

Optimal synthesis in the Reeds and Shepp problem with a free final direction

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Abstract. We consider a time-optimal problem for the Reeds and Shepp model describing a moving point on a plane, with a free final direction of velocity. Using Pontryagin Maximum Principle, we obtain all types of extremals and, analyzing them and discarding nonoptimal ones, construct the optimal synthesis.

Keywords. Time-optimal problem, Pontryagin Maximum Principle, extremals, reachability sets, optimal synthesis.

1 Statement of the problem

Consider the following problem for a moving point on a plane:

$$\left\{ \begin{array}{lll} \dot{x} = u \sin \varphi, & x(t_0) = 0, & x(T) = x_T, \\ \dot{y} = u \cos \varphi, & y(t_0) = 0, & y(T) = y_T, \\ \dot{\varphi} = v, & \varphi(t_0) = 0, & \varphi(T) \text{ is free,} \\ |u| \leq 1, & |v| \leq 1, & J = T \rightarrow \min. \end{array} \right. \quad (1)$$

Here we have three state variables x, y, φ , and two controls u, v . The pair (x, y) determines a position of the point on the plane, φ is the angle between its velocity (\dot{x}, \dot{y}) and the ordinate axis. The initial time instant t_0 is fixed.

This problem (with a fixed direction of the velocity both at the initial and at the final moment) was stated by Reeds and Shepp [2], where all possible extremals were described. It was then studied by many authors (e.g. [3, 4, 5, 7, 11, 13]); a complete synthesis for the problem with a fixed final direction was given in [5]. Here we consider this problem with a free final direction $\varphi(T)$, i.e., the point should be driven, in a shortest time, at a prescribed position, no matter from which direction it is done. No doubts, such a statement is physically reasonable. Our aim is to construct an optimal synthesis for problem (1).

Note that the problem with a free $\varphi(T)$ is essentially simpler than the problem with a fixed $\varphi(T)$ (which is a typical situation in optimal control); however, the optimal synthesis for problem (1) does not follow from the optimal synthesis for

the problem with a fixed $\varphi(T)$. At the same time, a relative simplicity of problem (1) allows one to study it very clearly, completely, and rather shortly, avoiding the cumbersome considerations of the case with a fixed final direction. Moreover, since the synthesis for problem (1) depend only on the two parameters (x_T, y_T) (unlike the synthesis for a fixed $\varphi(T)$, which depends on three parameters), it can be presented on the plane.

Note also that, for the frozen control $u = 1$, the problem (1) (with different choices of conditions on $\varphi(T)$) reduces to the well known Markov–Dubins problem [1], studied in that and many other papers (see e.g. [6, 8, 9, 10, 11, 12, 13, 14]).

Since the admissible control set in problem (1) is a convex compactum (a square in the plane), and the control system is linear in both controls, the classical Filippov theorem guarantees that a solution always exist.

Let $(x(t), y(t), \varphi(t), u(t), v(t), t \in [0, T])$ be an optimal process. According to the Pontryagin Maximum Principle (MP), there exist a number $\alpha \geq 0$ and Lipschitz functions $\psi_x(t), \psi_y(t), \psi_\varphi(t)$, not all identically zero, that generate *the Pontryagin function*

$$H = (\psi_x \sin \varphi + \psi_y \cos \varphi) u + \psi_\varphi v, \quad (2)$$

and satisfy the costate (adjoint) equations

$$-\dot{\psi}_x = H_x = 0, \quad -\dot{\psi}_y = H_y = 0, \quad (3)$$

$$-\dot{\psi}_\varphi = H_\varphi = (\psi_x \cos \varphi - \psi_y \sin \varphi) u, \quad (4)$$

the transversality conditions: $\psi_\varphi(T) = 0$,

the “energy conservation law”: $H(x, y, \varphi, u, v) \equiv \alpha \geq 0$,

and the maximality condition: for almost all t

$$\max_{|u'| \leq 1, |v'| \leq 1} H(x, y, \varphi, u', v') = H(x, y, \varphi, u, v). \quad (5)$$

In view of separability of H in u and v , the last condition splits onto the two separate conditions: for a.a. t

$$\max_{|u'| \leq 1} (\psi_x \sin \varphi + \psi_y \cos \varphi) u' = (\psi_x \sin \varphi + \psi_y \cos \varphi) u, \quad (6)$$

$$\max_{|v'| \leq 1} \psi_\varphi v' = \psi_\varphi v. \quad (7)$$

These conditions, in turn, mean that

$$u \in \text{Sign}(\psi_x \sin \varphi + \psi_y \cos \varphi), \quad v \in \text{Sign} \psi_\varphi, \quad (8)$$

where $\text{Sign } z = \partial|z|$ is a set-valued function, equal 1 for $z > 0$, -1 for $z < 0$, and to the interval $[-1, 1]$ for $z = 0$.

If $u(t) \in \text{Sign } z(t)$, and the function $z(t)$ vanishes only on a set of zero measure, one can write the “standard” relation $u(t) = \text{sign } z(t)$.

