

A Complementarity Partition Theorem for Multifold Conic Systems

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August 26, 2011

Abstract

Consider a homogeneous multifold convex conic system

$$Ax = 0, \quad x \in K_1 \times \cdots \times K_r$$

and its alternative system

$$A^T y \in K_1^* \times \cdots \times K_r^*,$$

where K_1, \dots, K_r are regular closed convex cones. We show that there is a canonical partition of the index set $\{1, \dots, r\}$ determined by certain complementarity sets associated to the most interior solutions to the two systems. Our results are inspired by and extend the Goldman-Tucker Theorem for linear programming.

Key words strict complementarity, Goldman-Tucker Theorem, conic feasibility system, multifold conic system

1 Introduction

Assume $K \subseteq \mathbb{R}^n$ is a closed convex cone and $A \in \mathbb{R}^{m \times n}$. Consider the homogeneous conic system

$$Ax = 0, \quad x \in K, \tag{P}$$

and its alternative system

$$A^T y \in K^*, \tag{D}$$

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where $K^* \subseteq \mathbb{R}^n$ is the dual of K . It is immediate that any solutions \bar{x} and \bar{y} to (P) and (D) respectively are *complementary*, that is, they satisfy $\bar{y}^T(A\bar{x}) = 0$. In particular, if either (P) or (D) has a strict feasible solution, then the other one only has trivial solutions. In the special case when $K = \mathbb{R}_+^n$, a stronger related property holds. As a consequence of Goldman-Tucker Theorem [4], there always exist solutions \bar{x} and \bar{y} to (P) and (D) respectively such that $\bar{x} + A^T\bar{y} \in \mathbb{R}_{++}^n$. Such pairs of *strictly complementary* solutions are associated to a canonical partition $B \cup N = \{1, \dots, n\}$ of the index set $\{1, \dots, n\}$ (see Proposition 1 below). The partition sets B and N correspond to the most interior solutions to (P) and (D) respectively. Furthermore, there is a nice geometric interpretation of the sets B, N (see Proposition 2 below). The existence of strictly complementary primal-dual solutions is a fundamental property of linear programming. It plays a key role in the analysis of interior point methods, particularly the behavior of the central path near optimality and the local convergence of interior-point algorithms, see [8].

We present a generalization of the above strict complementary results to more general conic systems. To that end, we consider the case when the cone K is the direct product of r lower-dimensional regular closed convex cones. That is, we assume

$$K = K_1 \times \dots \times K_r, \tag{1}$$

where $K_i \subseteq \mathbb{R}^{n_i}$ is a regular closed convex cone for $i = 1, \dots, r$. Throughout the sequel we shall use I to denote the set $I = \{1, \dots, r\}$ and n to denote the dimension $n = \sum_{i=1}^r n_i$.

Following the terminology introduced in [3] we call the conic systems (P) and (D) *multifold* when the cone K is as in (1). This type of multifold structure is common in optimization. Formulations for linear programming (LP), second-order conic programming (SOCP) and semidefinite programming (SDP) problems generally lead to feasibility problems of this form. Our first main result (Theorem 1) shows that there are some canonical subsets B, N and B_0, N_0 of I associated to certain geometric properties of the problems (P) and (D). These sets generalize the partition sets B, N in the case $K = \mathbb{R}_+^n$. Our second main result (Theorem 2) shows that there exists a unique canonical partition of the index set I associated to the most interior solutions to (P) and (D). It should be noted that unlike the case when $K = \mathbb{R}_+^n$, for multifold conic systems neither the pair B, N nor the pair B_0, N_0 necessarily partitions the index set I . The insight leading to the two different pairs B, N and B_0, N_0 , presented in detail in Section 3,

comes from the fact that the linear image of a closed convex cone is not necessarily closed. The interesting question of determining sufficient conditions for the closedness of the linear image of a closed convex cone has been thoroughly studied by Pataki [7]. These conditions are related to a recent characterization of badly behaved semidefinite programs [6]. Some related issues of complementarity and non-degeneracy for second-order and semidefinite programming have been studied by Alizadeh et al. [1, 2].

The paper is organized as follows. Section 2 provides the foundation for our work, namely the existence of strictly complementary solutions to (P), (D) when $K = \mathbb{R}_+^n$. Section 3 presents our main results, namely Theorem 1 and Theorem 2. Section 4 discusses in more detail the special case of second-order conic systems. Section 5 concludes the paper with some final remarks.

