LOWER BOUNDS FOR CHVÁTAL-GOMORY STYLE OPERATORS

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ABSTRACT. Valid inequalities or cutting planes for (mixed-) integer programming problems are an essential theoretical tool for studying combinatorial properties of polyhedra. They are also of utmost importance for solving optimization problems in practice; in fact any modern solver relies on several families of cutting planes. The Chvátal-Gomory procedure, one such approach, has a peculiarity that differentiates it essentially from all other known cutting-plane operators. There exists a family of polytopes in the 0/1 cube for which more than n rounds of Chvátal-Gomory cuts are needed to derive the integral hull where n is the dimension of the polytope. All other known operators achieve this in at most n rounds. We will prove that this behavior is not an inherent weakness of the Chvátal-Gomory operator but rather a consequence of deriving new inequalities solely from a *single inequality* (not to confuse with single row cuts). We will first introduce a generalization of the Chvátal-Gomory closure which is significantly stronger than the classical Chvátal-Gomory procedure. We will then provide a new bounding technique for rank lower bounds for operators that essentially derive new inequalities from examining a single inequality only. A construction of a family of polytopes whose rank exceeds n follows. Contrasting these results we will show that as soon as the operator can use at least two inequalities for its derivations the rank in 0/1 cube is bounded by n from above and we will construct a new cutting-plane operator, the transient closure that combines a strengthening of lift-and-project cuts and generalized Chvátal-Gomory cuts. We obtain several rank lower bounds for specific families of polytopes in the process.

1. INTRODUCTION

Deriving valid inequalities from polyhedral descriptions is an essential tool to approximate the integral hull of a polyhedron. In particular, the success of today's mixed-integer programming solver is largely due to the availability of strong general-purpose valid inequalities (also refereed to as *cutting planes*). The Chvátal-Gomory procedure (see Gomory [1958, 1960, 1963], Chvátal [1973]) is one such approach to derive new valid inequalities. It essentially takes positive

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combinations of valid inequalities and rounds (-down) left- and right-hand sides. Any feasible integral solution will satisfy this new inequality. However fractional solutions might be cut-off. The aim is to obtain strong valid inequalities in as few applications of this operation as possible. The number of applications of such a procedure, needed to derive a certain inequality valid for the integral hull is usually referred to as *rank* or *depth* of an inequality (with respect to some cutting-plane operator) and the *rank of a polyhedron (denoted by:* $\mathrm{rk}(\cdot)$) is the maximum rank over all facets of the integral hull.

While the Chvátal-Gomory cutting-plane operator is one of the most studied operators (see e.g., Schrijver [1986]) many theoretical questions remain unanswered. One of the most striking differences of the Chvátal-Gomory operator in contrast to any other known operator is that there exists a family of polytopes $P_n \subseteq [0,1]^n$ such that $\mathrm{rk}(P_n) > n$ (see Eisenbrand and Schulz [2003] or Pokutta and Stauffer [2011]); all other operators satisfy $\mathrm{rk}(P) \leq n$ whenever $P \subseteq [0,1]^n$. The currently best known upper bound for the Chvátal-Gomory operator is $O(n^2 \log n)$ in the ambient dimension n of the polytope which was established in Eisenbrand and Schulz [2003]. However a large gap between the two bounds remains.

Recently, several people proposed strengthenings of cutting-plane operators when applied to polytopes in the $[0,1]^n$-cube by restricting them to 0/1 feasible solutions (see e.g., Dunkel and Schulz [2010], Fischetti and Lodi [2010], Lodi et al. [2011]). Partially with the aim to resolve some of the aforementioned rank questions, partially with the aim to obtain stronger operators tailored to 0/1 programming, these strengthenings try to exploit the information that the solution are 0/1 vectors. For example in the case of the Chvátal-Gomory operator, a close inspection of the Chvátal-Gomory procedure reveals that it pushes valid inequalities towards the integral hull until they touch *any* integral point (not necessarily contained in the integral hull). In this sense, the Chvátal-Gomory operator rounds inequalities to the next point on the $\mathbb{Z}^n$-grid. However often we deal with polytopes stemming from combinatorial optimization problems so that the polytopes are contained in the $[0,1]^n$-cube. Therefore we would like to use this additional information to further push inequalities inside, until a 0/1 point is touched. In the process of deciding whether a cutting plane generated in this way is valid, we have to test whether certain hyperplanes contain 0/1 points which is a hard problem. By allowing the operator to solve harder problems, we hope for strengthened procedures that derive stronger inequalities in fewer rounds. (This is in particular of practical interest since, while hard in general, many practical separation problem might be easy to solve.) We will prove that this is not the case for the canonical 0/1 strengthening of the Chvátal-Gomory operator. For $P \subseteq [0,1]^n$ and $\{0,1\}^n \subseteq S \subseteq \mathbb{Z}^n$ we will consider the *generalized Chvátal-Gomory operator* defined as

$$GCG_S(P) := \bigcap_{\substack{(c,\delta)\in\mathbb{Q}^{n+1},\\ P\subseteq\{cx\leq\delta\}}} \{x \in [0,1]^n \mid cx \leq \max\{cx \mid cx \leq \delta, x \in S\}\}.$$

We will show that already relatively simple polytopes need a large number of applications to derive the integral hull. In fact, we prove that the generalized Chvátal-Gomory cutting-plane operators suffers from the same weaknesses (in terms of rank) as the classical Chvátal-Gomory operator.

Contribution. We will introduce a generalization of the Chvátal-Gomory closure (Definition 2.2) and establish a new lower bounding technique that can be used for operators that derive new valid inequalities by solely examining *single inequalities* valid for the original polytope (Proposition 3.1). The proposed technique, in its core, is based on invariant scalar products. However for the purpose of a more streamlined exposition and being self-contained it has been brought into a more explicit form, removing all links to representation theory and the essential idea has been captured in a remark after Proposition 3.1. It is inspired by the lower bounding techniques presented in the seminal work of Chvátal et al. [1989]. In contrast to Chvátal et al. [1989] where explicit knowledge about the cutting-plane procedure (in this case the Chvátal-Gomory procedure) has been used, our technique only relies on some local knowledge assumption that is satisfied by the Chvátal-Gomory-operator and the considered strengthenings. While also being based on some type of local knowledge assumption, the framework for establishing lower bounds on the rank in Pokutta and Schulz [2010a] is only partially applicable. In particular, it cannot be used for almost integral polytopes, our main building block.