2 Analysis of the Maximum Principle

Following A.A. Milyutin, we call *the control system* of the problem the collection of all its pointwise constraints, with no account of the endpoint constraints. For problem (1), the control system consists of the relations

$$\begin{aligned} \dot{x} &= u \sin \varphi, & \dot{y} &= u \cos \varphi, \\ \dot{\varphi} &= v, & |u| &\leq 1, & |v| &\leq 1. \end{aligned} \quad (9)$$

By *extremal* of a control system we call a collection of state, control, and costate variables satisfying the given control system as well as costate equations, energy conservation law, and maximality conditions (i.e. all pointwise conditions of MP). For system (9), the extremal is the collection of functions $(x, y, \varphi, u, v, \psi_x, \psi_y, \psi_\varphi)$, satisfying (9) and (3)-(7). An extremal is *nontrivial* if the collection of costate functions is not identically zero. For system (9), the nontriviality of an extremal is equivalent to the nontriviality of the total collection of Lagrange multipliers. (If $\psi_x = \psi_y = \psi_\varphi = 0$, then also $\alpha = 0$).

Let us find all nontrivial extremals of system (9). For convenience, we will assume that they are defined on the whole real axis $-\infty < t < \infty$, and a proper position of the interval $[t_0, T]$ will be determined later, with account of endpoint conditions.

From (3) it follows that $\psi_x = \text{const} = \beta_x$, $\psi_y = \text{const} = \beta_y$, and so, the only costate variable that essentially remains is ψ_φ , and we will further denote it simply ψ .

Notice that, if $\beta_x = \beta_y = 0$, then according to (4), $\psi = \text{const}$, and from $\psi(0) = 0$ we obtain $\psi \equiv 0$, then also $\alpha = 0$, i.e., all the collection is trivial.

Thus, $(\beta_x, \beta_y) \neq (0, 0)$, and without loss of generality we can set $\beta_x^2 + \beta_y^2 = 1$. Then $\beta_x \sin \varphi + \beta_y \cos \varphi = \sin(\varphi - \theta)$ for some θ , and $H = u \sin(\varphi - \theta) + v\psi$, so the functions $\psi(t)$ and $\varphi(t)$ satisfy the following conditions:

$$\dot{\psi} = -u \cos(\varphi - \theta), \quad \psi(T) = 0, \quad (10)$$

$$\dot{\varphi} = v, \quad \varphi(0) = 0. \quad (11)$$

$$u \in \text{Sign} \sin(\varphi - \theta), \quad v \in \text{Sign} \psi. \quad (12)$$

$$\hat{H} = |\sin(\varphi - \theta)| + |\psi| \equiv \alpha \geq 0. \quad (13)$$

Note that in view of (12), for almost all t

$$u(t) \sin(\varphi(t) - \theta) = |\sin(\varphi(t) - \theta)|.$$

Consider the so called *abnormal* case, when $\alpha = 0$. Here both $\sin(\varphi - \theta) \equiv 0$ and $\psi \equiv 0$. Since $\varphi(t)$ is continuous, and the sine vanishes at isolated points,

$\varphi(t) = \text{const}$, and hence, $v \equiv 0$. Moreover, since $\cos(\varphi - \theta) \neq 0$ in this case, and $\dot{\psi} \equiv 0$, relation (10) yields $u \equiv 0$. Thus, our moving point actually stays immobile at the same place (at the origin). This, however, is possible only in the case when $(x_T, y_T) = (0, 0)$. Therefore, we assume in what follows that $(x_T, y_T) \neq (0, 0)$, and then $\alpha > 0$.

Now, consider the case when $\alpha > 0$, but $\psi(t) \equiv 0$. Then $|\sin(\varphi - \theta)| \equiv \alpha$, so $\varphi(t) = \text{const}$, and hence $v \equiv 0$. Moreover, $\sin(\varphi - \theta) \equiv \text{const} \neq 0$, whence $u = \text{const} = \pm 1$. This means that the point moves along a straight line either forth or back with a speed 1. This is possible only if the final position lies on the ordinate axis (i.e., on the straight line generated by the vector of initial velocity).

Next we assume that $\psi(t)$ is not identically 0. As both of functions $\psi(t)$ and $\sin(\varphi(t) - \theta)$ determining the controls $u(t)$ and $v(t)$ by formulas (12) are continuous, the sets of their positivity and negativity consist of intervals.

Consider any maximal interval (connected component) $\omega = (t_1, t_2)$, where, for definiteness, $\psi > 0$ (then $v = 1$). Note that if either t_1 or t_2 is finite, then $\psi = 0$ at that point (otherwise the interval ω is not maximal). Since $\varphi(t) = t + C$ on ω in view of (10), then shifting the axis t , we may assume that $\varphi(t) - \theta = t$.

From (13) we have

$$0 < \psi(t) = \alpha - |\sin t|, \quad \text{and then} \quad |\sin t| < \alpha. \quad (14)$$

Consider first the case $\alpha > 1$. Then $\psi(t) \neq 0 \forall t$, and so, as we must satisfy the transversality condition $\psi(T) = 0$, we exclude this type from further considerations.

Now assume that $0 < \alpha \leq 1$. Then, obviously, t_1 and t_2 are finite. From the continuity of ψ and maximality of ω it follows that at the end points of this interval $\psi = 0$. Then by (14) we have $|\sin t| = \alpha$ at these endpoints, and hence, $\omega = (t_1, t_2) = (-\sigma, \sigma)$, where $\sigma = \arcsin \alpha$. On the segment $[-\sigma, \sigma]$ the function $\sin t$ either increases from $-\alpha$ to α , or decreases from α to $-\alpha$.

Suppose for definiteness that the increase is realized. Let us find the further behavior of $\psi(t)$ for $t > \sigma$. Since $\sin \sigma > 0$, we still have $\sin t > 0$ in a neighborhood of the point $t_2 = \sigma$, whence by (12) $u = 1$, and in view of (10) $\dot{\psi}$ is continuous. Further, different cases can happen, depending on the value of $\alpha \in (0, 1]$.