2 Strict Partition for Polyhedral Homogeneous Systems

To motivate and state our main results, we first consider the special case when $K = \mathbb{R}_+^n$ in (P), (D). In this case the conic systems become

$$Ax = 0, \quad x \geq 0; \tag{2}$$

and

$$A^T y \geq 0, \tag{3}$$

where $A \in \mathbb{R}^{m \times n}$. This can be considered as a special case of a multifold conic system with $r = n$ and $K_i = \mathbb{R}_+$ in (1). Hence throughout this section we have $I = \{1, \dots, n\}$. Furthermore, for notational convenience, we shall write $A = [a_1 \ \dots \ a_n] \in \mathbb{R}^{m \times n}$. In other words, $a_i \in \mathbb{R}^m$ is the i -th column of A . The following result is a consequence of the Goldman-Tucker Theorem for linear programming [4].

Proposition 1. *Consider the pair of feasibility problems (2) and (3) for a given $A \in \mathbb{R}^{m \times n}$. For a unique partition $B \cup N = I$ of the index set I there exist solutions \bar{x} to (2) and \bar{y} to (3) satisfying*

$$\bar{x}_B > 0, \quad A_N^T \bar{y} > 0,$$

where we have used the standard notation: $\bar{x}_B > 0$ means $\bar{x}_i > 0$ for all $i \in B$, and $A_N^T \bar{y} > 0$ means $a_i^T \bar{y} > 0$ for all $i \in N$.

The partition sets B, N in Proposition 1 can be described in several ways. The three descriptions of B, N displayed in Proposition 2 below lay the

foundation for our main work. In the sequel we use the following convenient notation. For a convex cone $C \subseteq \mathbb{R}^d$, let $\text{Lin}(C) \subseteq \mathbb{R}^d$ denote the *lineality space* of C , that is, the largest linear subspace contained in C . Observe that because C is a convex cone, $\text{Lin}(C) = \{x \mid x, -x \in C\}$.

Proposition 2. *The sets B, N in Proposition 1 can be described as*

$$\begin{aligned} B &= \{i \in I \mid \exists x : Ax = 0, x \geq 0, x_i > 0\}, \\ N &= \{i \in I \mid \exists y : A^T y \geq 0, a_i^T y > 0\}. \end{aligned} \tag{4}$$

These sets can also be described as

$$\begin{aligned} B &= \{i \in I \mid A^T y \geq 0 \Rightarrow a_i^T y = 0\}, \\ N &= \{i \in I \mid Ax = 0, x \geq 0 \Rightarrow x_i = 0\}. \end{aligned} \tag{5}$$

And they can also be described as

$$\begin{aligned} B &= \{i \in I \mid a_i \in \text{Lin}(\text{AIR}_+^n)\}, \\ N &= \{i \in I \mid a_i \notin \text{Lin}(\text{AIR}_+^n)\}. \end{aligned} \tag{6}$$

The description (6) of the sets B, N has an interesting geometric interpretation. What determines if a particular index i belongs to B or N is whether the corresponding i -th column a_i lies in the lineality space of the cone AIR_+^n . This geometric interpretation has an interesting extension to multifold conic systems as Theorem 1 below shows. Proposition 2 is a consequence of Farkas Lemma and is also a special case of Theorem 1 below.

3 A Canonical Partition for Multifold Conic Systems

Consider now the general conic systems (P), (D) where $A \in \mathbb{R}^{m \times n}$ and the cone $K \subseteq \mathbb{R}^n$ is as in (1). For notational convenience, write $A = [A_1 \ \cdots \ A_r]$, where $A_i \in \mathbb{R}^{m \times n_i}$ is the i -th block of the matrix A .

Our main results generalize Proposition 1 and Proposition 2 to multifold conic systems. Motivated by (4), define

$$\begin{aligned} B &= \{i \in I \mid \exists x : Ax = 0, x \in K, x_i \in \text{int } K_i\}, \\ N &= \{i \in I \mid \exists y : A^T y \in K^*, A_i^T y \in \text{int } K_i^*\}. \end{aligned} \tag{7}$$

Likewise, motivated by (5), define

$$\begin{aligned} B_0 &= \{i \in I \mid A^\top y \in K^* \Rightarrow A_i^\top y = 0\}, \\ N_0 &= \{i \in I \mid Ax = 0, x \in K \Rightarrow x_i = 0\}. \end{aligned} \tag{8}$$

We are now ready to state our main results. The following theorem establishes a characterization of the index sets B, B_0, N, N_0 in terms of the geometry of the sets AK and $A_i K_i$. In the statement below, \overline{AK} denotes the closure of AK .

