SAMPLING WITH RESPECT TO A CLASS OF MEASURES ARISING IN SECOND-ORDER CONE OPTIMIZATION WITH RANK CONSTRAINTS

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Abstract. We describe a class of measures on second-order cones as a push-forward of the Cartesian product of a probabilistic measure on positive semi-line corresponding to Gamma distribution and the uniform measure on the sphere.

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1. Introduction. In ([3]) we showed that certain type of probabilistic measures defined on cones of squares of Euclidean Jordan algebras play a crucial role in optimization problems on symmetric cones involving rank constraints. More precisely, these measures play a crucial role in analysis of accuracy of convex approximations of hard optimization problems of above mentioned type obtained by omitting rank constraints. Since symmetric cones are completely classified as direct sums of several concrete types of irreducible symmetric cones, it is important to have an explicit description of above mentioned measures for each type of irreducible symmetric cones. This note solves this problem for an important class of irreducible symmetric cones, the so-called second-order cones. The abstract description of these measures is given in terms of their Laplace transforms ([1]) which is convenient for the theoretical analysis ([3]). However, to find desired approximations to optimal solutions, it is important to have simple algorithmic procedures for finding random samples with respect to measures under consideration.

In the present note we provide an explicit description of our measures as a push-forward of a Cartesian product of the probabilistic measure on positive semi-line (corresponding to the Gamma distribution) and the uniform measure on the sphere of an appropriate dimension. Since sampling with respect to these measures is quite simple (see e.g. [4]), we completely solve the problem of finding random samples for the measures under consideration for the class of second-order symmetric cones.

Note that , since we work with a concrete class of cones, no knowledge of theory of Euclidean Jordan algebras is required for reading this note.

2. Main result . Let $\bar{\Omega}$ be a cone of squares in a simple Euclidean Jordan algebra V. Let, further, Ω_l be a subset of $\bar{\Omega}$ consisting of all elements of $\bar{\Omega}$ of rank l, where $l = 0, 1, 2, \dots r$ and r is the rank of V. The following Proposition is from ([1], Corollary VII.2.3 and Proposition VII.2.3).

PROPOSITION 1. *For any* $l = 1, 2, \dots r-1$, *there exists a probabilistic measure* μ_l *on* $\bar{\Omega}$ *which is uniquely characterized by the following properties:*

a) the support of μ_l *is* $\bar{\Omega}_l$,

b) $\int_{\bar{\Omega}} \exp\left(-\langle x, y\rangle\right) d\mu_l(x) = \det(id + y)^{-ld/2}$,

for any $y \in \Omega$. *Here* $\langle , \rangle$ *is the canonical scalar product on* V*;* det *is the corresponding determinant function; id is the identity element in* V*;* d *is the degree of the Jordan algebra* V.

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Since we restrict ourselves in this note to the specific type of a simple Jordan algebra, there is no need to explain all the concepts mentioned above (see e.g [1]). The concrete case that we consider here can be described as follows. Let $V = \mathbf{R} \times \mathbf{R^n}$, where $n \geq 2$ and let $(,)$ be the standard scalar product in $\mathbf{R^n}$. This vector space can be endowed with the structure of a simple Euclidean Jordan algebra of rank 2 and degree $n-1$ with the multiplication operation:

$$[s,x] \circ [t,y] = [st + (x,y), tx + sy],$$

where $s, t \in \mathbf{R}; \mathbf{x}, \mathbf{y} \in \mathbf{R^n}$. (see e.g. [1]). Moreover,

$$\det[s,x] = s^2 - (x,x); \Omega = \{[s,x] \in V : s > \sqrt{(x,x)}\};$$

$$\bar{\Omega} = \{[s,x] \in V : s \geq \sqrt{(x,x)}\}; \Omega_1 = \{[s,x] \in V : s = \sqrt{(x,x)}, s > 0\};$$

$$\bar{\Omega}_1 = \Omega_1 \cup \{[0,0]\}.$$

One can easily verify that the measure described in Proposition 1 can be in our case characterized as follows:

$$\int_{\Omega_1} \exp\left(-2st - 2(x,y)\right) d\mu_1(s,x) = [(1+t)^2 - (y,y)]^{-\frac{n-1}{2}}, \tag{2.1}$$

for any $[t,y] \in \Omega$, and by the property that the support of μ_1 is $\bar{\Omega}_1$.

Denote by S the unit sphere in $\mathbf{R^n}$ and let $\mathbf{R}^+ = \{\mathbf{s} \in \mathbf{R} : \mathbf{s} > \mathbf{0}\}$. Consider the map

$$\rho : \mathbf{R}^+ \times \mathbf{S} \to \mathbf{\Omega_1}$$

$$\rho[s,x] = [s, sx]. \tag{2.2}$$

It is obvious that ρ is a smooth diffeomorphism of $\mathbf{R}^+ \times \mathbf{S}$ onto Ω_1. Denote, further, by σ the uniform probabilistic measure on S(i.e. the probabilistic measure on S invariant under the action of orthogonal group $O(n)$ on S). Consider also the probabilistic measure ν on $\mathbf{R}^+$ defined as follows:

$$\nu = \frac{2^{n-1}}{(n-2)!} \lambda^{n-2} \exp\left(-2\lambda\right) d\lambda, \tag{2.3}$$

where $d\lambda$ is the standard Lebesgue measure. Our main result can be formulated as follows.