I) $\alpha < 1$. In this case, $\dot{\psi}(t_2 + 0) = \dot{\psi}(t_2 - 0) = -\cos \sigma < 0$, therefore we have $\psi < 0$ on some right hand interval (t_2, t_3) . As was found above, the maximal length of such interval is 2σ , i.e. $t_3 = t_2 + 2\sigma$. On this interval, $v = -1$, $\varphi(t) = -t + C'$ (i.e. $\varphi(t)$ is the inverted time), and by (14) $\psi(t) = \sin(\varphi - \theta) - \alpha$. It is easy to see that the graph of ψ on $[t_2, t_3]$ is the symmetric reflection of the same graph on $[t_1, t_2]$ with respect to the point $(t_2, 0)$, and the graph of $\sin(\varphi - \theta)$ is the symmetric reflection of the same graph on $[t_1, t_2]$ with respect to the line $t = t_1$. Thus, if $\alpha \in (0, 1)$, the graph of $\sin(\varphi - \theta)$ has the following form. On the real line \mathbf{R} we have to consider the graph of $\sin t$, discard the intervals where

$|\sin t| > \alpha$, and join the remaining segments end-to-end to each other, so that they are connected at their end points. The result is a kind of “pseudo-sinusoid”.

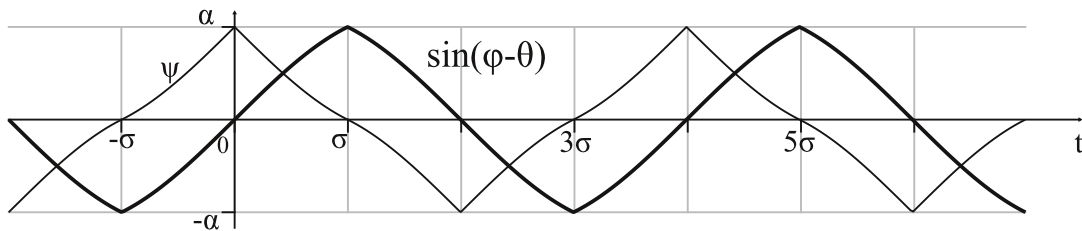


Figure 1: Case of increasing $\sin(\varphi - \theta)$ when $\psi > 0$

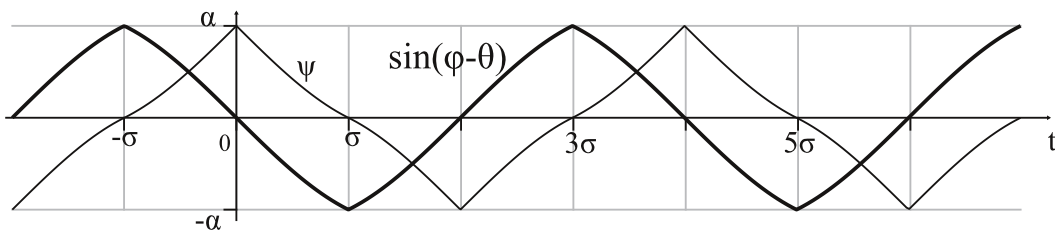


Figure 2: Case of decreasing $\sin(\varphi - \theta)$ when $\psi > 0$

The graph of ψ is obtained as follows. On the interval $[-\sigma, \sigma]$ it is $\psi(t) = \alpha - |\sin t|$ (peaked “cap”), on the next segment $[\sigma, 3\sigma]$, this cap is taken with the minus sign, then with the plus sign, and so on. We get a kind of sawtooth curve (see Fig. 1).

When these graphs are constructed, the controls u, v can be obtained from them by formulas (12), where the multi-valued Sign can be now replaced by the usual sign . It is easy to verify that here $u(t) = -\text{sign } \dot{\psi}(t)$, therefore, to determine u it is not necessary to use the graph of $\sin(\varphi - \theta)$, it is sufficient to know the graph of ψ . Thus, in the case when $\sin(\varphi - \theta)$ increases as $\psi > 0$, the whole extremal is defined by the graph of one function $\psi(t)$. If $\sin(\varphi - \theta)$ decreases as $\psi > 0$, the whole extremal is also defined by the graph of $\psi(t)$, but in this case $u(t) = \text{sign } \dot{\psi}(t)$ (see Fig. 2).

So, for all $\sigma \in (0, \pi/2)$ and the corresponding $\alpha = \sin \sigma \in (0, 1)$ we have the extremal, in which ψ has the above sawtooth type, $v = \text{sign } \psi$, and u is determined either from $u(t) = -\text{sign } \dot{\psi}(t)$, or has the opposite sign. We call them *extremals of type I*.

The interval $[t_0, T]$ can be placed anywhere on the axis t with the only condition that $\psi(T) = 0$. The left end of this interval can be arbitrary.

