Multi-target Linear-quadratic control problem: semi-infinite interval

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Abstract

We consider multi-target linear-quadratic control problem on semi-infinite interval. We show that the problem can be reduced to a simple convex optimization problem on the simplex

1 Introduction

Let $(H, <,>)$ be a Hilbert space, Z be its closed vector subspace, $h_1, \cdots, h_m$ and c be vectors in H. Consider the following optimization problem:

$$\max_{1 \leq i \leq m} \|h - h_i\| \to \min, h \in c + Z. \tag{1}$$

Here $\|\cdot\|$ is the norm in H induced by the scalar product $<,>$. In [FM], we analyzed (1) using duality theory for infinite-dimensional second-order cone programming. We obtained a reduction of this problem to a finite dimensional second-order cone programming and applied this result to a multi-target linear-quadratic control problem on a finite time interval. In this paper, we consider a reduction (1) to even simpler optimization problem of minimization of convex quadratic function on the $(m-1)$ dimensional simplex. We then apply this result to the analysis of a multi-target linear-quadratic control problem on semi-infinite time interval. We show that the coefficients of the quadratic function admit a simple expressions in term of the original data.

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2 Reduction to a simple quadratic programming problem

Let $f_i(h) = \|h - h_i\|^2, i = 1, 2, \cdots, m$. It is obvious that (1) is equivalent to the following optimization problem

$$z \to \min, \tag{2}$$

$$f_i(h) \leq z, i = 1, 2, \cdots, m, \tag{3}$$

$$h \in c + Z. \tag{4}$$

Consider the Lagrange function

$$\begin{aligned}\mathcal{L}(\lambda_1, \cdots, \lambda_m, h, z) &= z + \sum_{i=1}^{m} \lambda_i (f_i(h) - z) \\ &= z(1 - \sum_{i=1}^{m} \lambda_i) + \sum_{i=1}^{m} \lambda_i f_i(h).\end{aligned}$$

Notice that despite the fact that our original problem is infinite-dimensional, the usual KKT theorem holds true (see e.g. [MT],p.72). It is also clear that Slater conditions are satisfied. Hence, optimality condition for (2) - (4) take the form

$$\lambda_i \geq 0, \ \ \lambda_i(f_i(h) - z) = 0, \ i = 0, 1, 2, \cdots, m, \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial z} = 0, \ \sum_{i=1}^{m} \lambda_i \nabla f_i(h) \in Z^{\perp}, \tag{6}$$

where $\nabla f_i(h) = 2(h - h_i), \ i = 1, 2, \cdots, m, \ Z^{\perp}$ is the orthogonal complement of Z in H. Conditions (5), (6) lead to

$$\sum_{i=0}^{m} \lambda_i = 1, \ \lambda_i \geq 0, \ i = 1, 2, \cdots, m, \tag{7}$$

$$\pi_Z(h) = \sum_{i=1}^{m} \lambda_i (\pi_Z h_i). \tag{8}$$

Here $\pi_Z : H \to Z$ is the orthogonal projection. Let us form the Lagrange dual of (2), (3),(4). Consider

$$\varphi(\lambda_1, \lambda_2, \cdots, \lambda_m) = \min\{\mathcal{L}(\lambda_1, \cdots, \lambda_m, h, z) : h \in c + Z, \ z \in Z\}.$$

Using (7), (8), we obtain

$$\varphi(\lambda_1, \lambda_2, \cdots, \lambda_m) = \sum_{i=1}^{m} \lambda_i f_i(h(\lambda_1, \cdots, \lambda_m)) \tag{9}$$

where

$$h(\lambda_1, \cdots, \lambda_m) = \pi_{Z^\perp}(c) + \sum_{i=1}^{m} \lambda_i \pi_Z(h_i). \tag{10}$$

