Weak and Strong Convergence of Algorithms for the Split Common Null Point Problem

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Abstract

We introduce and study the Split Common Null Point Problem (SCNPP) for set-valued maximal monotone mappings in Hilbert space. This problem generalizes our Split Variational Inequality Problem (SVIP) [Y. Censor, A. Gibali and S. Reich, Algorithms for the split variational inequality problem, Numerical Algorithms, accepted for publication, DOI 10.1007/s11075-011-9490-5]. The SCNPP with only two set-valued mappings entails finding a zero of a maximal monotone mapping in one space, the image of which under a given bounded linear transformation is a zero of another maximal monotone mapping. We present three iterative algorithms that solve such problems in Hilbert

space, and establish weak convergence for one and strong convergence for the other two.

1 Introduction

In this paper we introduce and study the *Split Common Null Point Problem* (SCNPP) for set-valued maximal monotone mappings in Hilbert space. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces. Given set-valued mappings $B_i : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$, $1 \leq i \leq p$, and $F_j : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$, $1 \leq j \leq r$, respectively, and bounded linear operators $A_j : \mathcal{H}_1 \to \mathcal{H}_2$, $1 \leq j \leq r$, the SCNPP is formulated as follows:

$$\text{find a point } x^* \in \mathcal{H}_1 \text{ such that } 0 \in \cap_{i=1}^{p} B_i(x^*) \tag{1.1}$$

$$\text{and such that the points}$$

$$y_j^* = A_j x^* \in \mathcal{H}_2 \text{ solve } 0 \in F_j(y_j^*) \text{ for each } j = 1, \ldots, r. \tag{1.2}$$

In order to motivate this new problem and to understand its relationship with other problems, we first look at the prototypical *Split Inverse Problem* (SIP) formulated in [20, Section 2]. It concerns a model in which there are given two vector spaces X and Y and a linear operator $A : X \to Y$. In addition, two inverse problems are involved. The first one, denoted by IP_1, is formulated in the space X and the second one, denoted by IP_2, is formulated in the space Y. Given these data, the *Split Inverse Problem* (SIP) is formulated as follows:

$$\text{find a point } x^* \in X \text{ that solves } \mathrm{IP}_1 \tag{1.3}$$

$$\text{and such that}$$

$$\text{the point } y^* = Ax^* \in Y \text{ solves } \mathrm{IP}_2. \tag{1.4}$$

Real-world inverse problems can be cast into this framework by making different choices of the spaces X and Y (including the case $X = Y$), and by choosing appropriate inverse problems for IP_1 and IP_2. The *Split Convex Feasibility Problem* (SCFP) [18] is the first instance of an SIP. The two problems IP_1 and IP_2 there are of the *Convex Feasibility Problem* (CFP) type. This formulation was used for solving an inverse problem in radiation therapy treatment planning [19, 15]. The SCFP has been well studied for the last two decades both theoretically and practically; see, e.g., [10, 19] and the references therein. Two leading candidates for IP_1 and IP_2 are the mathematical models of the CFP and problems of constrained optimization. In particular, the CFP formalism is in itself at the core of the modeling of many inverse problems in various areas of mathematics and the physical sciences; see, e.g., [14] and references therein for an early example. Over the past four decades, the CFP has been used to model significant real-world inverse problems in sensor networks, radiation therapy treatment planning, resolution enhancement and in many others; see [16] for exact references to all of the above. More work on the CFP can be found in [9, 11, 17].

It is therefore natural to ask whether other inverse problems can be used for IP_1 and IP_2, besides the CFP, and be embedded in the SIP methodology. For example, can $\mathrm{IP}_1 = \mathrm{CFP}$ in the space X and can a constrained optimization problem be IP_2 in the space Y? In our recent paper [20] we have made a step in this direction by formulating an SIP with a *Variational Inequality Problem* (VIP) in each of the two spaces of the SIP. In the present paper we study an SIP with a *Null Point Problem* in each of the two spaces. As we explain below, this formulation includes the earlier formulation with VIPs and all its special cases such as the CFP and constrained optimization problems.

To further motivate our study, we now put our SCNPP in the context of other SIPs and related works. We first recall our *Split Variational Inequality Problem* (SVIP), which is an SIP with a VIP in each one of the two spaces [20]. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces, and assume that there are given two operators $f : \mathcal{H}_1 \to \mathcal{H}_1$ and $g : \mathcal{H}_2 \to \mathcal{H}_2$, a bounded linear operator $A : \mathcal{H}_1 \to \mathcal{H}_2$, and nonempty, closed and convex subsets $C \subset \mathcal{H}_1$ and $Q \subset \mathcal{H}_2$. The SVIP is then formulated as follows:

$$\text{find a point } x^* \in C \text{ such that } \langle f(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in C \tag{1.5}$$

$$\text{and such that}$$

$$\text{the point } y^* = Ax^* \in Q \text{ and solves } \langle g(y^*), y - y^* \rangle \geq 0 \text{ for all } y \in Q. \tag{1.6}$$

Denoting by $\mathrm{SOL}(f, C)$ and $\mathrm{SOL}(g, Q)$ the solution sets of the VIPs in (1.5) and (1.6), respectively, we can also write the SVIP in the following way:

$$\text{find a point } x^* \in \mathrm{SOL}(f, C) \text{ such that } Ax^* \in \mathrm{SOL}(g, Q). \tag{1.7}$$

Taking in (1.5)–(1.6) $C = \mathcal{H}_1$, $Q = \mathcal{H}_2$, and choosing $x := x^* - f(x^*) \in \mathcal{H}_1$ in (1.5) and $y = Ax^* - g(Ax^*) \in \mathcal{H}_2$ in (1.6), we obtain the *Split Zeros Problem*

(SZP) for two operators $f : \mathcal{H}_1 \to \mathcal{H}_1$ and $g : \mathcal{H}_2 \to \mathcal{H}_2$, which we introduced in [20, Subsection 7.3]. It is formulated as follows:

$$\text{find a point } x^* \in \mathcal{H}_1 \text{ such that } f(x^*) = 0 \text{ and } g(Ax^*) = 0. \tag{1.8}$$

Observe that if we denote by $N_C(v)$ the *normal cone* of some nonempty, closed and convex set C at a point $v \in C$,

$$N_C(v) := \{d \in \mathcal{H} \mid \langle d, y - v\rangle \leq 0 \text{ for all } y \in C\}, \tag{1.9}$$

and define the set-valued mapping B by

$$B(v) := \begin{cases} f(v) + N_C(v), & v \in C, \\ \emptyset, & \text{otherwise}, \end{cases} \tag{1.10}$$

where f is some given operator, then, under a certain continuity assumption on f, Rockafellar in [39, Theorem 3] showed that B was a maximal monotone mapping and $B^{-1}(0) = \mathrm{SOL}(f, C)$. Following this idea, Moudafi [36] introduced the *Split Monotone Variational Inclusion* (SMVI) which generalized the SVIP. Given two operators $f : \mathcal{H}_1 \to \mathcal{H}_1$ and $g : \mathcal{H}_2 \to \mathcal{H}_2$, a bounded linear operator $A : \mathcal{H}_1 \to \mathcal{H}_2$, and two set-valued mappings $B_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $B_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$, the SMVI is formulated as follows:

$$\text{find a point } x^* \in \mathcal{H}_1 \text{ such that } 0 \in f(x^*) + B_1(x^*) \tag{1.11}$$

$$\text{and such that the point}$$

$$y^* = Ax^* \in \mathcal{H}_2 \text{ solves } 0 \in g(y^*) + B_2(y^*). \tag{1.12}$$

Moudafi presented an algorithm that converged weakly to a solution of the SMVI under certain conditions. Asking if it is possible to obtain strong convergence under reasonable assumptions, we show in this paper that this is indeed the case. We note that our two-operator SZP (1.8) is obtained from (1.11) and (1.12) by letting B_1 and B_2 be the zero operators.