In a first step we will use this technique to construct a family of almost integral polytopes with non-constant rank (Theorem 3.2). The actual family of polytopes used for the lower bounds is the one that has been presented in Pokutta and Stauffer [2011]. It is quite surprising that this weakness of the Chvátal-Gomory operator carries over to the generalized Chvátal-Gomory operators considered here (recall that any other operator derives the integral hull of almost integral polytopes within one round via an elementary split.) We will use this family in turn to construct of a very basic family of polytopes $P_n \subseteq [0,1]^n$ that exhibits a rank of at least $n+\log n-2 > n$ (Theorem 5.1). It turns out that the reason why the classical Chvátal-Gomory procedure exhibits super-n rank is not due to its actual design but it is an *inherent lack of information* about the integral hull. In fact we will show that while typical operators are not limited by information but rather computational power, for the generalized Chvátal-Gomory operators the opposite is true: while having unlimited computational power they are limited by the information they can use. We will also make the difference to single-row cuts explicit showing that those in fact use more information than just a single inequality (Example 3.4). For example the Gomory-Mixed-Integer cuts (GMI cuts) (cf. Gomory [1958, 1960, 1963]) are single-row cuts, however they do rely on a disjunction and information of a larger number of inequalities as we will see.

To contrast the existence of this family of polytopes that exhibits super-n rank with respect to the generalized Chvátal-Gomory operators, we will construct a new cutting-plane operator, the transient closure, that can deduce new inequalities from at most two inequalities. We show that this operator refines the classical lift-and-project operator N_0 (see Lovász and Schrijver [1991]) as well as the generalized Chvátal-Gomory operators (Corollary 6.3) and as a consequence it can derive the integral hull of any polytope $P \subseteq [0,1]^n$ in at most n rounds.

In the process we will also prove that the classical lift-and-project operator N_0 is incompatible with the considered strengthening of the Chvátal-Gomory-operator and briefly sketch how the lower bound result for the subtour elimination polytope given in Chvátal et al. [1989] for the Chvátal-Gomory procedure and in Cook and Dash [2001] for the lift-and-project related procedures can be carried over to the generalized Chvátal-Gomory operators.

Outline. We begin with some preliminaries and definitions in Section 2. In particular we define the generalizations of the Chvátal-Gomory operator and we establish some basic properties. In Section 3 we show that there exists a family of almost integral polytopes $P_n \subseteq [0,1]^n$ with rank at least $\log n - 1$. This family will be of crucial importance for constructing a family of polytopes with super-n rank. In Section 4 we consider the classical example A_n which has been used so far for cutting-plane operators *with polynomial verification of validity* to provide a rank lower bound of n. We will show that this bound even holds for the generalized Chvátal-Gomory operators. In Section 5 we establish the super-n lower bound and in Section 6 we define the transient closure operator which can deduce new inequalities from two valid inequalities. We conclude with some final remarks in Section 7.

2. Generalized Chvátal-Gomory Operators

We introduce necessary notions and notation as well as we establish basic properties of the generalized Chvátal-Gomory operators. For convenience, let $[n] := \{1,\ldots,n\}$ for $n \in \mathbb{N}$. We will use $\{cx \le \delta\}$ as a shorthand for the half-space $\{x \in [0,1]^n \mid cx \le \delta\}$. All polytopes considered here are contained in $[0,1]^n$. For a polytope $P \subseteq [0,1]^n$ let $P_I := \operatorname{conv}(P \cap \mathbb{Z}^n)$ denote the *integral hull of* P and

$$P' := \bigcap_{\substack{c\in\mathbb{Z}^n,\delta\in\mathbb{Q},\\ P\subseteq\{cx\le\delta\}}} \{cx \le \lfloor \delta \rfloor\}$$

denote the Chvátal-Gomory-closure of P. In the following let $\log(\cdot)$ denote the logarithm to the basis of 2.

We will now introduce a generalized rounding operation.

Definition 2.1. Let $S \subseteq \mathbb{R}^n$ and let $cx \le \delta$ with $(c,\delta) \in \mathbb{Q}^{n+1}$. Then the *$S$-optimizing operator* is defined as

$$[\![c,\delta]\!]_S := \max\{cx \mid cx \le \delta, x \in S\}.$$

We set $[\![c,\delta]\!]_S = \infty$ if the maximum does not exist.

Observe that the actual optimization problem solved by the S-optimizing operator might be NP-hard. A *cutting-plane operator* is a map from polytopes to closed convex sets. It is defined by assigning every polytope $P \subseteq [0,1]^n$ its closure. We say that a cutting-plane operator M *deduces from single inequalities* if whenever $cx \le \delta$ is valid for $M(P)$ then there exists a inequality $\pi x \le \pi_0$ such that $P \subseteq [0,1]^n \cap \{\pi x \le \pi_0\}$ and $cx \le \delta$ is valid for $M([0,1]^n \cap \{\pi x \le \pi_0\})$. Put informally, the cutting-plane operator M uses only information contained in the single inequality $cx \le \delta$ (and the bounding cube). It is exactly this restricted use of information that characterizes the Chvátal-Gomory-operator. Using $[\![\cdot,\cdot]\!]_S$ we can define a generalization of the Chvátal-Gomory operator. In the following we will confine ourselves to the more interesting case $S \subseteq \mathbb{Z}^n$.

Definition 2.2. Let $P \subseteq [0,1]^n$ be a polytope and $S \subseteq \mathbb{Z}^n$. Then the *generalized Chvátal-Gomory closure* with respect to S is defined as

$$GCG_S(P) := \bigcap_{\substack{(c,\delta)\in\mathbb{Q}^{n+1},\\ P\subseteq\{cx\leq\delta\}}} \{cx \leq [\![c,\delta]\!]_S\}.$$

The GCG_S *operator* is the map $P \mapsto GCG_S(P)$. The definition readily generalizes to P being an arbitrary, closed convex set $C \subseteq [0,1]^n$.

It will turn out later that $GCG_{\mathbb{Z}^n}(P) = P'$ (see Proposition 2.7) which trivially implies the polyhedrality of $GCG_{\mathbb{Z}^n}(P)$. The polyhedrality of GCG_S with $S = \{0,1\}^n$ has been established in Dunkel and Schulz [2010] (in Dunkel and Schulz [2010] the closure is called M-closure). It is open whether GCG_S is polyhedral for general S in the case of P being a closed convex set; it is known to be true for $S = \mathbb{Z}^n$ as shown in Dadush et al. [2010]. We will drop S if it is clear from the context. The following properties are immediate.

Remark 2.3. Let $P \subseteq [0,1]^n$ be a polytope. Then the following hold

(1) $P_I \subseteq GCG_S(P) \subseteq P$ for all polytopes $P \subseteq [0,1]^n$ and $P_I \cap \{0,1\}^n \subseteq S$;
(2) $GCG_S(P) \subseteq GCG_{\tilde{S}}(P)$ whenever $S \subseteq \tilde{S}$;
(3) $GCG_S(P) \subseteq GCG_S(Q)$ for any $S \subseteq \mathbb{Z}^n$ and $P \subseteq Q \subseteq [0,1]^n$ polytopes.

