On Kusuoka representation of law invariant risk measures

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Abstract. In this paper we discuss representations of law invariant coherent risk measures in a form of integrals of the Average Value-at-Risk measures. We show that such integral representation exists iff the dual set of the considered risk measure is generated by one of its elements, and this representation is uniquely defined. On the other hand, representation of risk measures as maximum of such integral forms is not unique. The suggested approach gives a constructive way for writing such representations.

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1 Introduction

In this paper we discuss Kusuoka [8] representations of law invariant coherent risk measures. We give a somewhat different from the standard literature view of such representations. In particular we show that for comonotonic risk measures the corresponding representation is unique, while such uniqueness doesn't hold for general law invariant coherent risk measures. Let us recall some basic definitions and results.

Consider a probability space $(\Omega, \mathcal{F}, P)$ and the space $\mathcal{Z} := L_p(\Omega, \mathcal{F}, P)$, $p \in [1, \infty)$, of measurable functions $Z : \Omega \to \mathbb{R}$ (random variables) having finite p-th order moment; for $p = \infty$ the space $\mathcal{Z} := L_\infty(\Omega, \mathcal{F}, P)$ is formed by essentially bounded measurable functions. Unless stated otherwise all probabilistic statements, in particular expectations, will be with respect to the reference probability distribution P. A function $\rho : \mathcal{Z} \to \mathbb{R}$ is said to be a coherent risk measure if it satisfies the following axioms (Artzner et al [2])

(A1) *M*onotonicity: If $Z, Z' \in \mathcal{Z}$ and $Z \succeq Z'$, then $\rho(Z) \geq \rho(Z')$.

(A2) *C*onvexity:
$$\rho(tZ + (1-t)Z') \leq t\rho(Z) + (1-t)\rho(Z')$$
for all $Z, Z' \in \mathcal{Z}$ and all $t \in [0, 1]$.

(A3) *T*ranslation Equivariance: If $a \in \mathbb{R}$ and $Z \in \mathcal{Z}$, then $\rho(Z + a) = \rho(Z) + a$.

(A4) *P*ositive Homogeneity: If $t \geq 0$ and $Z \in \mathcal{Z}$, then $\rho(tZ) = t\rho(Z)$.

The notation $Z \succeq Z'$ means that $Z(\omega) \geq Z'(\omega)$ for a.e. $\omega \in \Omega$. For a thorough discussion of coherent risk measures we refer to Föllmer and Schield [6]. Risk measures $\rho : \mathcal{Z} \to \mathbb{R}$ satisfying axioms (A1)–(A3) are called convex. A risk measure $\rho : \mathcal{Z} \to \mathbb{R}$ is said to be law invariant if $\rho(Z)$ depends only on the distribution of Z, i.e., if Z and Z' are two distributionally equivalent random variables, then $\rho(Z) = \rho(Z')$.

For $p \in [1, \infty)$ the space $\mathcal{Z} = L_p(\Omega, \mathcal{F}, P)$ is paired with the space $\mathcal{Z}^* = L_q(\Omega, \mathcal{F}, P)$, where $q \in (1, \infty]$ is such that $1/p + 1/q = 1$, with the corresponding scalar product

$$\langle \zeta, Z \rangle = \int_\Omega \zeta(\omega) Z(\omega) dP(\omega), \;\; \zeta \in \mathcal{Z}^*, Z \in \mathcal{Z}. \tag{1.1}$$

In that case $\mathcal{Z}^*$ coincides with the linear space of continuous linear functionals on $\mathcal{Z}$, i.e., $\mathcal{Z}^*$ is the dual of the Banach space $\mathcal{Z}$. For $\mathcal{Z} = L_\infty(\Omega, \mathcal{F}, P)$ the situation is more involved, the dual of the Banach space $L_\infty(\Omega, \mathcal{F}, P)$ is quite complicated. The standard practice, which we follow, is to pair $\mathcal{Z} = L_\infty(\Omega, \mathcal{F}, P)$ with the space $\mathcal{Z}^* := L_1(\Omega, \mathcal{F}, P)$ by using the respective scalar product of the form (1.1).

It is known [12] that (real valued) convex risk measures are continuous in the norm topology of the space $L_p(\Omega, \mathcal{F}, P)$. Thus for $p \in [1, \infty)$ it follows by the Fenchel-Moreau Theorem that

$$\rho(Z) = \sup_{\zeta \in \mathcal{Z}^*} \big\{ \langle \zeta, Z \rangle - \rho^*(\zeta) \big\}, \;\; \forall Z \in \mathcal{Z}, \tag{1.2}$$

where $\rho^*(\zeta) := \sup_{Z \in \mathcal{Z}} \big\{ \langle \zeta, Z \rangle - \rho(Z) \big\}$ is the corresponding conjugate functional. For $p = \infty$ we pair $L_\infty(\Omega, \mathcal{F}, P)$ with the space $L_1(\Omega, \mathcal{F}, P)$, rather than with its dual space of continuous linear functionals. In that case we *assume* that $\rho(\cdot)$ is lower semicontinuous in the respective paired topology, which is the weak* topology of $L_\infty(\Omega, \mathcal{F}, P)$. Then the Fenchel-Moreau Theorem can be invoked in order to establish the corresponding dual representation (1.2).

If the risk measure ρ is convex and positively homogeneous (i.e., is coherent), then ρ^* is an indicator function of a convex and closed, in the respective paired topology, set $\mathfrak{A} \subset \mathcal{Z}^*$, and hence the dual representation (1.2) takes the form

$$\rho(Z) = \sup_{\zeta \in \mathfrak{A}} \langle \zeta, Z \rangle, \ \ \forall Z \in \mathcal{Z}. \tag{1.3}$$

The set $\mathfrak{A}$ is uniquely defined and is given by

$$\mathfrak{A} = \big\{ \zeta \in \mathcal{Z}^* : \langle \zeta, Z \rangle \le \rho(Z), \ \forall Z \in \mathcal{Z} \big\}. \tag{1.4}$$

Because of the axioms (A1) and (A3) the set $\mathfrak{A}$ consists of density functions $\zeta : \Omega \to \mathbb{R}$, i.e., if $\zeta \in \mathfrak{A}$, then $\zeta \succeq 0$ and $\int \zeta dP = 1$ (cf., [12]). We refer to the set $\mathfrak{A}$ as the *dual set* of the corresponding coherent risk measure ρ. For $p \in [1, \infty)$ the dual set $\mathfrak{A}$ is weakly* compact, and hence the maximum in (1.3) is attained for any $Z \in \mathcal{Z}$. On the other hand, for $p = \infty$ the space $L_\infty(\Omega, \mathcal{F}, P)$ is paired with $L_1(\Omega, \mathcal{F}, P)$. In that case the dual set $\mathfrak{A} \subset L_1(\Omega, \mathcal{F}, P)$ is not necessarily compact in the respective paired topology (which is the weak topology of the Banach space $L_1(\Omega, \mathcal{F}, P)$), and the corresponding maximum may be not attained (see Example 1 in the next section).

An important example of law invariant coherent risk measure is the *Average Value-at-Risk* measure (also called the Conditional Value-at-Risk or Expected Shortfall measure), which can be defined as

$$\mathsf{AV@R}_\alpha(Z) := \inf_{t \in \mathbb{R}} \left\{ t + (1-\alpha)^{-1} \mathbb{E}[Z - t]_+ \right\}, \tag{1.5}$$

or equivalently as

$$\mathsf{AV@R}_\alpha(Z) = (1-\alpha)^{-1} \int_\alpha^1 \mathsf{V@R}_\tau(Z) d\tau, \tag{1.6}$$

where $\alpha \in [0, 1)$, $F_Z(\cdot)$ is the cumulative distribution function of Z and

$$\mathsf{V@R}_\alpha(Z) := F_Z^{-1}(\alpha) = \inf\{t : F_Z(t) \ge \alpha\} \tag{1.7}$$

is the corresponding left side quantile. It is natural to use here the space $\mathcal{Z} := L_1(\Omega, \mathcal{F}, P)$. The dual set of $\mathsf{AV@R}_\alpha$ is

$$\mathfrak{A} = \big\{ \zeta : 0 \preceq \zeta \preceq (1-\alpha)^{-1}, \ \textstyle\int \zeta dP = 1 \big\} \subset L_\infty(\Omega, \mathcal{F}, P). \tag{1.8}$$

For a given $Z \in \mathcal{Z}$, $\mathsf{AV@R}_\alpha(Z)$ is monotonically nondecreasing in α, and it follows from (1.6) that $\mathsf{AV@R}_\alpha(Z)$ is a continuous function of $\alpha \in [0, 1)$.