For two set-valued mappings our SCNPP is a special case of Moudafi's SMVI, with $f = g = 0$. The applications presented in [36] only deal with this situation. However, the newly introduced SCNPP is not a special case of the SMVI and is yet another generalization of the SVIP. Before we introduce the case where more than two set-valued mappings and more than one bounded linear operator A are involved, we recall the problem Masad and Reich [34] called the *Constrained Multiple-Set Split Convex Feasibility Problem* (CMSSCFP). Let r and p be two natural numbers. Let C_i, $1 \leq i \leq p$, and Q_j, $1 \leq j \leq r$, be closed and convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively; further, for each $1 \leq j \leq r$, let $A_j : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Finally, let Ω be another closed and convex subset of $\mathcal{H}_1$. The CMSSCFP is formulated as follows:

$$\text{find a point } x^* \in \Omega \tag{1.13}$$

$$\text{such that}$$

$$x^* \in \cap_{i=1}^{p} C_i \text{ and } A_j x^* \in Q_j \text{ for each } j = 1, \ldots, r. \tag{1.14}$$

Motivated by this CMSSCFP, we suggest here our *Split Common Null Point Problem* (SCNPP) (see (1.1)–(1.2)), which is a generalization of the SZP.

Another related split problem is the *Split Common Fixed Point Problem* (SCFPP), first introduced in Euclidean spaces in [23] and later extended by Moudafi [35] to Hilbert spaces. Given operators $U_i : \mathcal{H}_1 \to \mathcal{H}_1$, $i = 1, 2, \ldots, p$, and $T_j : \mathcal{H}_2 \to \mathcal{H}_2$, $j = 1, 2, \ldots, r$, with nonempty fixed points sets C_i, $i = 1, 2, \ldots, p$, and Q_j, $j = 1, 2, \ldots, r$, respectively, and a bounded linear operator $A : \mathcal{H}_1 \to \mathcal{H}_2$, the SCFPP is formulated as follows:

$$\text{find a point } x^* \in C := \cap_{i=1}^{p} C_i \text{ such that } Ax^* \in Q := \cap_{j=1}^{r} Q_j. \tag{1.15}$$

The purpose of this paper is to introduce the SCNPP and present several algorithms for solving it. Following [34] and [28], we are able to establish strong convergence of two of the algorithms that we propose.

Our paper is organized as follows. In Section 2 we list several known facts about operators and set-valued mappings that we need in the sequel. In Section 3 we present an algorithm for solving the SCNPP and show its weak convergence. In Section 4 we present two additional algorithms for solving the SCNPP and present strong convergence theorems for them.

2 Preliminaries

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$, and let $D \subset \mathcal{H}$ be a nonempty, closed and convex subset of it. We write either $x^k \rightharpoonup x$ or $x^k \to x$ to indicate that the sequence $\left\{ x^k \right\}_{k=0}^{\infty}$ converges either weakly or strongly, respectively, to x. Next we present several properties of operators and set-valued mappings which will be useful later on. For more details on many of the notions and results quoted here see, e.g., the recent books [3, 8].

Definition 2.1 *Let $\mathcal{H}$ be a real Hilbert space. Let $D \subset \mathcal{H}$ be a subset of $\mathcal{H}$ and $h : D \to \mathcal{H}$ be an operator from D to $\mathcal{H}$.*

1. *h is called ν*-**inverse strongly monotone** *(ν-ism) on D if there exists a number $\nu > 0$ such that*

$$\langle h(x) - h(y), x - y\rangle \geq \nu\|h(x) - h(y)\|^2 \text{ for all } x, y \in D. \qquad (2.1)$$

2. *h is called* **firmly nonexpansive** *on D if*

$$\langle h(x) - h(y), x - y\rangle \geq \|h(x) - h(y)\|^2 \text{ for all } x, y \in D, \qquad (2.2)$$

   i.e., if it is 1-ism.

3. *h is called* **Lipschitz continuous** *with constant $\kappa > 0$ on D if*

$$\|h(x) - h(y)\| \leq \kappa\|x - y\| \text{ for all } x, y \in D. \qquad (2.3)$$

4. *h is called* **nonexpansive** *on D if*

$$\|h(x) - h(y)\| \leq \|x - y\| \text{ for all } x, y \in D, \qquad (2.4)$$

   i.e., if it is 1-Lipschitz.

5. *h is called a* **strict contraction** *if it is Lipschitz continuous with constant $\kappa < 1$.*

6. *h is called* **hemicontinuous** *if it is continuous along each line segment in D.*

7. *h is called* **asymptotically regular** *at $x \in D$ [6] if*

$$\lim_{k\to\infty}(h^k(x) - h^{k+1}(x)) = 0 \text{ for all } x \in \mathcal{H}, \qquad (2.5)$$

   where h^k denotes the k-th iterate of h.

8. *h is called* **demiclosed** *at $y \in \mathcal{H}$ if for any sequence $\{x^k\}_{k=0}^{\infty} \subset D$ such that $x^k \rightharpoonup \overline{x} \in D$ and $h(x^k) \to y$, we have $h(\overline{x}) = y$.*

9. *h is called* ***averaged*** *[1] if there exists a nonexpansive operator $N : D \to \mathcal{H}$ and a number $c \in (0,1)$ such that*

$$h = (1-c)I + cN. \tag{2.6}$$

In this case we also say that h is c-av [11].

10. *h is called* ***odd*** *if D is symmetric, i.e., $D = -D$, and if*

$$h(-x) = -h(x) \text{ for all } x \in D. \tag{2.7}$$

Remark 2.2 *(i) It can be verified that if h is ν-ism, then it is Lipschitz continuous with constant $\kappa = 1/\nu$.*

(ii) It is known that an operator h is averaged if and only if its complement $I - h$ is ν-ism for some $\nu > 1/2$; see, e.g., [11, Lemma 2.1].