Observe that $[\![.,.]\!]_S$ is invariant under scaling:

Observation 2.4. *Let $S \subseteq \mathbb{Z}^n$ and let $cx \leq \delta$ with $(c,\delta) \in \mathbb{R}^{n+1}$. For $\alpha \in \mathbb{R}_+$ it holds*

$$\alpha [\![c,\delta]\!]_S = [\![\alpha c, \alpha\delta]\!]_S.$$

Proof.

$$\begin{aligned}\alpha [\![c,\delta]\!]_S &= \alpha \max\{cx \mid cx \leq \delta, x \in S\}\\ &= \max\{\alpha cx \mid cx \leq \delta, x \in S\}\\ &= \max\{\alpha cx \mid \alpha cx \leq \alpha\delta, x \in S\} = [\![\alpha c, \alpha\delta]\!]_S.\end{aligned}$$

□

We obtain the classical Chvátal-Gomory operator in a natural way; actually we derive a slightly more general result. For this we will use the following lemma (see Dadush et al. [2010] for a similar result) and observation:

Lemma 2.5. *([Dey and Pokutta, 2011, Lemma 2]) Let $P,Q \subseteq \mathbb{R}^n$ be compact convex sets and let $\sigma_P(c) \leq \sigma_Q(c)$ for all $c \in \mathbb{Z}^n$ with $\sigma_M(c) := \sup\{cx \mid x \in M\}$. Then $P \subseteq Q$.*

In general it is not true that $\lambda\lfloor\delta\rfloor \geq \lfloor\lambda\delta\rfloor$ when $\delta \in \mathbb{R}$ and $\lambda \in [0,1]$. For example consider $\lambda = 2/3$ and $\delta = 3/2$. Then $2/3\lfloor 3/2\rfloor = 2/3$ whereas $\lfloor 2/3 \cdot 3/2\rfloor = 1$. However it holds $\frac{1}{k}\lfloor\delta\rfloor \geq \left\lfloor \frac{1}{k}\delta \right\rfloor$, i.e., the case where λ is chosen as $\lambda = 1/k$ with $k \in \mathbb{N}$.

Observation 2.6. *Let $\delta \in \mathbb{R}$ and $k \in \mathbb{N}$. Then*

$$\frac{1}{k}\lfloor\delta\rfloor \geq \left\lfloor \frac{1}{k}\delta \right\rfloor.$$

We will now use the above to recover the classical Chvátal-Gomory operator from the generalized Chvátal-Gomory operators. While this was already indicated for integral (and in consequence rational normals) in Schrijver [1986] one does not need to restrict the coefficients. In fact we prove a slightly stronger result, i.e., that $P' = GCG_{\mathbb{Z}^n}(P)$ even when the considered normals for GCG could take coefficients in $\mathbb{R}$. The reason for this is that the integrality condition is already incorporated implicitly into $[\![\cdot,\cdot]\!]_S$.

Proposition 2.7. *Let $P \subseteq [0,1]^n$ be a polytope then*

$$P' = GCG_{\mathbb{Z}^n}(P) = \bigcap_{\substack{(c,\delta)\in\mathbb{R}^{n+1},\\ P\subseteq\{cx\le\delta\}}} \left\{cx \le [\![c,\delta]\!]_{\mathbb{Z}^n}\right\}.$$

Proof. Observe that both P' and

$$\bigcap_{\substack{(c,\delta)\in\mathbb{R}^{n+1},\\ P\subseteq\{cx\le\delta\}}} \left\{cx \le [\![c,\delta]\!]_{\mathbb{Z}^n}\right\}$$

are compact convex sets. Therefore we can confine ourselves to normals $c \in \mathbb{Z}^n$ by Lemma 2.5

We first show that $P' \subseteq \bigcap_{\substack{(c,\delta)\in\mathbb{R}^{n+1},\\ P\subseteq\{cx\le\delta\}}} \left\{cx \le [\![c,\delta]\!]_{\mathbb{Z}^n}\right\}$. Let $cx \le \delta$ be valid for P with $c \in \mathbb{Z}^n$. Then $cx \le \lfloor\delta\rfloor$ is a CG-cut. We can further assume that $cx \le \lfloor\delta\rfloor$ is an undominated CG-cut and in particular that $\gcd(c) = 1$; otherwise there exists $k \in \mathbb{N}$ so that $\frac{1}{k}c$ is integral, $\gcd(\frac{1}{k}c) = 1$, and we have $\frac{1}{k}cx \le \left\lfloor\frac{1}{k}\delta\right\rfloor \le \frac{1}{k}\lfloor\delta\rfloor$ by Observation 2.6. As $\gcd(c) = 1$ there exists $z \in \mathbb{Z}^n$ such that $cz = \lfloor\delta\rfloor$ and so $[\![c,\delta]\!] \ge \lfloor\delta\rfloor$. On the other hand we trivially have $[\![c,\delta]\!] \le \lfloor\delta\rfloor$ so that $[\![c,\delta]\!] = \lfloor\delta\rfloor$ follows.

For the converse let $c \in \mathbb{Z}^n$ and consider $cx \le [\![c,\delta]\!]$ for some $\delta \in \mathbb{R}$ so that $cx \le \delta$ is valid for P. By Observation 2.4 we can rescale c and assume without loss of generality that $\gcd(c) = 1$. Set $z := \operatorname{argmax}\{cx \mid x \in \mathbb{Z}^n, cx \le \delta\}$ and observe that $cz \ge \lfloor\delta\rfloor$ since there exists (as before) $z' \in \mathbb{Z}^n$ with $cz' = \lfloor\delta\rfloor \le \delta$. Together with $cz \le \lfloor\delta\rfloor$ it follows $cz = \lfloor\delta\rfloor$, i.e., $[\![c,\delta]\!] = \lfloor\delta\rfloor$. □

We can iterate the GCG operator in the usual fashion by putting $GCG_S^{i+1}(P) := GCG_S(GCG_S^i(P))$; for consistency we put $GCG_S^0(P) := P$. We define the *rank of P with respect to S* as

$$\operatorname{rk}_S(P) := \min\left\{i \in \mathbb{N} \mid GCG_S^i(P) = P_I\right\}.$$

Note that the rank is finite whenever $P_I \cap \{0,1\}^n \subseteq S$ since $GCG_S(P) \subseteq GCG_{\mathbb{Z}^n}(P) = P'$. In the following we will consider the rank always with respect to GCG if not stated otherwise.

Remark 2.8. We have the following rank inequalities:

(1) Let $P \subseteq [0,1]^n$ be a polytope and $\{0,1\}^n \subseteq S \subseteq \tilde{S} \subseteq \mathbb{Z}^n$, then

$$\operatorname{rk}_S(P) \le \operatorname{rk}_{\tilde{S}}(P).$$

(2) Let $P, Q \subseteq [0,1]^n$ be polytopes with $P \subseteq Q$ and $P_I = Q_I$. Further let $\{0,1\}^n \subseteq S \subseteq \mathbb{Z}^n$, then

$$\operatorname{rk}_S(P) \le \operatorname{rk}_S(Q).$$

Proof. For (1) observe that we have $GCG_S(P) \subseteq GCG_{\tilde{S}}(P)$ by Remark 2.3. And so iterating the operators we have $GCG^i_S(P) \subseteq GCG^i_{\tilde{S}}(P)$ for all $i \in \mathbb{N}$. If now $P_I \neq GCG^i_S(P)$ for some i, then $P_I \neq GCG^i_{\tilde{S}}(P)$. The first inequality follows.