It is possible to show that $\mathsf{AV@R}_0(\cdot) = \mathbb{E}(\cdot)$, and $\mathsf{AV@R}_\alpha(Z)$ tends to $\operatorname{ess\,sup}(Z)$ as $\alpha \uparrow 1$. Therefore we set $\mathsf{AV@R}_1(Z) := \operatorname{ess\,sup}(Z)$. In that case, in order for $\mathsf{AV@R}_1(Z)$ to be real valued, we will have to use the space $\mathcal{Z} := L_\infty(\Omega, \mathcal{F}, P)$. The dual set of $\mathsf{AV@R}_1$, for $L_\infty(\Omega, \mathcal{F}, P)$ paired with $L_1(\Omega, \mathcal{F}, P)$, consists of density functions

$$\mathfrak{A} = \{ \zeta \succeq 0 : \textstyle\int \zeta dP = 1 \} \subset L_1(\Omega, \mathcal{F}, P). \tag{1.9}$$

It is interesting to note that, as it was pointed earlier, for $\rho := \mathsf{AV@R}_\alpha$ with $\alpha \in [0, 1)$ the dual set $\mathfrak{A}$ is weakly* compact, and hence the maximum in the dual representation (1.3) is always attained. On the other hand, for $\rho := \mathsf{AV@R}_1$ with $\mathcal{Z} := L_\infty(\Omega, \mathcal{F}, P)$, maximum in the respective dual representation may be not attained.

- Unless stated otherwise we assume in the remainder of this paper that the probability space $(\Omega, \mathcal{F}, P)$ is *nonatomic*.

It was shown by Kusuoka [8] that any law invariant coherent risk measure can be represented in the following form

$$\rho(Z) = \sup_{\mu \in \mathfrak{M}} \int_0^1 \mathsf{AV@R}_\alpha(Z) d\mu(\alpha), \tag{1.10}$$

where $\mathfrak{M}$ is a set of probability measures on the interval [0,1]. Actually the original proof in [8] was for the $L_\infty(\Omega, \mathcal{F}, P)$ space; for a discussion in $L_p(\Omega, \mathcal{F}, P)$ spaces see, e.g., [11]. It could be noted that if a measure $\mu \in \mathfrak{M}$ has a positive mass at $\alpha = 1$, then the right hand side of (1.10) is $+\infty$ for any unbounded from above function $Z(\omega)$. Therefore for $\rho : L_p(\Omega, \mathcal{F}, P) \to \mathbb{R}$ with $p \in [1, \infty)$, all measures $\mu \in \mathfrak{M}$ should be on the interval $[0, 1)$. Also if (1.10) holds for some set $\mathfrak{M}$, then it holds for the convex hull of $\mathfrak{M}$. Moreover, as the following proposition shows we can take the topological closure of this convex hull with respect to the weak topology of probability measures (see, e.g., [3] for a discussion of the weak topology). Therefore it can be assumed that the set $\mathfrak{M}$ is convex and closed, and hence by Prohorov's theorem is compact, in the weak topology of probability measures.

Proposition 1.1 *Let $\rho : L_p(\Omega, \mathcal{F}, P) \to \mathbb{R}$ be a coherent risk measure representable in the form* (1.10) *for some set $\mathfrak{M}$ of probability measures on the interval* $[0, 1]$*. Then*

$$\rho(Z) = \sup_{\mu \in \overline{\mathfrak{M}}} \int_0^1 \mathsf{AV@R}_\alpha(Z) d\mu(\alpha), \quad Z \in L_p(\Omega, \mathcal{F}, P), \tag{1.11}$$

where $\overline{\mathfrak{M}}$ is the closure of $\mathfrak{M}$ in the weak topology of probability measures.

Proof. For a given $Z \in L_p(\Omega, \mathcal{F}, P)$ we have that $\phi(\alpha) := \mathsf{AV@R}_\alpha(Z)$ is a real valued continuous and monotonically nondecreasing function on the interval $[0, 1)$. If $Z(\cdot)$ is essentially bounded, and hence $\phi : [0, 1] \to \mathbb{R}$ is bounded, then the assertion follows in a straightforward way from the definition of weak convergence (see [3, p.7]). In general, let $\mu_k \in \mathfrak{M}$, $k = 1, ...$, be a sequence of probability measures converging (in the weak topology) to a probability measure $\bar{\mu}$. Recall that the weak topology of probability measures on the interval [0,1] is metrizable and hence can be described in terms of convergent sequences. We need to show that

$$\int_0^1 \phi(\alpha) d\bar{\mu}(\alpha) \le \rho(Z). \tag{1.12}$$

Indeed, for any $\tau \in (0, 1)$ we have

$$\lim_{k \to \infty} \int_0^1 \phi(\alpha) \mathbf{1}_{[0,\tau]}(\alpha) d\mu_k(\alpha) = \int_0^1 \phi(\alpha) \mathbf{1}_{[0,\tau]}(\alpha) d\bar{\mu}(\alpha), \tag{1.13}$$

where $\mathbf{1}_A(\cdot)$ denotes the indicator function of set A. By adding a constant to the function $Z(\cdot)$ if necessary, we can assume without loss of generality that $\mathsf{AV@R}_0(Z) > 0$, and hence $\phi(\alpha) > 0$ for all $\alpha \in [0, 1)$, and thus

$$\int_0^1 \phi(\alpha) \mathbf{1}_{[0,\tau]}(\alpha) d\mu_k(\alpha) \le \int_0^1 \phi(\alpha) d\mu_k(\alpha) \le \rho(Z), \ k = 1, ... \tag{1.14}$$

It follows from (1.13) and (1.14) that

$$\int_0^1 \phi(\alpha) \mathbf{1}_{[0,\tau]} d\bar{\mu}(\alpha) \le \rho(Z). \tag{1.15}$$

Moreover, by the Monotone Convergence Theorem

$$\lim_{\tau \uparrow 1} \int_0^1 \phi(\alpha) \mathbf{1}_{[0,\tau]}(\alpha) d\bar{\mu}(\alpha) = \int_0^1 \phi(\alpha) d\bar{\mu}(\alpha), \tag{1.16}$$

and hence (1.12) follows. This completes the proof. ▮

2 A preliminary discussion

Let $T : \Omega \to \Omega$ be one-to-one onto mapping, i.e., $T(\omega) = T(\omega')$ iff $\omega = \omega'$ and $T(\Omega) = \Omega$. It is said that T is a *measure-preserving transformation* if image $T(A) = \{T(\omega) : \omega \in A\}$ of any measurable set $A \in \mathcal{F}$ is also measurable and $P(A) = P(T(A))$ (see, e.g., [3, p.311]). Let us denote by

$$\mathfrak{G} := \{\text{the set of one-to-one onto measure-preserving transformations } T : \Omega \to \Omega\}.$$

We have that if $T \in \mathfrak{G}$, then $T^{-1} \in \mathfrak{G}$; and if $T_1, T_2 \in \mathfrak{G}$, then their composition[1] $T_1 \circ T_2 \in \mathfrak{G}$. That is, $\mathfrak{G}$ forms a group of transformations.

