuts [2], [8])**.** *Consider any closed convex set M containing the point f in its interior, but no integer point in its interior. Let $K = M - f$. Then the inequality $\sum_{j=1}^{k} \psi_K(r^j)s_j \geq 1$ is valid for $\operatorname{conv}(R_f)$. Moreover, every valid inequality of $\operatorname{conv}(R_f)$ can be derived in this manner.*

For convenience, we also say that the function ψ_K is extreme when the corresponding inequality $\sum_{j=1}^{k} \psi_K(r^j)s_j \geq 1$ is extreme. We will refrain from using the terminology that ψ_K defines a facet of $\operatorname{conv}(R_f)$ as to not confuse these facets with facets of lattice-free polytopes. We will work with a fixed set of rays $\{r^1, \dots, r^k\} \subset \mathbb{R}^m$. The interior of any set $M \subseteq \mathbb{R}^m$ will be denoted by $\operatorname{int}(M)$.

It is also well-known (see [8]) that all minimal inequalities (and hence all extreme inequalities) can be derived using *maximal lattice-free convex sets*, i.e., convex sets containing no integer point in their interior that are maximal with respect to set inclusion. Moreover, it is known [4, 17] that maximal lattice-free convex sets are polyhedra whose recession cones are not full-dimensional. Since we will be concerned with maximal lattice-free convex sets with f in their interior, one can represent such sets in the following canonical manner.

Let $B \in \mathbb{R}^{n\times m}$ be a matrix with n rows $b^1, \dots, b^n \in \mathbb{R}^m$. We write $B = (b^1; \dots; b^n)$. Let

$$M(B) = \{\, x \in \mathbb{R}^m \mid b^i \cdot (x - f) \leq 1 \text{ for } i = 1, \dots, n \,\}. \tag{2}$$

This is a polyhedron with f in its interior. We will denote its vertices by $\operatorname{vert}(B)$. In fact, any polyhedron with f in its interior can be given such a description. We will mostly deal with matrices B such that $M(B)$ is a maximal lattice-free convex set in $\mathbb{R}^m$.

This description enables one to describe the Minkowski functional by a simple piecewise-linear formula:

Theorem 2.3 (see [3], Theorem 24)**.** *Let $B \in \mathbb{R}^{n\times m}$ such that the recession cone of $M(B)$ is not full-dimensional (i.e., $b^i \cdot r \leq 0$ has no solution satisfying all constraints at strict inequality). Then,*

$$\psi_{M(B)-f}(r) = \max_{i\in\{1,\dots,n\}} b^i \cdot r. \tag{3}$$

Therefore, all minimal inequalities for $\operatorname{conv}(R_f)$ can be derived using (3) from matrices B such that $M(B)$ is a maximal lattice-free convex set in $\mathbb{R}^m$. For convenience of notation, for any matrix $B \in \mathbb{R}^{n\times m}$ we define $\psi_B(r) = \psi_{M(B)-f}(r) = \max_{i\in\{1,\dots,n\}} b^i \cdot r$.

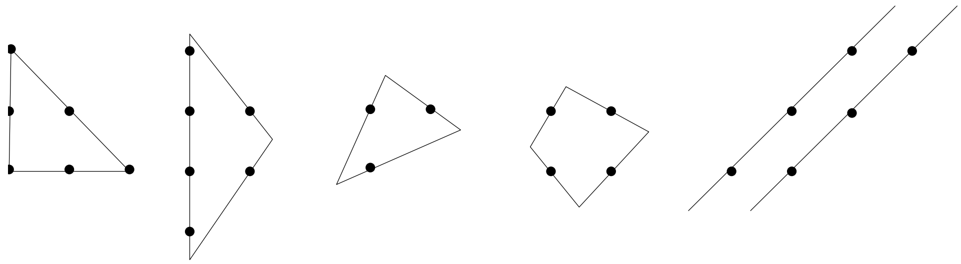

Figure 1: Types of maximal lattice-free convex sets in $\mathbb{R}^2$

For the case of $m = 2$, Lovász characterized the maximal lattice-free convex sets in $\mathbb{R}^2$ as follows.

Theorem 2.4 (Lovász [17])**.** *In the plane, a maximal lattice-free convex set with non-empty interior is one of the following:*

1. *A split* $c \leq ax_1 + bx_2 \leq c + 1$ *where* a *and* b *are co-prime integers and* c *is an integer;*

2. *A triangle with an integral point in the interior of each of its edges;*

3. *A quadrilateral containing exactly four integral points, with exactly one of them in the interior of each of its edges. Moreover, these four integral points are vertices of a parallelogram of area 1.*

Following Dey and Wolsey [11], the maximal lattice-free triangles can be further partitioned into three canonical types (see Figure 1):

- *Type 1 triangles*: triangles with integral vertices and exactly one integral point in the relative interior of each edge;

- *Type 2 triangles*: triangles with at least one fractional vertex v, exactly one integral point in the relative interior of the two edges incident to v and at least two integral points on the third edge;

- *Type 3 triangles*: triangles with exactly three integral points on the boundary, one in the relative interior of each edge.

Figure 1 shows these three types of triangles as well as a maximal lattice-free quadrilateral and a split satisfying the properties of Theorem 2.4.

3 Description and algorithmic results for the set of all valid inequalities for $\mathrm{conv}(R_f)$

For the results of this section, we will assume that the conical hull of the set of rays $\{r^1, \ldots, r^k\}$ is $\mathbb{R}^m$. This simplifies the arguments presented and implies $k > m$.

3.1 Polyhedral structure

As mentioned in Section 2, $\mathrm{conv}(R_f)$ is a polyhedron of blocking type. We will study the *blocking polyhedron* of $\mathrm{conv}(R_f)$, i.e.,

$$\mathrm{conv}(R_f)^\vee = \left\{\, \gamma \in \mathbb{R}^k_+ \;\middle|\; \gamma \cdot s \geq 1 \text{ for all } s \in \mathrm{conv}(R_f) \,\right\}.$$

This is the set of all normal vectors of nontrivial valid inequalities for $\mathrm{conv}(R_f)$. We refer to [19] for a discussion of polyhedra of blocking type and these related notions. It is well-known that for any polyhedron P of blocking type, the set $P^\vee$ is a polyhedron.

In this section, we give an explicit description of $\mathrm{conv}(R_f)^\vee$. Moreover, when m is fixed (not part of the input), our description of $\mathrm{conv}(R_f)^\vee$ will have polynomially many inequalities. From the definitions, it follows that the extreme inequalities for $\mathrm{conv}(R_f)$ are given by the extreme points of $\mathrm{conv}(R_f)^\vee$. It is well-known that for a full-dimensional polyhedron like $\mathrm{conv}(R_f)$, facets and extreme inequalities are equivalent concepts.

We start with the following version of Carathéodory's theorem.

Lemma 3.1. *Let P be a polyhedron given by $P = \mathrm{conv}(\{v^1, \ldots, v^p\}) + \mathrm{cone}(\{r^1, \ldots, r^q\})$ with $dim(P) = n$. For any $x \in P$, there exist subsets $I \subseteq \{1, \ldots, p\}$ and $J \subseteq \{1, \ldots, q\}$ such that*

(i) $|I| + |J| \leq n + 1$,

(ii) $x \in \mathrm{conv}(\{\, v^i \mid i \in I \,\}) + \mathrm{cone}(\{\, r^j \mid j \in J \,\})$.

The lemma follows immediately by the standard homogenization of P and then applying Carathéodory's theorem for cones.

Let $\mathcal{I}$ be the set of all subsets I of $\{1, \ldots, k\}$ such that $\{\, r^j \mid j \in I \,\}$ is a basis for $\mathbb{R}^m$. Given any $x \in \mathbb{Z}^m$ and $I \in \mathcal{I}$ such that $x - f \in \mathrm{cone}(\{\, r^j \mid r^j \in I \,\})$, let $s_j(x, I)$ be the (non-negative) coefficient of r^j when $x - f$ is expressed in the basis $\{\, r^j \mid j \in I \,\}$. Moreover, for any set $I \in \mathcal{I}$, $X(I)$ is the set of all $x \in \mathbb{Z}^m$ such that $x - f \in \mathrm{cone}(\{\, r^j \mid j \in I \,\})$.

Proposition 3.2.

$$\mathrm{conv}(R_f)^\vee = \left\{ \gamma \geq 0 \;\middle|\; \sum_{j \in I} \gamma_j s_j(x, I) \geq 1 \quad \forall x \in X(I), \quad \forall I \in \mathcal{I} \right\}. \tag{4}$$

Proof. Let γ be any vector in $\mathbb{R}^k_+$. Consider the convex set

$$M_\gamma = \operatorname{conv}\big(\{\, f + \tfrac{r^j}{\gamma_j} \mid \gamma_j > 0 \,\}\big) + \operatorname{cone}\big(\{\, r^j \mid \gamma_j = 0 \,\}\big). \tag{5}$$

Since $\operatorname{cone}(\{r^1, \ldots, r^k\}) = \mathbb{R}^m$, we have that f is in the interior of M_γ. Observe that $\gamma_j = \psi_{M_\gamma - f}(r^j)$. Using Theorem 2.2, it can be shown that $\sum_{i=1}^k \gamma_i s_i \geq 1$ is a valid inequality if and only if M_γ does not have any integer point in its interior. We denote the right hand side of (4) by

$$\Gamma = \Big\{\, \gamma \geq 0 \,\Big|\, \sum_{j \in I} \gamma_j s_j(x, I) \geq 1 \quad \forall x \in X(I), \quad \forall I \in \mathcal{I} \,\Big\}.$$

We first show that any $\gamma \in \Gamma$ gives the coefficients of a valid inequality. We will show that M_γ does not contain any integer point in its interior. Suppose to the contrary and let $\bar{x}$ be a point in the interior of M_γ. If $\bar{x} - f \in \operatorname{rec}(M_\gamma)$, where rec denotes the recession cone, then $\bar{x} - f \in \operatorname{cone}\{\, r^j \mid \gamma_j = 0 \,\}$. Carathéodory's theorem for cones then implies that there exists a subset I of $\{\, j \mid \gamma_j = 0 \,\}$ of size m such that $\bar{x} - f \in \operatorname{cone}\{\, r^j \mid j \in I \,\}$ and therefore $\bar{x} \in X(I)$. But then $\sum_{j \in I} \gamma_j s_j(\bar{x}, I) = 0 < 1$, which violates the inequality corresponding to I and $\bar{x}$ in the definition of Γ. If $\bar{x} - f \notin \operatorname{rec}(M_\gamma)$, then there exists $\mu > 1$ such that $\mu(\bar{x} - f) + f$ is on the boundary of M_γ because $\bar{x}$ is in the interior of M_γ. This implies that $\mu(\bar{x} - f) + f$ lies on a facet of M_γ and therefore, using Lemma 3.1, there exists a subset I of $\{\, j \mid \gamma_j > 0 \,\}$ and a subset J of $\{\, j \mid \gamma_j = 0 \,\}$, with $\mu(\bar{x} - f) + f \in \operatorname{conv}(\{\, f + \frac{r^j}{\gamma_j} \mid j \in I \,\}) + \operatorname{cone}(\{\, r^j \mid j \in J \,\})$ and $|I| + |J|$ is at most m. Since the number of rays is at least $m + 1$, we may assume that $|I| + |J| = m$. Without loss of generality, let us assume that $I = \{1, \ldots, |I|\}$ and $J = \{|I| + 1, \ldots, m\}$. This then implies that there exist $\lambda_1 \geq 0, \ldots, \lambda_m \geq 0$ satisfying $\sum_{j=1}^{|I|} \lambda_j = 1$ and

$$\mu(\bar{x} - f) + f = \textstyle\sum_{j=1}^{|I|} \lambda_j (f + \frac{r^j}{\gamma_j}) + \sum_{j=|I|+1}^{m} \lambda_j r^j,$$

thus

$$\mu(\bar{x} - f) = \textstyle\sum_{j=1}^{|I|} \lambda_j (\frac{r^j}{\gamma_j}) + \sum_{j=|I|+1}^{m} \lambda_j r^j,$$

and finally

$$\bar{x} - f = \textstyle\sum_{j=1}^{|I|} (\lambda_j/\mu)(\frac{r^j}{\gamma_j}) + \sum_{j=|I|+1}^{m} (\lambda_j/\mu) r^j.$$

The last equation shows that $\bar{x} \in X(I \cup J)$. Moreover, $s_j(\bar{x}, I \cup J) = \frac{\lambda_j}{\mu \gamma_j}$ for $1 \leq j \leq |I|$ and $s_j(\bar{x}, I \cup J) = \frac{\lambda_j}{\mu}$ for $|I| + 1 \leq j \leq m$. Substituting into the left-hand side of the constraint for Γ corresponding to $I \cup J$ and $\bar{x}$, we get

$$\textstyle\sum_{j=1}^{|I|} \gamma_j \cdot \frac{\lambda_j}{\mu \gamma_j} + \sum_{j=|I|+1}^{m} 0 \cdot \frac{\lambda_j}{\mu} = \sum_{j=1}^{|I|} \frac{\lambda_j}{\mu} < 1.$$

The inequality follows from the fact that $\sum_{j=1}^{|I|} \lambda_j = 1$ and $\mu > 1$. Therefore this constraint is violated by γ. So we reach a contradiction. Hence we conclude that $\operatorname{int}(M_\gamma) \cap \mathbb{Z}^m = \emptyset$.

We now show that if $\sum_{j=1}^{k} \gamma_j s_j \geq 1$ is a valid inequality, then $\gamma \in \Gamma$. If not, there exists $I \in \mathcal{I}$ and $x \in X(I)$ such that $\sum_{j\in I} \gamma_j s_j(x, I) < 1$. Let I_+ be the set $\{\, j \in I \mid \gamma_j > 0 \,\}$ and $I_0 = I \setminus I_+$. By definition,

$$x - f = \textstyle\sum_{j\in I} s_j(x, I) r^j = \sum_{j\in I_+} \gamma_j s_j(x, I) \frac{r^j}{\gamma_j} + \sum_{j\in I_0} s_j(x, I) r^j.$$

Thus,

$$x = \mu f + \textstyle\sum_{j\in I_+} \gamma_j s_j(x, I)(f + \frac{r^j}{\gamma_j}) + \sum_{j\in I_0} s_j(x, I) r^j,$$

where $\mu = 1 - \sum_{j\in I} \gamma_j s_j(x, I) > 0$. Since $f \in \operatorname{int}(M_\gamma)$, the last equation shows that x is in the interior of M_γ. This contradicts the validity of $\sum_{j=1}^{k} \gamma_j s_j \geq 1$. □

The description of $\operatorname{conv}(R_f)^\vee$ in Proposition 3.2 uses infinitely many inequalities. We now show that we need only finitely many of these inequalities. Given $I \in \mathcal{I}$, let $\operatorname{ext}(X(I))$ denote the extreme points of the convex hull of $X(I)$.