Statement (2) follows in a similar fashion from Remark 2.3. Again we have $GCG^i_S(P) \subseteq GCG^i_S(Q)$ for all $i \in \mathbb{N}$. Thus, if $P_I \neq GCG^i_S(P)$ then $Q_I = P_I \subsetneq GCG^i_S(P) \subseteq GCG^i_S(Q)$. □

For $c \in \mathbb{Q}^n$ let $\mathbb{A}[c] := \frac{1}{n}ce$ denote the average value of c with e being the all-1 vector. Using Observation 2.4 we obtain a simplified characterization of $GCG_S(P)$.

Lemma 2.9. *Let $P \subseteq [0,1]^n$ be a polytope and $S \subseteq \mathbb{Z}^n$. Then*

$$GCG_S(P) = \bigcap_{\substack{(c,\delta)\in\mathbb{Q}^{n+1}, \mathbb{A}[c]\in\{-1,0,1\},\\ P\subseteq\{cx\leq\delta\}}} \{cx \leq [\![c,\delta]\!]_S\},$$

i.e., it suffices to consider normalized normals.

Proof. Clearly, $GCG_S(P) \subseteq \bigcap_{(c,\delta)\in\mathbb{Q}^{n+1}, \mathbb{A}[c]\in\{-1,0,1\}, P\subseteq\{cx\leq\delta\}} \{cx \leq [\![c,\delta]\!]_S\}$. For the converse let $cx \leq \delta$ be valid for P. In case $\mathbb{A}[c] = 0$ there is nothing to be shown. Therefore let $|\mathbb{A}[c]| > 0$. We have that $1/|\mathbb{A}[c]| \cdot cx \leq 1/|\mathbb{A}[c]| \cdot \delta$ is valid for P which is equivalent to $cx \leq \delta$. Now $cx \leq [\![c,\delta]\!]_S$ is valid for $GCG_S(P)$. This is equivalent to $1/|\mathbb{A}[c]| \cdot cx \leq 1/|\mathbb{A}[c]| [\![c,\delta]\!]_S = [\![1/|\mathbb{A}[c]| \cdot c, 1/|\mathbb{A}[c]| \cdot \delta]\!]_S$ by Observation 2.4. Therefore we have an inequality $1/|\mathbb{A}[c]| \cdot cx \leq 1/|\mathbb{A}[c]| \cdot \delta$ with $\mathbb{A}(\cdot)$-value ± 1 that induces the same valid inequality. □

3. An almost integral family of polytope with non-constant rank

We will now prove that there exists a family of polytopes $P_n \subseteq [0,1]^n$ such that $\mathrm{rk}_S(P_n) \in \Omega(\log n)$ whenever $\{0,1\}^n \subseteq S$. By Remark 2.8 it suffices to consider the case $S = \{0,1\}^n$ for establishing lower bounds that are unconditional on S.

Let $P_\lambda := \mathrm{conv}(\{x \in [0,1]^n \mid ex \leq 1\} \cup \{\lambda e\})$ with $\lambda \in (0,1)$. In a first step we will show that $P_{\frac{1}{2}\lambda} \subseteq GCG(P_\lambda)$ whenever $\lambda > 1/n$.

Proposition 3.1. *Let $\lambda \in (0,1)$ and $\lambda > 1/n$ where n is the dimension of P_λ. Then*

$$P_{\frac{1}{2}\lambda} \subseteq GCG_S(P_\lambda)$$

for $\{0,1\}^n \subseteq S \subseteq \mathbb{Z}^n$.

Proof. With Remark 2.8 it suffices to consider $S = \{0,1\}^n$; we drop the index throughout the proof. In a first step we will consider $GCG(P_\lambda)$ and we will argue that it is sufficient to consider normals c with $\mathbb{A}[c] > 0$. Clearly, it suffices to consider inequalities $cx \leq \delta$ with $\delta = \max_{x\in P_\lambda} cx$. Moreover, if there exists $x_I \in P_I$ so that $cx_I = \delta$ then we obtain $\delta = [\![c,\delta]\!]$. Thus it suffices to consider normals c so that there exists no such integral vertex, and, in particular, the maximum δ is solely taken at the fractional vertex λe (note that P_λ has only one fractional vertex). Observe that $c\lambda e = \mathbb{A}[c]\lambda n$. If now $\mathbb{A}[c] < 0$, then $c\lambda e = \lambda\mathbb{A}[c]n < 0$ however $0 \in P_\lambda$ and so $c\lambda e < c0 = 0$; a contradiction. If $\mathbb{A}[c] = 0$ and $c \neq 0$ then there is at least one coordinate i of c such that $c_i > 0$. Now $c\lambda e = \lambda\mathbb{A}[c]n = 0 < ce_i = c_i$ where e_i denotes the i-th unit vector and so λe is not the (unique) maximizer. Therefore, in the following we will confine ourselves to normals c with $\mathbb{A}[c] > 0$. In view of Lemma 2.9 we can further restrict ourselves to normals $c \in \mathbb{Q}^n$ with $\mathbb{A}[c] = 1$. As c is (uniquely) maximized at λe (by assumption from above) we have $c_i < \lambda n$ for all $i \in [n]$. Moreover can assume that $c_i \geq c_{i+1}$ as P is symmetric with respect to coordinate permutations and that $c_1 \geq 1$ holds; otherwise $\mathbb{A}[c] < 1$.

Let c be a normal with the properties as described above, i.e.,

(1) $\mathbb{A}[c] = 1$;
(2) $c_i \geq c_{i+1}$ for all $i \in [n-1]$;
(3) $c_i < \lambda n$ for all $i \in [n]$ and $c_1 \geq 1$.

Choose ℓ maximal so that $\sum_{i \in [j]} c_i < \lambda n$ for all $j \leq \ell$ and define $\tilde{x}$ with the first ℓ entries being equal to 1 and 0 else. We claim that

$$\frac{1}{2}\lambda n \leq c\tilde{x} < \lambda n.$$

By the choice of ℓ we have $c\tilde{x} < \lambda n$. Now for showing $\frac{1}{2}\lambda n \leq c\tilde{x}$, observe that $\lambda n = c\tilde{x} + \alpha c_{\ell+1}$ with $\alpha \in [0, 1)$. Such an α indeed exists since $\mathbb{A}[c] = 1$ and therefore $ce = n > \lambda n$ and we also have $c_{\ell+1} > 0$ if $\alpha \neq 0$ as $c\tilde{x} < \lambda n$. Moreover $c_1 < \lambda n$ so that $\ell \geq 1$. We conclude

$$\lambda n \leq c_{\ell+1} + c\tilde{x} \leq 2c\tilde{x},$$

and so $\frac{1}{2}\lambda n \leq c\tilde{x}$. It follows $[\![c, \lambda n]\!] \geq \frac{1}{2}\lambda n$ and therefore $\frac{1}{2}\lambda e \in P_\lambda$ satisfies $cx \leq [\![c, \lambda n]\!]$. As the choice of c was arbitrary it follows that $\frac{1}{2}\lambda e \in GCG(P_\lambda)$ and so we obtain $P_{\frac{1}{2}\lambda} \subseteq GCG(P_\lambda)$ as claimed. □