(iii) The operator h is firmly nonexpansive if and only if its complement $I - h$ is firmly nonexpansive. The operator h is firmly nonexpansive if and only if h is (1/2)-av (see [27, Proposition 11.2] and [11, Lemma 2.3]).

(iv) If h_1 and h_2 are c_1-av and c_2-av, respectively, then their composition $S = h_1 h_2$ is $(c_1 + c_2 - c_1 c_2)$-av. See [11, Lemma 2.2].

Definition 2.3 *Let $\mathcal{H}$ be a real Hilbert space. Let $B : \mathcal{H} \to 2^{\mathcal{H}}$ and $\lambda > 0$.*

(i) B is called ***odd*** *if*

$$B(-x) = -B(x) \text{ for all } x \in \mathcal{H}. \tag{2.8}$$

(ii) B is called a ***maximal monotone mapping*** *if B is* ***monotone****, i.e.,*

$$\langle u - v, x - y \rangle \geq 0 \text{ for all } u \in B(x) \text{ and } v \in B(y), \tag{2.9}$$

and the ***graph*** *$G(B)$ of B,*

$$G(B) := \{(x,u) \in \mathcal{H} \times \mathcal{H} \mid u \in B(x)\}, \tag{2.10}$$

is not properly contained in the graph of any other monotone mapping.

(iii) The ***domain*** *of B is*

$$\operatorname{dom}(B) := \{x \in \mathcal{H} \mid B(x) \neq \emptyset\}. \tag{2.11}$$

(iv) The ***resolvent*** *of B with parameter λ is denoted and defined by $J_\lambda^B := (I + \lambda B)^{-1}$, where I is the identity operator.*

Remark 2.4 *It is well known that for* $\lambda > 0$*,*

(i) B *is monotone if and only if the resolvent* J_λ^B *of* B *is single-valued and firmly nonexpansive.*

(ii) B *is maximal monotone if and only if* J_λ^B *is single-valued, firmly nonexpansive and* $\operatorname{dom}(J_\lambda^B) = \mathcal{H}$*.*

(iii) The following equivalence holds:

$$0 \in B(x^*) \Leftrightarrow x^* \in \operatorname{Fix}(J_\lambda^B). \tag{2.12}$$

It follows from (2.12) that the SCNPP with two set-valued maximal monotone mappings can be seen as an SCFPP with respect to their resolvents. Now we present another known result; see, e.g., [36, Fact 2].

Remark 2.5 *Let* $\mathcal{H}$ *be a real Hilbert space, and let a maximal monotone mapping* $B : \mathcal{H} \to 2^{\mathcal{H}}$ *and an* α*-ism operator* $h : \mathcal{H} \to \mathcal{H}$ *be given. Then the operator* $J_\lambda^B(I - \lambda h)$ *is averaged for each* $\lambda \in (0, 2\alpha)$*.*

Next we present an important class of operators, the $\mathfrak{T}$-class operators. This class was introduced and investigated by Bauschke and Combettes in [2, Definition 2.2] and by Combettes in [25]. Operators in this class were named *directed operators* by Zaknoon [45] and further employed under this name by Segal [40], and by Censor and Segal [22, 23]. Cegielski [12, Def. 2.1] studied these operators under the name *separating operators*. Since both *directed* and *separating* are key words of other, widely-used, mathematical entities, Cegielski and Censor have recently introduced the term *cutter operators* [13]. This class coincides with the class $\mathcal{F}^\nu$ for $\nu = 1$ [26] and with the class $\mathrm{DC}_{\boldsymbol{p}}$ for $\boldsymbol{p} = -1$ [33]. The term *firmly quasi-nonexpansive* (FQNE) for $\mathcal{T}$-class operators was used by Yamada and Ogura [44] because every *firmly nonexpansive* (FNE) mapping [27, page 42] is obviously FQNE.

Definition 2.6 *Let* $\mathcal{H}$ *be a real Hilbert space. An operator* $h : \mathcal{H} \to \mathcal{H}$ *is called a* ***cutter operator*** *if* $\operatorname{dom}(h) = \mathcal{H}$ *and*

$$\langle h(x) - x, h(x) - q \rangle \leq 0 \text{ for all } (x, q) \in \mathcal{H} \times \operatorname{Fix}(h), \tag{2.13}$$

where the fixed point set $\operatorname{Fix}(h)$ *of* h *is defined by*

$$\operatorname{Fix}(h) := \{x \in \mathcal{H} \mid h(x) = x\}. \tag{2.14}$$

It can be seen that this class of operators coincides with the class of *firmly quasi-nonexpansive operators*, which satisfy the inequality

$$\|h(x)-q\|^2 \leq \|x-q\|^2 - \|x-h(x)\|^2 \text{ for all } (x,q) \in \mathcal{H} \times \operatorname{Fix}(h). \quad (2.15)$$

Note that the $\mathfrak{T}$-class operators include, among others, orthogonal projections, subgradient projectors, resolvents of maximal monotone mappings, and firmly nonexpansive operators. This last class was first introduced by Browder [5, Definition 6] under the name firmly contractive operators. Every $\mathfrak{T}$-class operator belongs to the class $\mathcal{F}^0$ of operators, defined by Crombez [26, p. 161]:

$$\mathcal{F}^0 := \{h : \mathcal{H} \to \mathcal{H} \mid \|h(x)-q\| \leq \|x-q\| \text{ for all } (x,q) \in \mathcal{H} \times \operatorname{Fix}(h)\}. \quad (2.16)$$

The elements of $\mathcal{F}^0$ are called quasi-nonexpansive or paracontracting operators. A more general class of operators is the class of demicontractive operators (see, e.g., [33]).

Definition 2.7 *Let $\mathcal{H}$ be a real Hilbert space and let $h : \mathcal{H} \to \mathcal{H}$ be an operator.*

*(i) h is called a **demicontractive operator** if there exists a number $\beta \in [0,1)$ such that*

$$\|h(x)-q\|^2 \leq \|x-q\|^2 + \beta \|x-h(x)\|^2 \text{ for all } (x,q) \in \mathcal{H} \times \operatorname{Fix}(h), \quad (2.17)$$

which is equivalent to

$$\langle x-h(x), x-q\rangle \geq \frac{1-\beta}{2} \|x-h(x)\|^2 \text{ for all } (x,q) \in \mathcal{H} \times \operatorname{Fix}(h). \quad (2.18)$$

Another useful observation is that if $h : \mathcal{H} \to \mathcal{H}$ is monotone and hemicontinuous on a nonempty, closed and convex subset D, then the set-valued mapping

$$M(v) = \begin{cases} h(v) + N_D(v), & v \in D, \\ \emptyset, & \text{otherwise}, \end{cases} \quad (2.19)$$

is, by [39, Theorem 3], maximal monotone and $M^{-1}(0) = \mathrm{SOL}(h,D)$. Therefore, as mentioned in [36], if we choose $B_1 = N_C$ and $B_2 = N_Q$ in (1.11) and (1.12), respectively, then we get the SVIP of (1.5)–(1.6). Of course, this result also holds for our SCNPP with two set-valued maximal monotone mappings

$(i = j = 1)$ when we take B_1 and F_1 to be similar to M in (2.19). This enables us to solve the SVIP for monotone and hemicontinuous operators (which constitute a larger class than the class of inverse strongly monotone operators) by using our convergence theorem for the SVIP [20, Theorem 6.3]. In [20, Theorem 6.3] we also assumed [20, Equation (5.9)] that for all $x^* \in \mathrm{SOL}\,(f, C)$,

$$\langle f(x), P_C(I - \lambda f)(x) - x^* \rangle \geq 0 \text{ for all } x \in \mathcal{H}_1, \tag{2.20}$$

an assumption which is not needed for the convergence theorems we establish in the present paper.