II) $\alpha = 1$. Here $|\sin(\varphi - \theta)| + |\psi| = 1$. Consider, as before, the maximal interval ω where $\psi > 0$ and $\sin(\varphi - \theta)$ increases from -1 to 1 . Then $v = 1$

and again, shifting the axis t , we can assume that $\varphi(t) - \theta = t$. In this case $\omega = (-\pi/2, \pi/2)$, and the graph of ψ has the form of “maximal cap”:

$$\psi(t) = 1 - |\sin(t)|.$$

For $t_2 = \pi/2$ we have $\dot{\psi}(t_2) = 0$ and therefore, unlike the case I, after t_2 the function ψ may not pass to the negative zone. Here the following variants of further behavior of ψ are possible:

a) the graph of ψ on $[t_1, t_1 + \pi]$ is a copy (translation) of its graph on $[t_1 - \pi, t_1]$ (the maximal peaked up cap),

b) the graph of ψ on $[t_1, t_1 + \pi]$ is a reflection of the graph of ψ on $[t_1 - \pi, t_1]$ with respect to t -axis (i.e., the shift of the graph of $-\psi$ on $[t_1 - \pi, t_1]$ (the maximal peaked down cap),

c) the values $\psi = 0$ and $\sin(\varphi - \theta) = const = \pm 1$ are preserved on an interval $[t_1, t_2]$ of arbitrary length, and then on the interval $[t_2, t_2 + \pi]$ one of the variants a) or b) are realized, i.e. ψ is a maximal cap either peaked up or down.

In all these cases, the extremal belong to *type II*.

Next, we need a more detailed decomposition of the above types I and II. In type I (for $0 < \alpha < 1$, i.e. $0 < \sigma < \pi/2$), the following cases are possible:

Type Ia) the extremal contains at least one “complete” cap, i.e., there is an interval of length 2σ where $\psi > 0$ or $\psi < 0$. As we have condition $\psi(T) = 0$, this requirement is equivalent to the condition $T - t_0 \geq 2\sigma$.

Type Ib) the extremal contains only an incomplete cap: $T - t_0 \in (\sigma, 2\sigma)$.

Type Ic) the extrememal contains no more than half of the cap: $T - t_0 \leq \sigma$.

In type II (i.e., when $\alpha = 1$, $\sigma = \pi/2$) the following cases are possible:

Type IIa) the extremal contains at least one complete cap, i.e., there is an interval of length π where $\psi > 0$ or $\psi < 0$. This is equivalent to that $T - t_0 \geq \pi$.

Type IIb) the extremal contains an incomplete cap: $T - t_0 \in (\pi/2, \pi)$.

Type IIb0) the extremal is an incomplete cap extended by zero: $\exists t_1 \in (t_0, \pi)$ such that $t_1 - t_0 \in (\pi/2, \pi)$; on the interval (t_0, t_1) the graph of ψ is an incomplete cap, and further $\psi \equiv 0$ on $[t_1, T]$.

Type IIc) the extremal contains no more than half a cap, i.e., $T - t_0 \leq \pi/2$.

Type IIc0) the extremal is an incomplete “semi-cap” extended by zero: $\exists t_1 \in (t_0, T)$ such that $t_1 - t_0 \leq \pi/2$ on (t_0, t_1) there holds either $\psi > 0$ or $\psi < 0$, and then $\psi \equiv 0$ on $[t_1, T]$.

Now we show that the extremals of some of these types are definitely not optimal (in the global sense).

3 Nonoptimality of extremals of type Ia and IIa

We show that for any extremal of these types there exists an admissible trajectory with a shorter time of motion. To prove it we make some geometrical constructions (that depend on the value of α).

Recall that $0 < \alpha \leq 1$, i.e. $0 < \sigma \leq \pi/2$. Take any extremal of type Ia and consider its final complete cap. Choose the coordinate system so that the beginning of this cap is at $t = 0$ and coincides with the origin, and the speed at this point is directed along the y -axis. For brevity, the circle of radius 1 centered at the point $(-1, 0)$ will be called *the left circle*, and the symmetric circle w.r.t. the ordinate axis *the right circle*.

Without loss of generality, we assume that in our cap, $\psi < 0$ and $\sin(\varphi - \theta)$ decreases. Then we have $v \equiv -1$ on $[0, 2\sigma]$, and $u = (1, -1)$ on the intervals $(0, \sigma)$ and $(\sigma, 2\sigma)$. This corresponds to the motion along the left circle in the positive direction (counter clockwise) during time σ , and then along the tangential circle still in the positive direction (i.e., the common point of the circles is a cusp point of the curve). If $\sigma < \pi/2$, we have a trajectory OAB of type Ia shown in Fig. 3a. Its endpoint B obviously lies in the first quadrant with $y(B) > 0$.

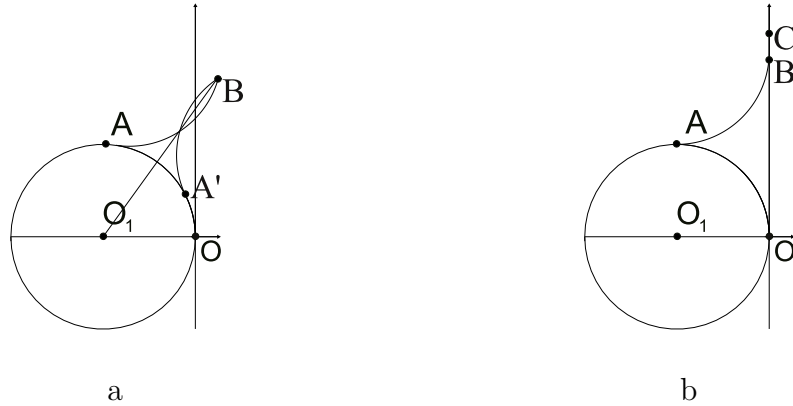


Figure 3:

Connect the center O_1 of left circle with the point B by a straight line. Let the arc $A'B$ be symmetric to the arc AB w.r.t. this line. The lengths of these arcs are obviously the same. However, the arc OA' has smaller length than the arc OA . This fact can be conveniently formulated as the following purely geometric assertion, which proof is easily seen from Fig. 4.