While hidden in the actual proof, what we did is to exploit symmetry of P_λ in the following way. Let S_n denote the symmetric group on n elements, here, acting by permuting coordinates. For a vector $c \in \mathbb{Q}^n$ let $c[S_n]$ denote the S_n-average of c, i.e., $c[S_n] = \frac{1}{|S_n|}\sum_{\pi \in S_n} \pi c$ and observe that $c[S_n] = e$ if $\mathbb{A}[c] = 1$. Suppose that $cx \leq [\![c, \delta]\!]$ is valid for $GCG(P_\lambda)$ with $\mathbb{A}[c] = 1$. Then by symmetry $\pi c x \leq [\![c, \delta]\!]$ is valid for $GCG(P_\lambda)$ for all $\pi \in S_n$. Thus $c[S_n]x = ex \leq [\![c, \delta]\!]$ is valid for $GCG(P_\lambda)$.

We obtain the main theorem of this section.

Theorem 3.2. *Let* $P = \operatorname{conv}\left(\{x \in [0,1]^n \mid ex \leq 1\} \cup \left\{\frac{1}{2}e\right\}\right)$ *with* $n \geq 3$ *and* $\{0,1\}^n \subseteq S \subseteq \mathbb{Z}^n$. *Then* $\operatorname{rk}_S(P) \geq \log(n) - 1$.

Proof. Clearly $P = P_\lambda$ with $\lambda = 1/2$. As before, we consider the case $S = \{0,1\}^n$. By applying Proposition 3.1 repeatedly we obtain a chain

$$P_{(\frac{1}{2})^{i+1}} \subseteq GCG(P_{(\frac{1}{2})^i}),$$

with $i \in \mathbb{N}$ and therefore, as long as $P_{(\frac{1}{2})^{i+1}} \not\subseteq P_I$ we have $P_I \neq GCG(P_{(\frac{1}{2})^i}) \subseteq GCG^i(P_{\frac{1}{2}})$ for all $i \in \mathbb{N}$: Clearly this is true for $i = 0$; suppose that $P_{(\frac{1}{2})^{i+1}} \subseteq GCG(P_{(\frac{1}{2})^i}) \subseteq GCG^i(P_{\frac{1}{2}})$ holds, then

$$GCG(P_{(\frac{1}{2})^{i+1}}) \subseteq GCG^2(P_{(\frac{1}{2})^i}) \subseteq GCG^{i+1}(P_{\frac{1}{2}}).$$

A necessary condition for $P_{(\frac{1}{2})^i} \subseteq P_I$ is that $e((\frac{1}{2})^i e) = (\frac{1}{2})^i n \leq 1$ and so $i \geq \log(n)$. With $P_{(\frac{1}{2})^i} \subseteq GCG^{i-1}(P_{\frac{1}{2}})$ we conclude $GCG^i(P) \neq P_I$ as long as $i \leq \log(n) - 2$. Thus $\operatorname{rk}_{\{0,1\}^n}(P) \geq \log(n) - 1$. □

A polytope $P \subseteq [0,1]^n$ is *almost integral* if $P \cap \{x_i = \ell\}$ is integral for all $(i,\ell) \in [n] \times \{0,1\}$. We have seen in Theorem 3.2 that there exists an almost integral family of polytopes with non-constant rank. While at first this might not sound surprising, in fact it is *very unexpected*. All known cutting-plane operators except for the classical Chvátal-Gomory operator do derive the integral hull of an almost integral polytope *within one application*; the reason is that any elementary split over any x_i derives the integral hull. In fact it will turn out that exactly this property is the one that allows us to construct a family of polytopes with rank $> n$ for the generalized Chvátal-Gomory operators (see Section 5). It can be easily shown that the existence of a family of almost integral polytopes with non-constant rank is necessary for the existence of a family of polytopes with rank $> n$. Let $P_n \subseteq [0,1]^n$ denote the polytope $P_{1/2}$ from Proposition 3.1 in dimension n. In Section 5 we will show that P_n having non-constant rank is also sufficient for the existence of a family of polytopes with rank $> n$. The following corollary is immediate:

Corollary 3.3. *Let K_n be the clique on n vertices and let*

$$FSTAB(K_n) := \{x \in [0,1]^n \mid x_u + x_v \leq 1 \quad \forall\, u < v \in [n]\}$$

denote the fractional stable set polytope of K_n. Then $\mathrm{rk}_S(FSTAB(K_n)) \geq \log(n) - 1$ *for all* $\{0,1\}^n \subseteq S \subseteq \mathbb{Z}^n$.

Proof. As before it suffices to consider the case $S = \{0,1\}^n$ by Remark 2.8. Consider $G = K_n$, the complete graph on n vertices. Then $P_n \subseteq FSTAB(G)$ with P_n as given above. We conclude that it takes at least $\log(n) - 1$ applications of GCG_S to derive the clique inequality $ex \leq 1$. □

As mentioned before P_λ is almost integral. Therefore we can optimize over P_λ in polynomial time (by fixing, say, $x_1 \in \{0,1\}$, solving the two LPs, and choosing the better solution), however it takes at least $\log(n) - 1$ rounds of the GCG-operators to derive the integral hull although GCG solves a hard separation problem (actually even verifying the validity of a cut is hard). Moreover P_λ can be described by a linear number of inequalities as already shown in Pokutta and Stauffer [2011]. The lift-and-project rank of P_λ is 1. This immediately follows from the definition of the lift-and-project closure as

$$N_0(P) := \bigcap_{i \in [n]} \operatorname{conv}\left((P \cap \{x_i = 0\}) \cup (P \cap \{x_i = 1\})\right)$$

and the fact that $P_\lambda \cap \{x_i = \ell\} = (P_\lambda)_I \cap \{x_i = \ell\}$ for all $(i,\ell) \in [n] \times \{0,1\}$. The reason why the lift-and-project operator (and almost any other operator) is able to derive $(P_\lambda)_I$ within a single application is the following. Reinterpreted in our framework the lift-and-project operator uses more information about the integral hull. In fact it does not work with a *static* set $\{0,1\}^n \subseteq S \subseteq \mathbb{Z}^n$ and a single inequality.