The next lemma is the well-known *Demiclosedness Principle* [4].

Lemma 2.8 *Let $\mathcal{H}$ be a Hilbert space, D a closed and convex subset of $\mathcal{H}$, and let $h : D \to \mathcal{H}$ be a nonexpansive operator. Then $I - h$ is demiclosed at any $y \in \mathcal{H}$.*

The next definition is due to Clarkson [24].

Definition 2.9 *A Banach space $\mathcal{B}$ is said to be* ***uniformly convex*** *if to each $\varepsilon \in (0, 2]$, there corresponds a positive $\delta(\varepsilon)$ such that the conditions $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$ imply that $\|(x + y)/2\| \leq 1 - \delta(\varepsilon)$.*

It follows from the Parallelogram Identity that every Hilbert space is uniformly convex. Next we present two known theorems, the Krasnosel'skiĭ-Mann-Opial theorem [30, 32, 37] and the Halpern-Suzuki theorem [28, 41].

Theorem 2.10 *[30, 32, 37] Let $\mathcal{H}$ be a real Hilbert space and $D \subset \mathcal{H}$ be a nonempty, closed and convex subset of $\mathcal{H}$. Given an averaged operator $h : D \to D$ with $\mathrm{Fix}(h) \neq \emptyset$ and an arbitrary $x^0 \in D$, the sequence generated by the recursion $x^{k+1} = h(x^k)$, $k \geq 0$, converges weakly to a point $z \in \mathrm{Fix}(h)$.*

Theorem 2.11 *[28, 41] Let $\mathcal{H}$ be a real Hilbert space and $D \subset \mathcal{H}$ be a closed and convex subset of $\mathcal{H}$. Given an averaged operator $h : D \to D$, and a sequence $\{\alpha_k\}_{k=0}^{\infty} \subset [0, 1]$ satisfying $\lim_{k\to\infty} \alpha_k = 0$ and $\sum_{k=0}^{\infty} \alpha_k = \infty$, the sequence $\{x^k\}_{k=0}^{\infty}$ generated by $x^0 \in D$ and $x^{k+1} = \alpha_k x^0 + (1 - \alpha_k)h(x^k)$, $k \geq 0$, converges strongly to a point $z \in \mathrm{Fix}(h)$.*

3 Weak convergence

In this section we first present an algorithm for solving the SCNPP for two set-valued maximal monotone mappings. Then, for the general case, we employ a product space formulation in order to transform the SCNPP into an SCFPP, in a similar fashion to what has been done in [23, Section 4] and [20, Subsection 6.1].

3.1 The SCNPP for two set-valued maximal monotone mappings

Consider the SCNPP (1.1)–(1.2) with $i = j = 1$. That is, given two set-valued mappings $B_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $F_1 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$, and a bounded linear operator $A : \mathcal{H}_1 \to \mathcal{H}_2$, we consider the following two-mapping SCNPP:

$$\text{find a point } x^* \in \mathcal{H}_1 \text{ such that } 0 \in B_1(x^*) \text{ and } 0 \in F_1(Ax^*). \tag{3.1}$$

Here is our algorithm for solving (3.1).

Algorithm 3.1
Initialization: *Let $\lambda > 0$ and select an arbitrary starting point $x^0 \in \mathcal{H}_1$.*
Iterative step: *Given the current iterate x^k, compute*

$$x^{k+1} = J_\lambda^{B_1}\left(x^k - \gamma A^*(I - J_\lambda^{F_1})Ax^k\right), \tag{3.2}$$

where A^* is the adjoint of A, $L = \|A^*A\|$ and $\gamma \in (0, 2/L)$.

Our convergence theorem for this algorithm is presented next. Its proof is similar to the proof of [36, Theorem 3.1], which is based on the Krasnosel'skiĭ-Mann-Opial theorem [30, 32, 37]. We denote by Γ the solution set of (3.1).

Theorem 3.2 *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces. Given two set-valued maximal monotone mappings $B_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $F_1 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$, and a bounded linear operator $A : \mathcal{H}_1 \to \mathcal{H}_2$, any sequence $\left\{x^k\right\}_{k=0}^{\infty}$, generated by Algorithm 3.1 converges weakly to a point $x^* \in \Gamma$, provided that $\Gamma \neq \emptyset$ and $\gamma \in (0, 2/L)$, where $L = \|A^*A\|$.*

Proof. First we prove that the operator $\gamma A^*(I - J_\lambda^{F_1})A$ is ν-ism for some $\nu > 1/2$ and therefore its complement $I - \gamma A^*(I - J_\lambda^{F_1})A$ is averaged. By

Remark 2.4(i), $J_\lambda^{F_1}$ is firmly nonexpansive and therefore (1/2)-av (Remark 2.2(iii)). So,

$$J_\lambda^{F_1} = \frac{(I+N)}{2} \tag{3.3}$$

for some nonexpansive operator $N : \mathcal{H}_2 \to \mathcal{H}_2$. Since $I - J_\lambda^{F_1} = (I-N)/2$, it follows that $I - J_\lambda^{F_1}$ is 1-ism. Hence

$$\begin{aligned}&\left\langle (I - J_\lambda^{F_1})Ax - (I - J_\lambda^{F_1})Ay, Ax - Ay \right\rangle \\ &\geq \|(I - J_\lambda^{F_1})Ax - (I - J_\lambda^{F_1})Ay\|^2. \end{aligned} \tag{3.4}$$

Now

$$\begin{aligned}&\|A^*(I - J_\lambda^{F_1})Ax - A^*(I - J_\lambda^{F_1})Ay\|^2 \\ &= \langle A^*(I - J_\lambda^{F_1})Ax - A^*(I - J_\lambda^{F_1})Ay, A^*(I - J_\lambda^{F_1})Ax - A^*(I - J_\lambda^{F_1})Ay\rangle \\ &= \langle A^*((I - J_\lambda^{F_1})Ax - (I - J_\lambda^{F_1})Ay), A^*((I - J_\lambda^{F_1})Ax - (I - J_\lambda^{F_1})Ay)\rangle \\ &= \langle (I - J_\lambda^{F_1})Ax - (I - J_\lambda^{F_1})Ay, AA^*((I - J_\lambda^{F_1})Ax - (I - J_\lambda^{F_1})Ay)\rangle \\ &\leq L\|(I - J_\lambda^{F_1})Ax - (I - J_\lambda^{F_1})Ay\|^2. \end{aligned} \tag{3.5}$$