Lemma 3.1. *Let a straight line l pass through the point $O_1 = (-1, 1)$ with a positive slope, and let a point A lie on the left circle above this line, with $x_A \geq -1$. Let A' be the point symmetric to A w.r.t. l (clearly, it lies on the same circle). Then $|\sphericalcap OA| > |\sphericalcap OA'|$.*

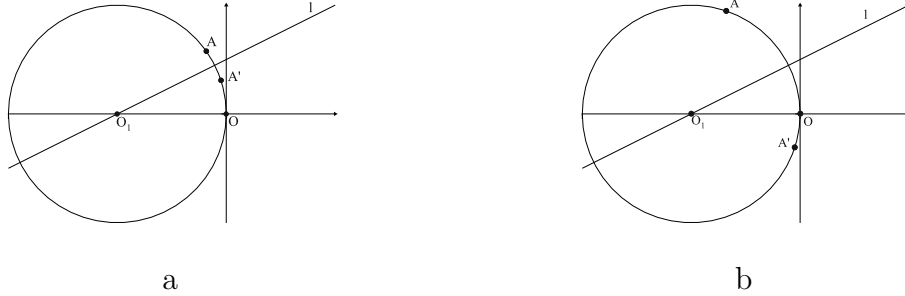


Figure 4: Arc OA' is shorter than OA .

So, the last cap of our extremal OAB has a bigger length than the trajectory consisting of the arc OA' of left circle and the arc $A'B$ of its tangential circle. Since the subarcs of this new trajectory are joined tangentially at A' , this trajectory is admissible in the problem, and since its length is shorter than that of our extremal, the last one is not optimal (in the global sense).

Consider now the case $\sigma = \pi/2$. Here the point B lies on the y -axis, and the extremal (of type IIa) can be continued by a segment BC lying on the y -axis, along which $(u, v) = (1, 0)$ (see Fig. 3b). Obviously, the trajectory $OABC$ is not optimal, because it has a bigger length than the straight line OBC , along which $(u, v) \equiv (1, 0)$.

Thus, only extremals of types Ib, Ic, IIb, IIb0, IIc, and IIc0 can pretend to be optimal.

Let us find the reachable set for each of these types, and if they intersect, figure out extremals of what type are optimal.

4 Construction of optimal synthesis

Note first that our problem (1) has two symmetries: replacing $u \mapsto -u$, we obtain the trajectory centrally symmetrical to the original one w.r.t. the origin, and the replacement of $v \mapsto -v$ in fact leads to the substitution $t \mapsto -t$ and then to $x \mapsto -x$, i.e. to the symmetric reflection of the original trajectory w.r.t. ordinate axis. Therefore, in the construction of reachability sets we can assume that the end point $p = (x(T), y(T))$ lies in the first quadrant \mathbf{R}_+^2 .

Consider now each of the above types of extremals.

Type Ib: incomplete nonmaximal cap. We assume that here $u = (-1, 1)$ on the intervals $(0, \gamma)$, $(\gamma, \gamma + \sigma)$, where $0 < \gamma < \sigma$, and $T = \gamma + \sigma$. Since in this case $v \equiv 1$, the point rotates all the time clockwise: first in the angle γ along the “left” circle, and then in the angle σ along its tangential “right” circle (i.e. not the original right circle, but the rotated one in the angle γ around the center O_1). Let us first fix any $\sigma \in (0, \pi/2)$. Then the limit values of γ are equal to 0

and σ . For $\gamma = 0$ the endpoint $p(\gamma, \sigma) = A'$ lies on the right circle, and when γ increases from 0 to σ it moves along the circle of radius $R(\sigma) = \sqrt{5 - 4 \cos \sigma}$ around the center O_1 , which we denote by $S_3(\sigma)$ (see Fig. 5). We claim that for $\gamma = \sigma$ the point $p(\gamma, \sigma) = B'$ lies below the horizontal axis.

Indeed, let G' be the point of the left circle, making the angle $-\sigma$ with the horizontal axis. The point $p(\sigma, \sigma) = B'$ is obtained by the motion along the arc OG' and then along the arc of $G'B'$ symmetric to OG' w.r.t. the common tangent line at the point G' . Then the point B' is symmetrical to the point O w.r.t. this tangent line, and therefore B' lies below the horizontal axis.

Thus, when γ runs over the interval $(0, \sigma)$, the endpoint $p(\gamma, \sigma)$ draws an open arc $A'B'$ of the circle $S_3(\sigma)$ with center O_1 and radius $R(\sigma)$.

Considering these arcs for all $\sigma \in (0, \pi/2)$, we get a set Q^+ , bounded by the arc OA of right circle, the arc AF of “big” circle $S_3(\pi/2)$ of radius $R(\pi/2) = \sqrt{5}$ and the curve $OB'F$, which is the trace of point $p(\sigma, \sigma)$. (It is easy to see that this curve is part of the cardioid with parametric equation $x = 2 \cos t (1 - \cos t)$, $y = 2 \sin t (1 - \cos t)$, $0 \leq t \leq 2\pi$).

So, Q^+ is an open set consisting of the endpoints of all the trajectories of type Ib in the case $\psi(t) > 0$, $u = (-1, 1)$.

If $\psi(t) > 0$ and $u = (1, -1)$ on the same intervals, then instead of Q^+ we obtain the centrally symmetrical set. Since it lies in the left semi-plane and we are interested in the first quadrant, we ignore this case.