To make the difference even more apparent we can (re-)interpret any cutting-plane operator M as one that optimizes right-hand sides. In its most naïve form we can formally write

$$M(P) = \bigcap_{(c,\delta) \in \mathbb{R}^{n+1}} cx \leq f(c,\delta,P),$$

with

$$f(c,\delta,P) := \max\{cx \mid x \in S(c,\delta,P,M)\},$$

where S is chosen depending on c,δ,P, and M. In the case of the lift-and-project closure N_0 we have

$$S(c,\delta,P,N_0) = \bigcap_{i\in[n]} \operatorname{conv}\left((P \cap \{x_i = 0\}) \cup (P \cap \{x_i = 1\})\right).$$

The role of the additional information becomes clear: In the case of GCG_S we have $S(c,\delta,P,GCG_S) = S \cap \{cx \leq \delta\}$, i.e., we do not use information about P except for the *single valid inequality*, whereas in the case of the lift-and-project closure, $S(c,\delta,P,N_0)$ includes many additional inequalities and their information about the integral hull. This is also the difference between cutting-plane operators using only a *single inequality* vs. those that use a *single row*; the latter uses additional information (e.g., implicitly derived from the tableau). We elicit this difference in view of P_λ in the following example.

Example 3.4 (N_0 uses more than one inequality)**.** Let P_λ be the polytope as defined before and let N_0 denote the lift-and-project closure. Consider the normal e. We have that $ex \leq 1$ is valid for $N_0(P_\lambda)$ because

$$S(c,\delta,P_\lambda,N_0) = \bigcap_{i\in[n]} \operatorname{conv}\left(P_\lambda \cap \{x_i = 0\}) \cup P_\lambda \cap \{x_i = 1\}\right) = [0,1]^n \cap \{ex \leq 1\}.$$

We say that two cutting-plane operators M and N are *incompatible* if there exist two polytopes P and Q, so that $M(P) \not\subseteq N(P)$ and $N(Q) \not\subseteq M(Q)$, i.e., non refines the other. From the previous discussion and Example 3.4 we obtain:

Corollary 3.5. *N_0 is incompatible to GCG_S for all $\{0,1\}^n \subseteq S \subseteq \mathbb{Z}^n$.*

Proof. First consider the polytope P_λ with $\lambda = \frac{1}{2}$. Then $\operatorname{rk}_S(P_\lambda) \geq \log(n) - 1$ and we have $\operatorname{rk}_{N_0}(P_\lambda) = 1$. Therefore $GCG_S \not\subseteq N_0$.

For the converse consider the polytope $P := [0,1]^n \cap \left\{ex \leq n - \frac{1}{2}\right\}$. Then $\operatorname{rk}_S(P) = 1$ as already its classical Chvátal-Gomory rank is 1, however we have $\operatorname{rk}_{N_0}(P) = n$ (see [Cook and Dash, 2001, Theorem 3.1]). ☐

The natural question that now arises is how the incompatibility shown in Corollary 3.5 goes together with the preceding discussion about N_0 having access to more information. However everything fits naturally together. While the N_0-operator is using more information *in principle*, it cannot exploit all information as it is polynomial time bounded; it solves a linear program. In contrast to this, a close inspection of the GCG operators show that even if they have *unbounded computational power* there is not enough information contained in the single inequality to be exploited. So in some sense N_0 is computationally bounded whereas GCG uses bounded information.

4. The rank of A_n is n

In this section we will show that the polytope A_n defined as

$$A_n := \left\{ x \in [0,1]^n \mid \sum_{i\in I} x_i + \sum_{i\notin I}(1-x_i) \geq \frac{1}{2} \quad \forall I \subseteq [n] \right\},$$

has rank n for all operators GCG_S with $\{0,1\}^n \subseteq S \subseteq \mathbb{Z}^n$. A similar result was indicated in Dunkel and Schulz [2010] using a different approach. Note that $(A_n)_I = \emptyset$. The upper bound of n follows immediately from the fact that GCG_S refines the classical Chvátal-Gomory operator (as shown in Proposition 2.7). The latter is known to derive the integral hull of integer-empty polytopes in at most n rounds (see Bockmayr et al. [1999]). Thus it suffices to establish the lower bound.

Let $F_\ell^n := \{x \in \{0,1/2,1\}^n \mid \text{exactly } \ell \text{ entries equal to } 1/2\}$. It is easy to see that $A_n = \operatorname{conv}\left(F_1^n\right)$ (see e.g., Chvátal et al. [1989] or Pokutta and Schulz [2010b]); we drop the index n if it is clear from the context. We will follow the classical route of establishing the lower bound on the rank of A_n as outlined in Chvátal et al. [1989].

Lemma 4.1. *Let $P \subseteq [0,1]^n$ be a polytope such that $F_\ell^n \subseteq P$ with $\ell \in [n-1]$. Then $F_{\ell+1}^n \subseteq GCG_S(P)$.*

Proof. As before we consider the case $S = \{0,1\}^n$. Let $cx \leq \delta$ be a valid inequality for P. We have to show that $c\bar{x} \leq [\![c, \delta]\!]$ for all $\bar{x} \in F_{\ell+1}^n$ which is equivalent to

$$\max_{x \in F_{\ell+1}} cx \leq [\![c, \delta]\!]. \tag{4.1}$$

By Remark 2.3, without loss of generality we can assume $P = \operatorname{conv}\left(F_\ell^n\right)$. As P is symmetric with respect to coordinate flips and permutations, we can further assume that $c \geq 0$ and $c_i \geq c_{i+1}$ for all $i \in [n-1]$. Therefore the maximizing element in (4.1) is given by $x^{\ell+1} = (1,\dots,1,\frac{1}{2},\dots\frac{1}{2})$ with $n-(\ell+1)$ entries being equal to 1 and $\ell+1$ entries being equal to $\frac{1}{2}$. Similarly, as $cx \leq \delta$ is valid for $P = \operatorname{conv}\left(F_\ell^n\right)$ we have that $x^\ell = (1,\dots,1,\frac{1}{2},\dots\frac{1}{2})$ with $n-\ell$ entries being equal to 1 and ℓ entries being equal to $\frac{1}{2}$ maximizes cx over P.

We will now derive two integral points $x^0, x^1 \in \{0,1\}^n$ from x^ℓ so that $cx^0 \leq \delta$ and $cx^1 \leq \delta$ and $x^{\ell+1} = \frac{1}{2}x^0 + \frac{1}{2}x^1$. As x^0, x^1 satisfy $cx \leq \delta$ it follows $cx^0 \leq [\![c, \delta]\!]$ and $cx^1 \leq [\![c, \delta]\!]$. We obtain that $cx^{\ell+1} \leq [\![c, \delta]\!]$. This finishes the proof.