Combining the above inequalities, we obtain

$$\begin{aligned}&\left\langle A^*(I - J_\lambda^{F_1})Ax - A^*(I - J_\lambda^{F_1})Ay, x - y \right\rangle \\ &= \left\langle (I - J_\lambda^{F_1})Ax - (I - J_\lambda^{F_1})Ay, Ax - Ay \right\rangle \\ &\geq \|(I - J_\lambda^{F_1})Ax - (I - J_\lambda^{F_1})Ay\|^2 \\ &\geq \frac{1}{L}\left\|A^*(I - J_\lambda^{F_1})Ax - A^*(I - J_\lambda^{F_1})Ay\right\|. \end{aligned} \tag{3.6}$$

It follows that the operator $A^*(I - J_\lambda^{F_1})A$ is $(1/L)$-ism and therefore the operator $\gamma A^*(I - J_\lambda^{F_1})A$ is $[1/(\gamma L)]$-ism. Now, $\gamma \in (0, 2/L)$ implies that $1/(\gamma L) > 1/2$. Thus the operator $I - \gamma A^*(I - J_\lambda^{F_1})A$ is averaged.

Since both $J_\lambda^{B_1}$ and $I - \gamma A^*(I - J_\lambda^{F_1})A$ are averaged, so is their composition $J_\lambda^{B_1}\left(I - \gamma A^*(I - J_\lambda^{F_1})A\right)$ (see Remark 2.2(iv)). Therefore, by the Krasnosel'skiĭ-Mann-Opial theorem [30, 32, 37], the sequence $\left\{x^k\right\}_{k=0}^\infty$ generated by Algorithm 3.1 converges weakly to a fixed point x^* of the operator $J_\lambda^{B_1}\left(I - \gamma A^*(I - J_\lambda^{F_1})A\right)$. It remains to show that $x^* \in \Gamma$. Let $z \in \Gamma$, i.e., $0 \in B_1(z)$ and $0 \in F_1(Az)$. So, by (2.12), $z \in \mathrm{Fix}(J_\lambda^{B_1})$ and $Az \in \mathrm{Fix}(J_\lambda^{F_1})$.

In addition, since

$$\begin{aligned}(I-\gamma A^*(I-J_\lambda^{F_1})A)(z) &= z-\gamma A^*(I-J_\lambda^{F_1})Az \\ &= z-\gamma A^*Az+\gamma A^*J_\lambda^{F_1}(Az) \\ &= z-\gamma A^*Az+\gamma A^*Az = z, \end{aligned} \tag{3.7}$$

we get $z\in \mathrm{Fix}(I-\gamma A^*(I-J_\lambda^{F_1})A)$. Observe that any $z\in\Gamma$ is a fixed point of the averaged operator $J_\lambda^{B_1}\left(I-\gamma A^*(I-J_\lambda^{F_1})A\right)$. Indeed, by the above equalities we get

$$\left(J_\lambda^{B_1}\left(I-\gamma A^*(I-J_\lambda^{F_1})A\right)\right)(z) = J_\lambda^{B_1}(z-\gamma A^*(I-J_\lambda^{F_1})Az) = J_\lambda^{B_1}(z) = z. \tag{3.8}$$

Since $\Gamma\neq\emptyset$, we get from [11, Proposition 2.2] (see also [7, Lemma 2.1]), with the averaged operators $I-\gamma A^*(I-J_\lambda^{F_1})A$ and $J_\lambda^{B_1}$, that

$$\begin{aligned}\mathrm{Fix}(J_\lambda^{B_1})\cap \mathrm{Fix}(I-\gamma A^*(I-J_\lambda^{F_1})A) &= \mathrm{Fix}\left(J_\lambda^{B_1}\left(I-\gamma A^*(I-J_\lambda^{F_1})A\right)\right) \\ &= \mathrm{Fix}\left(\left(I-\gamma A^*(I-J_\lambda^{F_1})A\right)J_\lambda^{B_1}\right). \end{aligned} \tag{3.9}$$

Since x^* is a fixed point of $J_\lambda^{B_1}\left(I-\gamma A^*(I-J_\lambda^{F_1})A\right)$, we have $x^*\in \mathrm{Fix}(J_\lambda^{B_1})$ and $x^*\in \mathrm{Fix}(I-\gamma A^*(I-J_\lambda^{F_1})A)$. Now we need to show that $Ax^*\in \mathrm{Fix}(J_\lambda^{F_1})$. Indeed, from $x^*\in \mathrm{Fix}(I-\gamma A^*(I-J_\lambda^{F_1})A)$, we get

$$A^*(I-J_\lambda^{F_1})Ax^* = 0, \tag{3.10}$$

or

$$J_\lambda^{F_1}(Ax^*) = Ax^*+w, \tag{3.11}$$

where $A^*w=0$. Since $J_\lambda^{F_1}(Az)=Az$, we get

$$J_\lambda^{F_1}(Ax^*)-J_\lambda^{F_1}(Az) = Ax^*+w-Az. \tag{3.12}$$

So,

$$\begin{aligned}\|Ax^*-Az\|^2 \geq \|J_\lambda^{F_1}(Ax^*)-J_\lambda^{F_1}(Az)\|^2 &= \|Ax^*+w-Az\|^2 \\ &= \|Ax^*-Az\|^2+2\langle Ax^*-Az, w\rangle+\|w\|^2 \\ &= \|Ax^*-Az\|^2+2\langle x^*-z, A^*w\rangle+\|w\|^2 \\ &= \|Ax^*-Az\|^2+\|w\|^2. \end{aligned} \tag{3.13}$$

Hence $w=0$, which means that $J_\lambda^{F_1}(Ax^*)=Ax^*$. This completes the proof of Theorem 3.2. ■

Remark 3.3 *Observe that in Theorem 3.2 we assume that* $\gamma \in (0, 2/L)$, *while in [20, Theorem 6.3],* γ *is assumed to be in* $(0, 1/L)$, *which obviously was a more restrictive assumption.*

3.2 The SCNPP as an SCFPP

We now show, using similar arguments to those in [23], how our SCNPP can be transformed into a split common fixed point problem (SCFPP) (see (1.15)) with two operators T and U in a product space. Then we apply Algorithm 3.1 to the resulting problem in the product space.