Theorem 3.3.

$$\operatorname{conv}(R_f)^\vee = \Big\{\, \gamma \geq 0 \,\Big|\, \sum_{j\in I} \gamma_j s_j(x, I) \geq 1 \quad \forall x \in \operatorname{ext}(X(I)), \quad \forall I \in \mathcal{I} \,\Big\}.$$

Proof. We show that for any $I \in \mathcal{I}$ and $x \in X(I)$, the inequality $\sum_{j\in I} \gamma_j s_j(x, I) \geq 1$ is dominated by a convex combination of inequalities corresponding to points in $\operatorname{ext}(X(I))$. Since $\{r^1, \ldots, r^k\}$ and f are all rational, the recession cone of the convex hull of $X(I)$ is the same as $\operatorname{cone}(\{\, r^j \mid j \in I \,\})$ (see, for example, Theorem 16.1 in [19]). In fact, the convex hull of $X(I)$ is a polyhedron. Therefore, x can be represented as $\sum_{p\in P} \mu_p x_p + \sum_{j\in I} \lambda_j r^j$ where $x_p \in \operatorname{ext}(X(I))$ for all $p \in P$ and μ_p are convex coefficients and λ_j's are nonnegative coefficients. This further implies that $x - f = \sum_{p\in P} \mu_p(x_p - f) + \sum_{j\in I} \lambda_j r^j$.

If we represent $x - f$, $x_p - f$ in the basis $\{\, r^j \mid j \in I \,\}$, we conclude that $s_j(x, I) = \sum_{p\in P} \mu_p s_j(x_p, I) + \lambda_j$. Since the λ_j's are nonnegative, this shows that the inequality corresponding to x is dominated by a convex combination of the inequalities corresponding to x_p, $p \in P$. □

3.2 Complexity of the inequality description of $\operatorname{conv}(\boldsymbol{R_f})^\vee$

We now turn to the study of the complexity of the inequality description of the polyhedron $\operatorname{conv}(R_f)^\vee$.

We use the following general result about the integer hull of a polyhedron. If P is a polyhedron, we denote by P_{I} its integer hull, i.e., the convex hull of all integer points contained in P. When the dimension is fixed, P_{I} has only a polynomial number of vertices, as Cook et al. [9] showed.

Theorem 3.4. *Let $P = \{ x \in \mathbb{R}^q \mid Ax \leq b \}$ be a rational polyhedron with $A \in \mathbb{Q}^{p\times q}$ and let ϕ be the largest binary encoding size of any of the rows of the system $Ax \leq b$. Let $P_\mathrm{I} = \mathrm{conv}(P \cap \mathbb{Z}^q)$ be the integer hull of P. Then the number of vertices of P_I is at most $2p^q(6q^2\phi)^{q-1}$.*

Moreover, Hartmann [14] gave an algorithm for enumerating all the vertices, which runs in polynomial time in fixed dimension.

We thus obtain:

Remark 3.5. *Let the dimension m be a fixed number. Since all the rays $r^1, \ldots, r^k$ and f are rational, by Theorem 3.4, the cardinality of $\mathrm{ext}(X(I))$ is bounded by a polynomial in the binary encoding length of the data $r^1, \ldots, r^k, f$ for any $I \in \mathcal{I}$. Moreover, the cardinality of $\mathcal{I}$ is at most $\binom{k}{m}$, which is a polynomial in k. Hence, $\mathrm{conv}(R_f)^\vee$ is a polyhedron which can be represented as the intersection of polynomially many half-spaces.*

3.3 Finding the strongest cuts

Let γ^* be the optimal solution to the following convex program.

$$\begin{aligned} \min\ & \|\gamma\|_p \\ \text{s.t.}\ & \sum_{j\in I} \gamma_j s_j(x, I) \geq 1 \qquad \forall x \in \mathrm{ext}(X(I)), \quad \forall I \in \mathcal{I}, \\ & \gamma \geq 0. \end{aligned} \tag{6}$$

Theorem 3.3 implies that γ^* gives the coefficients of a valid inequality with minimum l_p norm. There is an interesting interpretation for the optimal cut with respect to the l_2 norm. If we view (1) as the optimal LP tableau, then valid inequalities for $\mathrm{conv}(R_f)$ are cuts which separate the current LP solution, $x = f, s = 0$ from the integer hull. The valid inequality with minimum l_2 norm is then the "deepest" cut, i.e., the cut whose Euclidean distance from the current LP solution is the maximum. The other l_p norms are also often used as a criterion for choosing the "best" cut.

Remark 3.6. *Since the feasible region for the convex program* (6) *is described by polynomially many inequalities by Remark 3.5, we can solve these programs in polynomial time. However, from a practical point of view, it might be easier to solve these programs using a cutting-plane or separation approach. We present a polynomial time separation algorithm for the convex program when the dimension m is an arbitrary fixed number, which uses integer feasibility algorithms in fixed dimensions. This avoids explicitly enumerating $I \in \mathcal{I}$ and $\mathrm{ext}(X(I))$, which could be a nontrivial and time-consuming task.*

Given a point γ, we need to decide if it is feasible for (6)*. This is achieved by testing if the convex set M_γ defined in* (5) *has an integer point in its interior.*

If M_γ is tested to have no integer point in its interior, then Theorem 2.2 implies that the inequality $\sum_{j=1}^k \gamma_j s_j \geq 1$ is valid. The proof of Proposition 3.2 shows that γ is therefore feasible to (6)*.*

On the other hand, if M_γ is tested to have an integer point $\bar{x}$ in its interior, then the proof of Proposition 3.2 shows that some constraint corresponding to $I \in \mathcal{I}$ such that $\bar{x} \in X(I)$ is violated.

By testing each subset of $\{r^1, \dots, r^k\}$ of size m, we can find this violated constraint in $O(mk^m)$ calls to an integer feasibility oracle. When m is fixed, this is a polynomial in k.

Note that for the l_1 and l_∞ norms, the optimization problem (6) can be changed to a linear program by a standard reformulation.

Finding the valid inequality with minimum l_∞ norm admits an alternative algorithm, which avoids solving (6). This again utilizes only integer feasibility algorithms for fixed dimensions. This approach could be more practical than solving the linear program because it would avoid explicitly enumerating $I \in \mathcal{I}$ and $\operatorname{ext}(X(I))$ and also does not require to use a cutting-plane procedure.

Instead, we can use a simple search procedure as follows. For any scalar $\alpha > 0$, let

$$C(\alpha) = \operatorname{conv}(\{\, f + \alpha r^j \mid j = 1, \dots, k \,\}).$$

Let $\sum_{j=1}^k \gamma_j s_j \geq 1$ be a valid inequality. Let M_γ be defined as in (5). Observe that $C(1/\|\gamma\|_\infty) \subseteq M_\gamma$. Since M_γ does not contain any integer point in its interior, neither does $C(1/\|\gamma\|_\infty)$. Therefore, to find the inequality with optimal l_∞ norm, we need to find the maximum possible value of α, such that $\operatorname{int}(C(\alpha)) \cap \mathbb{Z}^m = \emptyset$. Let this maximum be α^*.

The maximum α^*, of course, corresponds to a set $C(\alpha^*)$ that has an integer point on one of its facets. This shows that α^* is a rational number, for which, using standard techniques, we can determine a bound on its numerator and denominator of polynomial binary encoding length.

Then we can use the asymptotically optimal algorithm by Kwek and Mehlhorn [15] for searching a rational number α^* of bounded numerator and denominator, using only queries of the type "Is $\alpha^* \leq \alpha$?" This is similar to a binary search algorithm. Each such query amounts to testing $\operatorname{int}(C(\alpha)) \cap \mathbb{Z}^m = \emptyset$ for some current estimate α for α^*. Thus, this query step can be solved by integer feasibility algorithms for fixed dimensions.

4 The Tilting Space

For any matrix $B = (b^1; \dots; b^n) \in \mathbb{R}^{n \times m}$, let $Y(B)$ be the set of integer points y^j contained in

$$M(B) = \{\, x \in \mathbb{R}^m \mid b^i \cdot (x - f) \leq 1 \text{ for } i = 1, \dots, n \,\}.$$

If $M(B)$ is a lattice-free convex set, all elements of $Y(B)$ of course lie on the boundary of $M(B)$, that is, on at least one facet F_i of $M(B)$, induced by a constraint $b^i \cdot (x - f) \leq 1$.

In the present paper, we prove necessary conditions for $\sum_{j=1}^k \psi_B(r^j) s_j \geq 1$ to be an extreme inequality mainly by perturbation arguments. Given a matrix B, we show

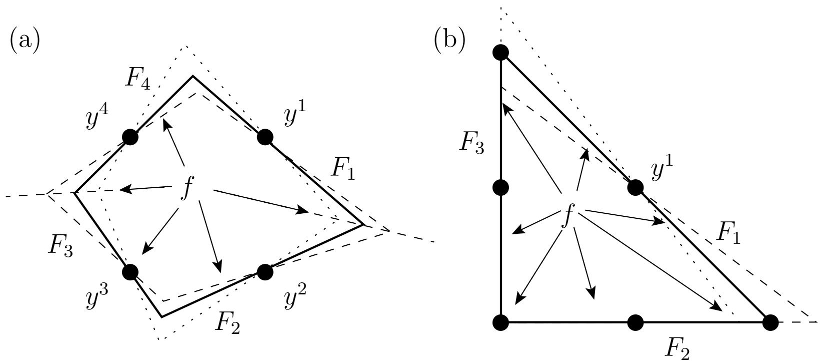

Figure 2: Tilting the facets of maximal lattice-free sets. (a) In this particular quadrilateral, setting $Y_1 = \{y^1\}, \ldots, Y_4 = \{y^4\}$ allows to tilt all facets $F_1, \ldots, F_4$. This still holds true if we ensure that all the corner rays remain corner rays for the perturbation (constraint (7b)). (b) In this Type-1 triangle, setting $Y_1 = \{y^1\}$ (a strict subset of $Y(B) \cap F_1$) and $Y_2 = Y(B) \cap F_2$, $Y_3 = Y(B) \cap F_3$, then facet F_1 can tilt, whereas facets F_2 and F_3 remain fixed. This still holds true if we ensure that all the non-corner rays remain non-corner rays for the perturbation (constraint (7c)). Note that choosing tilts from the set $\mathcal{S}(B)$ ensures that no new integer points enter. However, integer points may lie outside the set after tilting, such as the top and right vertices in this example.

under suitable hypotheses the existence of certain small perturbations A and C of B such that the inequality $\sum_{j=1}^{k} \psi_B(r^j)s_j \geq 1$ is a strict convex combination of the inequalities $\sum_{j=1}^{k} \psi_A(r^j)s_j \geq 1$ and $\sum_{j=1}^{k} \psi_C(r^j)s_j \geq 1$. Geometrically, these perturbations correspond to slightly 'tilting' the facets F_i of $M(B)$. In our proofs, it is convenient to choose, for every $i = 1, \ldots, n$, a certain subset $Y_i \subseteq Y(B) \cap F_i$ of the integer points on the facet F_i. When we tilt the facet F_i, we require that this subset Y_i continues to lie in the tilted facet; this obviously restricts how we can change the facet. This is illustrated in Figure 2.

We also need to control the interaction of the rays r^j and the facets. We will often refer to the set of *ray intersections*

$$P = \left\{ p^j \in \mathbb{R}^2 \;\middle|\; p^j = f + \tfrac{1}{\psi_B(r^j)} r^j,\ \psi_B(r^j) > 0,\ j = 1, \ldots, k \right\},$$

that is, the points p^j where the rays r^j meet the boundary of the set $M(B)$.

It is easy to see that whenever $\psi_B(r^j) > 0$, the set $I_B(r^j) = \arg\max_{i=1,\ldots,n} b^i \cdot r$ is the index set of all inequalities of $M(B)$ that the ray intersection $p^j = f + \frac{1}{\psi_B(r^j)} r^j$ satisfies with equality.

In particular, for $m = 2$, when all the inequalities corresponding to the rows of B are facets of $M(B)$, we have $|I_B(r^j)| = 1$ when r^j points to the relative interior of a facet, and $|I_B(r^j)| = 2$ when r^j points to a vertex of $M(B)$. In this second case, we call r a *corner ray* of $M(B)$. Again see Figure 2. When $M(B)$ is a split in $\mathbb{R}^2$, $|I_B(r^j)| = 1$ if r^j

is *not* in the recession cone of $M(B)$ and $|I_B(r^j)| = 2$ when r^j is in the recession cone.

Definition 4.1. *Let $\mathcal{Y}$ denote the tuple $(Y_1, \ldots, Y_n)$. The* tilting space $\mathcal{T}(B, \mathcal{Y}) \subset \mathbb{R}^{n\times m}$ *is defined as the set of matrices $A = (a^1; \ldots; a^n) \in \mathbb{R}^{n\times m}$ that satisfy the following conditions:*

$$a^i \cdot (y - f) = 1 \qquad \text{for } y \in Y_i,\ i = 1, \ldots, n, \tag{7a}$$

$$a^i \cdot r^j = a^{i'} \cdot r^j \qquad \text{for } i, i' \in I_B(r^j), \tag{7b}$$

$$a^i \cdot r^j > a^{i'} \cdot r^j \qquad \text{for } i \in I_B(r^j),\ i' \notin I_B(r^j). \tag{7c}$$

Constraint (7b) implies that if r^j hits a facet F_i of $M(B)$, then it also needs to hit the same facet of $M(A)$. In particular, for $m = 2$, this means that if r^j is a corner ray of $M(B)$, then r^j must also be a corner ray for $M(A)$ if $A \in \mathcal{T}(B, \mathcal{Y})$. Constraint (7c) enforces that if r^j does not hit a facet F_i of $M(B)$, then it also does not hit the same facet of $M(A)$. Thus we have $I_A(r^j) = I_B(r^j)$ for all rays r^j if $A \in \mathcal{T}(B, \mathcal{Y})$.

Note that $\mathcal{T}(B, \mathcal{Y})$ is cut out by linear equations and strict linear inequalities only and, since we always have $B \in \mathcal{T}(B, \mathcal{Y})$, it is non-empty. Thus it is a convex set whose dimension is the same as that of the affine space defined by the equations, (7a) and (7b), only. By $\mathcal{N}(B, \mathcal{Y}) \subset \mathbb{R}^{n\times m}$ we denote the linear space parallel to this affine space, in other words the null space of these equations.

If $\dim \mathcal{T}(B, \mathcal{Y}) \geq 1$, we can find two other matrices A and C in $T(B, \mathcal{Y})$ such that B is a strict convex combination of A and C. This will have the following important consequence which says that the inequality derived using $M(B)$ is a convex combination of the inequalities derived using $M(A)$ and $M(C)$.