The integral points will be constructed by fractionally shifting coefficients to the right. We iteratively construct x^1 from x^ℓ: Let $i \in [n]$ be the smallest coefficient of x^ℓ being equal to $\frac{1}{2}$ (in the first iteration $i = n-\ell$). Then replace the i-th coefficient with 0 and the $(i+1)$-th coefficient with 1, where the latter replacement is done for $i \leq n-1$. We repeat these replacements until we obtain an integral vector. Observe that for the next iteration i increases by two. Let the integral vector be x^1 which is of the form

$$x^1 = (1,\dots,1;1,0,1,0,\dots),$$

where the block of consecutive 1's is of length $n-\ell$. Observe that $cx^1 \leq \delta$ as x^1 arose from x^ℓ with changes i of the form $\frac{1}{2}(e_{i+1}-e_i)$ and $c(e_{i+1}-e_i) \leq 0$ as c was in non-increasing order. Let x^0 be defined as

$$x^0 := \sum_{i \in [n-(\ell+1)]} x_i^1 e_i + \sum_{n-\ell \leq i \leq n-1} x_i^1(i) e_{i+1},$$

i.e., x^0 arises from x^1 by keeping the first $n-(\ell+1)$ entries and shifting the remaining ones to the right by 1 (without wrapping). Therefore x^0 is of the form

$$x^0 = (1,\dots,1;0,1,0,1,\dots),$$

where the block of consecutive 1's is of length $n-(\ell+1)$. With a similar argument as before we obtain that $cx^0 \le cx^1 \le \delta$; again we shift entries in x^1 in the direction of smaller entries in c. Due to the alternation on the last $\ell+1$ coordinates we obtain $x^{\ell+1} = \frac{1}{2}x^0 + \frac{1}{2}x^1$. This completes the proof. □

Using Lemma 4.1 we obtain:

Theorem 4.2. *Let $n \in \mathbb{N}$ and $\{0,1\}^n \subseteq S \subseteq \mathbb{Z}^n$. Then*

$$\mathrm{rk}_S(A_n) = n.$$

Proof. Let $S = \{0,1\}^n$. Iterating Lemma 4.1 we obtain

$$F_{\ell+1} \subseteq GCG^{\ell}(A_n).$$

As $F_n = \frac{1}{2}e$ we have $\emptyset \neq GCG^{n-1}(A_n)$. Therefore $\mathrm{rk}_{\{0,1\}^n}(A_n) \ge n$. As $GCG^n(A_n) \subseteq GCG^n_{\mathbb{Z}^n}(A_n) = \emptyset$ we obtain $\mathrm{rk}_{\{0,1\}^n}(A_n) = n$. □

Let H_n denote the *subtour elimination polytope* on the complete graph with n vertices given by the inequalities

$$\begin{aligned} x(\delta(\{v\})) &= 2 && \forall\, v \in [n] \\ x(\delta(W)) &\ge 2 && \forall\, \emptyset \subsetneq W \subsetneq [n] \\ x_i &\in [0,1]^n && \forall\, i \in [\tfrac{1}{2}n(n-1)]. \end{aligned}$$

With the arguments in Chvátal et al. [1989] or [Cook and Dash, 2001, Section 4] we immediately obtain $\mathrm{rk}_S(H_n) \in \Omega(n)$ for $\{0,1\}^n \subseteq S \subseteq \mathbb{Z}^n$. Note that for S being 'asymmetric' we do not necessarily obtain that GCG commutes with coordinate flips, coordinate duplications, and embeddings, so that the arguments in Chvátal et al. [1989] or [Cook and Dash, 2001, Section 4] might not apply. However by Remark 2.8 it suffices to establish the result for $S = \{0,1\}^n$ for which we can easily check that GCG commutes with the above operations.

5. SUPER-n LOWER BOUNDS ON THE RANK

We will now combine Theorem 3.2 and Theorem 4.2 to construct a class of polytopes $Q_n \subseteq [0,1]^n$ such that $\mathrm{rk}_S(Q_n) > n$ for all $\{0,1\}^n \subseteq S \subseteq \mathbb{Z}^n$. The construction is analogous to Eisenbrand and Schulz [2003] and Pokutta and Stauffer [2011]. Let $P_{\lambda,n}$ denote the polytope $P_\lambda \subseteq [0,1]^n$ in dimension n as defined in Section 3. We define $Q_n := \mathrm{conv}\left(A_n \cup \{ex \le 1\}\right) \subseteq [0,1]^n$.

Theorem 5.1. *Let $n \in \mathbb{N}$ with $n \ge 8$ and $\{0,1\}^n \subseteq S \subseteq \mathbb{Z}^n$. Then*

$$\mathrm{rk}_S(Q_n) \ge n + \log(n) - 2 > n.$$

Proof. Let $S = \{0,1\}^n$. Observe that $\frac{1}{2}e \in GCG^{n-1}(A_n)$ as $GCG^{n-1}(A_n) \neq \emptyset$ by Theorem 4.2 and $GCG^{n-1}(A_n)$ is symmetric with respect to coordinate flips and permutations. By Remark 2.3 we have that $\frac{1}{2}e \in GCG^{n-1}(Q_n)$. We can conclude that $P_{\frac{1}{2},n} \subseteq GCG^{n-1}(Q_n)$. Therefore, by Theorem 3.2, we need at least $\log(n)-1$ more applications of GCG to derive $(Q_n)_I$. Thus $\mathrm{rk}(Q_n) \ge n + \log(n) - 2$. □

6. Operators using more than one inequality

As a consequence of the discussion in Section 3 and having seen that deductions from a single inequality might lead to super-n rank, one might wonder whether this is true in general for any *fixed* number of inequalities used in the derivation. The answer is in the negative: as soon as the operator can use (at least) two inequalities it can simulate elementary splits and so it refines the N_0 closure.

More precisely, we say that a cutting-plane operator M *deduces from two inequalities* if whenever $cx \leq \delta$ is valid for $M(P)$ then there exist inequalities $\pi^1 x \leq \pi^1_0$ and $\pi^2 x \leq \pi^2_0$ such that $P \subseteq [0,1]^n \cap \left\{\pi^1 x \leq \pi^1_0\right\} \cap \left\{\pi^2 x \leq \pi^2_0\right\}$ and $cx \leq \delta$ is valid for $M([0,1]^n \cap \left\{\pi^1 x \leq \pi^1_0\right\} \cap \left\{\pi^2 x \leq \pi^2_0\right\})$.

We define a simply cutting-plane operator M that deduces from two inequalities.

Definition 6.1. Let $P \subseteq [0,1]^n$ be a polytope. Then the *transient closure* $\daleth(P)$ is defined as

$$\daleth(P) := \bigcap_{\substack{(c,\delta,\tau,\delta,i)\in\mathbb{Q}^{n+1}\times\mathbb{R}^2_+\times\mathbb{N},\\ P\subseteq\{cx-\tau x_i\leq\delta\}\cap\{cx-\lambda(1-x_i)\leq\delta\}}} \left\{cx \leq \overline{[\![c,\delta,\tau,\lambda,i]\!]}\right\}$$

where

$$\overline{[\![c,\delta,\tau,\lambda,i]\!]} := \max\left\{cx \mid cx - \tau x_i \leq \delta, cx - \lambda(1-x_i) \leq \delta, x \in \{0,1\}^n\right\}.$$

This operator is similar to $GCG_{\{0,1\}^n}$ however now it can optimize over the intersection of two half spaces, i.e., it deduces from two inequalities. Moreover, these two inequalities are not arbitrary but related to each other by rotation.