The measure-preserving transformation group $\mathfrak{G}$ was used in Jouini, Schachermayer and Touzi [7] to study coherent risk measures. Let us observe that since the probability space $(\Omega, \mathcal{F}, P)$ is nonatomic, we have that two random variables $Z, Z' \in \mathcal{Z}$ have the same distribution iff there exists a measure-preserving transformation $T \in \mathfrak{G}$ such that $Z' := Z \circ T$ (see, e.g., [7, Lemma A.4]). This implies that a risk measure $\rho : \mathcal{Z} \to \mathbb{R}$ is law invariant iff it is invariant with respect to measure-preserving transformations, i.e., iff for any $T \in \mathfrak{G}$ it follows that $\rho(Z \circ T) = \rho(Z)$ for all $Z \in \mathcal{Z}$. For a measurable function $\zeta : \Omega \to \mathbb{R}$ denote by

$$\mathcal{O}(\zeta) := \{\zeta \circ T : T \in \mathfrak{G}\} \tag{2.1}$$

the corresponding orbit of ζ. By the above discussion, $\mathcal{O}(\zeta)$ forms a class of distributionally equivalent functions.

Proposition 2.1 *A convex risk measure $\rho : \mathcal{Z} \to \mathbb{R}$ is law invariant iff its conjugate function ρ^* is invariant with respect to measure-preserving transformations, i.e., iff for any $T \in \mathfrak{G}$ it follows that $\rho^*(\zeta \circ T) = \rho^*(\zeta)$ for all $\zeta \in \mathcal{Z}^*$.*

Proof. For any $\zeta \in \mathcal{Z}^*$ and $T \in \mathfrak{G}$ we have

$$\begin{array}{rcl} \rho^*(\zeta \circ T) & = & \sup_{Z \in \mathcal{Z}} \left\{ \int \zeta(T(\omega)) Z(\omega) dP(\omega) - \rho(Z) \right\} \\ & = & \sup_{Z \in \mathcal{Z}} \left\{ \int \zeta(\omega) Z(T^{-1}(\omega)) dP(\omega) - \rho(Z) \right\} \\ & = & \sup_{Z' \in \mathcal{Z}} \left\{ \int \zeta(\omega) Z'(\omega) dP(\omega) - \rho(Z' \circ T) \right\}, \end{array}$$

where we made change of variables $Z' = Z \circ T^{-1}$. If ρ is law invariant and hence is invariant with respect to measure-preserving transformations, then $\rho(Z' \circ T) = \rho(Z')$ and thus it follows that $\rho^*(\zeta \circ T) = \rho^*(\zeta)$.

Conversely, recall that we have that $\rho^{**} = \rho$. Therefore if ρ^* is invariant with respect to measure-preserving transformations, then by the same arguments it follows that ρ is invariant with respect to measure-preserving transformations. ∎

Corollary 2.1 *A coherent risk measure $\rho : \mathcal{Z} \to \mathbb{R}$ is law invariant iff its dual set $\mathfrak{A}$ is invariant with respect to measure-preserving transformations, i.e., iff for any $\zeta \in \mathfrak{A}$ it follows that $\mathcal{O}(\zeta) \subset \mathfrak{A}$.*

Definition 2.1 *We say that the dual set $\mathfrak{A}$, of a law invariant coherent risk measure $\rho : \mathcal{Z} \to \mathbb{R}$, is generated by a set $\Upsilon \subset \mathfrak{A}$ if*

$$\rho(Z) = \sup_{\zeta \in \mathcal{O}(\Upsilon)} \langle \zeta, Z \rangle, \;\; \forall Z \in \mathcal{Z}, \tag{2.2}$$

where $\mathcal{O}(\Upsilon) := \cup_{\eta \in \Upsilon} \mathcal{O}(\eta)$. In particular, if $\Upsilon = \{\eta\}$ is a singleton, we say that $\mathfrak{A}$ is generated by η.

[1] Composition $T = T_1 \circ T_2$ of two mappings is the mapping $T(\omega) = T_1(T_2(\omega))$.

Note that since ρ is law invariant, we have by Proposition 2.1 that the set $\mathcal{O}(\Upsilon)$ forms a subset of the dual set $\mathfrak{A}$. It follows from (2.2) that the topological closure (in the respective paired topology) of the convex hull of set $\mathcal{O}(\Upsilon)$ coincides with the dual set $\mathfrak{A}$.

Recall that an element $\zeta \in \mathfrak{A}$ is said to be an *extreme point* of $\mathfrak{A}$, if $\zeta_1, \zeta_2 \in \mathfrak{A}$, $t \in (0,1)$ and $\zeta = t\zeta_1 + (1-t)\zeta_2$, implies that $\zeta_1 = \zeta_2 = \zeta$. For $p \in [1, \infty)$ the dual set $\mathfrak{A}$ is convex and weakly* compact and hence by the Krein-Milman Theorem coincides with the topological closure (in the weak* topology) of the convex hull of the set of its extreme points.

Example 1 Consider $\rho := \mathsf{AV@R}_\alpha$ with $\alpha \in [0,1)$. The corresponding dual set $\mathfrak{A}$ is described in equation (1.8). Let us show that the dual set $\mathfrak{A}$ is generated by its element[2] $\eta := (1-\alpha)^{-1}\mathbf{1}_A$, where $A \in \mathcal{F}$ is a measurable set such that $P(A) = 1-\alpha$. The corresponding set $\mathcal{O}(\eta)$ can be written as

$$\mathcal{O}(\eta) = \{(1-\alpha)^{-1}\mathbf{1}_B : B \in \mathcal{F},\ P(B) = 1-\alpha\}.$$

Let $Z \in \mathcal{Z}$ and

$$t^* := \inf\big\{t : P\{\omega : Z(\omega) \geq t\} \leq 1-\alpha\big\} = \mathsf{V@R}_{1-\alpha}(Z).$$

Suppose for the moment that $P\{\omega : Z(\omega) = t^*\} = 0$, i.e., cumulative distribution function (cdf) $F(t)$ of Z is continuous at $t = t^*$. Then for $B^* := \{\omega : Z(\omega) \geq t^*\}$ we have that $P(B^*) = 1-\alpha$. It follows that

$$\sup_{\zeta \in \mathfrak{A}(\eta)} \langle Z, \zeta\rangle = \sup_{B \in \mathcal{F}} \left\{(1-\alpha)^{-1}\int_B Z dP : P(B) = 1-\alpha\right\} = (1-\alpha)^{-1}\int_{B^*} Z dP.$$

Moreover, $\int_{B^*} Z dP = \int_{t^*}^{+\infty} t dF(t)$, and hence

$$\sup_{\zeta \in \mathcal{O}(\eta)} \langle Z, \zeta\rangle = \mathsf{AV@R}_\alpha(Z). \tag{2.3}$$

Since the set of random variables $Z \in \mathcal{Z}$ having continuous cdf forms a dense subset of $\mathcal{Z}$, it follows that formula (2.3) holds for all $Z \in \mathcal{Z}$. We obtain that the set $\mathfrak{A}$ is generated by η.

We also have that $\mathcal{O}(\eta)$ coincides with the set of extreme points of the set $\mathfrak{A}$. Indeed, if $\zeta \in \mathfrak{A}$, then $\zeta(\omega) \leq \eta(\omega)$ for a.e. $\omega \in A$ and $\zeta(\omega) \geq \eta(\omega)$ for a.e. $\omega \in \Omega \setminus A$. It follows that if $\zeta_1, \zeta_2 \in \mathfrak{A}$, $t \in (0,1)$ and $\eta = t\zeta_1 + (1-t)\zeta_2$, then $\zeta_1 = \zeta_2 = \eta$, i.e., η is an extreme point of $\mathfrak{A}$. The same arguments can be applied to every element of $\mathcal{O}(\eta)$ to show that all points of $\mathcal{O}(\eta)$ are extreme points of $\mathfrak{A}$. Now let $\zeta \in \mathfrak{A}$ be such that $\zeta \notin \mathcal{O}(\eta)$. Then there exists $\varepsilon > 0$ and a set $B \in \mathcal{F}$ of positive measure such that for a.e. $\omega \in B$ it holds that $\zeta(\omega) - \varepsilon \geq 0$ and $\zeta(\omega) + \varepsilon \leq (1-\alpha)^{-1}$. Let us partition B into two disjoint sets B_1 and B_2 such that $P(B_1) = P(B_2) = P(B)/2$. Then $\zeta_1 := \zeta + \varepsilon\mathbf{1}_{B_1} - \varepsilon\mathbf{1}_{B_2} \in \mathfrak{A}$, $\zeta_2 := \zeta + \varepsilon\mathbf{1}_{B_2} - \varepsilon\mathbf{1}_{B_1} \in \mathfrak{A}$ and $\zeta = (\zeta_1 + \zeta_2)/2$. That is, ζ is not an extreme point of $\mathfrak{A}$.