Theorem 1. (i) *The sets B, N defined in (7) can also be described as*

$$\begin{aligned} B &= \{i \in I \mid \text{ri } A_i K_i \cap \text{Lin}(AK) \neq \emptyset\}, \\ N &= \{i \in I \mid A_i(K_i \setminus \{0\}) \cap \text{Lin}(\overline{AK}) = \emptyset\}. \end{aligned} \tag{9}$$

(ii) *The sets B_0, N_0 defined in (8) can also be described as*

$$\begin{aligned} B_0 &= \{i \in I \mid \text{ri } A_i K_i \cap \text{Lin}(\overline{AK}) \neq \emptyset\}, \\ N_0 &= \{i \in I \mid A_i(K_i \setminus \{0\}) \cap \text{Lin}(AK) = \emptyset\}. \end{aligned} \tag{10}$$

To ease exposition, we defer the proof of Theorem 1 to the end of this Section.

Observe that in the case when K is a polyhedral cone, we have $AK = \overline{AK}$. Thus for K polyhedral Theorem 1 yields $B = B_0$ and $N = N_0$. In particular Proposition 2 readily follows from Theorem 1.

The next theorem generalizes Proposition 1. It shows that there is a unique canonical partition of the index set I into six complementarity subsets of indices.

Theorem 2. *For a unique partition $B \cup B' \cup N \cup N' \cup C \cup O = I$ of the index set I the following three properties hold:*

(i) *There exists a solution \bar{x} to (P) such that*

$$\bar{x}_i \in \text{int } K \text{ for all } i \in B \text{ and } \bar{x}_i \neq 0 \text{ for all } i \in B' \cup C,$$

(ii) *There exists a solution \bar{y} to (D) such that*

$$A_i^\top \bar{y} \in \text{int } K_i^* \text{ for all } i \in N, \text{ and } A_i^\top \bar{y} \neq 0 \text{ for all } i \in N' \cup C.$$

(iii) For any solutions x to (P) and y to (D) we have

$$x_i = 0 \text{ for all } i \in N' \cup N \cup O \text{ and } A_i^T y = 0 \text{ for all } i \in B \cup B' \cup O.$$

Proof. Take B, N and B_0, N_0 as in (7) and (8) respectively, and let

$$B' := B_0 \setminus (B \cup N_0); \quad N' = N_0 \setminus (N \cup B_0); \quad O = B_0 \cap N_0; \quad C = I \setminus (B_0 \cup N_0). \quad (11)$$

The sets B, B', C, N, N', O comprise a partition of I because by Theorem 1 $B \subseteq B_0$, $N \subseteq N_0$, and also $B \cap N_0 = N \cap B_0 = \emptyset$.

We next prove part (i). By Theorem 1(ii), for every $i \notin N_0$ there exists a solution $x^{(i)}$ to (P) such that $x_i^{(i)} \in K_i \setminus \{0\}$. Hence $x_{N_0} = \sum_{i \in I \setminus N_0} x^{(i)}$ is solution to (P) and for every $i \notin N_0$ we have $x_i \neq 0$ (since K_i is pointed). By the definition of B , for each $i \in B$ there exists a solution $\bar{x}^{(i)}$ to (P) such that $x_i^{(i)} \in \text{int } K_i$. Then $x_B = \sum_{i \in B} x^{(i)}$ is solution to (P) and $(x_B)_i \in \text{int } K_i$ (by [5, Lemma A.2.1.6]). Therefore, again by the pointedness of each K_i and by [5, Lemma A.2.1.6], the point $\bar{x} = x_B + x_{N_0}$ is a solution to (P) such that $\bar{x}_i \in \text{int } K$ for all $i \in B$ and $\bar{x}_i \neq 0$ for all $i \in B' \cup C$. An analogous argument proves part (ii). Part (iii) follows directly from the definition (8) of $B_0 = B \cup B' \cup O$ and $N_0 = N \cup N' \cup O$.