Lemma 4.2. *Suppose $A, C \in \mathcal{T}(B, \mathcal{Y})$ with $B = \alpha A + (1 - \alpha)C$, $\alpha \in (0, 1)$. Then*

$$\psi_B(r^j) = \alpha\psi_A(r^j) + (1 - \alpha)\psi_C(r^j) \quad \textit{for } j = 1, \ldots, k.$$

Proof. Let $j \in \{1, \ldots, k\}$. Since $A, C \in \mathcal{T}(B, \mathcal{Y})$ we know that $I_B(r^j) = I_A(r^j) = I_C(r^j)$. Hence, let $i \in I_B(r^j)$. Then

$$\begin{aligned}\alpha\psi_A(r^j) + (1 - \alpha)\psi_C(r^j) &= \alpha a^i \cdot r^j + (1 - \alpha)c^i \cdot r^j \\ &= (\alpha a^i + (1 - \alpha)c^i) \cdot r^j = b^i \cdot r^j = \psi_B(r^j).\end{aligned}$$ □

Following the definition of extreme inequality, we see that finding such lattice-free polytopes $M(A)$ and $M(C)$ would imply that $\sum_{j=1}^{k} \psi_B(r^j)s_j \geq 1$ is not extreme provided that $\psi_A(r^j) \neq \psi_C(r^j)$ for some $j = 1, \ldots, k$. We will first handle the lattice-free condition, and later, via case analysis, we will argue that we can find distinct inequalities.

Next we introduce a tool that helps to ensure that no extra lattice points lie in the set after tilting the facets. To this end, consider the set

$$\mathcal{S}(B) := \{\, A = (a^1; \ldots; a^n) \in \mathbb{R}^{n\times m} \mid Y(A) \subseteq Y(B) \,\}.$$

Lemma 4.3. *Let $B \in \mathbb{R}^{n\times m}$ be such that $M(B)$ is a bounded maximal lattice-free set. Then $\mathcal{S}(B)$ contains an open neighborhood of B in the topology of $\mathbb{R}^{n\times m}$.*

This follows from now-classic results in the theory of parametric linear programming. Specifically, consider a parametric linear program,

$$\sup\{\, c(t)x : A(t)x \le b(t)\,\} \in \mathbb{R} \cup \{\pm\infty\},$$

where all coefficients depend continuously on a parameter vector t within some parameter region $\mathcal{R} \subseteq \mathbb{R}^q$. It is a theorem by D. H. Martin [18] that the optimal value function is upper semicontinuous in every parameter point t_0 such that the solution set (optimal face) is bounded, relative to the set of parameters where the supremum is finite. Here we only make use of a lemma used in the proof:

Theorem 4.4 (D. H. Martin [18], Lemma 3.1)**.** *Suppose that the solution set for $t = t_0$ is non-empty and bounded. Then, in parameter space, there is an open neighborhood $\mathcal{O}$ of t_0 such that the union of all solution sets for $t \in \mathcal{O}$ is bounded.*

Proof of Lemma 4.3. Consider the parametric linear program

$$\max\{\, 0 \mid a^i \cdot (x - f) \le 1,\ i = 1, \dots, n \,\}$$

with parameters $t = A = (a^1; \dots; a^n) \in \mathbb{R}^{n\times m}$. By the assumption of the lemma, the solution set for $t_0 = B = (b^1; \dots; b^n)$ is bounded. Let $\mathcal{O}$ be the open neighborhood of t_0 from Theorem 4.4, and let $\hat{S}$ be the union of all solution sets for $t \in \mathcal{O}$, which is by the theorem a bounded set.

For each of the finitely many lattice points $y \in \hat{S} \setminus M(B)$, let $i(y) \in \{1, \dots, n\}$ be an index of an inequality that cuts off y, that is, $b^{i(y)} \cdot (y - f) > 1$. Then

$$\mathcal{O}' = \{\, A = (a^1; \dots; a^n) \in \mathcal{O} \mid a^{i(y)} \cdot (y - f) > 1 \text{ for all } y \in \hat{S} \setminus M(B) \,\}$$

is an open set containing $B = (b^1; \dots; b^n)$. For $A = (a^1; \dots; a^n) \in \mathcal{O}'$ we have $Y(A) \subseteq Y(B)$, and thus $\mathcal{O}'$ is the desired open neighborhood of B contained in $\mathcal{S}(B)$. ☐

Observation 4.5. *Suppose* $\dim \mathcal{T}(B, \mathcal{Y}) \ge 1$*. By virtue of Lemma 4.3, for any $\bar{A} \in \mathcal{N}(B, \mathcal{Y})$, there exists $0 < \delta < 1$ such that both $B \pm \epsilon\bar{A} \in \mathcal{T}(B, \mathcal{Y}) \cap \mathcal{S}(B)$ for all $0 < \epsilon \le \delta$.*

Observation 4.6. *If $\mathcal{Y} = (Y_1, \dots, Y_n)$ is a covering of $Y(B)$, then $M(A)$ is lattice-free for every $A \in \mathcal{T}(B, \mathcal{Y}) \cap \mathcal{S}(B)$.*

Observation 4.5 and 4.6 are very useful because when we can ensure that $\mathcal{Y} = (Y_1, \dots, Y_n)$ is a covering of $Y(B)$, we no longer have to worry about finding explicit lattice-free convex sets. Rather, we can concentrate on simply showing that $\dim \mathcal{T}(B, \mathcal{Y}) \ge 1$ and that there exist matrices in that space such that there is a change in the coefficient of at least one of the rays.

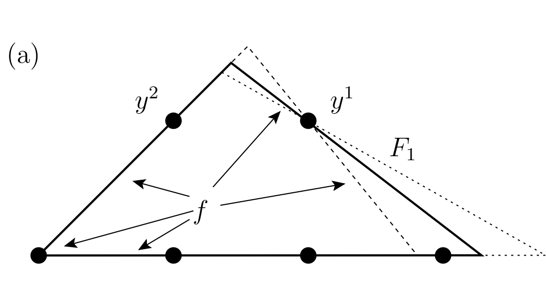

Figure 3: Simple tilts: Tilting one facet of a polytope to generate new inequalities. In both examples, there is a ray pointing to a non-integer point on the interior of the facet being tilted. This ensures that the inequalities from the tilted sets are distinct, and therefore we see that the original inequality $\sum_{j=1}^{k} \psi_B(r^j)s_j \geq 1$ is not extreme because it is the strict convex combination of two other inequalities. This is the assertion of Lemma 4.7.

A simple application of this principle is to tilt one facet of a polytope to show that the corresponding inequality is not extreme, as shown in Figure 3. This is summarized in the following lemma.

Lemma 4.7 (Simple tilts)**.** *Let $m \geq 2$. Let $M(B)$ be a maximal lattice-free polytope for some matrix $B \in \mathbb{R}^{n\times m}$. Let F_1 be a facet of $M(B)$ such that* $\operatorname{rel\,int}(F_1) \cap \mathbb{Z}^m = \{y^1\}$ *and $P \cap F_1 \subset \operatorname{rel\,int}(F_1)$, i.e., there are no ray intersections on the lower-dimensional faces of F_1. If* $\operatorname{rel\,int}(F_1) \cap P \setminus \mathbb{Z}^m \neq \emptyset$, *then $\sum_{j=1}^{k} \psi_B(r^j)s_j \geq 1$ is not extreme.*

Proof. Let $F_1, \dots, F_n$ be the facets of $M(B)$. Let $Y_1 = \{y^1\}$ and $Y_i = Y(B) \cap F_i$, $i = 2, \dots, n$, so that $\mathcal{Y} = (Y_1, \dots, Y_n)$ is a covering of the set $Y(B)$ of integer points in $M(B)$.

Let us analyze $\dim \mathcal{T}(B, \mathcal{Y})$. Since $P \cap F_1 \subset \operatorname{rel\,int}(F_1)$, there are no equalities in $\mathcal{T}(B, \mathcal{Y})$ corresponding to some $I_B(r^j)$ which involve a^1. Moreover, Y_1 is a singleton set consisting of y^1. Hence, there is only one equation in $\mathcal{T}(B, \mathcal{Y})$ which involves a^1, and that is $a^1 \cdot (y^1 - f) = 1$. This implies that $\dim \mathcal{T}(B, \mathcal{Y}) \geq m - 1 \geq 1$ for $m \geq 2$. We will now select a particular element in $\mathcal{N}(B, \mathcal{Y}) \setminus \{0\}$.

By the hypothesis, there exists $j \in \{1, \dots, k\}$ such that $p^j \in (\operatorname{rel\,int}(F_1) \cap P) \setminus \mathbb{Z}^m$. Since $\operatorname{rel\,int}(F_1) \cap \mathbb{Z}^m = \{y^1\}$, this implies r^j and $y^1 - f$ are linearly independent. Since $a^1 \cdot (y^1 - f) = 0$ is the only equation involving a^1 in $\mathcal{N}(B, \mathcal{Y})$, and $y^1 - f$ and r^j are linearly independent, $\mathcal{N}(B, \mathcal{Y}) \cap \{\, (a^1; \dots; a^n) \mid a^1 \cdot r^j = 0 \,\} \subsetneq \mathcal{N}(B, \mathcal{Y})$. Pick any $\bar{A} \in \mathcal{N}(B, \mathcal{Y}) \setminus \{\, (a^1; \dots; a^n) \mid a^1 \cdot r^j = 0 \,\}$.

By Observation 4.5, there exists an $\epsilon > 0$ such that both $B \pm \epsilon\bar{A} \in \mathcal{T}(B, \mathcal{Y}) \cap \mathcal{S}(B)$. By our choice of $\mathcal{Y}$, the hypothesis of Observation 4.6 is satisfied and therefore $M(B \pm \epsilon\bar{A})$ are both lattice-free. Moreover, since $\bar{A} \notin \{\, (a^1; \dots; a^n) \mid a^1 \cdot r^j = 0 \,\}$, we have $\bar{a}^1 \cdot r^j \neq 0$. Therefore, $\psi_{B+\epsilon\bar{A}}(r^j) = (b^1 + \epsilon\bar{a}^1) \cdot r^j \neq (b^1 - \epsilon\bar{a}^1) \cdot r^j = \psi_{B-\epsilon\bar{A}}(r^j)$; the equalities follow from the fact that $B \pm \epsilon A \in \mathcal{T}(B, \mathcal{Y})$ and so $I_{B+\epsilon\bar{A}}(r^j) = I_{B-\epsilon\bar{A}}(r^j) = I_B(r^j) = \{1\}$.

Moreover, since $B = \frac{1}{2}(B+\epsilon\bar{A}) + \frac{1}{2}(B-\epsilon\bar{A})$, one can now apply Lemma 4.2 to show that the inequality from $M(B)$ is a convex combination of the two different valid inequalities coming from $M(B \pm \epsilon\bar{A})$. □

In the next section, we will use this lemma and more complicated applications of the tilting space.

5 New Necessary Conditions for $m = 2$

In this section, we prove necessary conditions for $\sum_{j=1}^{k} \psi_B(r^j)s_j \geq 1$ to be an extreme inequality for any matrix B such that $M(B)$ is a maximal lattice-free set in $\mathbb{R}^2$. These conditions can also be shown using the complete characterization of facets for $m = 2$ in [10]. Our proofs primarily use geometrically motivated tilting arguments which illuminate why certain inequalities are not extreme.

We find only three cases when a non-extreme inequality is a convex combination of inequalities derived from convex sets of a different combinatorial type: splits can be convex combinations of two Type 2 triangle inequalities; Type 2 triangles can, in some instances, be convex combinations of a Type 3 triangle and a quadrilateral inequality; and in some other cases, Type 2 inequalities can be convex combinations of two quadrilaterals. In Section 6, we will use these conditions to show that there are only polynomially many extreme inequalities for $\operatorname{conv}(R_f)$.

Notation. The integer points will typically be labeled such that $y^1 \in \operatorname{rel\,int}(F_1), y^2 \in \operatorname{rel\,int}(F_2)$. The closed line segment between two points x^1 and x^2 will be denoted by $[x^1, x^2]$, and the open line segment will be denoted by (x^1, x^2). Within the case analysis of some of the proofs, we will refer to certain points lying within splits. For convenience, for $i = 1, 2, 3$, we define S_i as the split such that one facet of S_i contains F_i and $S_i \cap \operatorname{int}(M(B)) \neq \emptyset$. For any facet F_i, we will need to consider the sub-lattice of $\mathbb{Z}^2$ contained in the linear space parallel to F_i. We use the notation $v(F_i)$ to denote the primitive lattice vector which generates this one-dimensional lattice.

We begin with a lemma regarding corner rays for triangles and quadrilaterals in $\mathbb{R}^2$.

Lemma 5.1. *Let $B \in \mathbb{R}^{n\times 2}$ be such that $M(B)$ is a triangle ($n = 3$) or a quadrilateral ($n = 4$). Let $Y_i = \{y^i\}$, for any $y^i \in \operatorname{rel\,int}(F_i) \cap \mathbb{Z}^2$. If $P \not\subset \mathbb{Z}^2$ and $M(B)$ has fewer than n corner rays, then there exists $\bar{A} \in \mathcal{N}(B, \mathcal{Y}) \setminus \{0\}$ such that for all $0 < \epsilon < 1$ $\psi_{B+\epsilon\bar{A}}(r^j) \neq \psi_{B-\epsilon\bar{A}}(r^j)$ for some $j = 1, \ldots, k$ and $\psi_B(r^j) = \frac{1}{2}\psi_{B-\epsilon\bar{A}}(r^j) + \frac{1}{2}\psi_{B+\epsilon\bar{A}}(r^j)$ for all $j = 1, \ldots, k$.*

Proof. We examine the tilting space of B with at most $n-1$ corner rays. We only need to examine the tilting space of exactly $n-1$ corner rays, as it is a subspace of the other cases. With $n-1$ corner rays, $\mathcal{T}(B, \mathcal{Y})$ is the set of matrices $A = (a^1; \ldots; a^n)$ satisfying the following system of equations, where, for convenience, we define $\bar{y}^i := y^i - f$:

$$a^i \cdot \bar{y}^i = 1 \text{ for } i = 1, \ldots, n \qquad \text{and} \qquad a^i \cdot r^i = a^{i+1} \cdot r^i \text{ for } i = 1, \ldots, n-1,$$

and a number of strict inequalities, which we do not list here.

We have assumed, without loss of generality, that the rays and facets are numbered such that we have corner rays $r^i \in F_i \cap F_{i+1}$ for $i = 1, \ldots, n-1$, so the remaining ray r^n is not a corner ray. As usual, $y^i \in F_i \cap \mathbb{Z}^2$ for $i = 1, \ldots, n$. Note that $\bar{y}^i$ is linearly independent from r^i for $i = 1, \ldots, n-1$ and linearly independent from r^{i-1} for $i = 2, \ldots, n$, because y^i lies in the relative interior of F_i and the rays point to the vertices.

We now study the linear subspace $\mathcal{N}(B, \mathcal{Y})$ that lies parallel to the affine hull of $\mathcal{T}(B, \mathcal{Y})$, so that $\mathcal{N}(B, \mathcal{Y})$ is described by the homogeneous equations

$$a^i \cdot \bar{y}^i = 0 \text{ for } i = 1, \ldots, n \qquad \text{and} \qquad a^i \cdot r^i = a^{i+1} \cdot r^i \text{ for } i = 1, \ldots, n-1. \tag{8}$$

There are $2n-1$ equations and $2n$ variables, so $\dim \mathcal{N}(B, \mathcal{Y}) \geq 1$. Moreover, observe that B satisfies all the strict inequalties of $\mathcal{T}(B, \mathcal{Y})$ and therefore, we can choose $\bar{A} = (\bar{a}^1; \ldots; \bar{a}^n) \in \mathcal{N}(B, \mathcal{Y}) \setminus \{0\}$ such that $B \pm \epsilon \bar{A} \in \mathcal{T}(B, \mathcal{Y})$ for all $0 < \epsilon < 1$.