On the other hand, consider $\rho := \mathsf{AV@R}_1$. Its dual set $\mathfrak{A}$ is the set of density functions (see (1.9)). This dual set does not have extreme points and is not generated by one of its elements. ∎

Since $(\Omega, \mathcal{F}, P)$ is nonatomic, we can assume without loss of generality that Ω is the interval $[0,1]$ equipped with its Borel sigma algebra and uniform reference distribution P.

Lemma 2.1 *Let Ω be the interval $[0,1]$ equipped with its Borel sigma algebra and uniform distribution P, and $\mathfrak{G}$ be the corresponding group of measure-preserving transformations. Then for any measurable function $\eta : (0,1) \to \mathbb{R}$ there exists monotonically nondecreasing right hand side continuous function $\zeta : (0,1) \to \mathbb{R}$ and $T \in \mathfrak{G}$ such that $\eta = \zeta \circ T$ a.e.*

[2]By $\mathbf{1}_A(\cdot)$ we denote indicator function of set A, i.e., $\mathbf{1}_A(\omega) = 1$ if $\omega \in A$, and $\mathbf{1}_A(\omega) = 0$ if $\omega \notin A$.

Proof. We can view η as a random variable defined on the probability space $(\Omega, \mathcal{F}, P)$. Consider its cumulative distribution function $F(t) = P(\eta \le t)$ and the left-side quantile $F^{-1}(\omega) = \inf\{t : F(t) \ge \omega\}$, $\omega \in (0,1)$. Note that the function $\zeta(\cdot) := F^{-1}(\cdot)$ is monotonically nondecreasing right hand side continuous and, since the distribution P is uniform on $[0,1]$, the random variables η and ζ have the same distribution. Therefore, as it was pointed above there exists $T \in \mathfrak{G}$ such that $\eta = \zeta \circ T$ a.e. ∎

If $\mathfrak{A}$ is generated by an element $\eta \in \mathfrak{A}$, then $\mathfrak{A}$ is also generated by any $\zeta \in \mathcal{O}(\eta)$. It could be remarked that an element $\eta \in \mathfrak{A}$, viewed as a function $\eta(\omega)$ of $\omega \in [0,1]$, is defined up to a set of measure zero. Therefore by making transformation $\eta \mapsto \eta \circ T$ for an appropriate $T \in \mathfrak{G}$, we can assume that the generating element η of $\mathfrak{A}$ is a *monotonically nondecreasing* right hand side continuous function on the interval (0,1). More accurately it should be said that $\eta(\cdot)$ is a member of the class of functions corresponding to the element $\eta \in \mathfrak{A}$. By assuming that $\eta(\cdot)$ is right hand side continuous we uniquely specify such monotonically nondecreasing member function.

Definition 2.2 *Let $\eta, \zeta : [0,1] \to \mathbb{R}$ be two integrable functions. We say that ζ is* majorized *by η, denoted $\eta \gg \zeta$, if*

$$\int_\gamma^1 \eta(\omega)d\omega \ge \int_\gamma^1 \zeta(\omega)d\omega,\ \forall \gamma \in [0,1];\ \text{and}\ \int_0^1 \eta(\omega)d\omega = \int_0^1 \zeta(\omega)d\omega. \tag{2.4}$$

The relation " $\gg$ " defines a partial order, i.e., for $\xi, \eta, \zeta \in \mathcal{Z}$ it holds that: (i) $\xi \gg \xi$, (ii) if $\xi \gg \eta$ and $\eta \gg \xi$, then $\xi = \eta$, (iii) if $\xi \gg \eta$ and $\eta \gg \zeta$, then $\xi \gg \zeta$.

Remark 2.1 For monotonically nondecreasing functions $\eta, \zeta : [0,1] \to \mathbb{R}$ the above concept of majorization is closely related to the concept of dominance in the convex order. That is, if η and ζ are monotonically nondecreasing right hand side continuous functions, then they can be viewed as quantile functions $\eta = F_X^{-1}$ and $\zeta = F_Y^{-1}$ of some respective random variables X and Y. It is said that X dominates Y in the convex order if $\mathbb{E}[u(X)] \ge \mathbb{E}[u(Y)]$ for all convex functions $u : \mathbb{R} \to \mathbb{R}$ such that the expectations exist. Equivalently this can written as (see, e.g., [10])

$$\int_\gamma^1 F_X^{-1}(\omega)d\omega \ge \int_\gamma^1 F_Y^{-1}(\omega)d\omega,\ \forall \gamma \in [0,1];\ \text{and}\ \mathbb{E}[X] = \mathbb{E}[Y].$$

Note that $\mathbb{E}[X] = \int_0^1 F_X^{-1}(\omega)d\omega$ and $\mathbb{E}[Y] = \int_0^1 F_Y^{-1}(\omega)d\omega$. The dominance in the convex (concave) order was used in studying risk measures in Föllmer and Schield [6] and Dana [4], for example.

It is also related to the concept of majorization used in the theory of Shur convexity. That is, consider a finite set $\Omega_n := \{\omega_1, ..., \omega_n\}$ with the sigma algebra of all its subsets and assigned equal probabilities $1/n$. This probability space can be viewed as a discretization of the probability space $\Omega = [0,1]$ equipped with uniform distribution. Random variables $x : \Omega_n \to \mathbb{R}$ on that space can be identified with vectors $x = (x_1, ..., x_n) \in \mathbb{R}^n$, and the set of measure-preserving transformations $\mathfrak{G}$ with the group of permutations of the set $\{\omega_1, ..., \omega_n\}$. It is said that vector $y \in \mathbb{R}^n$ is majorized by vector $x \in \mathbb{R}^n$ if

$$\sum_{i=k}^n x_{[i]} \ge \sum_{i=k}^n y_{[i]},\quad k = 1, ..., n-1;\ \ \text{and}\ \sum_{i=1}^n x_i = \sum_{i=1}^n y_i, \tag{2.5}$$

where $x_{[1]} \le \cdots \le x_{[n]}$ are components of vector x arranged in the increasing order (see, e.g., [9]).

It is not difficult to show that the following properties hold.

(P1) If $\eta \gg \zeta$, then for a monotonically nondecreasing on [0,1] function $Z(\cdot)$ it holds that

$$\int_0^1 Z(\omega)\eta(\omega)d\omega \ge \int_0^1 Z(\omega)\zeta(\omega)d\omega. \tag{2.6}$$

(P2) If $Z(\cdot)$ and $\eta(\cdot)$ are monotonically nondecreasing on the interval [0,1] functions, then

$$\sup_{T\in\mathfrak{G}}\int_0^1 Z(\omega)\eta(T(\omega))d\omega=\int_0^1 Z(\omega)\eta(\omega)d\omega. \tag{2.7}$$

Theorem 2.1 *Let $\rho:\mathcal{Z}\to\mathbb{R}$ be a law invariant coherent risk measure. Suppose that the dual set $\mathfrak{A}$ is generated by (monotonically nondecreasing) $\eta\in\mathfrak{A}$. Then:* (i) *for every $Z\in\mathcal{Z}$ maximum in the dual representation* (1.3) *is attained at some element of the set $\mathcal{O}(\eta)$,* (ii) *if $\zeta\in\mathfrak{A}$, then $\eta\gg\zeta$,* (iii) *if $\zeta\in\mathcal{Z}^*$ is monotonically nondecreasing and $\eta\gg\zeta$, then $\zeta\in\mathfrak{A}$,* (iv) *$\mathfrak{A}$ is generated by an element ζ iff $\zeta\in\mathcal{O}(\eta)$,* (v) *the set $\mathcal{O}(\eta)$ coincides with the set of extreme points of $\mathfrak{A}$.*

Proof. Note that if a function $Z\in\mathcal{Z}$ is monotonically nondecreasing, then by (P2)

$$\rho(Z)=\sup_{\zeta\in\mathcal{O}(\eta)}\langle\zeta,Z\rangle=\sup_{T\in\mathfrak{G}}\int_0^1 Z(\omega)\eta(T(\omega))d\omega=\int_0^1 Z(\omega)\eta(\omega)d\omega=\langle\eta,Z\rangle. \tag{2.8}$$

Now consider an element $Z\in\mathcal{Z}$. We can choose $T\in\mathfrak{G}$ such that $Z\circ T$ is monotonically nondecreasing on the interval [0,1]. By (2.8) we have then that

$$\langle\eta\circ T^{-1},Z\rangle=\langle\eta,Z\circ T\rangle=\rho(Z\circ T)=\rho(Z), \tag{2.9}$$

where the last equality follows since ρ is law invariant. That is, $\eta\circ T^{-1}$ is a maximizer in the dual representation (1.3). This proves the assertion (i).