Notice that for any $h \in c + Z, \pi_{Z^\perp}(h) = \pi_{Z^\perp}(c)$. Here $\pi_{Z^\perp} : H \to Z^\perp$ is the orthogonal projection of H onto orthogonal complement $Z^\perp$ of Z. To further simplify (9), introduce the notation:

$$h(\lambda) = \sum_{i=1}^{m} \lambda_i h_i.$$

Then

$$\begin{aligned} f_j(h(\lambda_1, \cdots, \lambda_m)) &= \|\pi_Z(h(\lambda) - h_j) + \pi_{Z^\perp}(c - h_j)\|^2 \\ &= \|\pi_Z(h(\lambda) - \pi_Z(h_j))\|^2 + \|\pi_{Z^\perp}(c - h_j)\|^2 \\ &= \|\pi_Z(h(\lambda)\|^2 + \|\pi_Z(h_j))\|^2 - 2 < \pi_Z(h(\lambda)), \pi_Z(h_j) > \\ &\quad + \|\pi_{Z^\perp}(c - h_j)\|^2. \end{aligned}$$

Hence, according to (9)

$$\begin{aligned} \varphi(\lambda_1, \cdots, \lambda_m) &= \|\pi_Z(h(\lambda))\|^2 + \sum_{j=1}^{m} \lambda_j \|\pi_Z(h_j)\|^2 \\ &\quad -2 < \pi_Z(h(\lambda)), \pi_Z(h(\lambda)) > + \sum_{j=1}^{m} \lambda_j \|\pi_{Z^\perp}(c - h_j)\|^2. \end{aligned}$$

We, hence, arrive at the following expression of φ:

$$\varphi(\lambda_1, \cdots, \lambda_m) = -\|\pi_Z(\sum_{i=1}^{m} \lambda_i h_i)\|^2 + \sum_{j=1}^{m} \lambda_j (\|\pi_Z(h_j)\|^2 + \pi_{Z^\perp}(c - h_j)\|^2). \tag{11}$$

We can simplify (11) somewhat. Notice that

$$\|\pi_{Z^\perp}(c - h_j)\|^2 = \|\pi_{Z^\perp}(c)\|^2 + \|\pi_{Z^\perp}(h_j)\|^2 - 2 < \pi_{Z^\perp}(c), \pi_{Z^\perp}(h_j) > .$$

Consequently,

$$\begin{aligned} \varphi(\lambda_1, \cdots, \lambda_m) &= -\|\pi_Z(h(\lambda))\|^2 + \sum_{j=1}^{m} \lambda_j \|h_j\|^2 \\ &\quad -2 < \pi_{Z^\perp}(c), \pi_{Z^\perp}(h(\lambda)) > + \|\pi_{Z^\perp}(c)\|^2 \\ &= -\|h(\lambda)\|^2 + \|\pi_{Z^\perp}(h(\lambda) - c)\|^2 + \sum_{j=1}^{m} \lambda_j \|h_j\|^2 \end{aligned} \tag{12}$$

Here,

$$h(\lambda) = \sum_{i=1}^{m} \lambda_i h_i.$$

Hence, the Lagrange dual to (2), (3), (4) takes the form:

$$\varphi(\lambda_1, \cdots, \lambda_m) \to \max, \tag{13}$$

$$\sum_{i=1}^{m} \lambda_i = 1,\ \lambda_i \geq 0,\ i = 1, 2, \cdots, m. \tag{14}$$

If $(\lambda_1^*, \cdots, \lambda_m^*)$ is an optimal solution to (13), (14), we can recover the optimal solution of the original problem using the relation (10) and $\varphi(\lambda_1^*, \cdots, \lambda_m^*)$ gives the optimal value for the original problem (1).

3 Linear-quadratic case

Denoted by $L_2^n[0, \infty)$ the vector space of square integrable functions $f : [0, \infty) \to R^n$. Let $H = L_2^n[0, \infty) \times L_2^m[0, \infty)$, and

$$Z = \{(\alpha, \beta) \in H : \alpha \text{ is absolutely continuous on } [0, \infty),\ \dot{\alpha} = A\alpha + B\beta, \alpha(0) = 0\}.$$

Here A (respectively B) is an n by n (respectively n by m) matrix. Observe that

$$< (\alpha_1, \beta_1), (\alpha_2, \beta_2) >= \int_0^\infty [\alpha_1(t)^T \alpha_2(t) + \beta_1(t)^T \beta_2(t)] dt,$$

$$(\alpha_i, \beta_i) \in H,\ i = 1, 2.$$

In this setting, the problem (1) admits a natural interpretation as a linear-quadratic multi-target control problem. Its solution requires explicit computation of the coefficients of the objective function (12) which in turn requires an explicit description of orthogonal projection π_Z. Such a description has been found in [FM1]. We briefly describe it here.