Notice that for $i = 1, \ldots, n-1$, if $\bar{a}^i = 0$, then $\bar{a}^{i+1}$ must satisfy $\bar{a}^{i+1} \cdot r^i = 0$ and $\bar{a}^{i+1} \cdot \bar{y}^{i+1} = 0$, which implies that $\bar{a}^{i+1} = 0$, since $\bar{y}^{i+1}$ and r^i are linearly independent. Similarly, for $i = 2, \ldots, n$, if $\bar{a}^i = 0$, then $\bar{a}^{i-1}$ must satisfy $\bar{a}^{i-1} \cdot r^{i-1} = 0$ and $\bar{a}^{i-1} \cdot \bar{y}^{i-1} = 0$, which implies that $\bar{a}^{i-1} = 0$. By induction, this shows that if $\bar{a}^i = 0$ for any $i = 1, \ldots, n$, then $\bar{A} = 0$, which contradicts our assumption. Hence, $a^i \neq 0$ for $i = 1, \ldots, n$.

Now suppose the ray $r \in \{r^1, \ldots, r^k\}$ points to $F_i \setminus \mathbb{Z}^2$ for some $i \in \{1, \ldots, n\}$. This ray must exist by the assumption that $P \not\subset \mathbb{Z}^2$. If r is parallel to $\bar{y}^i$, then it either points to y^i from f, or it does not point to F_i. Since we assumed that r points to $F_i \setminus \mathbb{Z}^2$, neither of these is possible, so r is not parallel to $\bar{y}^i$. Now since $\bar{a}^i \cdot \bar{y}^i = 0$ and neither is the zero vector, $\bar{y}^i$ and $\bar{a}^i$ are linearly independent and thus span $\mathbb{R}^2$. Pick α, β such that $r = \alpha \bar{y}^i + \beta \bar{a}^i$. Then $\bar{a}^i \cdot r = \bar{a}^i \cdot (\alpha \bar{y}^i + \beta \bar{a}^i) = \beta \|\bar{a}^i\|_2^2$. Note $\beta \neq 0$ since r is not parallel to $\bar{y}^i$. Since $B \pm \epsilon \bar{A} \in \mathcal{T}(B, \mathcal{Y})$ for every $0 < \epsilon < 1$, $I_{B+\epsilon \bar{A}}(r) = I_B(r) = I_{B-\epsilon \bar{A}}(r)$. Therefore, $\psi_{B+\epsilon \bar{A}}(r) = (b^i + \epsilon \bar{a}^i) \cdot r \neq (b^i - \epsilon \bar{a}^i) \cdot r = \psi_{B-\epsilon \bar{A}}(r)$. Since $B = \frac{1}{2}(B + \epsilon \bar{A}) + \frac{1}{2}(B - \epsilon \bar{A})$, applying Lemma 4.2 finishes the result. $\square$

We comment here that in the statement of Lemma 5.1, we do not insist that $M(B)$ is a lattice-free convex set. Therefore, the statement does not mention anything about valid or extreme inequalities for $\operatorname{conv}(R_f)$. This generality will be needed in our results in the coming subsections.

5.1 Type 3 triangles and quadrilaterals

For this section on Type 3 triangles and quadrilaterals, we will be using a specific $\mathcal{Y} = (Y_1, \ldots, Y_n)$ where Y_i will consist of the unique integer point in the relative interior of facet F_i. This would mean that $\mathcal{Y} = (Y_1, \ldots, Y_n)$ is a covering of $Y(B)$ for Type 3 triangles and quadrilaterals. We will now apply Lemma 5.1 to matrices B such that $M(B)$ is a maximal lattice-free set that is either a Type 3 triangle or a quadrilateral.

Corollary 5.2. *Suppose that $M(B)$ has n facets and is a maximal lattice-free set that is either a Type 3 triangle ($n = 3$) or a quadrilateral ($n = 4$), and that $P \not\subset \mathbb{Z}^2$. If $M(B)$ has fewer than n corner rays, then $\sum_{j=1}^{k} \psi_B(r^j) s_j \geq 1$ is not extreme.*

Proof. Apply Lemma 5.1 on $M(B)$ with $\mathcal{Y}$ to obtain $\bar{A} \in \mathcal{N}(B, \mathcal{Y}) \setminus \{0\}$ with the stated properties. Since $\mathcal{Y}$ is a covering of $Y(B)$, by Observation 4.5, there exists $0 < \epsilon < 1$ such that $B \pm \epsilon\bar{A} \in \mathcal{T}(B, \mathcal{Y}) \cap S(B)$; so by Observation 4.6, $M(B \pm \epsilon\bar{A})$ are both lattice-free. From the conclusion of Lemma 5.1, we see that $\sum_{j=1}^{k} \psi_B(r^j) s_j \geq 1$ is not extreme as it is the convex combination of two distinct valid inequalities derived from the lattice-free sets $M(B \pm \epsilon\bar{A})$. □

Lemma 5.3 (Type 3 Triangles)**.** *Suppose $M(B)$ is a Type 3 triangle. If $\sum_{j=1}^{k} \psi_B(r^j) s_j \geq 1$ is extreme, then one of the following holds:*
Case a. $P \subset \mathbb{Z}^2$.
Case b. $\operatorname{vert}(B) \subseteq P$.

Proof. This follows from Corollary 5.2. □

For quadrilaterals, Cornuéjols and Margot defined the *ratio condition* as a necessary and sufficient condition to yield an extreme inequality when all corner rays are present. Suppose p^1, p^2, p^3, p^4 are the corner ray intersections assigned in a counter-clockwise orientation, and y^i is the integer point contained in $[p^i, p^{i+1}]$. The ratio condition holds if there does not exist a scalar $t > 0$ such that

$$\frac{\|y^i - p^i\|}{\|y^i - p^{i+1}\|} = \begin{cases} t & \text{for } i = 1, 3 \\ \frac{1}{t} & \text{for } i = 2, 4. \end{cases} \tag{9}$$

This is illustrated in Figure 4. We will now show the relation between the ratio condition and the tilting space.

Lemma 5.4. *Suppose $M(B)$ is a quadrilateral with four corner rays. If the ratio condition does not hold, i.e., there exists a scalar $t > 0$ with (9), then* $\dim \mathcal{T}(B, \mathcal{Y}) \neq 0$.

Proof. We will first analyze the tilting space equations with four corner rays, and then apply the assumption that the ratio condition does not hold. For convenience we define $\bar{y}^i := y^i - f$ and $\bar{p}^i := p^i - f$, where p^i are the ray intersections. Then $\bar{p}^i = \frac{1}{\psi_B(r^i)} r^i$.

We want to determine when there is not a unique solution to the following system of

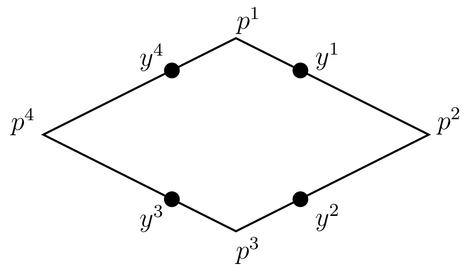

Figure 4: Example of a quadrilateral for which the ratio condition does *not* hold, i.e., there exists a $t > 0$ satisfying (9). Here $\dim \mathcal{T}(B, \mathcal{Y}) \neq 0$.

equations that come from the tilting space:

$$\begin{aligned} a^1 \cdot \bar{y}^1 &= 1 \\ a^1 \cdot \bar{p}^2 &= a^2 \cdot \bar{p}^2 \\ a^2 \cdot \bar{y}^2 &= 1 \\ a^2 \cdot \bar{p}^3 &= a^3 \cdot \bar{p}^3 \\ a^3 \cdot \bar{y}^3 &= 1 \\ a^3 \cdot \bar{p}^4 &= a^4 \cdot \bar{p}^4 \\ a^4 \cdot \bar{y}^4 &= 1 \\ a^4 \cdot \bar{p}^1 &= a^1 \cdot \bar{p}^1 \end{aligned} \qquad \text{or} \qquad \begin{bmatrix} \bar{y}^1 & & & \\ \bar{p}^2 & -\bar{p}^2 & & \\ & \bar{y}^2 & & \\ & \bar{p}^3 & -\bar{p}^3 & \\ & & \bar{y}^3 & \\ & & \bar{p}^4 & -\bar{p}^4 \\ & & & \bar{y}^4 \\ -\bar{p}^1 & & & \bar{p}^1 \end{bmatrix} \begin{bmatrix} a^1 \\ a^2 \\ a^3 \\ a^4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

as an 8×8 matrix equation where every vector shown in the matrix is a row vector of size 2. We will analyze the determinant of the matrix.

Since the points $\bar{y}^1, \bar{y}^2, \bar{y}^3, \bar{y}^4$ are on the interior of each facet, they can be written as certain convex combinations of $\bar{p}^1, \bar{p}^2, \bar{p}^3, \bar{p}^4$. We write this in a complicated form at first to simplify resulting calculations. Here, $\alpha' = 1 + \alpha$, and $\alpha > 0$, and similarly for β, γ, and δ.

$$\begin{aligned} \bar{y}^1 &= \tfrac{1}{\alpha'}\bar{p}^1 + \tfrac{\alpha}{\alpha'}\bar{p}^2 \\ \bar{y}^2 &= \tfrac{1}{\beta'}\bar{p}^2 + \tfrac{\beta}{\beta'}\bar{p}^3 \\ \bar{y}^3 &= \tfrac{1}{\gamma'}\bar{p}^3 + \tfrac{\gamma}{\gamma'}\bar{p}^4 \\ \bar{y}^4 &= \tfrac{1}{\delta'}\bar{p}^4 + \tfrac{\delta}{\delta'}\bar{p}^1 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \bar{p}^1 &= \alpha'\bar{y}^1 - \alpha\bar{p}^2 \\ \bar{p}^2 &= \beta'\bar{y}^2 - \beta\bar{p}^3 \\ \bar{p}^3 &= \gamma'\bar{y}^3 - \gamma\bar{p}^4 \\ \bar{p}^4 &= \delta'\bar{y}^4 - \delta\bar{p}^1 \end{aligned}$$

Now just changing the last row using the above columns

$$\begin{bmatrix} -\bar{p}^1 & 0 & 0 & \bar{p}^1 \end{bmatrix} \to \begin{bmatrix} 0 & \alpha\bar{p}^2 & 0 & \bar{p}^1 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & -\alpha\beta\bar{p}^3 & \bar{p}^1 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 & \alpha\beta\gamma\bar{p}^4 + \bar{p}^1 \end{bmatrix}$$

The resulting matrix, after adding this last row and substituting in $\bar{y}^4$, is

$$\begin{bmatrix} \bar{y}^1 & & & \\ \bar{p}^2 & -\bar{p}^2 & & \\ & \bar{y}^2 & & \\ & \bar{p}^3 & -\bar{p}^3 & \\ & & \bar{y}^3 & \\ & & \bar{p}^4 & -\bar{p}^4 \\ & & & \frac{1}{\delta'}\bar{p}^4 + \frac{\delta}{\delta'}\bar{p}^1 \\ & & & \alpha\beta\gamma\bar{p}^4 + \bar{p}^1 \end{bmatrix}$$

This is now an upper block triangular matrix. The first three blocks are all non-singular, and the last block is non-singular if and only if there does not exist a t such that

$$\frac{1}{\delta'}\bar{p}^4 + \frac{\delta}{\delta'}\bar{p}^1 = t(\alpha\beta\gamma\bar{p}^4 + \bar{p}^1) \;\Rightarrow\; \Big(\frac{\delta}{\delta'} - t\Big)\bar{p}^1 + \Big(\frac{1}{\delta'} - t\alpha\beta\gamma\Big)\bar{p}^4 = 0.$$

If such a t exists, then $t = \frac{\delta}{\delta'}$ since $\bar{p}^1$ and $\bar{p}^4$ are linearly independent. It follows that $\alpha\beta\gamma\delta = 1$ if and only if $\dim \mathcal{T}(B, \mathcal{Y}) \neq 0$. If the ratio condition does not hold, then it is easy to see that $\alpha = \frac{1}{\beta} = \gamma = \frac{1}{\delta}$, and hence $\alpha\beta\gamma\delta = 1$ and $\dim \mathcal{T}(B, \mathcal{Y}) \neq 0$. $\square$

Lemma 5.5 (Quadrilaterals)**.** *Suppose $M(B)$ is a quadrilateral. If $\sum_{j=1}^k \psi_B(r^j)s_j \geq 1$ is extreme, then one of the following holds:*
Case a. *$P \subset \mathbb{Z}^2$.*
Case b. *$\mathrm{vert}(B) \subseteq P$ and the ratio condition holds. Moreover, $M(B)$ is the unique quadrilateral with these four corner rays and these four integer points.*

Proof. Suppose that we are not in Case a. Corollary 5.2 shows that all four corner rays must exist. Lemma 5.4 shows that if the ratio condition does not hold, then $\dim \mathcal{T}(B, \mathcal{Y}) \geq 1$ and so one of the equalities in $\mathcal{T}(B, \mathcal{Y})$ corresponding to a corner ray is redundant. This means that N is a subspace of $\mathcal{N}(B, \mathcal{Y})$ where N is the subspace given by the equations (8). Since we suppose $P \not\subset \mathbb{Z}^2$, the proof of Lemma 5.1 shows that there exists $\bar{A} \in N \setminus \{0\}$ such that for every $0 < \epsilon < 1$, $\psi_{B+\epsilon\bar{A}}(r^j) \neq \psi_{B-\epsilon\bar{A}}(r^j)$ for some $j = 1, \ldots, k$ and $\psi_B(r^j) = \frac{1}{2}\psi_{B-\epsilon\bar{A}}(r^j) + \frac{1}{2}\psi_{B+\epsilon\bar{A}}(r^j)$ for all $j = 1, \ldots, k$. Since N is a subspace of $\mathcal{N}(B, \mathcal{Y})$, we have that $\bar{A} \in \mathcal{N}(B, \mathcal{Y}) \setminus \{0\}$. We can again use Observations 4.5 and 4.6 to show that $\sum_{j=1}^k \psi_B(r^j)s_j \geq 1$ is not extreme.