In the case $\psi(t) < 0$, $u = (-1, 1)$, instead of $v = 1$ we have $v = -1$, and then, making substitution $x \mapsto -x$, we obtain a symmetric reflection of the set Q^+ w.r.t. ordinate axis. This case can be also ignored.

In the case $\psi(t) < 0$, $u = (1, -1)$ we get a set Q^- symmetric to Q^+ w.r.t. horizontal axis. Therefore, the full reachability set for the type Ib in the right semi-plane is the union of these two sets. This union is also the union of two open sets, one of which is bounded by the “big” circle $S_3(\pi/2)$ and the arcs OCE and $OB'F$ of cardioid, and the other is the intersection of the big and the right discs. In the first quadrant this set can be represented as the union of three sets Q_1, Q_2, Q_3 , where Q_1 is bounded by the arcs $OA'C$ and $\overline{OC'C}$, the set Q_2 is bounded by the arcs $OC'C, CA, AD$ and \overline{OD} , and the set Q_3 (which arose from Q^-) is bounded by the arcs EC, \overline{CA} (including the point C) and AE . (The bar over an arc means that this arc without its ends is also included in the given set.)

Now note that the points $p \in Q_1$ can be terminal points of the trajectories of type Ib only for $v = 1$, $u = (-1, 1)$ (we call them “lower” trajectories), while the points $p \in Q_3$ – only for $v = -1$, $u = (1, -1)$ (we call them “upper” trajectories). On the other hand, any point $p \in Q_2$ can be obtained by both upper and lower trajectory. However, by Lemma 3.1 the upper trajectory gives a greater time of motion, so it can be discarded, and then for the points from Q_2 only lower trajectories of type Ib remain. Thus, we proved the following

Proposition 4.1. *For any point from $Q_1 \cup Q_2$ the lower extremal of type Ib is optimal.*

Type IIb. Here the function $\psi(t)$ has the same form as in the case Ib with $\sigma = \pi/2$, $0 < \gamma < \sigma$.

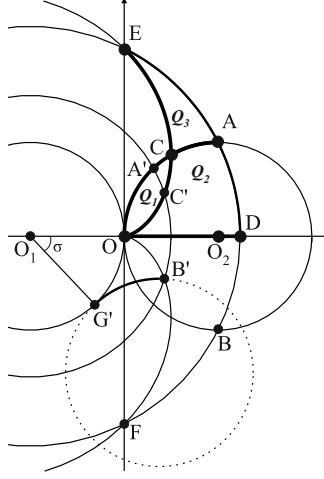


Figure 5:

In this case the set of all endpoints $p(\gamma, \pi/2)$ is the arc ADF of the “big” circle, without points A, F . Considering also the case $\psi < 0$, the full set of terminal points in the first quadrant is the arc $(ED]$ of the “big” circle, without point E .

As before, the points of the arc $(AD]$ can be obtained both by the upper and lower trajectories, and again by Lemma 3.1, the upper trajectory can be discarded. Thus, we proved the following

Proposition 4.2. *For the points of arc $(AD]$ only lower trajectories of type IIb are optimal. For the points of arc $(EA]$ only upper trajectories are optimal.*

Denote by Z_1 the closure of $Q_1 \cup Q_2$ without the origin, and by Z_3 the closure of Q_3 without the point E . Thus, $Z_1 \cup Z_3$ is the united reachability set for trajectories of type Ib and IIb in the first quadrant. The points of the “big” circle $S_3(\pi/2)$ in this set and only they correspond to trajectories of type IIb.

Types Ic and IIc. Here $T = \gamma \in (0, \pi/2]$; the case $\gamma < \pi/2$ corresponds to type Ic, and $\gamma = \pi/2$ to type IIc. If $u = 1$, the terminal point $p(\gamma)$ draws the arc $(OA]$ of right circle. The point $p(\pi/2) = A = (1, 1)$ and only it corresponds to a trajectory of type IIc.

The cases $u = -1$ or $\psi < 0$, $u = \pm 1$ give a symmetric reflection of the arc $(OA]$ w.r.t. either the origin or the coordinate axes, which nothing adds in the first quadrant.

Type IIb0. Trajectories of this type are obtained from trajectories of type IIb by adding a new section, along which $v = 0$ and u is the same as on the previous interval.

The reachability set is obtained as follows. Every lower trajectory of type IIb ending on the arc $(AD]$ should be continued by its tangent line. In the first quadrant, these tangents cover the set bounded by the arc AD and semi-lines AA_* $\overline{DD_*}$. Let us denote it by Z_5 .

Every upper trajectory of type IIb ending on the arc (EB) should be also continued by its tangent line. In the first quadrant these tangents cover the open

exterior of the big circle plus the ray (DD_*) . The difference of the obtained set and Z_5 we denote by Z_4 – this is an open set bounded by the ray EE_* , the arc EA and the ray AA_* , plus the last ray itself.

Thus, the complete reachability set for the trajectories of type IIb0 in the first quadrant is $Z_4 \cup Z_5$. Here, each point of Z_4 can be obtained by a single (upper) trajectory, and each point of Z_5 can be obtained by two – upper and lower – trajectories. As before, applying lemma 3.1, we see that the upper trajectory gives a bigger time of motion, so it can be discarded. This allows us to state the following

Proposition 4.3. *For the points of Z_5 , the lower trajectories of type IIb0 and only they are optimal.*

Type IIc0. Trajectories of this type are obtained from the trajectories of type IIc by adding a new section, along which $v = 0$ and u is the same as on the previous interval.