Theorem 1 *Let C be an anti-stable n by n matrix (i.e. real parts of all eigenvalues of C are positive). Consider the following system of linear differential equations:*

$$\dot{x} = Cx + f, \tag{15}$$

where $f \in L_2^n[0, \infty)$. Then there exists a unique solution $L(f)$ of (15) belonging to $L_2^n[0, \infty)$. Moreover, the map $L : L_2^n[0, \infty) \to L_2^n[0, \infty)$ is linear and bounded. Explicitly:

$$L(f)(t) = -\int_0^\infty e^{-C\tau} f(t + \tau) d\tau.$$

For the proof, see [FM1]

Consider the algebraic Riccati equation

$$KBB^TK + A^TK + KA - I = 0. \tag{16}$$

We assume that (16) has a real symmetric solution K_{st} such that the matrix

$$F = A + BB^TK_{st}$$

is stable (i.e. real parts of all eigenvalues of F are negative). Notice that such a solution exists if and only if the pair (A, B) is stabilizable. See e.g. [fai].

Theorem 2

$Z^\perp = \{(\dot{p} + A^Tp, B^Tp); p \in L_2^n[0,\infty), p$ *is absolutely continuous,* $\dot{p} \in L_2^n[0,\infty)\}$.

Given $(\psi, \varphi) \in H$, *we have*

$$\begin{aligned} \psi &= x - (\dot{p} + A^Tp), & (17)\\ \varphi &= u - B^Tp, & (18) \end{aligned}$$

x *is the solution of the differential equation*

$$\dot{x} = (A + BB^TK_{st})x + BB^T\rho + B\varphi,\ x(0) = 0, \tag{19}$$

$$u = B^TK_{st}x + B^T\rho + \varphi, \tag{20}$$

$$p = K_{st}x + \rho, \tag{21}$$

and ρ *is a unique solution to the differential equation*

$$\dot{\rho} = -(A + BB^TK_{st})^T\rho - K_{st}B\varphi - \psi \tag{22}$$

belonging to $L_2^n[0,\infty)$.

In particular, $(x, u) \in Z$, $-(\dot{p} + A^Tp, B^Tp) \in Z^\perp$ *and consequently* Z *is a closed subspace in* H *with*

$$\pi_Z(\psi, \varphi) = (x, u),\ \ \pi_{Z^\perp}(\psi, \varphi) = -(\dot{p} + A^Tp, B^Tp).$$

Remark: The required solution ρ exists and unique by Theorem 1, since the matrix $-(A + BB^TK_{st})$ is anti-stable.

Sketch of the proof

Let $p \in L_2^n[0,\infty)$ be absolutely continuous and such that $\dot{p} \in L_2^n[0,\infty)$. Suppose that $(x, u) \in Z$. Then

$$\begin{aligned} < (x,u),(\dot p + A^T p, B^T p) > &= \int_0^\infty (x^T \dot p + x^T A^T p + u B^T p)\, dt \\ &= \int_0^\infty [x^T \dot p + (Ax+Bu)^T p]\, dt \\ &= \int_0^\infty (x^T \dot p + \dot x^T p)\, dt \\ &= \int_0^\infty \frac{d}{dt}(x^T p)\, dt \\ &= \lim_{\tau\to\infty} x^T(\tau)p(\tau) - x(0)^T p(0). \end{aligned}$$

But $x(\tau), p(\tau) \to 0$, as $\tau \to \infty$ (see e.g.[FM1] for details) and $x(0) = 0$. Hence

$$< (x,u),(\dot p + A^T p, B^T p) >= 0.$$

Let us now show that the decomposition (16), (18) takes place for an arbitrary $(\psi, \varphi) \in H$. Indeed, using (21)

$$\dot p = K_{st}\dot x + \dot \rho.$$

Hence by (19), (22)

$$\begin{aligned} \dot p &= K_{st}(A + BB^T K_{st})x + K_{st}BB^T\rho + K_{st}B\varphi \\ &\quad -(A + BB^T K_{st})^T\rho - K_{st}B\varphi - \psi. \end{aligned}$$