The uniqueness of the partition is proven as follows. First, observe that if (i) and (ii) hold, then by construction B, N must be as in (7). Likewise if (iii) holds, then B_0, N_0 must be as in (8). Therefore if (i), (ii), and (iii) hold, the sets B, B', C, N, N', O must be as in (11). \square

The Venn diagram representing the relations between the subsets of B, B', N, N', C and O of I is given in Fig. 1. In Section 5.1 we provide

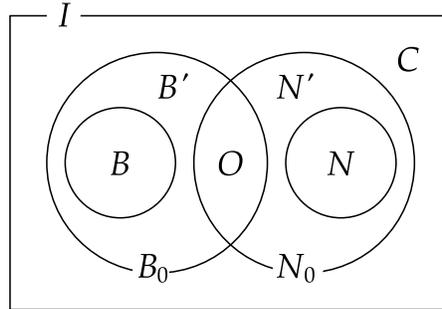


Figure 1: Partition of I into six disjoint sets based on B, N, B_0 and N_0
an example of a second-order conic programming problem for which all six

sets are nonempty. It should be noted that a six-set partition for second-order conic programs similar to the one suggested here was mentioned in [1, Section 6]. However, there was no prior characterization of this partition along the lines of Theorem 1.

We conclude this section with the proof of Theorem 1. Our proof relies on the following separation lemma. Although this result is likely known, we were not able to locate it in the literature in this exact form.

Lemma 1. *Let $K_1, K_2 \subseteq \mathbb{R}^n$ be closed convex cones such that $K_1 \cap K_2 = \{0\}$ and $\text{Lin}(K_2) = \{0\}$. Then K_1 and K_2 can be strictly separated in the following sense. There exists $s \in \mathbb{R}^n$ such that*

$$\langle s, y \rangle \leq 0 \quad \forall y \in K_1, \quad \langle s, y \rangle > 0 \quad \forall y \in K_2 \setminus \{0\}. \quad (12)$$

Proof. Let $C := \{x \in K_2 \mid \|x\| = 1\}$. Since K_2 is closed and $\text{Lin} K_2 = \{0\}$, the set $\text{co } C$ is compact and $0 \notin \text{co } C$. In particular $K_1 \cap \text{co } C = \emptyset$. Hence, by [5, Corol. A.4.1.3], there exists a point $s \in \mathbb{R}^n$ such that

$$\sup_{y \in K_1} \langle s, y \rangle < \min_{y \in \text{co } C} \langle s, y \rangle. \quad (13)$$

Since $0 \in K_1$ we have $\sup_{y \in K_1} \langle s, y \rangle \geq \langle s, 0 \rangle = 0$. Thus from (13) and the fact that K_1 is a cone it follows that

$$\sup_{y \in K_1} \langle s, y \rangle = 0 < \min_{y \in C} \langle s, y \rangle,$$

and (12) readily follows. \square

Proof of Theorem 1.

(i) We first show $B \supseteq \{i \in I \mid \text{ri } A_i K_i \cap \text{Lin}(AK) \neq \emptyset\}$. Assume $i \in I$ is such that $\text{ri}(A_i K_i) \cap \text{Lin}(AK) \neq \emptyset$. By [5, Prop. A.2.1.12], $\text{ri}(A_i K_i) = A_i(\text{ri } K_i) = A_i(\text{int } K_i)$. Hence there exists $\bar{x}_i \in \text{int } K_i$ such that $A_i \bar{x}_i \in \text{Lin}(AK)$. Thus $-A_i \bar{x}_i = Ax'$ for some $x' \in K$. Let $x \in K$ be defined by putting $x_j = x'_j$ for $j \neq i$ and $x_i = x'_i + \bar{x}_i$. By [5, Lemma A.2.1.6], it follows that x is a solution to (P) and $x_i \in \text{int } K_i$. Thus $i \in B$.

Next, we show $B \subseteq \{i \in I \mid \text{ri } A_i K_i \cap \text{Lin}(AK) \neq \emptyset\}$. Assume $i \in B$. Hence there exists $x \in K$ such that $x_i \in \text{int } K_i = \text{ri } K_i$ and $Ax = 0$. By [5, Prop. A.2.1.12],

$$A_i x_i \in \text{ri } A_i K_i. \quad (14)$$

Let $x' \in \mathbb{R}^n$ be defined by putting $x'_j = 0$ for $j \neq i$ and $x'_i = x_i$. We have $\bar{x} = x - x' \in K$ and so $-A_i x_i = A \bar{x} \in AK$. But $A_i x_i \in A_i K_i \subset AK$ as well, therefore

$$A_i x_i \in \text{Lin}(AK). \quad (15)$$

From (14) and (15) we have $\text{ri } A_i K_i \cap \text{Lin}(AK) \neq \emptyset$.