The reachability set is constructed as follows (see Fig. 5). Every arc OA' (i.e. a trajectory of type IIc) should be continued by its tangent line. These tangents cover an open set bounded by the ordinate axis, the arc OA of right circle and the ray AA_* , plus this ray. The part of this set, bounded by the ordinate axis and the arcs \overline{EC} and \overline{OC} , including the point C , we denote by Z_2 . (The case $\psi < 0$ adds nothing in the first quadrant.)

Thus, we divided the first quadrant into reachability sets Z_1, Z_2, Z_3, Z_4, Z_5 plus, separately, the positive ordinate axis (see Fig. 6).

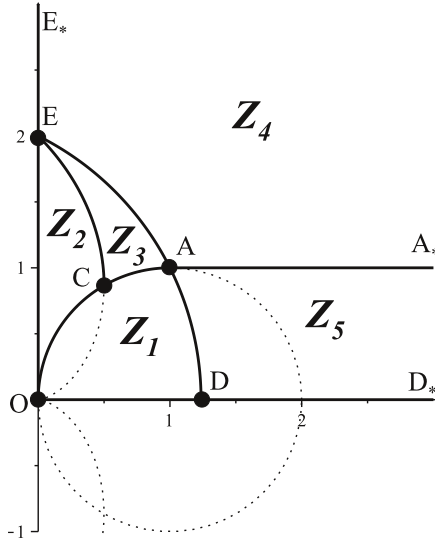


Figure 6: Reachability sets in the first quadrant.

Among the above sets, only in Z_3 and Z_4 we have not yet discarded the non-uniqueness of extremals. A point of Z_3 can be obtained both by the “upper”

trajectory of type Ib and by the trajectory of type IIc0. A point of Z_4 (except the ray AA_*) can be obtained both by the “upper” trajectory of type IIb0 and again by the trajectory of type IIc0. Consider these cases in more details.

Proposition 4.4. *For any point of $Z_3 \cup Z_4$, an upper trajectory of type IIc0 is optimal.*

Proof. Set Z_3 (Fig. 7a). For each upper trajectory OKN of type Ib, where $u = (1, -1)$, $v \equiv 1$, we can construct a trajectory $OK'N$ (which has an arc $K'N$ symmetric to the arc KN w.r.t. the line O_1N) with $u \equiv 1$, $v = (-1, 1)$, leading to the same point. By Lemma 3.1 this new trajectory gives a smaller time of motion, hence, the original upper trajectory is not optimal and can be discarded. But since the new trajectory does not satisfy the maximum principle (for it does not belong to the above found types of extremals), it is not optimal too, so only the trajectory OMN of type IIc0 remains.

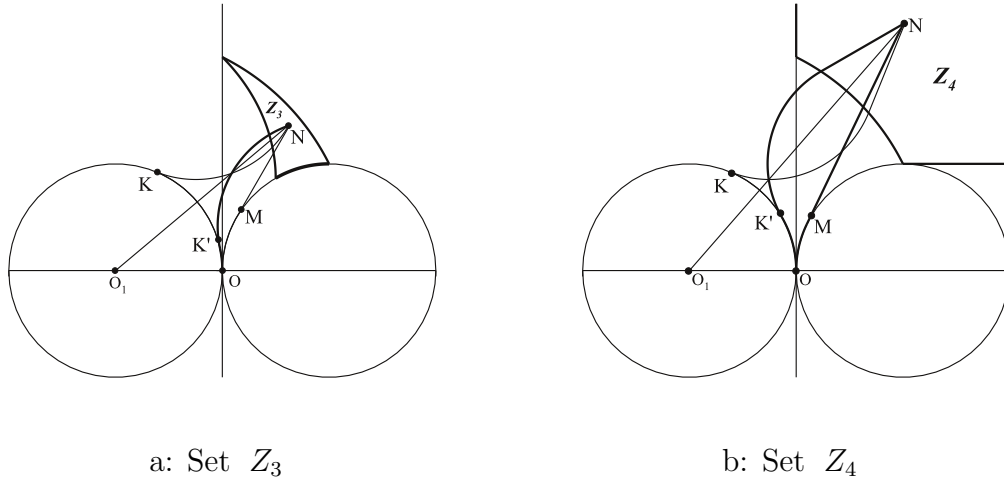


Figure 7: Optimal trajectory selection in sets Z_3 and Z_4

Set Z_4 (Fig. 7b). Here the situation is similar. For each trajectory OKN of type IIb0, where $u = (1, -1)$, $v = (-1, 0)$, we can construct a trajectory $OK'N$ with $u \equiv 1$, $v = (-1, 1, 0)$ leading to the same point. This new trajectory does not satisfy the maximum principle and, again by Lemma 3.1, gives a smaller time of motion than the original one. Therefore, both of this and the initial trajectory of type IIb0 can be discarded. So, only the trajectory OMN of type IIc0 remains, which will be optimal. (Note that for points lying on the ray AA_* , trajectories OMN and $OK'N$ coincide.) \square

Thus, for all three sets Z_2, Z_3, Z_4 , the optimal trajectories are unique and belong to type IIc0: first the point moves along the right circle, and then along its tangent line.

The above consideration allows us to construct a synthesis of optimal trajectories (Fig. 8). Let be given a terminal point $p = (x_T, y_T) \in \mathbf{R}_+^2$, $p \neq 0$.

Theorem 4.1.

If p lies inside Z_1 or on the arc (AD) , i.e. $(x_T - 1)^2 + y_T^2 < 1$, $y_T > 0$, $(x_T + 1)^2 + y_T^2 \leq 5$, then the optimal control is $(u, v) = ((-1, 1), (1, 1))$ with one switching point.