Combining all terms with x and all terms with ρ in two separate groups, we obtain

$$\begin{aligned} \dot p + A^T p &= \dot p + A^T K_{st}x + A^T\rho \\ &= (K_{st}A + K_{st}BB^T K_{st} + A^T K_{st})x \\ &\quad +(K_{st}BB^T - A^T - K_{st}BB^T + A^T)\rho - \psi. \end{aligned}$$

Using now the fact that K_{st} satisfies (16), we obtain:

$$\dot p + A^T p = x - \psi$$

which is (17). Using (20), (21), we obtain

$$\begin{aligned} u - B^T p &= B^T K_{st}x + B^T\rho + \varphi - B^T K_{st}x - B^T\rho \\ &= \varphi \end{aligned}$$

which is (18). Finally, it is clear that for x and u defined by (20), (21), we have

$$\dot x = Ax + Bu$$

and consequently $(x, u) \in Z$. This completes the proof of theorem 2.

Looking at (12), we see that the evaluation of coefficients of the quadratic function requires the knowledge of expressions of the type $\|\pi_{Z^\perp}(h)\|^2$, where $h \in H$.

Theorem 3 *Let $h = (\psi, \varphi) \in H$ and $\rho \in L_2^n[0, \infty)$ is the function entering the decomposition (17) and (18) and described in (22). Then*

$$\|\pi_Z(h)\|^2 = \|B^T\rho + \varphi\|^2, \tag{23}$$

$$\|\pi_{Z^\perp}(h)\|^2 = \|h\|^2 - \|B^T\rho + \varphi\|^2. \tag{24}$$

Proof: Let $(y, \nu) \in Z$. Let,further,

$$\Delta(y, \nu) = (\nu - B^T K_{st} y - B^T \rho - \varphi)^T (\nu - B^T K_{st} y - B^T \rho - \varphi)$$

Here for simplicity of notations we suppressed the dependence on t. Then

$$\Delta(x, u) = \Delta_1 + \Delta_2 + \Delta_3,$$

where
$\Delta_1 = (\nu - \varphi)^T (\nu - \varphi)$,
$\Delta_2 = (K_{st} y + \rho)^T BB^T (K_{st} y + \rho)$,
$\Delta_3 = -2(\nu - \varphi)^T (B^T K_{st} y + \rho)$.
Since $(y, \nu) \in Z$, we have

$$\dot{y} = Ay + B\nu, \; y(0) = 0.$$

Hence,

$$\Delta_2 = y^T (K_{st} BB^T K_{st}) y + \rho^T BB^T \rho + 2\rho^T BB^T K_{st} y,$$

$$\begin{aligned}
\Delta_3 &= -2(B\nu - B\varphi)^T (K_{st} y + \rho) \\
&= -2(\dot{y} - Ay - B\varphi)^T (K_{st} y + \rho) \\
&= -2\dot{y} K_{st} y + y^T (A^T K_{st} + K_{st} A) y + 2(B\varphi)^T K_{st} y \\
&\quad -2\dot{y}^T \rho + 2(Ay)^T \rho + 2(B\varphi)^T \rho.
\end{aligned}$$

Notice that $\dot{y}^T \rho + y^T \dot{\rho} = \frac{d}{dt}(y^T \rho)$. Hence,

$$\begin{aligned}
\Delta(y, \nu) &= (\nu - \varphi)^T (\nu - \varphi) + y^T (K_{st} BB^T K_{st} + A^T K_{st} + K_{st} A) y \\
&\quad +2y^T (\dot{\rho} + K_{st} B\varphi + K_{st} BB^T \rho + A^T \rho) + (B^T \rho)^T (B^T \rho) \\
&\quad +2\varphi^T (B^T \rho) - \frac{d}{dt}(y^T \rho) - \frac{d}{dt}(y^T K_{st} y).
\end{aligned}$$