Theorem 6.2. *Let $\daleth$ be transient closure operator, then $\daleth$ refines N_0, i.e., for any polytope $P \subseteq [0,1]^n$ we have*

$$\daleth(P) \subseteq N_0(P).$$

Proof. Let $P \subseteq [0,1]^n$ be a polytope and let $cx \leq \delta$ be valid for N_0. By the definition of N_0 therefore there exists $i \in [n]$ so that

$$(P \cap \{x_i = 0\}) \cup (P \cap \{x_i = 1\}) \subseteq \{cx \leq \delta\}.$$

If $P \cap \{x_i = \ell\} = \emptyset$ for some $\ell \in \{0,1\}$, then $\daleth(P) \subseteq \{x_i = 1-\ell\} \cap P$ and so $cx \leq \delta$ is valid for $\daleth(P)$. Thus suppose $P \cap \{x_i = \ell\} \neq \emptyset$ for all $\ell \in \{0,1\}$. Then, by Farkas' lemma, there exist $\tau, \lambda \geq 0$, so that

$$cx - \tau x_i \leq \delta \quad \text{and} \quad cx - \lambda(1-x_i) \leq \delta$$

are valid for P. Now consider $\overline{[\![c,\delta,\tau,\lambda,i]\!]}$ and a potential $0/1$ point $\tilde{x} \in \{0,1\}^n$ so that $c\tilde{x} - \tau\tilde{x}_i \leq \delta$ and $c\tilde{x} - \lambda(1-\tilde{x}_i) \leq \delta$. We have $\tilde{x}_i \in \{0,1\}$ and so $c\tilde{x} \leq \delta$ holds. Therefore

$$cx \leq \overline{[\![c,\delta,\tau,\lambda,i]\!]} \leq \delta$$

is valid for $\daleth(P)$ which completes the proof. □

Clearly $\daleth$ also refines $GCG_{\{0,1\}}$ (choosing $\tau = \lambda = 0$ and $i \in [n]$ arbitrary) so that we obtain:

Corollary 6.3. *Let $\daleth$ be the transient closure. Then $\daleth$ refines $GCG_{\{0,1\}^n}$ and N_0, i.e.,*

$$\daleth(P) \subseteq GCG_{\{0,1\}^n}(P) \cap N_0(P)$$

for any polytope $P \subseteq [0,1]^n$. Moreover $\mathrm{rk}_\daleth(P) \leq n$ for all polytopes $P \subseteq [0,1]^n$.

If we relax the integrality condition in the definition of $\overline{[\![\cdot,\cdot,\cdot,\cdot,\cdot]\!]}$ we obtain the classical N_0 closure. We therefore obtain a nicer hierarchy in the presence of 0/1 conditions: whereas CG and N_0 are incompatible, their 0/1 extended counterparts $GCG_{\{0,1\}^n}$ and $\daleth$ are actually compatible.

7. Conclusions

We have seen that even the generalized Chvátal-Gomory operators suffer from the same *inherent weakness* as the classical Chvátal-Gomory operator. What might be surprising is the fact that this is the case, although the verification of the validity of an inequality is already hard, i.e., the actual operator is solving *significantly harder problems* and is much more powerful (in terms of computational complexity). In contrast, for all other known operators we can easily verify the validity in polynomial time.

So what does this imply? First of all, one does not automatically gain a significantly stronger operator by solving harder separation problems. It also seems that the actual geometry and the *information* available to an operator (i.e., whether the provided inequalities do allow to decide membership over P_I *with unlimited computational power*) might be crucial. Moreover it also highlights that the notion of rank cannot be a good measure for the complexity of the integral hull in general. After all, the family P_n from Theorem 3.2 exhibits rank in $\Theta(\log n)$ for the generalized Chvátal-Gomory operators whereas the actual optimization problem is in P. The lower bound on A_n for the generalized Chvátal-Gomory operators naturally carries over to verification schemes (see Dey and Pokutta [2011]), i.e., we obtain $\mathrm{rk}_{\partial GCG_S}(A_n) \in \Omega(n)$.

We would also like to point out that a further generalization to an analogous lower bounding technique as the one in Pokutta and Stauffer [2011] is unlikely, as our construction depends on the symmetry of P_λ, whereas in Pokutta and Stauffer [2011] it is mostly the integrality gap and the protection derived from the integral hull that is important. In fact we do not believe that it is possible to reduce the speed of the geometric progression by adding more protection via a larger integral hull.

A natural question is whether the lower bound is tight for the generalized Chvátal-Gomory operator. We believe that some crucial insights of Proposition 3.1 might allow for providing at least new, stronger upper bounds on the rank.

We have seen that as soon as we allow deductions from at least two inequalities, then there exists a cutting-plane operator, the transient closure, which refines N_0 and therefore we have $\mathrm{rk}_\daleth(P) \leq n$ whenever $P \subseteq [0,1]^n$. However it is not clear how strong $\daleth$ is. We conjecture that GSC $\subsetneq \daleth$ should hold. The inclusion is clear, however we do not know about any separating family of polytopes.

One gap still remains. We considered generalized Chvátal-Gomory operators on the one hand and operators deducing from two inequalities on the other hand. However it is not clear whether the generalized Chvátal-Gomory operators capture the whole class of operators deducing from a single inequality. Provided a single inequality $cx \leq \delta$ such an operator can virtually generate all valid inequalities of the knapsack polytope $[0,1]^n \cap \{cx \leq \delta\}$ (see Fischetti and Lodi [2010] for a formal definition of the *knapsack closure*). We believe that essentially the same proof of Proposition 3.1 should work to establish the existence of a family of almost integral polytopes with non-contant rank. The only difference seems to be, that now we would work with a weighted knapsack problem rather than an unweighted one. We still get the same approximation guarantees using the density of elements. Similarly it should be possible to carry over Theorem 4.2.

It would be also worthwhile to consider 0/1 strengthenings of other operators, for example the split cut operator. Given $P \subseteq [0,1]^n$ and $cx \leq \delta$ let $P(c,\delta)^+ := P \cap \{cx \leq \delta\}$ and similarly $P(c,\delta)^- := P \cap \{cx \geq \delta\}$. We can define the *generalized split closure* of a polytope $P \subseteq [0,1]^n$ with respect to $\{0,1\}^n \subseteq S \subseteq \mathbb{Z}^n$ as

$$\mathrm{GSC}_S(P) := \bigcap_{\substack{(\pi,\pi_0,\delta)\in\mathbb{Z}^{n+2} \\ S=S(\pi,\pi_0)^+ \cup S(\pi,\pi_0+\delta)^-}} \mathrm{conv}\left(P(\pi,\pi_0)^+ \cup P(\pi,\pi_0+\delta)^-\right).$$

Thus in the case of this 0/1 strengthening, we allow wider disjunctions as long as they partition S. Is this operator significantly stronger than the classical split operator in terms of rank? If so, can we obtain an advantage in the order for, e.g., the rank?

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