A Proof by the Simplex Method for the Diameter of a (0,1)-Polytope

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Abstract

Naddef [3] shows that the Hirsch conjecture is true for (0,1)-polytopes by proving that the diameter of any $(0, 1)$-polytope in d-dimensional Euclidean space is at most d. In this short paper, we give a simple proof for the diameter. The proof is based on the number of solutions generated by the simplex method for a linear programming problem. Our work is motivated by Kitahara and Mizuno [2], in which they get upper bounds for the number of different solutions generated by the simplex method.

1 Introduction

Let d be a positive integer and P be a polytope in the d-dimensional Euclidean space $\Re^d$ with f facets. Let $V(P)$ be a set of its vertices. The diameter $\delta(P)$ of the polytope P is the minimum value δ with the following property: for any two vertices $\hat{\boldsymbol{y}}$, $\tilde{\boldsymbol{y}}$ in $V(P)$, there exists a path from $\hat{\boldsymbol{y}}$ to $\tilde{\boldsymbol{y}}$ with the length at most δ. Hirsch conjectures that $\delta(P) \leq (f - d)$ in the context of the simplex method [1]. In a celebrated paper [4], Santos presents a counterexample to the conjecture. However, the conjecture is true for some special cases. We call P a $(0, 1)$-polytope when any element of $\boldsymbol{y} \in V(P)$ is 0

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or 1. Naddef [3] shows that the Hirsch conjecture is true for $(0, 1)$-polytopes by proving that the inequality

$$\delta(P) \leq d \tag{1}$$

holds for any $(0, 1)$-polytope $P \subset \Re^d$.

In this short paper, we give a simple proof of the inequality (1) by using the number of different solutions generated by the simplex method. Our work is motivated by Kitahara and Mizuno [2], in which they get upper bounds for the number of different solutions generated by the simplex method.

2 Proof by the simplex method

Let $P \subset \Re^d$ be a $(0, 1)$-polytope with f facets and $V(P)$ be a set of its vertices. It is well known that P can be expressed by using f linear inequalities like

$$P = \{\boldsymbol{y} \in \Re^d | \boldsymbol{Q}^T \boldsymbol{y} \leq \boldsymbol{r}\}$$

for some $\boldsymbol{Q} \in \Re^{d \times f}$ and $\boldsymbol{r} \in \Re^f$. For any fixed vertex $\hat{\boldsymbol{y}} = (\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_d)^T \in V(P)$, we define a linear function

$$l(\boldsymbol{y}; \hat{\boldsymbol{y}}) = \sum_{i:\ \hat{y}_i = 0} (1 - y_i) + \sum_{i:\ \hat{y}_i = 1} y_i, \tag{2}$$

where $\boldsymbol{y} = (y_1, y_2, \ldots, y_d)^T$ is a vector of variables. A linear programming problem, which maximize the linear function (2) on P, is written as

$$\begin{array}{ll} \max & l(\boldsymbol{y}; \hat{\boldsymbol{y}}), \\ \text{subject to} & \boldsymbol{Q}^T \boldsymbol{y} \leq \boldsymbol{r}. \end{array} \tag{3}$$

Obviously, this problem has the unique optimal solution $\hat{\boldsymbol{y}}$ whose objective function value is d.

Theorem 1 *When we generate a sequence of vertices from any initial vertex $\tilde{\boldsymbol{y}} \in V(P)$ by the simplex method for solving (3) so that the objective function value increases whenever an iterate is updated, the number of different vertices generated is at most d.*

Proof: From the definition (2), it is easy to see that

$$0 \leq l(\boldsymbol{y}; \hat{\boldsymbol{y}}) \leq d$$

for any $\boldsymbol{y} \in V(P)$. Since the objective function value is integral and increases at least one whenever an iterate is updated, the number of different vertices generated is at most d. ■

From Theorem 1, we can easily get the next result.

Theorem 2 *The diameter of any* $(0,1)$*-polytope* $P \subset \Re^d$ *is at most* d*.*

Proof: Let $\hat{\boldsymbol{y}}$ and $\tilde{\boldsymbol{y}}$ be any two vertices of P. We define the linear function (2) and the linear programming problem (3). If we generate a sequence of vertices from the initial vertex $\tilde{\boldsymbol{y}}$ by the simplex method with Bland's pivoting rule, we can find the optimal solution $\hat{\boldsymbol{y}}$ in a finite number of iterations. From Theorem 1, the number of vertices generated is at most d. Hence there exists a path from $\hat{\boldsymbol{y}}$ to $\tilde{\boldsymbol{y}}$ with the length at most d. ∎

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