Observe that the set of matrices A such that $M(A)$ contains the same set of integer points as $M(B)$ and has the same four corner rays as $M(B)$ is given by all solutions to the equality system in $\mathcal{T}(B, \mathcal{Y})$. If this system had non unique solutions, then $\dim \mathcal{T}(B, \mathcal{Y}) \geq 1$ and following the same reasoning as above, we would conclude that $\sum_{j=1}^k \psi_B(r^j)s_j \geq 1$ is not extreme. $\square$

Remark 5.6. *The ratio condition is indeed equivalent to* $\dim \mathcal{T}(B, \mathcal{Y}) = 0$*. We can see this by showing that* $\dim \mathcal{T}(B, \mathcal{Y}) \neq 0$ *if and only if the ratio condition does not*

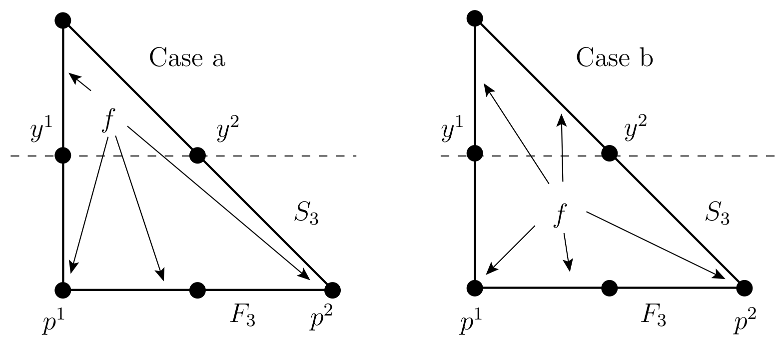

Figure 5: Cases for extreme Type 1 triangles in Lemma 5.7

hold. Lemma 5.4 shows that if the ratio condition does not hold, then $\dim \mathcal{T}(B, \mathcal{Y}) \neq 0$*. On the other hand, if* $\dim \mathcal{T}(B, \mathcal{Y}) \neq 0$*, then* $\sum_{j=1}^{k} \psi_B(r^j)s_j \geq 1$ *is not extreme using similar arguments as in the proof above of Lemma 5.5. Cornuéjols and Margot [10] show that the ratio condition holds if and only if* $\sum_{j=1}^{k} \psi_B(r^j)s_j \geq 1$ *is extreme, and so since* $\sum_{j=1}^{k} \psi_B(r^j)s_j \geq 1$ *is not extreme, the ratio condition does not hold.*

5.2 Type 1 triangles

Lemma 5.7 (Type 1 Triangles)**.** *Suppose* $M(B)$ *is a Type 1 triangle and suppose that* $\sum_{j=1}^{k} \psi_B(r^j)s_j \geq 1$ *cannot be realized or dominated by an inequality derived from either a Type 2 triangle or a split. If* $\sum_{j=1}^{k} \psi_B(r^j)s_j \geq 1$ *is extreme, then there exist* $p^1, p^2 \in \mathrm{vert}(B) \cap P$*. Moreover, labeling the facet containing* p^1, p^2 *as* F_3*, one of the following holds:*
Case a. $f \notin S_3$*.*
Case b. $f \in S_3$*, and* $P \not\subset S_3$*.*

Figure 5 illustrates the two cases of the lemma.

Proof. **Step 1.** We will show that if $\#(\mathrm{vert}(B) \cap P) \leq 1$, then either $\sum_{j=1}^{k} \psi_B(r^j)s_j \geq 1$ is not extreme, or it is realized by a Type 2 inequality.

If $\#(\mathrm{vert}(B) \cap P) \leq 1$, then there is a facet whose vertices are not contained in P; without loss of generality, let this facet be F_1. We now consider a simple tilt of facet F_1. Lemma 4.7 shows that if $P \cap \mathrm{rel\,int}(F_1) \setminus \mathbb{Z}^2 \neq \emptyset$, then $\sum_{j=1}^{k} \psi_B(r^j)s_j \geq 1$ is not extreme. Otherwise, if $P \cap \mathrm{rel\,int}(F_1) \setminus \mathbb{Z}^2 = \emptyset$, then since there are no corner rays, we can tilt F_1 with y^1 as a fulcrum and create a Type 2 triangle that realizes the same inequality as $\sum_{j=1}^{k} \psi_B(r^j)s_j \geq 1$ (see Figure 6).

Step 2. From Step 1, if $\sum_{j=1}^{k} \psi_B(r^j)s_j \geq 1$ is extreme, then $\#(\mathrm{vert}(B) \cap P) \geq 2$, i.e., there exist $p^1, p^2 \in \mathrm{vert}(B) \cap P$. As in the statement of this lemma, $p^1, p^2 \in F_3$. If $P \cup \{f\} \subset S_3$, then $\sum_{j=1}^{k} \psi_B(r^j)s_j \geq 1$ is dominated or realized by the valid inequality derived from S_3. Therefore either Case a or Case b occurs. □

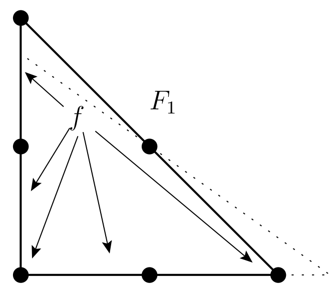

Figure 6: In the proof of Lemma 5.7, Step 1, a Type 1 triangle can be replaced by a Type 2 triangle (dotted) that gives the same inequality.

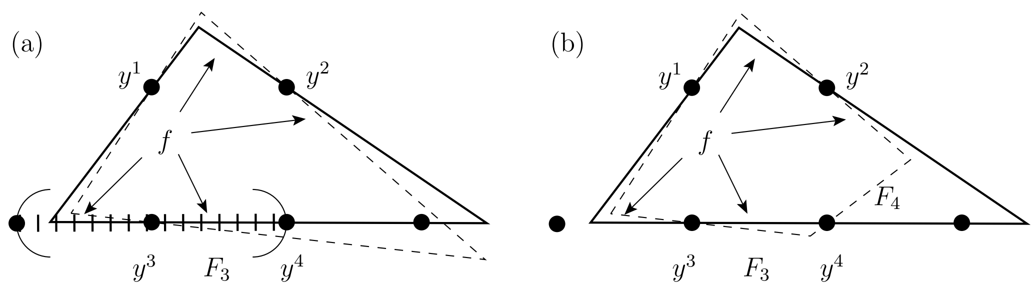

Figure 7: The geometry of Lemma 5.8. (a) The hypothesis of the lemma regarding the ray intersections on F_3. (b) A new edge is constructed such that no rays point to it, turning the triangle to a quadrilateral.

5.3 Type 2 triangles and splits

For these two types of maximal lattice-free sets, we allow tilts where $\mathcal{Y} = (Y_1, \ldots, Y_n)$ may not be a covering of $Y(B)$. This may create non-lattice-free sets in $\mathcal{T}(B, \mathcal{Y}) \cap \mathcal{S}(B)$ as the hypothesis of Observation 4.6 is not satisfied. We handle this by adding an additional edge to take care of the conflicting lattice points in the interior. Recall the notation $v(F_i)$ for the lattice vector which generates the sub-lattice of $\mathbb{Z}^2$ parallel to F_i. Moreover, we recall that (x^1, x^2) denotes the open line segment between x^1 and x^2.

Lemma 5.8. *Let $M(B)$ be a Type 2 triangle with $\#(\operatorname{conv}(P \cap F_3) \cap \mathbb{Z}^2) \leq 1$. Suppose there exists a point $y^3 \in F_3 \cap \mathbb{Z}^2$ such that $P \cap F_3 \subset (y^3 - v(F_3), y^3 + v(F_3))$. Let $Y_i = \{y^i\}$, and suppose that $\dim \mathcal{T}(B, \mathcal{Y}) \geq 1$.*

For any $\bar{A} \in \mathcal{N}(B, \mathcal{Y}) \setminus \{0\}$, there exists an $0 < \epsilon_1 < 1$ such that $\sum_{j=1}^{k} \psi_{B+\epsilon \bar{A}}(r^j) s_j \geq 1$ is a valid inequality for $\operatorname{conv}(R_f)$ for every $0 < \epsilon \leq \epsilon_1$.

The geometry of this lemma is illustrated in Figure 7 (a).

Proof. Recall that a lattice-free set containing f in its interior yields a valid inequality for $\operatorname{conv}(R_f)$. We will construct $0 < \epsilon_1 < 1$ such that for every $0 < \epsilon \leq \epsilon_1$ there exists a matrix $C = (c^1; c^2; c^3)$ with three rows or a matrix $C = (c^1; c^2; c^3; c^4)$ with four rows, such that $M(C)$ is a lattice-free set and $\psi_C(r^j) = \psi_{B+\epsilon\bar{A}}(r^j)$ for $j = 1, \ldots, k$. Of course, in the case when C has four rows, the set $M(C)$ will contain an additional edge.

By Observation 4.5, there exists $0 < \delta < 1$ such that $B + \epsilon\bar{A} \in \mathcal{T}(B, \mathcal{Y}) \cap \mathcal{S}(B)$ for all $0 < \epsilon \leq \delta$. From the definition of $\mathcal{S}(B)$ it follows that $M(B + \epsilon\bar{A}) \cap \mathbb{Z}^2 \subseteq Y(B)$ for all $0 < \epsilon \leq \delta$. Since $Y_1 = \{y^1\}$ and $Y_2 = \{y^2\}$, y^1 and y^2 are not contained in $\operatorname{int}(M(B + \epsilon\bar{A}))$. This implies that $\operatorname{int}(M(B + \epsilon\bar{A})) \cap \mathbb{Z}^2 \subset F_3$.

If $\operatorname{int}(M(B + \epsilon\bar{A})) \cap \mathbb{Z}^2 = \emptyset$ for every $0 < \epsilon \leq \delta$, then $M(B + \epsilon\bar{A})$ is lattice-free for every such ϵ. So we let $\epsilon_1 = \delta$ and let $C = B + \epsilon\bar{A}$ for every $0 < \epsilon \leq \delta$ and we are done.

Otherwise, let $0 < \epsilon' \leq \delta$ be such that $\operatorname{int}(M(B + \epsilon'\bar{A})) \cap \mathbb{Z}^2 \neq \emptyset$. Let y^4 be the closest integer point on F_3 to y^3 such that $y^4 \in \operatorname{int}(M(B + \epsilon'\bar{A}))$. Note that one can then assume $y^4 = y^3 + v(F_3)$. Next, pick $c^4 \in \mathbb{R}^2$ such that $c^4 \cdot (x - f) \leq 1$ is a halfspace containing $P \cup \{y^1, y^2, y^3\}$ and such that $c^4 \cdot (y^4 - f) = 1$. This exists because there are only finitely many ray intersections, y^4 is on the boundary, and $P \cap F_3 \subset \{\, y^4 + t(y^3 - y^4) \mid t > 0 \,\}$ since $P \cap F_3 \subset (y^3 - v(F_3), y^3 + v(F_3))$.

Consider the set

$$\mathcal{V} := \{\, (a^1; a^2; a^3) \in \mathbb{R}^{3\times 2} \mid a^i \cdot r^j > c^4 \cdot r^j \text{ for } j = 1, \ldots, k,\ i \in I_B(r^j) \,\}.$$

Since $\mathcal{V}$ is an open set containing B, there exists $0 < \epsilon_1 \leq \epsilon'$ such that $B + \epsilon\bar{A} \in \mathcal{V}$ for every $0 < \epsilon \leq \epsilon_1$. For any $0 < \epsilon \leq \epsilon_1$, let $(c^1; c^2; c^3) = B + \epsilon\bar{A}$. Then $C = (c^1; c^2; c^3; c^4)$ has the property that $M(C)$ is a lattice-free quadrilateral. This is because $\epsilon \leq \delta$ implies $\operatorname{int}(M(B + \epsilon\bar{A})) \cap \mathbb{Z}^2 \subset F_3$. But all these integer points violate the inequality $c^4 \cdot (x - f) \leq 1$. See Figure 7 (b).

Moreover, $\psi_C(r^j) = \psi_{B+\epsilon\bar{A}}(r^j)$ for $j = 1, \ldots, k$. This is because $I_C(r^j) = I_B(r^j) = I_{B+\epsilon\bar{A}}(r^j)$ for all j; the first equality follows because $B + \epsilon\bar{A} \in \mathcal{V}$ and the second equality follows from the fact that $B + \epsilon\bar{A} \in \mathcal{T}(B, \mathcal{Y})$, since $\epsilon \leq \delta$. □

One can prove an analogous lemma for splits. Although the statement and the proof are very similar to Lemma 5.8, there are some subtle differences. For example, $\mathcal{S}(B)$ is not full-dimensional when $M(B)$ is a split; Lemma 4.3 applies only when $M(B)$ is bounded. Hence, more work needs to be done to create a lattice-free set in this case.

Lemma 5.9. *Let $M(B)$ be a split with $\#(\operatorname{conv}(P \cap F_1) \cap \mathbb{Z}^2) \leq 1$. Let $y^1 \in F_1 \cap \mathbb{Z}^2$ such that $P \cap F_1 \subset (y^1 - v(F_1), y^1 + v(F_1))$. Let $Y_1 = \{y^1\}$ and $Y_2 = \{y^2, y^3\}$, where y^2, y^3 are two arbitrary integer points on F_2. Suppose that $\dim \mathcal{T}(B, \mathcal{Y}) \geq 1$.*

For any $\bar{A} \in \mathcal{N}(B, \mathcal{Y}) \setminus \{0\}$, there exists $0 < \epsilon_1 < 1$ such that $\sum_{j=1}^{k} \psi_{B+\epsilon\bar{A}}(r^j) s_j \geq 1$ is a valid inequality for $\operatorname{conv}(R_f)$ for every $0 < \epsilon \leq \epsilon_1$.

Proof. Similar to the proof of Lemma 5.8, we will construct $0 < \epsilon_1 < 1$ such that for every $0 < \epsilon < \epsilon_1$, there exists a matrix $C = (c^1; c^2; c^3)$ such that $M(C)$ is a lattice-free set containing one additional edge (so $M(C)$ is a triangle) and $\psi_C(r^j) = \psi_{B+\epsilon\bar{A}}(r^j)$ for $j = 1, \ldots, k$.

First, since B satisfies the strict inequalities in $\mathcal{T}(B, \mathcal{Y})$, there exists $0 < \delta < 1$ such that $B + \epsilon\bar{A} \in \mathcal{T}(B, \mathcal{Y})$ for every $0 < \epsilon \leq \delta$.

Observe that setting $Y_2 = \{\bar{y}^2, y^3\}$ implies that F_2 is fixed as the equalities in $\mathcal{T}(B, \mathcal{Y})$ corresponding to y^2, y^3 force F_2 to lie on the line passing through y^2, y^3. Therefore, for any $\bar{A} \in \mathcal{N}(B, \mathcal{Y}) \setminus \{0\}$, F_1 is tilted for $M(B + \bar{A})$ and hence $M(B + \bar{A})$ will contain lattice points in its interior. Let y^4 be the closest integer point on F_1 to y^1 such that $y^4 \in \operatorname{int}(M(B + \bar{A}))$. Note that one can then assume $y^4 = y^1 + v(F_1)$. Choose $\hat{y}^2, \hat{y}^3 \in M(B+\bar{A}) \cap F_2$ such that $\hat{y}^2 - y^1$ and $v(F_1)$ form a lattice basis for $\mathbb{Z}^2$ and $\hat{y}^3 = \hat{y}^2 + v(F_1)$. This can be done because the equality conditions in $\mathcal{T}(B, \mathcal{Y})$ from Y_2 fix the side F_2 of $M(B)$ and so it remains parallel to $v(F_1)$. Next, pick $c^3 \in \mathbb{R}^2$ such that $c^3 \cdot (x - f) \leq 1$ is a halfspace containing $P \cup \{y^1, \hat{y}^2, \hat{y}^3\}$ and such that $c^3 \cdot (y^4 - f) = 1$. This exists because there are only finitely many ray intersections, y^4 is on the boundary, and $P \cap F^1 \subset \{\, y^4 + t(y^1 - y^4) \mid t > 0 \,\}$ since $P \cap F_1 \subset (y^1 - v(F_1), y^1 + v(F_1))$.