Let Ψ be the solution set of the SCNPP:

$$\Psi := \{z \in \mathcal{H}_1 \mid 0 \in \cap_{i=1}^{p} B_i(z) \text{ and } 0 \in F_j(A_j z) \text{ for each } j = 1, \dots, r\}. \tag{3.14}$$

Consider the sets $C_i := \mathrm{Fix}(J_\lambda^{B_i})$ and $Q_j := \mathrm{Fix}(J_\lambda^{F_j})$, where $i = 1, 2, \dots, p$ and $j = 1, 2, \dots, r$, respectively. We introduce the spaces $\boldsymbol{W}_1 := \mathcal{H}_1$ and $\boldsymbol{W}_2 := \mathcal{H}_1^p \times \mathcal{H}_2^r$ and adopt the notational convention that the product spaces and objects in them are represented in boldface type. Define the following sets in the product spaces:

$$\boldsymbol{C} := \mathcal{H}_1 \tag{3.15}$$

and

$$\boldsymbol{Q} := \left(\prod_{i=1}^{p} \sqrt{\alpha_i} C_i \right) \times \left(\prod_{j=1}^{r} \sqrt{\beta_j} Q_j \right), \tag{3.16}$$

where $\{\alpha_i\}_{i=1}^p$ and $\{\beta_j\}_{j=1}^r$ are positive real numbers. Define the operator $\boldsymbol{A} : \boldsymbol{W}_1 \to \boldsymbol{W}_2$ by

$$\boldsymbol{A} := \left(\sqrt{\alpha_1} I, \dots, \sqrt{\alpha_p} I, \sqrt{\beta_1} A_1^*, \dots, \sqrt{\beta_r} A_r^* \right)^*, \tag{3.17}$$

where A^* denotes the adjoint of the operator A. Denote $U_i := J_\lambda^{B_i}$ and $T_j := J_\lambda^{F_j}$ for $i = 1, 2, \dots, p$ and $j = 1, 2, \dots, r$, respectively, and finally, define the operator $\boldsymbol{T}:\boldsymbol{W}_2\to\boldsymbol{W}_2$ by

$$\begin{aligned}\boldsymbol{T}(\boldsymbol{y}) &= \boldsymbol{T}\begin{pmatrix} y^1 \\ y^2 \\ \vdots \\ y^{p+r}\end{pmatrix} \\ &= \left(\left(U_1\left(y^1\right)\right)^*,\ldots,\left(U_p\left(y^p\right)\right)^*,\left(T_1\left(y^{p+1}\right)\right)^*,\ldots,\left(T_r(y^{p+r})\right)^*\right)^*,\end{aligned} \tag{3.18}$$

where $y^1,y^2,...,y^p\in\mathcal{H}_1$ and $y^{p+1},y^{p+2},...,y^{p+r}\in\mathcal{H}_2$. It is easy to verify that the following equivalence holds:

$$x\in\Psi \text{ if and only if } \boldsymbol{A}x\in\boldsymbol{Q}. \tag{3.19}$$

When Algorithm 3.1 is applied to the above problem in the product space and then translated back to the original spaces, it takes the following form.

Algorithm 3.4

Initialization: *Select an arbitrary starting point* $x^0\in\mathcal{H}_1$.

Iterative step: *Given the current iterate* x^k, *compute*

$$x^{k+1}=x^k+\gamma\left(\sum_{i=1}^{p}\alpha_i\left(J_\lambda^{B_i}(x^k)-x^k\right)+\sum_{j=1}^{r}\beta_jA_j^*(J_\lambda^{F_j}-I)A_jx^k\right), \tag{3.20}$$

where $\gamma\in(0,2/L)$, *with* $L=\sum_{i=1}^p\alpha_i+\sum_{j=1}^r\beta_j\|A_j\|^2$.

The following convergence result follows from Theorem 3.2.

Theorem 3.5 *Let* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *be two real Hilbert spaces. Given set-valued maximal monotone mappings* $B_i:\mathcal{H}_1\to 2^{\mathcal{H}_1}$, $1\le i\le p$, *and* $F_j:\mathcal{H}_2\to 2^{\mathcal{H}_2}$, $1\le j\le r$, *respectively, and bounded linear operators* $A_j:\mathcal{H}_1\to\mathcal{H}_2$, $1\le j\le r$, *then any sequence* $\left\{x^k\right\}_{k=0}^\infty$ *generated by Algorithm 3.4 converges weakly to a solution* $x^*\in\Psi$, *provided that* $\Psi\neq\emptyset$ *and* $\gamma\in(0,2/L)$, *where* $L=\sum_{i=1}^p\alpha_i+\sum_{j=1}^r\beta_j\|A_j\|^2$.

Proof. Apply Theorem 3.2 to the SCNPP with the following two set-valued maximal monotone mappings in the product space: $\boldsymbol{M}_1:\mathcal{H}_1\to 2^{\mathcal{H}_1}$ is taken to be the zero operator, and

$$\boldsymbol{M}_2:\mathcal{H}_1^p\times\mathcal{H}_2^r\to\underbrace{2^{\mathcal{H}_1}\times\cdots\times 2^{\mathcal{H}_1}}_{p\text{ times}}\times\underbrace{2^{\mathcal{H}_2}\times\cdots\times 2^{\mathcal{H}_2}}_{r\text{ times}} \tag{3.21}$$

is defined by

$$\boldsymbol{M}_2 := (B_1, \ldots, B_p, F_1, \ldots, F_r)\,. \tag{3.22}$$

The operator $\boldsymbol{A} : \boldsymbol{W}_1 \to \boldsymbol{W}_2$ is defined by (3.17). ■

We also may introduce relaxation parameters into the above algorithm as has been done in the relaxed version of [35, equation 2.10]. In [23, Algorithm 3.2] the authors assumed that the operators T and U were cutters in Euclidean spaces, while in [35] the operators T and U are assumed to be demicontractive in Hilbert spaces. The convergence theorems in [23, Algorithm 3.2] and in [35] require that the operators $I-T$ and $I-U$ be demiclosed at zero. By Remark 2.4, the resolvent of a maximal monotone mapping is firmly nonexpansive, hence nonexpansive, and then by the Demiclosedness Principle (Lemma 2.8), their complements are indeed demiclosed at zero.

4 Strong convergence

In this section we first present a strong convergence theorem for Algorithm 3.1 under an additional assumption. This result relies on the work of Browder and Petryshyn [6, Theorem 5], and on that of Baillon, Bruck and Reich [1, Theorem 1.1] (see also [34, Lemma 7]). The second algorithm is a modification of Algorithm 3.1 that results in a Halpern-type algorithm.

4.1 Strong convergence of Algorithm 3.1

The next two theorems are needed for our proof of Theorem 4.3. We present their full proofs for the reader's convenience.