Now we show $N \supseteq \{i \in I \mid A_i(K_i \setminus \{0\}) \cap \text{Lin}(\overline{AK}) = \emptyset\}$. Assume $i \in I$ is such that $A_i(K_i \setminus \{0\}) \cap \text{Lin}(\overline{AK}) = \emptyset$. Since $A_i K_i \subseteq \overline{AK}$, this yields

$$\text{Lin}(A_i K_i) = \{0\} \quad \text{and} \quad -A_i(K_i \setminus \{0\}) \cap \overline{AK} = \emptyset.$$

Therefore by Lemma 1 applied to $K_1 = \overline{AK}$ and $K_2 = -A_i K_i$, there exists a nonzero $y \in \mathbb{R}^m$ such that $y^T A x \geq 0 \quad \forall x \in K$ and $y^T(-A_i x_i) < 0 \quad \forall x_i \in K_i \setminus \{0\}$. In particular, y is a solution to (D) and $A_i^T y \in \text{int } K_i^*$.

Next we show $N \subseteq \{i \in I \mid A_i(K_i \setminus \{0\}) \cap \text{Lin}(\overline{AK}) = \emptyset\}$. To that end, we show the contrapositive. Assume $i \in I$ is such that $A_i(K_i \setminus \{0\}) \cap \text{Lin}(\overline{AK}) \neq \emptyset$. Then there exists $x_i \in K_i \setminus \{0\}$ such that $A_i x_i, -A_i x_i \in \overline{AK}$. Hence for any solution y to (D) we have $y^T A_i x_i \geq 0$ and $y^T(-A_i x_i) \geq 0$ so $y^T A_i x_i = 0$. Since $x_i \in K_i \setminus \{0\}$, this implies that $A_i^T y \notin \text{int } K_i^*$. Consequently $i \notin N$.

- (ii) We first show $B_0 \supseteq \{i \in I : \text{ri } A_i K_i \cap \text{Lin}(\overline{AK}) \neq \emptyset\}$. Assume that $\text{ri } A_i K_i \cap \text{Lin}(\overline{AK}) \neq \emptyset$. Then by [5, Prop. A.2.1.12] there exists $x_i \in \text{int } K_i$ such that $A_i x_i, -A_i x_i \in \overline{AK}$. Therefore, as in the previous paragraph, it follows that $y^T A_i x_i = 0$ for any solution y to (D). Since $x_i \in \text{int } K_i$, this implies that $A_i^T y = 0$ for any solution y to (D). Thus $i \in B_0$.

We next show $B_0 \subseteq \{i \in I : \text{ri } A_i K_i \cap \text{Lin}(\overline{AK}) \neq \emptyset\}$. Assume $i \in B_0$. Then for all solutions y to (D) and all $x_i \in K_i$ we have $(A_i x_i)^T y = 0$. Thus $A_i x_i, -A_i x_i \in \overline{AK}$ for all $x_i \in K_i$, and hence $A_i K_i \subset \text{Lin}(\overline{AK})$.

We now show $N_0 \supseteq \{i \in I : A_i(K_i \setminus \{0\}) \cap \text{Lin}(AK) = \emptyset\}$. To that end, we show the contrapositive. Assume $i \in I$ is such that there exists a solution x to (P) with $x_i \neq 0$. Since $-A_i x_i = \sum_{j \neq i} A_j x_j \in AK$, we have $A_i x_i, -A_i x_i \in AK$ with $x_i \neq 0$. Hence $A_i x_i \in A_i(K_i \setminus \{0\}) \cap \text{Lin}(AK)$.

We finally show $N_0 \subseteq \{i \in I : A_i(K_i \setminus \{0\}) \cap \text{Lin}(AK) = \emptyset\}$. Again we show the contrapositive. Assume $i \in I$ is such that $A_i(K_i \setminus \{0\}) \cap$

$\text{Lin}(AK) \neq \emptyset$. Then there exists $x_i \in K_i \setminus \{0\}$ such that $A_i x_i, -A_i x_i \in AK$. In particular, for some $x' \in K$ we have $-A_i x_i = Ax'$. Then the point $\bar{x} \in K$ defined by putting $\bar{x}_j = x'_j$ for $j \neq i$ and $\bar{x}_i = x'_i + x_i$ is a solution to (P) with $\bar{x}_i \neq 0$ (because K_i is pointed).