Consider $\zeta\in\mathfrak{A}$. It follows that ζ is a density function, i.e., $\zeta\succeq 0$ and $\int_0^1\zeta(\omega)d\omega=1$. By (1.4) and taking $Z:=\mathbf{1}_{[\gamma,1]}$ we obtain that

$$\int_\gamma^1\zeta(\omega)d\omega=\langle\zeta,Z\rangle\le\rho(Z)=\langle\eta,Z\rangle=\int_\gamma^1\eta(\omega)d\omega.$$

Since $\int_0^1\eta(\omega)d\omega=\int_0^1\zeta(\omega)d\omega=1$, it follows that $\eta\gg\zeta$. This proves (ii).

Conversely, suppose that $\zeta\in\mathcal{Z}^*$ is monotonically nondecreasing and $\eta\gg\zeta$. Then for a $Z\in\mathcal{Z}$ choose $T\in\mathfrak{G}$ such that $Z':=Z\circ T$ is monotonically nondecreasing on the interval [0,1]. Since ρ is law invariant we have that $\rho(Z)=\rho(Z')$. By (2.6) and (2.8) it follows that

$$\rho(Z')=\int_0^1 Z'(\omega)\eta(\omega)d\omega\ge\int_0^1 Z'(\omega)\zeta(\omega)d\omega.$$

Since ζ is monotonically nondecreasing we also have that

$$\int_0^1 Z'(\omega)\zeta(\omega)d\omega\ge\int_0^1 Z'(T^{-1}(\omega))\zeta(\omega)d\omega=\int_0^1 Z(\omega)\zeta(\omega)d\omega.$$

It follows that $\rho(Z)\ge\langle\zeta,Z\rangle$, and hence by (1.4) that $\zeta\in\mathfrak{A}$. This proves (iii).

Now suppose that there exists ζ generating $\mathfrak{A}$ which does not belong to $\mathcal{O}(\eta)$. We can assume that ζ is monotonically nondecreasing. Consider

$$\bar\omega:=\sup\left\{\omega\in[0,1]:\zeta(\omega)\ne\eta(\omega)\right\}. \tag{2.10}$$

Since $\zeta\ne\eta$, the set $\{\omega\in[0,1]:\zeta(\omega)\ne\eta(\omega)\}$ is nonempty, and $\bar\omega>0$. Since $\eta\gg\zeta$ we have that there is $\gamma\in[0,\bar\omega)$ such that $\zeta(\omega)<\eta(\omega)$ for all $\omega\in[\gamma,\bar\omega)$. For $Z:=\mathbf{1}_{[\gamma,1]}$ it follows that

$$\rho(Z)=\int_\gamma^1\eta(\omega)d\omega>\int_\gamma^1\zeta(\omega)d\omega=\langle\zeta,Z\rangle. \tag{2.11}$$

Since $\zeta(\cdot)$ is monotonically nondecreasing, we also have that $\langle \zeta, Z\rangle \geq \langle \zeta \circ T, Z\rangle$ for any $T \in \mathfrak{G}$. It follows that

$$\sup_{T\in\mathfrak{G}} \langle \zeta \circ T, Z\rangle < \rho(Z), \tag{2.12}$$

and hence ζ cannot be a generating element of $\mathfrak{A}$. This proves (iv).

Let us prove the assertion (v). We have that if $\zeta, \zeta_1, \zeta_2 \in \mathfrak{A}$ and $\zeta = t\zeta_1 + (1-t)\zeta_2$, then $\zeta\circ T, \zeta_1 \circ T, \zeta_2 \circ T \in \mathfrak{A}$ and $\zeta \circ T = t(\zeta_1 \circ T) + (1-t)(\zeta_2 \circ T)$ for any $T \in \mathfrak{G}$. Therefore if one of the points of $\mathcal{O}(\eta)$ is extreme, then all point of $\mathcal{O}(\eta)$ are extreme.

Suppose that $\eta = t\zeta_1 + (1-t)\zeta_2$ for some $\zeta_1, \zeta_2 \in \mathfrak{A}$ and $t \in (0,1)$, and that $\zeta_1 \neq \eta$. Then there exists a subinterval $I = [\gamma, 1]$ of $[0,1]$ such that either for $i = 1$ or $i = 2$, we have that $\zeta_i(\omega) \geq \eta(\omega)$ for all $\omega \in I$ and $\zeta_i(\cdot) > \eta(\cdot)$ on some subset of I of positive measure. For $Z := \mathbf{1}_{[\gamma,1]}$ we have

$$\langle \zeta_i, Z\rangle = \int_0^1 Z(\omega)\zeta_i(\omega)d\omega = \int_\gamma^1 \zeta_i(\omega)d\omega > \int_\gamma^1 \eta(\omega)d\omega = \int_0^1 Z(\omega)\eta(\omega)d\omega, \tag{2.13}$$

and since $\rho(Z) \geq \langle \zeta_i, Z\rangle$, it follows that $\rho(Z) > \int Z\eta dP$. This, however, contradicts (2.8). This shows that η is an extreme point of $\mathfrak{A}$ and hence all points of $\mathcal{O}(\eta)$ are extreme points.

Finally we need to show that if ζ is an extreme point of $\mathfrak{A}$, then $\zeta \in \mathcal{O}(\eta)$. We argue by a contradiction. Suppose that ζ is an extreme point of $\mathfrak{A}$ and $\zeta \notin \mathcal{O}(\eta)$. Consider $\zeta' = (2\zeta - \eta)\circ T$ for such $T \in \mathfrak{G}$ that ζ' is monotonically nondecreasing. By (ii) we have that $\eta \gg \zeta$ and hence

$$\int_\gamma^1 \zeta'(\omega)d\omega = 2\int_\gamma^1 \zeta(\omega)d\omega - \int_\gamma^1 \eta(\omega)d\omega \leq \int_\gamma^1 \eta(\omega)$$

for any $\gamma \in [0.1]$. Since $\int_0^1 \zeta(\omega)d\omega = \int_0^1 \eta(\omega)d\omega = 1$, we also have that $\int_0^1 \zeta'(\omega)d\omega = 1$. It follows that $\eta \gg \zeta'$, and hence by (iii) we have that $\zeta' \in \mathfrak{A}$ and thus $\zeta' \circ T^{-1} \in \mathfrak{A}$. It remains to note that $(\zeta' \circ T^{-1} + \eta)/2 = \zeta$, and hence since ζ is an extreme point of $\mathfrak{A}$ it follows that $\zeta = \eta$, a contradiction. ∎

It is also possible to give the following characterization of generation of the dual set by its element. It is said that an element η of $\mathfrak{A}$ is the *greatest element* of $\mathfrak{A}$ if for any $\zeta \in \mathfrak{A}$ it follows that $\eta \gg \zeta$. Clearly if the greatest element exists, it is unique.