Theorem 1.

$$\mu_1 = \rho_*(\nu \otimes \sigma). \tag{2.4}$$

Remark The formula (2.4) means that for any function f on Ω_1 integrable with respect to μ_1 we have:

$$\int_{\Omega_1} f(\xi) d\mu_1(\xi) = \frac{2^{n-1}}{(n-2)!} \int_0^{+\infty} \exp\left(-2s\right) s^{n-2} ds \int_S f(\rho(s,x)) d\sigma(x). \tag{2.5}$$

Remark To find a random sample on Ω_1 with respect to μ_1 it suffices to find a random sample s on $\mathbf{R}^+$ with respect to ν and random sample x on S with respect to σ and compute $\rho(s,x) = [s, sx]$. Since both tasks are standard (see e.g. [4]), this completely solves the random sampling problem with respect to μ_1.

Before we proceed with the proof of Theorem 1 we need a couple of auxiliary statements.

PROPOSITION 2. *Let F be a function defined on the interval $[-1,1]$. Suppose that $f(x) = F(x_1)$ is a function on S integrable with respect to the measure σ. Then*

$$\int_S f(x)d\sigma(x) = \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-1}{2})}\int_{-1}^{1} F(\xi)(1-\xi^2)^{\frac{n-3}{2}}d\xi. \tag{2.6}$$

For the proof of Proposition 2 see e.g. ([2], Proposition 9.1.2).

PROPOSITION 3.

$$\psi_n = \int_0^{+\infty} \frac{r^{n-2}dr}{(r^2+\alpha^2)^{n-1}} = \frac{1}{2^{n-1}}\frac{\sqrt{\pi}\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2}) \mid \alpha \mid^{n-1}}, \alpha \neq 0, n \geq 2. \tag{2.7}$$

Proof of Proposition 3. Without loss of generality we can assume $\alpha > 0$. An easy calculation shows:

$$\psi_2(\alpha) = \frac{\pi}{2\alpha}, \psi_3(\alpha) = \frac{1}{2\alpha^2}.$$

Differentiating with respect to parameter α, we obtain:

$$\psi_n^{'}(\alpha) = -2(n-1)\alpha \int_0^{+\infty} \frac{r^{n-2}dr}{(r^2+\alpha^2)^n}.$$

Integrating now by parts ($du = r^{n-2}dr, v = \frac{1}{(r^2+\alpha^2)^n}$),we obtain a key relation:

$$\psi_n^{'}(\alpha) = -4\alpha n \psi_{n+2}(\alpha).$$

We can now easily prove the required formula by induction on n using the classical relation $\Gamma(z+1) = z\Gamma(z)$, along with $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Proof of Theorem 1 According to (2.1), we need to show

$$I = \int_{\mathbf{R}^+\times\mathbf{S}} f_{(t,y)}(\rho(\lambda,x))d(\nu\otimes\sigma)(\lambda,x) = [(1+t)^2 - (y,y)]^{-\frac{(n-1)}{2}} \tag{2.8}$$

for any $f_{(t,y)}$ of the form

$$f_{(t,y)}(\xi,z) = \exp(-2\xi t - 2(z,y)),$$

where $[\xi, z] \in \mathbf{R}\times\mathbf{R^n}, \xi > \sqrt{(z,z)}$. Using (2.2), (2.3), we can rewrite (2.8) in the form:

$$I = \frac{2^{n-1}}{(n-2)!}\int_S d\sigma(x)\int_0^{+\infty} \lambda^{n-2}\exp(-2\lambda)\exp(-2\lambda t - 2\lambda(x,y))d\lambda.$$

Making the change of variables $\tilde{\lambda} = 2\lambda(1+t+(x,y))$ in the inner integral and using the definition of the Gamma function, we obtain:

$$I = \int_S \frac{d\sigma(x)}{[1+t+(x,y)]^{n-1}}.$$

The invariance of σ with respect to the orthogonal group yields:

$$I = \int_S \frac{d\sigma(x)}{[1 + t + \parallel y \parallel x_1]^{n-1}}, \tag{2.9}$$

where $\parallel y \parallel = \sqrt{(y, y)}$. It remains to calculate (2.9). Using Proposition 2, we can reduce (2.9) to:

$$I = \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-1}{2})} \int_{-1}^{+1} \frac{(1 - \xi^2)^{\frac{n-3}{2}}}{(1 + t + \parallel y \parallel \xi)^{n-1}} d\xi. \tag{2.10}$$

Consider the following change of variables in (2.10):

$$r = \sqrt{\frac{1 + \xi}{1 - \xi}}.$$

One can easily see that in new variables (2.10) takes the form:

$$I = \frac{2^{n-1}\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-1}{2})} \int_0^{+\infty} \frac{r^{n-2} dr}{[(1 + t + \parallel y \parallel)r^2 + (1 + t - \parallel y \parallel)]^{n-1}}.$$

The result now follows from Proposition 3.

3. Concluding remarks. This work was supported in part by NSF grant DMS07-12809.

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