Theorem 4.1 *[6, Theorem 5], [29] Let $\mathcal{B}$ be a uniformly convex Banach space. If the operator $S : \mathcal{B} \to \mathcal{B}$ is nonexpansive with a nonempty fixed point set* $\operatorname{Fix}(S) \neq \emptyset$*, then for any given constant $c \in (0,1)$, the operator $S_c := cI + (1-c)S$ is asymptotically regular and has the same fixed points as S.*

Proof. It is obvious that $\operatorname{Fix}(S) = \operatorname{Fix}(S_c)$ and that S_c is also a nonexpansive self-mapping of $\mathcal{B}$. Let $u \in \operatorname{Fix}(S_c)$ and for a given $x \in \mathcal{B}$, let $x^k = S_c^k(x)$. Since S_c is nonexpansive and $u \in \operatorname{Fix}(S_c)$, it follows that

$$\left\|x^{k+1} - u\right\| \leq \left\|x^k - u\right\| \text{ for all } k \geq 0. \tag{4.1}$$

Therefore there exists $\lim_{k\to\infty} \left\| x^k - u \right\| = \ell \geq 0$. Assume that $\ell > 0$. Then

$$\begin{aligned} x^{k+1} - u = S_c^{k+1}(x) - u &= S_c(x^k) - u \\ &= \left(cI + (1-c)S\right)(x^k) - u \\ &= c(x^k - u) + (1-c)\left(S(x^k) - u\right). \end{aligned} \tag{4.2}$$

Since

$$\lim_{k\to\infty} \left\| x^k - u \right\| = \lim_{k\to\infty} \left\| x^{k+1} - u \right\| = \ell \tag{4.3}$$

and

$$\left\| x^{k+1} - u \right\| = \left\| S(x^k) - u \right\| \leq \left\| x^k - u \right\|, \tag{4.4}$$

the uniform convexity of $\mathcal{B}$ implies that

$$\lim_{k\to\infty} \left\| \left(x^k - u\right) - \left(S(x^k) - u\right) \right\| = 0, \tag{4.5}$$

i.e., $x^k - S(x^k) \to 0$. Hence $x^{k+1} - x^k \to 0$, which means that S_c is asymptotically regular, as claimed. ■

Theorem 4.2 *[1, Theorem 1.1] Let $\mathcal{B}$ be a uniformly convex Banach space. If the operator $S : \mathcal{B} \to \mathcal{B}$ is nonexpansive, odd and asymptotically regular at $x \in \mathcal{B}$, then the sequence $\left\{S^k(x)\right\}_{k=0}^{\infty}$ converges strongly to a fixed point of S.*

Proof. Since S is odd, $S(0) = -S(0)$ and $S(0) = 0$. Since S is nonexpansive, we have by the triangle inequality,

$$\begin{aligned} \left\| S^k(x) \right\| &= \left\| S^k(x) \right\| - \left\| S^k(0) \right\| \leq \left\| S^k(x) - S^k(0) \right\| \\ &\leq \left\| S^{k-1}(x) - S^{k-1}(0) \right\| = \left\| S^{k-1}(x) \right\| \leq \cdots \leq \left\| x - 0 \right\| = \left\| x \right\|, \end{aligned} \tag{4.6}$$

which means that the sequence $\left\{\left\| S^k(x) \right\|\right\}_{k=0}^{\infty}$ is decreasing and bounded. Therefore the limit $\lim_{k\to\infty} \left\| S^k(x) \right\|$ exists and for a fixed i the sequence $\left\{\left\| S^{k+i}(x) + S^k(x) \right\|\right\}_{k=0}^{\infty}$ is decreasing. Let $\lim_{k\to\infty} \left\| S^k(x) \right\| = d$. Then by the triangle inequality,

$$\begin{aligned} 2d &\leq \left\| 2S^k(x) \right\| = \left\| S^k(x) - S^{k+i}(x) + S^{k+i}(x) + S^k(x) \right\| \\ &\leq \left\| S^k(x) - S^{k+i}(x) \right\| + \left\| S^k(x) + S^{k+i}(x) \right\|. \end{aligned} \tag{4.7}$$

Since S is asymptotically regular at x, $\lim_{k\to\infty}\left\|S^k(x)-S^{k+i}(x)\right\|=0$. Thus $\lim_{k\to\infty}\left\|S^k(x)+S^{k+i}(x)\right\|\geq 2d$. But the sequence $\left\{\left\|S^{k+i}(x)+S^k(x)\right\|\right\}_{k=0}^{\infty}$ is decreasing, so that $\left\|S^k(x)+S^{k+i}(x)\right\|\geq 2d$ for all k and i. We now have $\lim_{k\to\infty}\left\|S^k(x)\right\|=d$ and $\lim_{m,n\to\infty}\|S^n(x)+S^m(x)\|=2d$. The uniform convexity of $\mathcal{B}$ implies that $\lim_{m,n\to\infty}\|S^n(x)-S^m(x)\|=0$, whence $\left\{S^k(x)\right\}_{k=0}^{\infty}$ converges strongly to a fixed point of S. ■

In Theorem 4.3 we need the resolvent J_λ^B to be odd, which means that

$$\left((I+\lambda B)^{-1}\right)(-x)=-\left((I+\lambda B)^{-1}\right)(x) \text{ for all } x\in\mathcal{H}. \tag{4.8}$$

Denote

$$\left((I+\lambda B)^{-1}\right)(-x)=y \text{ and } \left((I+\lambda B)^{-1}\right)(x)=z. \tag{4.9}$$

Then

$$-x\in y+\lambda B(y) \text{ and } x\in z+\lambda B(z). \tag{4.10}$$

If B is odd, then

$$x\in -y+\lambda B(-y). \tag{4.11}$$

Hence $-y=z$, which is (4.8). Therefore we assume in the following theorem that both B_1 and F_1 are odd.

Now we are ready to present the strong convergence theorem for Algorithm 3.1. Its proof relies on Theorem 4.2.

Theorem 4.3 *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces. Let two set-valued, odd and maximal monotone mappings $B_1:\mathcal{H}_1\to 2^{\mathcal{H}_1}$ and $F_1:\mathcal{H}_2\to 2^{\mathcal{H}_2}$, and a bounded linear operator $A:\mathcal{H}_1\to\mathcal{H}_2$ be given. If $\gamma\in(0,2/L)$, then any sequence $\left\{x^k\right\}_{k=0}^{\infty}$ generated by Algorithm 3.1 converges strongly to $x^*\in\Gamma$.*

Proof. The operator $J_\lambda^{B_1}\left(I-\gamma A^*(I-J_\lambda^{F_1})A\right)$ is averaged by the proof of Theorem 3.2 (see also [36, Theorem 3.1]). Therefore by [6, Theorem 5] and [29] (see Theorem 4.1), the operator $J_\lambda^{B_1}\left(I-\gamma A^*(I-J_\lambda^{F_1})A\right)$ is also asymptotically regular. Since B_1 and F_1 are odd, so are their resolvents $J_\lambda^{B_1}$ and $J_\lambda^{F_1}$, and therefore $J_\lambda^{B_1}\left(I-\gamma A^*(I-J_\lambda^{F_1})A\right)$ is odd. Finally, the strong convergence of Algorithm 3.1 is now seen to follow from [1, Theorem 1.1] (see Theorem 4.2). ■

For the general SCNPP we could again employ a product space formulation as in Subsection 3.2 and under the additional oddness assumption also get strong convergence.