Using the fact that K_{st} is a solution to (16) and (22), we obtain:

$$\begin{aligned}
\Delta(y,\nu) &= (\nu-\varphi)^T(\nu-\varphi) + y^T y - 2y^T\psi + (B^T\rho+\varphi)^T(B^T\rho+\varphi) \\
&\quad -\varphi^T\varphi - \frac{d}{dt}(y^T\rho) - \frac{d}{dt}(y^T K_{st} y) \\
&= (\nu-\varphi)^T(\nu-\varphi) + (y-\psi)^T(y-\psi) + (B^T\rho+\varphi)^T(B^T\rho+\varphi) \\
&\quad -\varphi^T\varphi - \psi^T\psi - \frac{d}{dt}(y^T\rho) - \frac{d}{dt}(y^T K_{st} y). \qquad (25)
\end{aligned}$$

Integrating (25) from 0 to $+\infty$ and using the fact that $y(0) = 0, y(t), \rho(t) \to 0$ as $t \to \infty$, we obtain:

$$\int_0^\infty \Delta(y,\nu)dt = \|(y-\psi,\nu-\varphi)\|^2 - \|(\psi,\varphi)\|^2 + \|B^T\rho+\varphi\|^2. \qquad (26)$$

Notice that $\Delta(y,\nu) \geq 0$ and $\Delta(y,\nu) = 0$ provided $(y,\nu) = \pi_Z(\psi,\varphi)$. See (20). Consequently, (26) implies that

$$\| (\psi,\varphi) \|^2 = \| B^T\rho + \varphi \|^2 + \| \pi_{Z^\perp}(\psi,\varphi) \|^2 .$$

Hence,

$$\| \pi_Z(\psi,\varphi) \|^2 = \| B^T\rho + \varphi \|^2 .$$

This completes the proof of Theorem 3.

We can now easily compute the coefficients of the objective function (11). Assuming $h_i = (\psi_i,\varphi_i) \in L_2^n[0,\infty) \times L_2^m[0,\infty),\ i = 1,2,\cdots,m,\ c = (\alpha,\beta) \in L_2^n[0,\infty) \times L_2^m[0,\infty)$ and noticing that by Theorem 3

$$\|\pi_Z(h(\lambda)-c)\|^2 = \int_0^\infty [B^T\rho(\lambda)+\varphi(\lambda)]^T[B^T\rho(\lambda)+\varphi(\lambda)]\, dt,$$

where $\rho(\lambda)$ is the solution of the differential equation

$$\frac{d}{dt}\rho(\lambda) = -(A+BB^TK_{st})^T\rho(\lambda) - K_{st}B(\varphi(\lambda)-\psi(\lambda)),$$

belonging to $L_2^n[0,\infty)$ and

$$\varphi(\lambda) = \sum_{i=1}^m \lambda_i(\varphi_i - \beta),\ \psi(\lambda) = \sum_{i=1}^m \lambda_i(\psi_i - \alpha).$$

Consequently,

$$\rho(\lambda) = \sum_{i=1}^m \lambda_i(\rho_i - \rho_c),$$

where ρ_i and ρ_c are $L_2^n[0,\infty)$ solutions of differential equations

$$\dot{\rho}_i = -(A+BB^TK_{st})\rho_i - K_{st}B\varphi_i - \psi_i,\ i = 1,2,\cdots,m,$$

$$\dot{\rho}_c = -(A + BB^T K_{st})\rho_c - K_{st} B\beta - \alpha,$$

respectively.

Hence,

$$\|\pi_Z(h(\lambda) - c)\|^2 = \int_0^\infty \Gamma(\lambda)^T \Gamma(\lambda)\, dt,$$

where

$$\Gamma(\lambda) = \sum_{i=1}^{m} \lambda_i [B^T(\rho_i - \rho_c) + (\varphi_i - \beta)],$$

which allows us to easily express the objective function (12) in terms of integrals of ρ_i and ρ_c.

4 Concluding remarks

In this paper we have shown that multi-target linear-quadratic control problem on semi-infinite interval can be reduced to solving a simple convex optimization on the simplex. The reduction involves solving one standard algebraic Riccati equation and $m+1$ linear differential equations where m is the number of targets.Notice that our results can be easily extended to discrete-time systems.

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