Proposition 2.2 *Let $\mathfrak{A}$ be the dual set of a law invariant coherent risk measure. If $\mathfrak{A}$ possesses greatest element $\eta \in \mathfrak{A}$, then $\mathfrak{A}$ is generated by η. Conversely if $\mathfrak{A}$ is generated by η and η is monotonically nondecreasing, then η is the greatest element of $\mathfrak{A}$.*

Proof. Suppose that $\mathfrak{A}$ possesses a greatest element $\eta \in \mathfrak{A}$. We have that if $\zeta \in \mathcal{Z}^*$ and $\zeta' = \zeta \circ T$ is monotonically nondecreasing on [0,1] for some $T \in \mathfrak{G}$, then $\zeta' \gg \zeta$. Therefore η is monotonically nondecreasing on [0,1]. We also have by (P1) that if $Z \in \mathcal{Z}$ is monotonically nondecreasing and $\eta \gg \zeta$, then $\langle \eta, Z\rangle \geq \langle \zeta, Z\rangle$. It follows that the maximum of $\langle \zeta, Z\rangle$ over $\zeta \in \mathfrak{A}$ is attained at η, and hence $\mathfrak{A}$ is generated by η.

The converse assertion follows from Theorem 2.1(ii). ∎

3 AV@R representations

Recall that random variables X and Y are said to be comonotonic if (X, Y) is distributionally equivalent to $(F_X^{-1}(U), F_Y^{-1}(U))$, where U is a random variable uniformly distributed on the interval $[0, 1]$ (see, e.g., [5] for a discussion of comonotonic random variables). A risk measure $\rho : \mathcal{Z} \to \mathbb{R}$ is said to be *comonotonic* if for any two comonotonic random variables $X, Y \in \mathcal{Z}$ it follows that $\rho(X + Y) = \rho(X) + \rho(Y)$.

Theorem 3.1 *Let $\mathcal{Z} = L_p(\Omega, \mathcal{F}, P)$, $p \in [1, \infty)$, and $\rho : \mathcal{Z} \to \mathbb{R}$ be a law invariant coherent risk measure with the respective dual set $\mathfrak{A}$. Then:* (i) *the set $\mathfrak{A}$ is generated by an element $\eta \in \mathcal{Z}^*$ iff there exists a probability measure μ on the interval $[0, 1)$ such that*

$$\rho(Z) = \int_0^1 \mathsf{AV@R}_\alpha(Z) d\mu(\alpha), \;\; \forall Z \in \mathcal{Z}, \tag{3.1}$$

(ii) *the probability measure μ in representation* (3.1) *is defined uniquely,* (iii) *the risk measure ρ is comonotonic iff the set $\mathfrak{A}$ is generated by a single element η,* (iv) *the risk measure ρ is comonotonic iff the set of extreme points of $\mathfrak{A}$ consists of a single distributionally equivalent class $\mathcal{O}(\eta)$.*

Note that the measure μ in the above theorem is defined on the interval [0,1), i.e., it has mass zero at the point $\alpha = 1$. In order to proof this theorem we need the following lemma.

Lemma 3.1 *Let $\eta : [0, 1) \to \mathbb{R}_+$ be a monotonically nondecreasing, right hand side continuous function such that $\int_0^1 \eta(\omega) d\omega = 1$. Then there exists a uniquely defined probability measure μ on $[0, 1)$ such that*

$$\eta(\omega) = \int_0^\omega (1 - \alpha)^{-1} d\mu(\alpha), \;\; \omega \in [0, 1]. \tag{3.2}$$

Proof. We can view $\mu(\cdot)$ in (3.2) as a right hand side continuous, monotonically nondecreasing on the interval [0,1) function and the corresponding integral as the Lebesgue-Stieltjes integral. Suppose for the moment that $\eta(0) = 0$ and define

$$\mu(\alpha) := (1 - \alpha)\eta(\alpha) + \int_0^\alpha \eta(\tau) d\tau. \tag{3.3}$$

Then

$$d\mu(\alpha) = -\eta(\alpha) d\alpha + (1 - \alpha) d\eta(\alpha) + \eta(\alpha) d\alpha = (1 - \alpha) d\eta(\alpha),$$

and hence (3.2) follows. If $\eta(0) > 0$, then we can add the mass $\eta(0)$ to the measure μ at $\alpha = 0$. Note that if (3.2) holds for some measure μ, then

$$1 = \int_0^1 \eta(\omega) d\omega = \int_0^1 \int_0^\omega (1 - \alpha)^{-1} d\mu(\alpha) d\omega = \int_0^1 \int_\alpha^1 (1 - \alpha)^{-1} d\omega d\mu(\alpha) = \int_0^1 d\mu(\alpha),$$

and hence μ is a probability measure.

In order to show uniqueness of the measure μ we proceed as follows. Using integration by parts we have

$$\int_0^\omega (1 - \alpha)^{-1} d\mu(\alpha) = \frac{\mu(\omega)}{1 - \omega} - \mu(0) - \int_0^\omega \mu(\alpha) d(1 - \alpha)^{-1} = \frac{\mu(\omega)}{1 - \omega} - \mu(0) - \int_0^\omega \frac{\mu(\alpha)}{(1 - \alpha)^2} d\alpha.$$

Suppose that there are two measures μ_1 and μ_2 which give the same function η. Since measure μ_1 (and measure μ_2) is defined by the corresponding function up to a constant we can set $\mu_1(0) = \mu_2(0)$. Then for the function $\psi(\omega) := [\mu_1(\omega) - \mu_2(\omega)]/(1 - \omega)$ we have the following equation

$$\psi(\omega) - \int_0^\omega \frac{\psi(\alpha)}{1 - \alpha} d\alpha = 0, \; \omega \in [0, 1), \tag{3.4}$$

with $\psi(0) = 0$. It follows that $\psi(\cdot)$ is continuous and differentiable and satisfies the equation

$$\frac{d\psi(\omega)}{d\omega} - \frac{\psi(\omega)}{1 - \omega} = 0. \tag{3.5}$$

The last equation has solutions of the form $\psi(\omega) = c(1-\omega)^{-1}$ for some constant c. Since $\psi(0) = 0$ it follows that $c = 0$, and thus $\psi(\cdot) = 0$ and hence $\mu_1(\cdot) = \mu_2(\cdot)$. ∎

Existence of probability measure μ satisfying (3.2) is shown in [6, Lemma 4.63]. In the above lemma we also prove its uniqueness.

If $\eta(\cdot)$ is a step function, i.e., $\eta = \sum_{i=1}^n a_i \mathbf{1}_{[t_i,1]}$, where a_i, $i = 1, ..., n$, are positive numbers and $0 \le t_1 < t_2 < \cdots < t_n < 1$, then we can take

$$\mu(\omega) = \sum_{i=1}^n a_i (1-t_i) \mathbf{1}_{[t_i,1]}(\omega), \quad \omega \in [0,1]. \tag{3.6}$$

Proof of Theorem 3.1. As it was pointed earlier, since $(\Omega, \mathcal{F}, P)$ is nonatomic, we can assume without loss of generality that Ω is the interval $[0,1]$ equipped with its Borel sigma algebra and uniform reference distribution. Suppose that $\mathfrak{A}$ is generated by some element $\eta \in \mathfrak{A}$. Then $\mathfrak{A}$ is also generated by $\eta' = \eta \circ T$ for any $T \in \mathfrak{G}$. Therefore by making an appropriate transformation we can assume that $\eta(\cdot)$ is monotonically nondecreasing on the interval $[0,1)$ and right hand side continuous.