□

4 Second-Order Conic Systems

Consider the special case when the cone K in (P),(D) is a cartesian product of Lorentz cones. In other words,

$$K = \mathcal{L}_{n_1-1} \times \cdots \times \mathcal{L}_{n_r-1}, \quad (16)$$

where

$$\mathcal{L}_{n_i-1} = \{(x_0, \bar{x}) \in \mathbb{R}^{n_i} \mid x_0 \geq \|\bar{x}\|\}, \quad i = 1, \dots, r.$$

Here $\|\cdot\|$ is the Euclidean norm in \mathbb{R}^{n_i} . We shall put, by convention, $\mathcal{L}_0 = \mathbb{R}_+$ when $n_i = 1$. Also, for $d \geq 1$ we will let $\mathbb{B}_d \subseteq \mathbb{R}^d$ denote the Euclidean closed unit ball in \mathbb{R}^d centered at zero.

For each $i \in I$ assume the i -th block $A_i \in \mathbb{R}^{m \times n_i}$ of A is of the form

$$A = [A_{i0} \quad \bar{A}_i], \quad A_{i0} \in \mathbb{R}^m, \quad \bar{A}_i \in \mathbb{R}^{m \times (n_i-1)}.$$

In other words, A_{i0} denotes the first column of A_i , and \bar{A}_i denotes the block of remaining $n_i - 1$ columns. Put

$$E_i = \begin{cases} A_{i0} + \bar{A}_i \mathbb{B}_{n_i-1}, & \text{if } n_i > 1, \\ A_{i0}, & \text{if } n_i = 1. \end{cases} \quad (17)$$

Observe that $AK = \text{cone co}_{i \in I} \{E_i\}$. Theorem 1 can now be stated in a way that more closely resembles (6) in Proposition 2.

Proposition 3. *Consider the pair of multifold conic systems (P), (D). Assume K is as in (16) and E_i , $i \in I$ are as in (17). Then*

(i) *The sets B , N defined in (7) satisfy*

$$B = \{i \in I \mid \text{ri } E_i \cap \text{Lin}(AK) \neq \emptyset\},$$

$$N = \{i \in I \mid E_i \cap \text{Lin}(\overline{AK}) = \emptyset\}.$$

(ii) The sets B_0, N_0 defined in (8) satisfy

$$B_0 = \{i \in I \mid \text{ri } E_i \cap \text{Lin}(\overline{AK}) \neq \emptyset\},$$

$$N_0 = \{i \in I \mid E_i \cap \text{Lin}(AK) = \emptyset\}.$$

Proof. This readily follows from Theorem 1 and the construction of the sets $E_i, i \in I$. \square

We now discuss an example of a second-order feasibility system where all six sets B, N, B', N', C, O in the partition of Theorem 2 are nonempty.

Example 1. Let $K = \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{L}_1 \times \mathcal{L}_1 \times \mathcal{L}_3 \subseteq \mathbb{R}^{11}$ and

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

In this case,

$$E_1 = \{(1, 0, 0)\}, \quad E_2 = \{(0, 1, 0)\}, \quad E_3 = \{(0, 0, 1)\}, \quad E_4 = \text{co}\{(1, 0, 0), (1, 0, 2)\},$$

$$E_5 = \text{co}\{(0, -1, 0), (2, -1, 0)\}, \quad E_6 = \{(0, 0, 1)\} + \mathbb{B}_3.$$

Thus

$$AK = \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\} \cup \{(x, y, z) \in \mathbb{R}^3 \mid z = 0, x \geq 0\},$$

$$\overline{AK} = \{(x, y, z) \in \mathbb{R}^3 \mid z \geq 0\},$$

and

$$\text{Lin}(AK) = \{0\} \times \mathbb{R} \times \{0\}; \quad \text{Lin}(\overline{AK}) = \mathbb{R} \times \mathbb{R} \times \{0\}.$$

Figure 2 shows the sets $\text{Lin}(AK), \text{Lin}(\overline{AK}), E_1, \dots, E_6$.