If p lies on the semi-interval $(OD]$, i.e. $x_T \leq \sqrt{5} - 1$, $y_T = 0$, then there are exactly two optimal trajectories: $(u, v) = ((1, -1), (-1, -1))$ and $(u, v) = ((-1, 1), (1, 1))$, symmetric to each other w.r.t horizontal axis and giving the same value of the cost.

If p lies on the arc $(OA]$, i.e. $(x_T - 1)^2 + y_T^2 = 1$, $x_T \leq 1$, then $u \equiv 1$, $v \equiv 1$ without switchings.

If p lies inside Z_5 , i.e. $(x_T + 1)^2 + y_T^2 > 5$, $0 < y_T < 1$, then the optimal control is $(u, v) = ((-1, 1), (1, 1), (1, 0))$ with two switching points.

If p lies on the ray (DD_*) , i.e. $x_T > \sqrt{5} - 1$, $y_T = 0$, then there are exactly two optimal trajectories: $(u, v) = ((1, -1), (-1, -1), (-1, 0))$ and $(u, v) = ((-1, 1), (1, 1), (1, 0))$ with two switching points, symmetric to each other w.r.t horizontal axis and giving the same value of the cost.

If p lies inside $Z_2 \cup Z_3 \cup Z_4$ (i.e. $(x_T - 1)^2 + y_T^2 > 1$, and, if moreover, $x_T \geq 1$, then $y_T \geq 1$), then the optimal control is $u \equiv 1$, $v = (1, 0)$ with one switching point on the arc $(OA]$.

Finally, if p lies “right ahead” (i.e. $x_T = 0$, $y_T > 0$), then $u \equiv 1$, $v \equiv 0$.

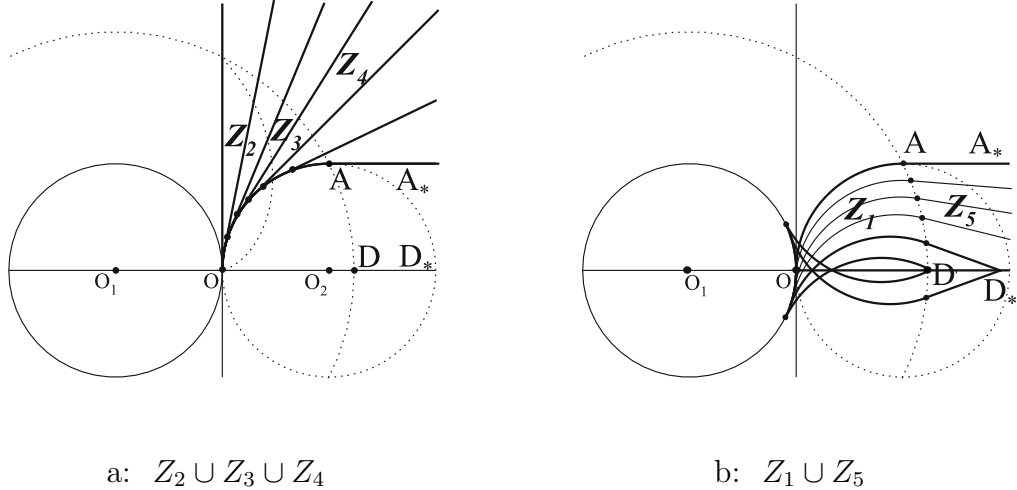


Figure 8: Synthesis of optimal trajectories

If we avoid the non-uniqueness of the optimal trajectory for $y_T = 0$ by discarding the “upper” trajectories, then, as a consequence of Theorem 5.1, we obtain an

expression for the instant value of optimal control in terms of the position of the target point (x_T, y_T) in the current coordinate system (the origin is placed at the current position and the velocity is directed along the ordinate axis), i.e. we obtain the control in the feedback form $(u, v) = \mathcal{F}(x_T, y_T)$.

Corollary.

If $p \in Z_1 \cup Z_5$, i.e. $x_T > 0$, $(x_T - 1)^2 + y_T^2 < 1$, or $x_T > 1$, $y_T < 1$, then $(u, v) = (-1, 1)$.

If $p \in Z_2 \cup Z_3 \cup Z_4$, i.e. $x_T > 0$, $(x_T - 1)^2 + y_T^2 \geq 1$, or $x_T > 0$, $y_T \geq 1$, then $(u, v) = (1, 1)$.

If, finally, p lies on the positive ordinate axis, i.e. $x_T = 0$, $y_T > 0$, then $(u, v) = (1, 0)$.

Note that this feedback relation is discontinuous on the common boundary of sets $Z_1 \cup Z_5$ and $Z_2 \cup Z_3 \cup Z_4$ (i.e. on the arc OA and on the ray AA_*), on the ordinate axis, and (with account of expansion of this feedback to the other quadrants due to symmetry), and on the x -axis. It is well known that such discontinuity is a characteristic feature of optimal control problems.

However, it is easy to verify that the optimal control, as a function $(u(t), v(t))$ of $t \in [0, T]$, continuously depends on the endpoint (x_T, y_T) with respect to the norm $\int_0^T (|u| + |v|) dt$, and the optimal time of motion — the problem value, or the Bellman function $T(x_T, y_T)$ — is also continuous. Moreover, one can show that this function is Hölder continuous, but its smoothness can be guaranteed only inside the sets $Z_1 \cup Z_5$ and $Z_2 \cup Z_3 \cup Z_4$.

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