Consider $Z \in \mathcal{Z}$. We have that $\rho(Z) = \rho(Z \circ T)$ for any $T \in \mathfrak{G}$. We can choose $\bar{T} \in \mathfrak{G}$ such that $Z(\bar{T}(\cdot))$ is monotonically nondecreasing on the interval $[0,1)$. Then

$$\rho(Z) = \sup_{T\in\mathfrak{G}} \int_0^1 Z(\omega)\eta(T(\omega))d\omega = \sup_{T\in\mathfrak{G}} \int_0^1 Z(T(\omega))\eta(\omega)d\omega = \int_0^1 Z(\bar{T}(\omega))\eta(\omega)d\omega. \tag{3.7}$$

Now let μ be a probability measure on $[0,1)$ such that equation (3.2) holds. It follows by (3.7) that

$$\rho(Z) = \int_0^1 \int_0^\omega Z(\bar{T}(\omega))(1-\alpha)^{-1} d\mu(\alpha) d\omega = \int_0^1 \int_\alpha^1 Z(\bar{T}(\omega))(1-\alpha)^{-1} d\omega d\mu(\alpha). \tag{3.8}$$

Since the dual set of $\mathsf{AV@R}_\alpha$ is generated by the function $(1-\alpha)^{-1}\mathbf{1}_{[\alpha,1]}(\cdot)$ and $Z(\bar{T}(\cdot))$ is monotonically nondecreasing, we have that

$$\int_\alpha^1 Z(\bar{T}(\omega))(1-\alpha)^{-1} d\omega = \mathsf{AV@R}_\alpha(Z \circ \bar{T}), \tag{3.9}$$

and since measure $\mathsf{AV@R}_\alpha$ is law invariant we also have that $\mathsf{AV@R}_\alpha(Z \circ \bar{T}) = \mathsf{AV@R}_\alpha(Z)$. Together with (3.8) this implies (3.1).

Conversely, suppose that the representation (3.1) holds for some probability measure μ. Consider $Z \in \mathcal{Z}$ and let $\bar{T} \in \mathfrak{G}$ be such that $Z(\bar{T}(\cdot))$ is monotonically nondecreasing on the interval $[0,1)$. Then equation (3.9) holds and hence (3.8) follows. Consequently for $\eta(\cdot)$ defined in (3.2) we obtain that (3.7) holds. This completes the proof of the assertion (i)

As far as uniqueness of μ is concerned, lets make the following observations. By Proposition 2.2 the nondecreasing right hand side continuous function $\eta(\cdot)$ generating the set $\mathfrak{A}$ is defined uniquely. Moreover, the representation (3.1) holds for some probability measure μ iff equation (3.2) holds. Hence the uniqueness of μ follows by Lemma 3.1. This proves the assertion (ii)

Now suppose that $\mathfrak{A}$ is generated by some element $\eta \in \mathfrak{A}$. Let $X, Y \in \mathcal{Z}$ be comonotonic variables, i.e., (X, Y) is distributionally equivalent to $(F_X^{-1}(U), F_Y^{-1}(U))$, where U is a random variable uniformly distributed on the interval $[0,1]$. This means in the considered setting that both functions $X, Y : [0,1] \to \mathbb{R}$ are monotonically nondecreasing, and hence $X + Y$ is also monotonically nondecreasing. Let $T \in \mathfrak{G}$ be such that $\bar{\zeta} := \eta \circ T$ is monotonically nondecreasing on the interval [0,1]. Then the maximum in (1.3) is attained at $\bar{\zeta}$ for both X and Y and for $X+Y$, i.e., $\rho(X) = \langle X, \bar{\zeta}\rangle$, $\rho(Y) = \langle Y, \bar{\zeta}\rangle$ and $\rho(X+Y) = \langle X+Y, \bar{\zeta}\rangle$. It follows that $\rho(X+Y) = \rho(X) + \rho(Y)$, and hence ρ is comonotonic.

Conversely, suppose that ρ is comonotonic. Let us argue by a contradiction, i.e., assume that $\mathfrak{A}$ is not generated by one of its elements. Then there are two monotonically nondecreasing variables $X, Y \in \mathcal{Z}$ such that the maximum in the right hand side of (1.3), for $Z = X$ and $Z = Y$, is not attained at a same $\zeta \in \mathfrak{A}$. We have that there is $\bar{\zeta} \in \mathfrak{A}$ such that

$$\rho(X + Y) = \langle \bar{\zeta}, X + Y \rangle = \langle \bar{\zeta}, X \rangle + \langle \bar{\zeta}, Y \rangle. \tag{3.10}$$

Moreover, it follows that

$$\langle \bar{\zeta}, X \rangle \leq \rho(X) \text{ and } \langle \bar{\zeta}, Y \rangle \leq \rho(Y) \tag{3.11}$$

with at least one of the inequalities in (3.11) being strict. Thus $\rho(X + Y) < \rho(X) + \rho(Y)$, and hence ρ is not comonotonic. This completes the proof of (iii).

Denote by $\mathfrak{E}$ the set of extreme points of $\mathfrak{A}$. Note that since the set $\mathfrak{A}$ is convex and weakly* compact, we have that for any $Z \in \mathcal{Z}$ the maximum of $\langle \zeta, Z \rangle$ over $\zeta \in \mathfrak{A}$ is attained at an extreme point of $\mathfrak{A}$ (this follows from the Krein-Milman Theorem), and hence

$$\rho(Z) = \sup_{\eta \in \mathfrak{E}} \langle \zeta, Z \rangle. \tag{3.12}$$

Suppose that ρ is comonotonic. Then by (iii) we have that $\mathfrak{A}$ is generated by an element $\eta \in \mathfrak{A}$. Thus by Theorem 2.1(v) we obtain that $\mathcal{O}(\eta) = \mathfrak{E}$.

Conversely, suppose that $\mathfrak{E} = \mathcal{O}(\eta)$. Then by (3.12) we obtain that $\rho(Z) = \sup_{\eta \in \mathcal{O}(\eta)} \langle \zeta, Z \rangle$, i.e., $\mathfrak{A}$ is generated by η. It follows by (iii) that ρ is comonotonic. This completes the proof of (iv). ∎

Remark 3.1 If the representation (3.1) holds with the measure μ having positive mass λ at $\alpha = 1$, then we can write the corresponding risk measure as $\rho = (1 - \lambda)\rho' + \lambda \mathsf{AV@R}_1$, where $\rho'(Z) := \int_0^1 \mathsf{AV@R}_\alpha(Z) d\mu'(\alpha)$ with μ' being the part of the measure μ restricted to the interval $[0, 1)$ and normalized by factor $(1 - \lambda)^{-1}$. In that case we will have to use the space $\mathcal{Z} = L_\infty(\Omega, \mathcal{F}, P)$ in order for $\rho(\cdot)$ to be real valued. The corresponding dual set $\mathfrak{A} = (1 - \lambda)\mathfrak{A}_1 + \lambda \mathfrak{A}_2$, where $\mathfrak{A}_1$ is the dual set of ρ' and $\mathfrak{A}_2$ is the dual set of $\mathsf{AV@R}_1$. The set $\mathfrak{A}_1$ is generated by one of its elements, while the set $\mathfrak{A}_2$ is not (see the discussion of Example 1). Therefore it is essential in the assertions (i),(iii) and (iv) of Theorem 3.1 that $p < \infty$.

By combining the assertions (i) and (iii) of the above theorem we obtain (for $p < \infty$) that a law invariant coherent risk measure is comonotonic iff it can be represented in the integral form (3.1). For $L_\infty(\Omega, \mathcal{F}, P)$ spaces this was first shown by Kusuoka [8]. Uniqueness of the measure μ in representation (3.1) also holds for $L_\infty(\Omega, \mathcal{F}, P)$ spaces.

Remark 3.2 By (1.6) we have that the representation (3.1) can be written as

$$\rho(Z) = \int_0^1 \int_\alpha^1 (1 - \alpha)^{-1} \mathsf{V@R}_\tau(Z) d\tau d\mu(\alpha) = \int_0^1 \eta(\tau) \mathsf{V@R}_\tau(Z) d\tau, \tag{3.13}$$

where $\eta(\cdot)$ is given in (3.2), i.e., $\eta(\cdot)$ is monotonically nondecreasing on [0,1] function generating the dual set of ρ. Risk measures of the form (3.13) were called *spectral risk measures* in Acerbi [1].