From Proposition 3 we readily get

$$B = \{2\}, \quad N = \{3\}, \quad B_0 = \{1, 2, 5\}, \quad N_0 = \{1, 3, 4\}.$$

Hence in this case the partition sets of Theorem 2 are

$$O = \{1\}, \quad B = \{2\}, \quad N = \{3\}, \quad N' = \{4\}, \quad B' = \{5\}, \quad C = \{6\}.$$

We note that in this small example the systems $Ax = 0, x \in K$ and $A^T y \in K$ can be solved directly. We obtain the following parametric families of solutions to (P) and (D) respectively:

$$x = (0, \lambda, 0, 0, 0, \lambda, -\lambda, \mu, 0, 0, -\mu), \quad \lambda \geq 0, \mu \geq 0;$$

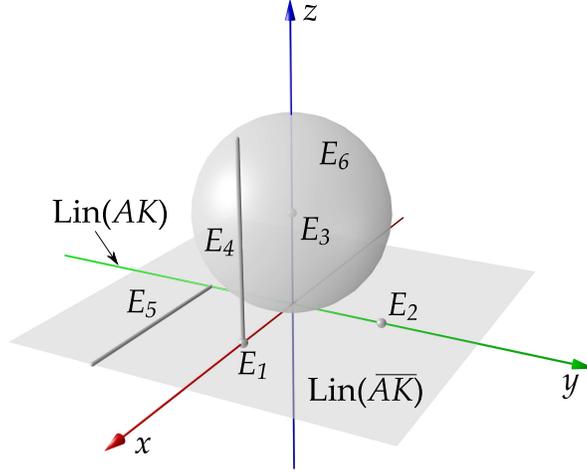


Figure 2: Geometric interpretation of the partition in Example 1

and

$$y = (0, 0, \gamma), \quad \gamma \geq 0.$$

The correctness of the partition $O = \{1\}$, $B = \{2\}$, $N = \{3\}$, $N' = \{4\}$, $B' = \{5\}$, $C = \{6\}$ can then be directly verified.

5 Some Final Remarks

5.1 Geometric interpretation of Theorem 1

Proposition 3 can be stated in a form that holds more generally. Consider the general multifold systems (P), (D). Assume K is as in (1) where each $K_i \subseteq \mathbb{R}^{n_i}$, $i \in I$ is regular. Furthermore, assume B_i be a compact convex subset of K_i such that $0 \notin B_i$ and $K_i = \text{cone } B_i$ for $i \in I$. Put

$$E_i = A_i B_i, \quad i \in I. \quad (18)$$

Observe that $AK = \text{cone } \text{co}_{i \in I} \{E_i\}$. Theorem 1 can now be stated as follows.

Theorem 3. *Consider the pair of multifold conic systems (P), (D). Assume K is as in (1) and E_i , $i \in I$ are as in (18). Then*

(i) *The sets B , N defined in (7) satisfy*

$$B = \{i \in I \mid \text{ri } E_i \cap \text{Lin}(AK) \neq \emptyset\},$$

$$N = \{i \in I \mid E_i \cap \text{Lin}(\overline{AK}) = \emptyset\}.$$

(ii) The sets B_0, N_0 defined in (8) satisfy

$$B_0 = \{i \in I \mid \text{ri } E_i \cap \text{Lin}(\overline{AK}) \neq \emptyset\},$$

$$N_0 = \{i \in I \mid E_i \cap \text{Lin}(AK) = \emptyset\}.$$

Remark 1. The alternate descriptions for the sets B, B_0 in Theorem 3 can also be stated as follows.

$$\text{ri } E_i \cap \text{Lin}(AK) \neq \emptyset \Leftrightarrow \text{ri } E_i \subseteq \text{Lin}(AK),$$

$$\text{ri } E_i \cap \text{Lin}(\overline{AK}) \neq \emptyset \Leftrightarrow \text{ri } E_i \subseteq \text{Lin}(\overline{AK}).$$

5.2 Some observations on polyhedral systems

While for the polyhedral feasibility problem strict complementarity always holds (Proposition 1), one might ask: what happens if each lower-dimensional cone in a multifold system is itself a product of nonnegative orthants? Since a linear image of a polyhedral set is closed, from Theorem 2 it follows that $B = B_0$ and $N = N_0$. Hence, we have only three possible complementarity sets: B, N and $C = I \setminus (B \cup N)$. Any problem with both B and N nonempty could alternatively be considered as a multifold problem with a single cone. In this case its only index would be in C . Therefore, there are polyhedral systems with nonempty C . However, for any polyhedral system the partition sets B', N' and O are always empty.

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