Consider the set

$$\mathcal{V} := \{\, (a^1; a^2) \in \mathbb{R}^{2\times 2} \mid a^i \cdot r^j > c^3 \cdot r^j \text{ for } j = 1, \dots, k,\ i \in I_B(r^j) \,\}.$$

Since $\mathcal{V}$ is an open set containing B, there exists an $0 < \epsilon_1 \leq \delta$ such that $B + \epsilon\bar{A} \in \mathcal{V}$ for every $0 < \epsilon \leq \epsilon_1$. For any such ϵ, let $(c^1; c^2) = B + \epsilon\bar{A}$.

We show that $C = (c^1; c^2; c^3)$ has the property that $M(C)$ is a lattice-free triangle. Let S be the split defined by the line passing through $y^1, \hat{y}^2$ and the line passing through $y^4, \hat{y}^3$ (this defines a split because $\hat{y}^2, \hat{y}^3, y^1$ and y^4 form a parallelogram of area 1). Since $M(C) \cap M(B) \subseteq M(B)$, $M(C) \cap M(B)$ is lattice-free. Also, $M(C) \setminus \operatorname{int}(M(B)) \subseteq S$ and hence $M(C) \setminus M(B)$ is lattice-free. Moreover the boundary shared by these two sets $M(C) \cap M(B)$ and $M(C) \setminus \operatorname{int}(M(B))$ is the line segment $[y^1, y^4]$, which contains no integer points in its relative interior. Therefore, $M(C)$ is lattice-free.

Moreover, $\psi_C(r^j) = \psi_{B+\epsilon\bar{A}}(r^j)$ for $j = 1, \dots, k$ because $I_C(r^j) = I_B(r^j) = I_{B+\epsilon\bar{A}}(r^j)$ for all j. The first equality follows because $B + \epsilon\bar{A} \in \mathcal{V}$ and the second equality is because $\epsilon \leq \delta$ and so $B + \epsilon\bar{A} \in \mathcal{T}(B, \mathcal{Y})$. □

With the above lemma, the necessary conditions for splits are easy to show.

Lemma 5.10 (Splits)**.** *Suppose $M(B)$ is a split. If $\sum_{j=1}^k \psi_B(r^j)s_j \geq 1$ is extreme, then one of the following holds:*
Case a. $P \subset \mathbb{Z}^2$.
Case b. *There exists $j \in \{1, \dots, k\}$ such that r^j lies in the recession cone of the split.*
Case c. $\#(\operatorname{conv}(P \cap F_i) \cap \mathbb{Z}^2) \geq 2$ *for at least one of $i = 1$ or $i = 2$.*

Proof. We suppose that we are not in Case a, Case b, or Case c and show that $\sum_{j=1}^k \psi_B(r^j)s_j \geq 1$ is not extreme. So we suppose, possibly by exchanging the labels on F_1 and F_2, that $F_1 \cap P \setminus \mathbb{Z}^2 \neq \emptyset$, no ray in $\{r^1, \dots, r^k\}$ lies in the recession cone of the split, and $\#(\operatorname{conv}(P \cap F_1) \cap \mathbb{Z}^2) \leq 1$.

Let $y^1 \in F_1$ such that $P \cap F_1 \subset (y^1 - v(F_1), y^1 + v(F_1))$. Choose any $y^2, y^3 \in F_2 \cap \mathbb{Z}^2$. Let $Y_1 = \{y^1\}, Y_2 = \{y^2, y^3\}$. Note that since we assumed that no ray lies in the recession cone, we have $|I_B(r^j)| = 1$, for every $j = 1, \dots, k$. Hence, there are no equalities in $\mathcal{T}(B, \mathcal{Y})$ for $I_B(r^j)$. Then $\dim \mathcal{T}(B, \mathcal{Y}) \geq 4 - 3 = 1$. Pick any $\bar{A} \in \mathcal{N}(B, \mathcal{Y}) \setminus \{0\}$.

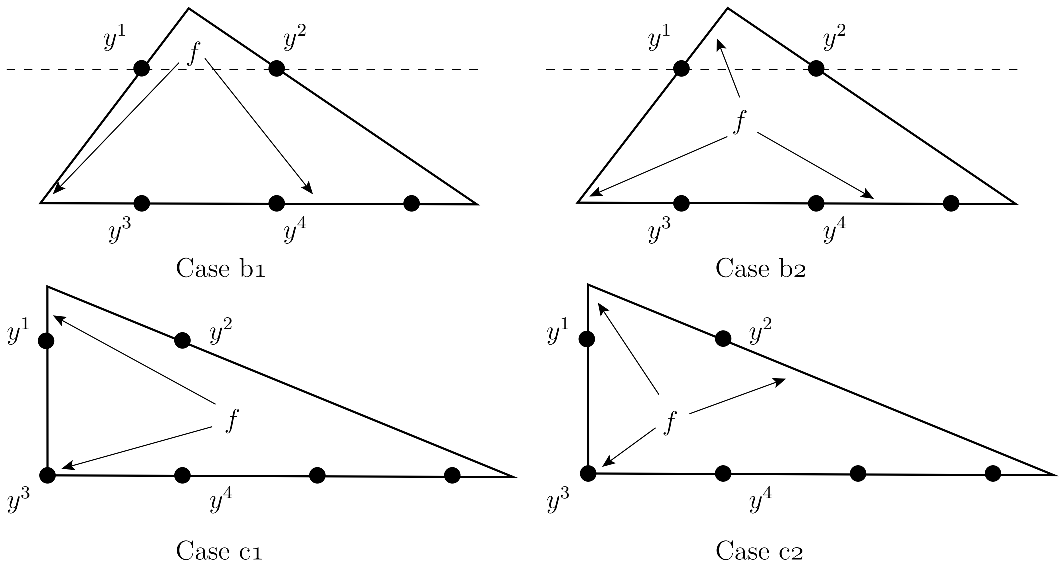

Figure 8: Cases of extreme Type 2 triangles in Lemma 5.11.

Notice that the equalities defining $\mathcal{T}(B,\mathcal{Y})$ corresponding to y^2 and y^3 fix F_2 completely because they force it to be the line going through y^2 and y^3. In other words, $\bar{a}^2 = 0$. Therefore $\bar{a}^1 \neq 0$.

Since B satisfies the strict inequalities of $\mathcal{T}(B,\mathcal{Y})$, there exists $\delta > 0$ such that $B \pm \epsilon\bar{A} \in \mathcal{T}(B,\mathcal{Y})$ for all $0 < \epsilon \leq \delta$, implying (amongst other things) that $I_{B\pm\epsilon\bar{A}}(r^j) = I_B(r^j)$ for all $j = 1,\ldots,k$. Using Lemma 5.9 with $\bar{A}$, we know that there exists an $0 < \epsilon_1 < 1$ such that $\sum_{j=1}^k \psi_{B+\epsilon\bar{A}}(r^j)s_j \geq 1$ is a valid inequality for every $0 < \epsilon \leq \epsilon_1$. Similarly, using Lemma 5.9 with $-\bar{A}$, there exists an $0 < \epsilon_2 < 1$ such that $\sum_{j=1}^k \psi_{B-\epsilon\bar{A}}(r^j)s_j \geq 1$ is a valid inequality for every $0 < \epsilon \leq \epsilon_2$. Let $\epsilon = \min\{\delta, \epsilon_1, \epsilon_2\}$. Thus, $\sum_{j=1}^k \psi_{B\pm\epsilon\bar{A}}(r^j) \geq 1$ are both valid inequalities.

Since $\bar{A} \in \mathcal{N}(B,\mathcal{Y})$, $\bar{a}^1 \cdot (y^1 - f) = 0$. Since $F_1 \cap P \setminus \mathbb{Z}^2 \neq \emptyset$, there exists r^j with $I_B(r^j) = \{1\}$ and $p^j \notin \mathbb{Z}^2$ and so r^j and $y^1 - f$ are linearly independent. This implies that $\bar{a}^1 \cdot r^j \neq 0$ since $\bar{a}^1 \cdot y^1 = 0$ and $\bar{a}^1 \neq 0$. Hence, $\psi_{B+\epsilon\bar{A}}(r^j) = (b^1 + \epsilon\bar{a}^1)\cdot r^j \neq (b^1 - \epsilon\bar{a}^1)\cdot r^j = \psi_{B-\epsilon\bar{A}}(r^j)$. The equalities follow because $\epsilon \leq \delta$ and so $I_{B\pm\epsilon\bar{A}}(r^j) = I_B(r^j) = \{1\}$. Moreover, since $B \pm \epsilon\bar{A} \in \mathcal{T}(B,\mathcal{Y})$, Lemma 4.2 implies that $\sum_{j=1}^k \psi_B(r^j) \geq 1$ is a convex combination of the two valid inequalities $\sum_{j=1}^k \psi_{B\pm\epsilon\bar{A}}(r^j) \geq 1$. Hence, we have shown that $\sum_{j=1}^k \psi_B(r^j)s_j \geq 1$ is not extreme by using two Type 2 triangles (note that the triangle $M(C)$ in the proof of Lemma 5.9 is a Type 2 triangle). $\square$

Lemma 5.11 (Type 2 Triangles). *Let $M(B)$ be a Type 2 triangle with facets F_1, F_2, F_3 where F_3 is the facet containing multiple integer points. Let y^1, y^2 be the unique integer points on the relative interiors of F_1 and F_2, respectively.*

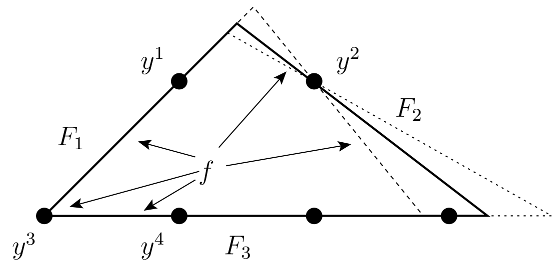

Figure 9: In the proof of Lemma 5.11, Step 1, a simple tilt from Lemma 4.7 shows that $\sum_{j=1}^k \psi_B(r^j)s_j \geq 1$ is not extreme.

If $\sum_{j=1}^k \psi_B(r^j)s_j \geq 1$ is extreme and not dominated or realized by a split inequality, then one of the following holds:
Case a. *$P \subset \mathbb{Z}^2$.*
Case b. *There exist $p^1, p^2 \in P \cap F_3$ with $\#([p^1, p^2] \cap \mathbb{Z}^2) \geq 2$, and there exists a matrix B' such that $M(B')$ is a Type 2 triangle, $\psi_{B'}(r^j) = \psi_B(r^j)$ for all $j = 1, \dots, k$, and has at least one of p^1 or p^2 in $\operatorname{vert}(B')$. If there exist non-integer-pointing rays on the relative interior of both F_1, F_2, then there exist two corner rays. Also, one of the following holds:*
 Case b1. *$f \notin S_3$.*
 Case b2. *$f \in S_3$ and $P \not\subset F_3$.*
Case c. *There exist $p^1, p^2 \in P \cap F_i$ with $i = 1$ or $i = 2$, with $\#([p^1, p^2] \cap \mathbb{Z}^2) \geq 2$, such that $p^1 \in F_3 \cap \mathbb{Z}^2$ and if $P \setminus (F_i \cup F_3 \cup \mathbb{Z}^2) \neq \emptyset$, then p^2 can be taken to be a corner ray. Also, one of the following holds:*
 Case c1. *$f \notin S_i$.*
 Case c2. *$f \in S_i$ and $P \not\subset S_i$.*

The cases of the lemma are illustrated in Figure 8.

Proof. **Step 1.** Suppose $P \not\subset \mathbb{Z}^2$ and there do not exist $p^1, p^2 \in P$ such that $\#([p^1, p^2] \cap \mathbb{Z}^2) \geq 2$. We will show that $\sum_{j=1}^k \psi_B(r^j)s_j \geq 1$ is then not extreme.

First note that there is at most one corner ray in F_3 because there are multiple integer points on F_3. Let $y^3 \in F_3$ such that $P \cap F_3 \subset (y^3 - v(F_3), y^3 + v(F_3))$. Let $Y_i = \{y^i\}$.

Suppose first that $y^3 \in \operatorname{vert}(B) \cap P$ and, without loss of generality, $y^3 \in F_1 \cap F_3$. Note that this implies that there are no corner rays on F_2, because $\#([p^1, p^2] \cap \mathbb{Z}^2) \leq 1$ and so $P \cap F_2 \subset \operatorname{rel\,int}(F_2)$. If $P \cap F_2 \setminus \mathbb{Z}^2 \neq \emptyset$, then a simple tilt from Lemma 4.7 shows that $\sum_{j=1}^k \psi_B(r^j)s_j \geq 1$ is not extreme, as shown in Figure 9. If instead $P \cap F_2 \setminus \mathbb{Z}^2 = \emptyset$, then $P \subset \operatorname{conv}(\{y^1, y^2, y^3, y^4\})$, where y^4 is the integer point adjacent to y^3 on F_3, since no two elements of P contain two integer points between them. Hence, $P \cup \{f\} \subset S_i$ for either $i = 1$ or 3, and hence $\sum_{j=1}^k \psi_B(r^j)s_j \geq 1$ is dominated by the inequality derived from S_i, contradicting the hypothesis of this lemma.

Suppose now that $y^3 \in \operatorname{rel\,int}(F_3)$. Since there are at most 2 corner rays, Lemma 5.1 shows that there exists $\bar{A} \in \mathcal{N}(B, \mathcal{Y}) \setminus \{0\}$ such that for every $0 < \epsilon < 1$, $\psi_{B+\epsilon\bar{A}}(r^j) \neq$

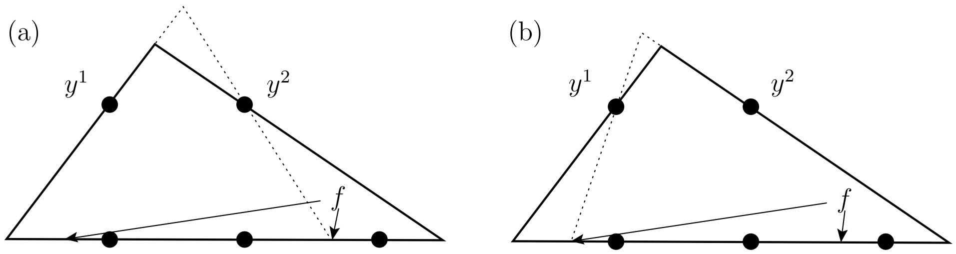

Figure 10: In the proof of Lemma 5.11, Step 2a, either F_1 or F_2 is tilted to give a new triangle $M(B')$ (dotted). (a) Here F_2 cannot be used because tilting would remove f from the interior. (b) Instead, F_1 needs to be used.