Unless stated otherwise we assume in the remainder of this section that $\mathcal{Z} = L_p(\Omega, \mathcal{F}, P)$, with $p \in [1, \infty)$. Let $\mathfrak{A}$ be the dual set of a law invariant coherent risk measure $\rho : \mathcal{Z} \to \mathbb{R}$ and $\mathfrak{E}$ be the set of extreme points of $\mathfrak{A}$. As it was pointed earlier the set $\mathfrak{E}$ is invariant with respect to measure preserving transformations $T \in \mathfrak{G}$, and hence is given by the union of distributionally equivalent classes. In each such class we can choose a monotonically nondecreasing right hand side continuous function $\eta(\omega)$. Let Υ be the collection of all such functions η, and thus $\mathfrak{E} = \cup_{\eta \in \Upsilon} \mathcal{O}(\eta)$. By (3.12 it follows that

$$\rho(Z) = \sup_{\eta \in \Upsilon} \sup_{\zeta \in \mathcal{O}(\eta)} \langle \zeta, Z \rangle, \tag{3.14}$$

i.e., $\mathfrak{A}$ is generated by the set Υ. For $\eta \in \Upsilon$ we have by Theorem 3.1 that the risk measure $\varrho_\eta := \sup_{\zeta\in\mathcal{O}(\eta)}\langle\zeta, Z\rangle$ is comonotonic and

$$\varrho_\eta(Z) = \int_0^1 \mathsf{AV@R}_\alpha(Z)d\mu_\eta(\alpha), \;\; \forall Z \in \mathcal{Z}, \tag{3.15}$$

with probability measure μ_η given by (3.3). Combination of (3.14) and (3.15) gives a constructive way for writing the Kusuoka representation (1.10) with $\mathfrak{M} := \cup_{\eta\in\Upsilon}\mu_\eta$. Furthermore, we can take the topological closure (in the weak topology) of the convex hull of $\cup_{\eta\in\Upsilon}\mu_\eta$ (see Proposition 1.1).

Example 2 Consider the absolute semideviation risk measure

$$\rho(Z) := \mathbb{E}[Z] + c\mathbb{E}\left\{\left[Z - \mathbb{E}(Z)\right]_+\right\}, \tag{3.16}$$

with constant $c \in [0, 1]$ and $\mathcal{Z} = L_1(\Omega, \mathcal{F}, P)$. If probability of the event "$Z(\omega) = \mathbb{E}[Z]$" is zero, then the maximizer of $\langle\zeta, Z\rangle$ over $\zeta \in \mathfrak{A}$ is given by

$$\eta(\omega) = \begin{cases} 1 - c\kappa & \text{if} \quad Z(\omega) < \mathbb{E}[Z], \\ 1 + c(1-\kappa) & \text{if} \quad Z(\omega) > \mathbb{E}[Z], \end{cases} \tag{3.17}$$

where $\kappa := \mathsf{Pr}\{Z > \mathbb{E}[Z]\}$ (e.g., [13, p. 278]), and this η is an extreme point of the dual set $\mathfrak{A}$. For $Z(\cdot)$ monotonically nondecreasing on [0,1] the corresponding maximizer (which is monotonically nondecreasing) is

$$\eta(\omega) = \begin{cases} 1 - c\kappa & \text{if} \quad 0 \le \omega < 1-\kappa, \\ 1 + c(1-\kappa) & \text{if} \quad 1-\kappa \le \omega \le 1. \end{cases} \tag{3.18}$$

Since variables $Z \in \mathcal{Z}$ such that $Z \neq \mathbb{E}[Z]$ w.p.1 form a dense set in the space $\mathcal{Z}$, it is sufficient to consider maximizers (extreme points) of the form (3.18). Clearly the set of extreme points here is not generated by one element of $\mathfrak{A}$, and hence this risk measure is not comonotonic. For an appropriate choice of $Z \in \mathcal{Z}$ the parameter κ can be any number of the interval (0,1). Therefore we can take here the generating set Υ to be the set of functions η of the form (3.18) for $\kappa \in (0, 1)$. By applying transformation (3.6) to an η of the form (3.18) we obtain the corresponding probability measure $\mu(\alpha)$ with mass $1 - c\kappa$ at $\alpha = 0$ and mass $c\kappa$ at $\alpha = 1-\kappa$ (note that here the function $\eta(\cdot)$ is piecewise constant and recall that mass $\eta(0)$ is assigned to μ at $\alpha = 0$). Consequently the risk measure ρ of the form (3.16) has the following Kusuoka representation

$$\rho(Z) = \sup_{\kappa\in(0,1)} \left\{(1-c\kappa)\mathsf{AV@R}_0(Z) + c\kappa\mathsf{AV@R}_{1-\kappa}(Z)\right\}. \tag{3.19}$$

That is, the corresponding set $\mathfrak{M}$ is can be taking as

$$\mathfrak{M} := \cup_{\kappa\in(0,1)}\{(1-c\kappa)\Delta(0) + c\kappa\Delta(1-\kappa)\},$$

where $\Delta(\alpha)$ denotes measure of mass one at $\alpha \in [0, 1]$ (as it was discussed above one can also take the topological closure of the convex hull of that set).

Recalling that $\mathsf{AV@R}_0(Z) = \mathbb{E}(\cdot)$ and using definition (1.5), we can write representation (3.19) in the form

$$\begin{aligned} \rho(Z) &= \sup_{\kappa\in[0,1]} \inf_{t\in\mathbb{R}} \mathbb{E}\big\{Z + c\kappa(t - Z) + c[Z-t]_+\big\} \\ &= \inf_{t\in\mathbb{R}} \sup_{\kappa\in[0,1]} \mathbb{E}\big\{Z + c\kappa(t - Z) + c[Z-t]_+\big\}. \end{aligned} \tag{3.20}$$

The above representation (3.20) can be also derived directly (e.g., [13, p.302]).

While the integral representation (3.1) is unique, the max-representation (1.10) is not necessarily unique. Consider for example a risk measure representable in the integral form (3.1). Then we can take $\mathfrak{M} = \{\mu\}$. On the other hand, $\mathbb{E}(\cdot) = \mathsf{AV@R}_0(\cdot)$ and for any law invariant coherent risk measure ρ it holds that $\rho(\cdot) \geq \mathbb{E}(\cdot)$. Therefore we always can add measure of mass one at the point 0, of the interval [0,1), to $\mathfrak{M}$. This will not change the maximum in the max-representation (1.10). Therefore it makes sense to talk about minimal representations of the form (1.10).

Let $\rho : \mathcal{Z} \to \mathbb{R}$ be a law invariant coherent risk measure and $\mathfrak{M}_1 \supset \mathfrak{M}_2 \supset \cdots$, be a sequence of convex compact (in the weak topology) sets such that the representation (1.10) holds for every $\mathfrak{M}_i$. Consider $\mathfrak{M} := \cap_{i=1}^{\infty} \mathfrak{M}_i$. Since $\mathfrak{M}$ is the intersection of a sequence of nonempty nested convex compact sets, it is nonempty convex and compact. Recall that the weak topology on the space of probability measures on [0,1] can be described in terms of Prohorov metric (and the obtained metric space is separable), e.g., [3, pp.72-73]. Therefore the weak topology can be described in terms of convergent sequences. It follows that the representation (1.10) also holds for the set $\mathfrak{M}$. Indeed, consider an element $Z \in \mathcal{Z}$ and let $\mu_i \in \mathfrak{M}_i$, $i = 1, ...,$ be a measure at which the maximum in the right hand side of (1.10) is attained. By passing to a subsequence if necessary, we can assume that μ_i converge to some $\mu \in \mathfrak{M}$. It follows that $\rho(Z) = \int_0^1 \mathsf{AV@R}_\alpha(Z) d\mu(\alpha)$, and since $\mathfrak{M} \subset \mathfrak{M}_1$ this implies that the representation (1.10) holds for $\mathfrak{M}$ as well. This implies that there exists a set $\mathfrak{M}^*$ for which the representation (1.10) holds, while it does not hold for any convex compact strict subset of $\mathfrak{M}^*$, i.e., there exists a "minimal" representation of the form (1.10). If the dual set $\mathfrak{A}$ is generated by one of its elements, then $\mathfrak{M}^* = \{\mu\}$ is a singleton and this minimal set is uniquely defined.

It is an open question whether such minimal representation is unique in general.

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