4.2 A Halpern-type algorithm

Next we consider a modification of Algorithm 3.1 inspired by the Halpern iterative method and prove its strong convergence. Let $T : C \to C$ be a nonexpansive operator, where C is a nonempty, closed and convex subset of a Banach space $\mathcal{B}$. A classical way to study nonexpansive mappings is to use strict contractions to approximate T, i.e., for $t \in (0,1)$, we define the strict contraction $T_t : C \to C$ by

$$T_t(x) = tu + (1-t)T(x) \text{ for } x \in C, \tag{4.12}$$

where $u \in C$ is fixed. Banach's Contraction Mapping Principle guarantees that each T_t has a unique fixed point $x_t \in C$. In case $\mathrm{Fix}(T) \neq \emptyset$, Browder [4] proved that if $\mathcal{B}$ is a Hilbert space, then x_t converges strongly as $t \to 0^+$ to the fixed point of T nearest to u. Motivated by Browder's results, Halpern [28] proposed an explicit iterative scheme and proved its strong convergence to a point $z \in \mathrm{Fix}(T)$. In the last decades many authors modified Halpern's iterative scheme and found necessary and sufficient conditions concerning the control sequence that guarantee the strong convergence of Halpern-type schemes (see, e.g., [31, 38, 42, 43, 41]). Our algorithm for the SCNPP with two set-valued maximal monotone mappings is presented next.

Algorithm 4.4

Initialization: *Select some* $\lambda > 0$ *and an arbitrary starting point* $x^0 \in \mathcal{H}_1$.

Iterative step: *Given the current iterate* x^k, *compute*

$$x^{k+1} = \alpha_k x^0 + (1-\alpha_k) J_\lambda^{B_1}\left(I - \gamma A^*(I - J_\lambda^{F_1})A\right) x^k, \tag{4.13}$$

where $\gamma \in (0, 2/L)$ *with* $L = \|A^*A\|$ *and the sequence* $\{\alpha_k\}_{k=0}^\infty \subset [0,1]$ *satisfies* $\lim_{k\to\infty} \alpha_k = 0$ *and* $\sum_{k=0}^{\infty} \alpha_k = \infty$.

Here is our strong convergence theorem for this algorithm.

Theorem 4.5 *Let* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *be two real Hilbert spaces. Let there be given two set-valued maximal monotone mappings* $B_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ *and* $F_1 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$, *and a bounded linear operator* $A : \mathcal{H}_1 \to \mathcal{H}_2$. *If* $\Gamma \neq \emptyset$, $\gamma \in (0, 2/L)$ *and* $\{\alpha_k\}_{k=0}^\infty \subset [0,1]$ *satisfies* $\lim_{k\to\infty} \alpha_k = 0$ *and* $\sum_{k=0}^{\infty} \alpha_k = \infty$, *then any sequence* $\{x^k\}_{k=0}^\infty$ *generated by Algorithm 4.4 converges strongly to* $x^* \in \Gamma$.

Proof. As proved in Theorem 3.2, the operator $J_{\lambda}^{B_1}\left(I-\gamma A^*(I-J_{\lambda}^{F_1})A\right)$ is averaged. So, according to the Halpern-Suzuki theorem [28, 41] (see Theorem 2.11), any sequence $\left\{x^k\right\}_{k=0}^{\infty}$ generated by Algorithm 4.4 converges strongly to a point $x^* \in \operatorname{Fix}\left(J_{\lambda}^{B_1}\left(I-\gamma A^*(I-J_{\lambda}^{F_1})A\right)\right)$ as long as this set is nonempty. Following the proof of Theorem 3.2, we conclude that $x^* \in \Gamma$, as claimed. ■

Remark 4.6 1. *Since the SCNPP generalizes the SVIP, it includes all the applications of the SVIP (see [20, Section 7]). In particular, it includes the Split Feasibility Problem (SFP) and the Convex Feasibility Problem (CFP). Since the Common Solutions to Variational Inequalities Problem (CSVIP) [21] with operators is a special case of the SVIP, the SCNPP includes its applications as well. In addition, since in [36] all the applications of the SMVI presented there are for $f = g = 0$ in (1.11) and (1.12), it follows that these applications are also covered by our SCNPP. They include the Split Minimization Problem (SMP), which had already been presented in [20, Subsection 7.3] with continuously differentiable convex functions, for which we can now drop this assumption, the Split Saddle-Point Problem (SSPP), the Split Minimax Problem (SMMP) and the Split Equilibrium Problem (SEP). Observe that if $\mathcal{H}_1 = \mathcal{H}_2$ and $A_j = I$ for for all $j = 1, 2, \ldots, r$, then we can deal with all of the above applications with "Split" replaced by "Common". We can even study mixtures of "split" and "common" applications.*

2. *According to Remark 2.5, the operator $J_{\lambda}^{B}(I-\lambda f)$ is averaged, where $B : \mathcal{H} \to 2^{\mathcal{H}}$ is maximal monotone, the operator $f : \mathcal{H} \to \mathcal{H}$ is α-ism and $\lambda \in (0, 2\alpha)$. Since our convergence theorems rely on the averagedness of the operators involved, we could modify our algorithms and obtain strong convergence for Moudafi's SMVI ((1.11) and (1.12)). In addition, our algorithms allow us to solve Moudafi's SMVI ((1.11) and (1.12)) with monotone and hemicontinuous operators f and g (which is a larger class than the class of inverse strongly monotone operators).*

3. *Assuming that the set-valued mappings $B_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $B_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ are maximal monotone, and $f : \mathcal{H}_1 \to \mathcal{H}_1$ and $g : \mathcal{H}_2 \to \mathcal{H}_2$ are ism operators in the SMVI (1.11) and (1.12). Moudafi presented an algorithm that converged weakly to a solution of the SMVI. By [39, Theorem 3], the sum of a maximal monotone mapping and an ism*

operator is maximal monotone. Therefore the SMVI reduces to our set-valued two-mapping SCNPP. In addition, we can phrase the set-valued SVIP for maximal monotone mappings in the following way. Given two maximal monotone mappings $B_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $B_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$, a bounded linear operator $A : \mathcal{H}_1 \to \mathcal{H}_2$, and nonempty, closed and convex subsets $C \subset \mathcal{H}_1$ and $Q \subset \mathcal{H}_2$, the set-valued SVIP is formulated as follows:

$$\begin{gathered} \text{find a point } x^* \in C \text{ and a point } u^* \in B_1(x^*) \\ \text{such that } \langle u^*, x - x^* \rangle \geq 0 \text{ for all } x \in C, \\ \text{and such that} \\ \text{the points } y^* = Ax^* \in Q \text{ and } v^* \in B_2(y^*) \\ \text{solve } \langle v^*, y - y^* \rangle \geq 0 \text{ for all } y \in Q. \end{gathered} \tag{4.14}$$

It is clear that if the zeros of the set-valued mappings B_1 and B_2 are in C and Q, respectively, then they are solutions of the set-valued SVIP, but in general not all solutions are zeros.

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