$\psi_{B-\epsilon\bar{A}}(r^j)$ for some $j = 1,\dots,k$ and $\psi_B(r^j) = \frac{1}{2}\psi_{B-\epsilon\bar{A}}(r^j) + \frac{1}{2}\psi_{B+\epsilon\bar{A}}(r^j)$ for every $j = 1,\dots,k$. If we pick ϵ arbitrarily, it is possible that $M(B+\epsilon\bar{A})$ or $M(B-\epsilon\bar{A})$ is not lattice-free. However, using Lemma 5.8 with $\bar{A}$ and $-\bar{A}$, we know that there exist $0 < \epsilon_1 < 1$ and $0 < \epsilon_2 < 1$ such that for $\epsilon = \min\{\epsilon_1, \epsilon_2\}$, both the inequalities $\sum_{j=1}^k \psi_{B\pm\epsilon\bar{A}}(r^j)s_j \geq 1$ are valid for $\mathrm{conv}(R_f)$. Therefore $\sum_{j=1}^k \psi_B(r^j)s_j \geq 1$ is not extreme.

We comment here that, due to Lemma 5.8, we may be using inequalities that come from quadrilaterals to show that $\sum_{j=1}^k \psi_B(r^j)s_j \geq 1$ is not extreme.

Therefore, if $\sum_{j=1}^k \psi_B(r^j)s_j \geq 1$ is extreme, we are either in Case a with $P \subset \mathbb{Z}^2$, or there exist $p^1, p^2 \in P$ with $\#([p^1,p^2] \cap \mathbb{Z}^2) \geq 2$. In the latter case, we now show that we must be in either Case b1, b2, c1, or c2.

Step 2. Suppose $P \not\subset \mathbb{Z}^2$ and there exist $p^1, p^2 \in P \cap F_3$ with $\#([p^1,p^2] \cap \mathbb{Z}^2) \geq 2$. Without loss of generality, we label p^1, p^2 such that $P \cap F_3 \subset [p^1, p^2]$.

Step 2a. We will show that there exists a matrix B' such that $M(B')$ is a lattice-free Type 2 triangle that has at least one corner ray in F_3, and $\psi_{B'}(r^j) = \psi_B(r^j)$ for all $j = 1,\dots,k$.

If either p^1 or p^2 is a vertex of $M(B)$, then we let $B' = B$ and move to Step 2b. We now deal with the case that $p^1 \notin \mathrm{vert}(B)$ and $p^2 \notin \mathrm{vert}(B)$.

Suppose there exists $\hat{r} \in \{r^1,\dots,r^k\}$ such that $\hat{p} \in F_1 \cap F_2$, i.e., $\hat{r}$ is a corner ray on F_1 and F_2. We now make a tilting space argument to argue that $\sum_{j=1}^k \psi_B(r^j)s_j \geq 1$ is not extreme. We define $\mathcal{Y} = (Y_1, Y_2, Y_3)$ as $Y_1 = \{y^1\}$, $Y_2 = \{y^2\}$ and $Y_3 = F_3 \cap Y(B)$. Hence, $\mathcal{Y}$ is a covering of $Y(B)$. Since there is only one corner ray ($p^1 \notin \mathrm{vert}(B)$ and $p^2 \notin \mathrm{vert}(B)$), only one equation in $\mathcal{N}(B,\mathcal{Y})$ comes from a corner ray condition. Y_1 and Y_2 each contribute one equation. Y_3 contributes a system of equalities involving a^3 with rank 2. Therefore, $\dim \mathcal{N}(B,\mathcal{Y}) = 6 - 5 = 1$. We pick any $\bar{A} \in \mathcal{N}(B,\mathcal{Y}) \setminus \{0\}$. From Observation 4.5 and Observation 4.6, there exists $\epsilon > 0$ such that $\sum_{i=1}^k \psi_{B\pm\epsilon\bar{A}}(r^j)s_j \geq 1$ are both valid inequalities and Lemma 4.2 implies that $\sum_{j=1}^k \psi_B(r^j)s_j \geq 1$ is a convex combination of these two valid inequalities. We now show that $\psi_{B+\epsilon\bar{A}}(\hat{r}) \neq \psi_{B-\epsilon\bar{A}}(\hat{r})$. Note that the equations from Y_3 impose that $\bar{a}^3 = 0$. Therefore, either $\bar{a}^1 \neq 0$ or

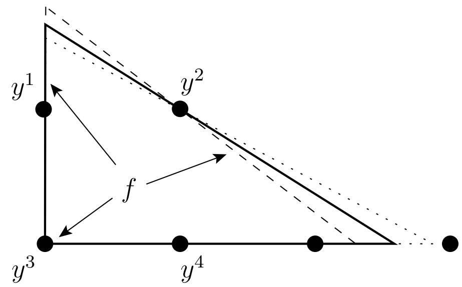

Figure 11: In the proof of Lemma 5.11, Step 3, a simple tilting argument (Lemma 4.7) shows that the inequality is not extreme.

$\bar{a}^2 \neq 0$. Without loss of generality, assume $\bar{a}^1 \neq 0$. Observe now that $y^1 - f$ and $\hat{r}$ are linearly independent since y^1 is in the relative interior of F_1 and $\hat{p}$ is a vertex of F_1. Since Y_1 imposes $\bar{a}^1 \cdot (y^1 - f) = 0$, this implies that $\bar{a}^1 \cdot \hat{r} \neq 0$. Therefore, $\psi_{B+\epsilon\bar{A}}(\hat{r}) = (b^1 + \epsilon\bar{a}^1) \cdot \hat{r} \neq (b^1 - \epsilon\bar{a}^1) \cdot \hat{r} = \psi_{B-\epsilon\bar{A}}(\hat{r})$; the equalities follow from the fact that $B \pm \epsilon\bar{A} \in \mathcal{T}(B, \mathcal{Y})$ implying that $I_{B\pm\epsilon\bar{A}}(\hat{r}) = I_B(\hat{r})$.

So we can assume that $p^1 \notin \operatorname{vert}(B)$, $p^2 \notin \operatorname{vert}(B)$ and $F_1 \cap F_2 \notin P$, i.e., there is no corner ray in $M(B)$. Since F_1 and F_2 do not have corner rays, then we must have $\operatorname{rel\,int}(F_i) \cap P \setminus \mathbb{Z}^2 = \emptyset$ for $i = 1, 2$ because otherwise Lemma 4.7 shows that $\sum_{j=1}^k \psi_B(r^j)s_j \geq 1$ is not extreme, by a simple tilt of F_1 or F_2. For $i = 1, 2$, since $\operatorname{rel\,int}(F_i) \cap (P \setminus \mathbb{Z}^2) = \emptyset$, changing F_i to now lie on the line through p^i and y^i does not change $\sum_{j=1}^k \psi_B(r^j)s_j \geq 1$, unless f is no longer in the interior of the set. At most one of these facet tilts puts f outside the perturbed set, thus at least one of them is possible. This is illustrated in Figure 10. Without loss of generality, we assume that the tilt of facet F_1 is possible. Let the set after tilting be $M(B')$ and B' be the corresponding matrix.

We claim that $M(B')$ is lattice-free. To see this, let $y^3, y^4 \in [p^1, p^2] \cap \mathbb{Z}^2$ be distinct integer points adjacent to each other. Then consider the split S with facets through $[y^3, y^1]$ and $[y^4, y^2]$. Since $[y^3, y^4] \subset [p^1, F_1 \cap F_3]$ is a strict subset, the new intersection at $F_1 \cap F_2$ is a subset of the split, and hence $M(B') \setminus M(B) \subset S$, and therefore no new integer points are introduced.

Step 2b. Suppose now that $p^1 \in F_1 \cap F_3$ and there exists a point $p \in \operatorname{rel\,int}(F_2) \setminus \mathbb{Z}^2$. If there are no corner rays on F_2, then Lemma 4.7 shows that $\sum_{j=1}^k \psi_B(r^j)s_j \geq 1$ is not extreme. Therefore the conditions of Case b are met. If $P \cup \{f\} \subset S_3$ then $\sum_{j=1}^k \psi_B(r^j)s_j \geq 1$ is dominated or realized by the split inequality from S_3, hence either Case b1 or Case b2 occurs.

Step 3. Suppose $P \not\subset \mathbb{Z}^2$ and there exist $p^1, p^2 \in P \cap F_i$ with $\#([p^1, p^2] \cap \mathbb{Z}^2) \geq 2$, for $i = 1$ or $i = 2$. Without loss of generality, we assume that $i = 1$. In order for $\#([p^1, p^2] \cap \mathbb{Z}^2) \geq 2$, it has to equal exactly two, and one of the points, say p^1, must lie in $p^1 \in F_1 \cap F_3 \cap \mathbb{Z}^2$. Thus, p^1 is the corner ray.

If there exists a point $p \in \operatorname{rel\,int}(F_2)\backslash\mathbb{Z}^2$, then again, there must be a corner ray on F_2; otherwise, Lemma 4.7 shows that $\sum_{j=1}^{k} \psi_B(r^j)s_j \geq 1$ is not extreme. See Figure 11. Since we are not in Case b, this must be the corner ray pointing to $F_1 \cap F_2$. Thus p^2 can be taken to be this corner ray.

As in Case b, if $P \cup \{f\} \subset S_1$, then $\sum_{j=1}^{k} \psi_B(r^j)s_j \geq 1$ is dominated or realized by the inequality derived from S_1. Hence, we are either in Case c1 or Case c2.

This concludes the proof. □

6 Number of facets of the integer hull

We recall that we have k rays $r^1, \dots, r^k$.

Remark 6.1. *Given two rays r^1 and r^2 in $\mathbb{R}^2$, we denote by $C(r^1, r^2)$ the cone $\{ x \in \mathbb{R}^2 \mid x = f + s_1 r^1 + s_2 r^2, \text{with } s_1, s_2 \geq 0 \}$. By Theorem 3.4, we get that $(C(r^1, r^2))_{\mathrm{I}}$ has a polynomial number of facets and vertices.*

Theorem 6.2. *The number of facets of* $\operatorname{conv}(R_f)$ *is polynomial in the size of the encoding of the problem for $m = 2$.*

Proof. We will follow the cases from section 5 for each type of maximal lattice-free convex set in $\mathbb{R}^2$.

We will first handle the case where $P \subset \mathbb{Z}^2$. That is, let P be the set of closest integer points that the rays point to from f. If $\operatorname{conv}(P)$ is a lattice-free set, then it is contained within a maximal lattice-free set. Choose any particular maximal lattice-free set containing P. This covers Case a for Type 2 and 3 triangles, quadrilaterals, and splits. We will no longer refer to this Case a for these types of lattice-free sets.

Splits. The necessary conditions are given in Lemma 5.10. We consider the two remaining cases, which are illustrated in Figure 12.

Case b. A ray direction r^j is parallel to the split. There are at most k such ray directions, and thus at most k splits in this case.

Case c. There exist p^1, p^2 such that $[p^1, p^2] \cap \mathbb{Z}^2 \geq 2$, and therefore, the split must run parallel to a facet of $(C(r^1, r^2))_{\mathrm{I}}$, of which there are only polynomially many. There are only $\binom{k}{2}$ ways to choose two rays for this possibility.

Type 1 triangles. We assume that the inequality cannot be realized or dominated by a Type 2 triangle or split, because in this case we will use the analysis for these two types. We now apply Lemma 5.7 and refer to Figure 13.

There are two corner rays, call them r^1, r^2; there are $\binom{k}{2}$ ways to choose them. Since these rays both point directly to integer points, they uniquely define F_3.

Case a. Since f does not lie in the split S_3, the integer points y^1, y^2 are uniquely determined.

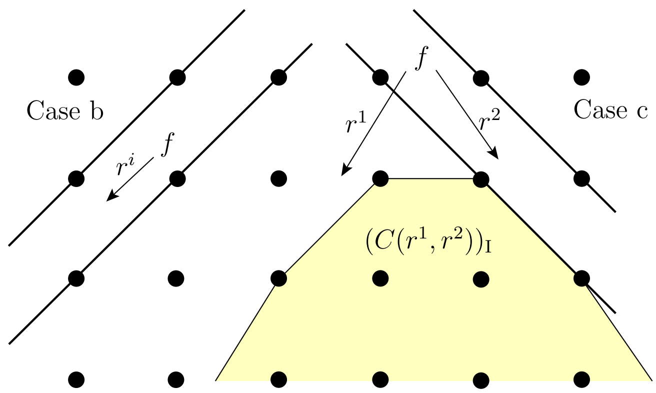

Figure 12: Counting a polynomial number of splits

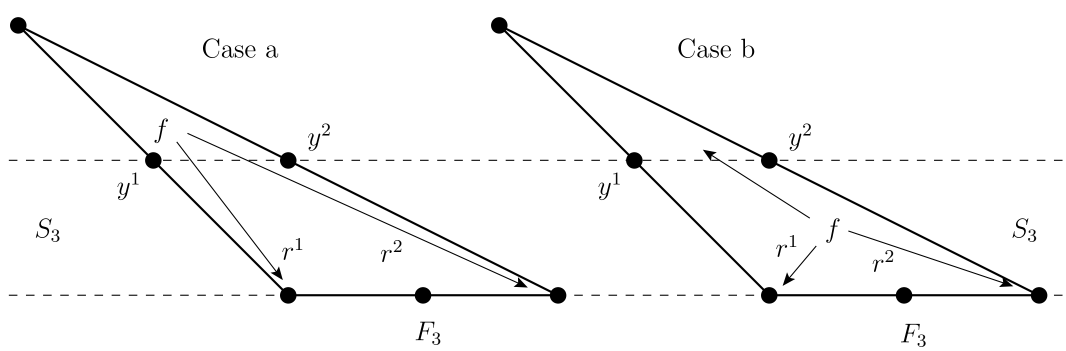

Figure 13: Counting a polynomial number of Type 1 triangles

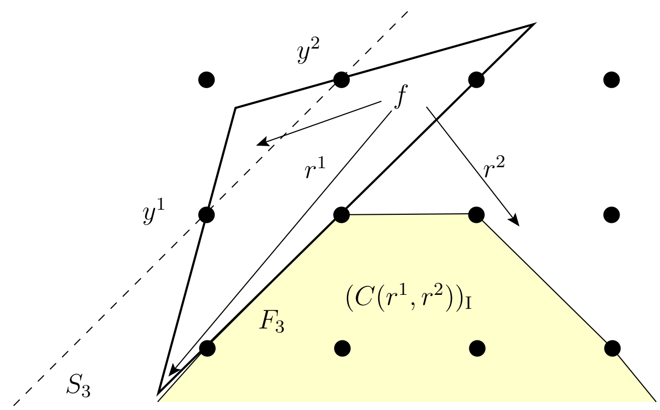

Figure 14: Counting a polynomial number of Type 2 triangles in Case b

Case b. Since f lies in the split S_3 and there exists a ray intersection p^3 outside the split, the integer points y^1, y^2 are uniquely determined.

In both cases, since F_3, y^1, y^2, and the corner rays r^1, r^2 uniquely determine the triangle, there are only polynomially many Type 1 triangles that we must consider.

Type 2 triangles. The necessary conditions are given in Lemma 5.11.

Case b. We first pick the two rays r^1, r^2 to be the rays that are closest to $F_1 \cap F_3$ and $F_2 \cap F_3$, respectively. This can be done in $\binom{k}{2}$ ways. See Figure 14.

We next pick the facet F_3 as a facet of $(C(r^1, r^2))_{\mathrm{I}}$, which can be done only polynomially many ways.

Now we choose y^1, y^2. In Case b1, where $f \notin S_3$, they are given uniquely by where f is. In Case b2, when $P \not\subset S_3$, we first pick a ray r^3 such that the corresponding ray intersection p^3 will be the one that is not contained in S_3, and so r^3 points between y^1 and y^2. This would imply that y^i is one of the vertices of $(C(r^i, r^3))_{\mathrm{I}}$. Moreover, since y^1, y^2 have to lie on the lattice plane adjacent to F_3, we have a unique choice for y^1, y^2 once we choose r^3. Now r^3 can be chosen in $O(k)$ ways and so there are $O(k)$ ways to pick y^1, y^2.

If we choose there to be a second corner ray somewhere (we can do this in $O(k)$ ways), then the triangle is uniquely determined by the two corner rays, F_3, y^1, and y^2.

On the other hand, if we choose that there is only one corner ray, then we pick r^1 or r^2 to be the only corner ray (2 choices), and the facet opposite of this corner ray cannot have any rays pointing to it that do not point to an integer point. This is because that facet has no corner rays. Therefore, any particular choice of this facet with no rays pointing to it will suffice (although one may not exist).

Hence, there are only polynomially many possibilities for Case b.

Case c. We first choose r^1, r^2 to be the two rays such that $\#([p^1, p^2] \cap \mathbb{Z}^2) \geq 2$. One of them must point to an integer point on the facet F_3. There are $2 \times \binom{k}{2}$ ways to choose this. Without loss of generality, let r^1 point to the integer point on F_3. See Figure 15.

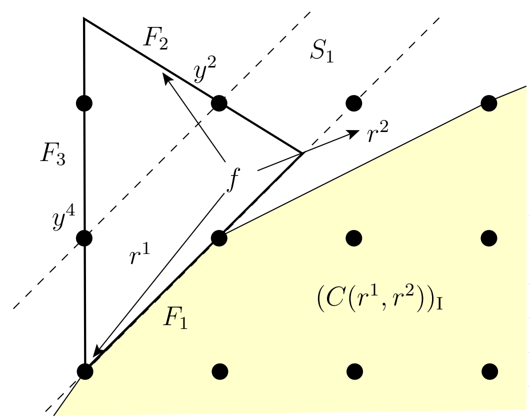

Figure 15: Counting a polynomial number of Type 2 triangles in Case c

We next choose the facet F_1 from $(C(r^1, r^2))_{\mathrm{I}}$. There is a unique choice for F_1 because p^3 is an integer point and so p^3 will be *the* vertex of $(C(r^1, r^2))_{\mathrm{I}}$ (if one exists) that lies on the facet of $C(r^1, r^2)$ defined by the ray r^1. Hence F_1 can be the unique facet that is adjacent to this vertex but not lying on the facet of $C(r^1, r^2)$ defined by the ray r^1.

Now we pick y^2, y^4. This analysis is the same as with Cases b1 and b2. In Case c1, these points are uniquely determined by f. In Case c2, these are uniquely determined by one of the rays pointing between them. Thus, y^2, y^4 can be chosen in $O(k)$ ways after choosing this ray.

If we assume there are two corner rays (r^1 and r^2), then the triangle is uniquely determined by these corner rays, F_1, y^2, and y^4.

On the other hand, if we assume that r^1 is the only corner ray, then there cannot be any rays pointing to the interior of the opposite facet F_2. Therefore, this facet can be chosen to be any particular facet (if one exists) that does not have rays pointing to it. Then the triangle is uniquely determined by r^1, F_1, F_2, y^2, and y^4.

Therefore, there are only polynomially many Type 2 triangles of Case c, and hence there are only polynomially many Type 2 triangles that we need to consider.

Type 3 triangles. The necessary conditions are given in Lemma 5.3.

Case b. We only need to consider Case b, where there are three corner rays. Now we pick any triplet of rays, say r^1, r^2, r^3, and require that each side of $M(B)$ passes through a vertex of $(C(r^i, r^{i+1}))_{\mathrm{I}}$, $i = 1, 2, 3$ and $r^4 = r^1$. There are only polynomially such triplets of integer vertices y^1, y^2, y^3 to choose.

We note that a triangle whose 3 corner rays and a point on the relative interior of each facet are known is already uniquely determined. In the appendix, we prove this claim (Proposition A.1). Thus, we can use a triplet of rays and a vertex from each integral hull of the three cones spanned by consecutive rays to define the triangle. These are polynomial in number.

Quadrilaterals. The necessary conditions are given in Lemma 5.5.

Case b. We first pick four rays r^1, r^2, r^3, r^4 to be corner rays, which can be done in $\binom{k}{4}$ ways. We next pick four integer points y^1, y^2, y^3, y^4, with y^i a vertex of $(C(r^i, r^{i+1}))_{\mathrm{I}}$, with $i = 1, 2, 3$ and y^4 a vertex of $(C(r^4, r^1))_{\mathrm{I}}$. This can be done in polynomially many ways.

Lemma 5.5 Case b says that if $\sum_{j=1}^{k} \psi_B(r^j)s_j \geq 1$ is extreme, then it is the unique quadrilateral with these corner rays and integer points. Therefore, we count at most one quadrilateral for each set of corner rays and integer points.

Therefore, there are only polynomially many quadrilaterals that must be considered.

We have enumerated all the types of maximal lattice-free convex sets in $\mathbb{R}^2$ and shown that there are only polynomially many sets of each type that must be considered. Hence, for the case of $m = 2$, we have shown that R_f has only polynomially many facets. □

We obtain the following result as a direct consequence of our proof for Theorem 6.2.

Theorem 6.3. *There exists a polynomial time algorithm to enumerate all the facets of* $\mathrm{conv}(R_f)$ *when* $m = 2$.

Proof. For each of the five types of maximal lattice-free sets in the plane, the proof for Theorem 6.2 shows how to generate in polynomial time the ones that are potentially facet defining. However, since we only ensure that the necessary conditions from Section 5 are not violated, we can potentially generate a set of valid inequalities (of polynomial size) which is a superset of all the facets. We can then use standard LP techniques to select the facet defining ones from these. □

A Appendix: Uniqueness of a triangle defined by 3 corner rays and a point on the relative interior of each facet

Proposition A.1. *Any triangle defined by 3 corner rays and 3 points (one on the relative interior of each facet) is uniquely defined.*

Proof. The space of these three corner rays and 3 points is exactly the tilting space of any such triangle satisfying this. For convenience we define $\bar{y}^i := y^i - f$ and $\bar{p}^i := p^i - f$, where p^i are the ray intersections. Then $\bar{p}^i = \frac{1}{\psi_B(r^i)} r^i$.

We want to show that the solution to the following systems of equations is unique.

$$\begin{aligned} a^1 \cdot \bar{y}^1 &= 1 \\ a^1 \cdot \bar{p}^2 &= a^2 \cdot \bar{p}^2 \\ a^2 \cdot \bar{y}^2 &= 1 \\ a^2 \cdot \bar{p}^3 &= a^3 \cdot \bar{p}^3 \\ a^3 \cdot \bar{y}^3 &= 1 \\ a^3 \cdot \bar{p}^1 &= a^1 \cdot \bar{p}^1 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} \bar{y}^1 & & \\ \bar{p}^2 & -\bar{p}^2 & \\ & \bar{y}^2 & \\ & \bar{p}^3 & -\bar{p}^3 \\ & & \bar{y}^3 \\ -\bar{p}^1 & & \bar{p}^1 \end{bmatrix} \begin{bmatrix} a^1 \\ a^2 \\ a^3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

We then write this down as a matrix equation where every vector in the matrix is a row vector of size 2, therefore we have a 6×6 matrix. We will analyze the determinant of the matrix.

Since the points $\bar{y}^1, \bar{y}^2, \bar{y}^3$ are on the interior of each facet, they can be written as convex combinations of $\bar{p}^1, \bar{p}^2, \bar{p}^3$.

$$\begin{aligned} \bar{y}^1 &= \tfrac{1}{\alpha'}\bar{p}^1 + \tfrac{\alpha}{\alpha'}\bar{p}^2 \\ \bar{y}^2 &= \tfrac{1}{\beta'}\bar{p}^2 + \tfrac{\beta}{\beta'}\bar{p}^3 \\ \bar{y}^3 &= \tfrac{1}{\gamma'}\bar{p}^3 + \tfrac{\gamma}{\gamma'}\bar{p}^1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \bar{p}^1 &= \alpha'\bar{y}^1 - \alpha\bar{p}^2 \\ \bar{p}^2 &= \beta'\bar{y}^2 - \beta\bar{p}^3 \\ \bar{p}^3 &= \gamma'\bar{y}^3 - \gamma\bar{p}^1 \end{aligned}$$

Therefore, we can perform row reduction on the last row. Just tracking the last row, we have

$$\begin{bmatrix} -\bar{p}^1 & 0 & \bar{p}^1 \end{bmatrix} \to \begin{bmatrix} 0 & \alpha\bar{p}^2 & \bar{p}^1 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & \bar{p}^1 - \alpha\beta\bar{p}^3 \end{bmatrix}.$$

This matrix now has an upper block triangular form, and the determinant is easily computed as

$$\det(\bar{y}^1; \bar{p}^2) \det(\bar{y}^2; \bar{p}^3) \det(\bar{y}^3; \bar{p}^1 - \alpha\beta\bar{p}^3).$$

The first two determinants are non-zero because those vectors are linearly independent. The last determinant requires some work:

$$\begin{bmatrix} \bar{y}^3 \\ \bar{p}^1 + \alpha\beta\bar{p}^3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\gamma'}\bar{p}^3 + \frac{\gamma}{\gamma'}\bar{p}^1 \\ \bar{p}^1 - \alpha\beta\bar{p}^3 \end{bmatrix} = \begin{bmatrix} \frac{\gamma}{\gamma'} & \frac{1}{\gamma'} \\ 1 & -\alpha\beta \end{bmatrix} \begin{bmatrix} \bar{p}^1 \\ \bar{p}^3 \end{bmatrix}.$$

Since all the coefficients are positive, the determinant of the first matrix is strictly negative, and since $\bar{p}^1, \bar{p}^3$ are linearly independent, the determinant of the second matrix is non-zero.

Hence, the determinant of the original matrix is non-zero, and therefore the system of equations has a unique solution. □

Acknowledgments. A. Basu wishes to thank Gérard Cornuéjols and François Margot for many discussions which led to an earlier manuscript [5], which the present article is based on. During the completion of this work, R. Hildebrand was supported by grant DMS-0636297 (VIGRE), and R. Hildebrand and M. Köppe were supported by grant DMS-0914873 of the National Science Foundation.

References

[1] K. Andersen, Q. Louveaux, R. Weismantel, and L. Wolsey, *Inequalities from two rows of a simplex tableau*, Integer Programming and Combinatorial Optimization. 12th International IPCO Conference, Ithaca, NY, USA, June 25–27, 2007. Proceedings (M. Fischetti and D. Williamson, eds.), Lecture Notes in Computer Science, vol. 4513, Springer Berlin / Heidelberg, 2007, pp. 1–15.

[2] E. Balas, *Intersection cuts – a new type of cutting planes for integer programming*, Operations Research **19** (1971), 19–39.

[3] A. Basu, *Lectures on modern approaches to cutting planes*, available from `http://www.math.ucdavis.edu/~abasu/RFG/lecture_notes.pdf`, 2010.

[4] A. Basu, M. Conforti, G. Cornuéjols, and G. Zambelli, *Maximal lattice-free convex sets in linear subspaces*, Mathematics of Operations Research **35** (2010), 704–720.

[5] A. Basu, G. Cornuéjols, and F. Margot, *Finding strongest two-row cuts*, Unpublished manuscript, 6 pages, May 2008.

[6] A. Basu, R. Hildebrand, and M. Köppe, *The triangle closure is a polyhedron*, manuscript, 2011.

[7] V. Borozan and G. Cornuéjols, *Minimal valid inequalities for integer constraints*, Mathematics of Operations Research **34** (2009), 538–546.

[8] M. Conforti, G. Cornuéjols, and G. Zambelli, *Corner polyhedra and intersection cuts*, Surveys in Operations Research and Management Science **16** (2011), 105–120.

[9] W. J. Cook, M. E. Hartmann, R. Kannan, and C. McDiarmid, *On integer points in polyhedra*, Combinatorica **12** (1992), no. 1, 27–37.

[10] G. Cornuéjols and F. Margot, *On the facets of mixed integer programs with two integer variables and two constraints*, Mathematical Programming **120** (2009), 429–456, DOI: 10.1007/s10107-008-0221-1.

[11] S. Dey and L. Wolsey, *Lifting integer variables in minimal inequalities corresponding to lattice-free triangles*, Integer Programming and Combinatorial Optimization.

13th International Conference, IPCO 2008, Bertinoro, Italy, May 26–28, 2008. Proceedings (A. Lodi, A. Panconesi, and G. Rinaldi, eds.), Lecture Notes in Computer Science, vol. 5035, Springer Berlin / Heidelberg, 2008, pp. 463–475.

[12] R. Fukasawa, 2011, Personal communication.

[13] R. E. Gomory, *Some polyhedra related to combinatorial problems*, Linear Algebra and its Applications **2(4)** (1969), 451–558.

[14] M. E. Hartmann, *Cutting planes and the complexity of the integer hull*, Phd thesis, Cornell University, Department of Operations Research and Industrial Engineering, Ithaca, NY, 1989.

[15] S. Kwek and K. Mehlhorn, *Optimal search for rationals*, Information Processing Letters **86** (2003), no. 1, 23–26.

[16] Q. Louveaux and L. Poirrier, *An algorithm for the separation of two-row cuts*, working paper, available from `http://www.montefiore.ulg.ac.be/~poirrier/download/pub/separation.pdf`, 2011.

[17] L. Lovász, *Geometry of numbers and integer programming*, Mathematical Programming: State of the Art (M. Iri and K. Tanabe, eds.), Mathematical Programming Society, 1989, pp. 177–201.

[18] D. H. Martin, *On the continuity of the maximum in parametric linear programming*, Journal of Optimization Theory and Applications **17** (1975), 205–210.

[19] A. Schrijver, *Theory of linear and integer programming*, John Wiley and Sons, New York, 1986.