Managing Operational and Financing Decisions to Meet Consumption Targets

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Abstract

We study dynamic operational decision problems where risky cash flows are being resolved over a finite planning horizon. Financing decisions via lending and borrowing are available to smooth out consumptions over time with the goal of achieving some prescribed consumption targets. Our target-oriented decision criterion is based on the aggregation of Aumann and Serrano (2008) riskiness indices of the consumption excesses over targets, which has salient properties of subadditivity, convexity and respecting second-order stochastic dominance. We show that if borrowing and lending are unrestricted, the optimal policy of this criterion is to finance consumptions at the target levels for all periods except the last. Moreover, the optimal policy has the same control state as the optimal risk neutral policy and could be achieved with relatively modest computational effort. Under restricted financing, we show that for convex dynamic decision problems, the optimal policies correspond to those that maximize expected additive-exponential utilities, and can be obtained by an efficient algorithm. We also analyze the optimal policies of joint inventory-pricing decision problems under the target-oriented criterion and provide optimal policy structures. With a numerical study for inventory control problems, we report favorable computational results for using targets in regulating uncertain consumptions over time.

Keywords: dynamic programming, targets, riskiness index, inventory control

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1 Introduction

Firms plan their operational decisions such as procurement, production, and inventory replenishment to generate cash flow for day-to-day expenditures (e.g., wages, dividend payments, and R&D costs) and more importantly, to grow the company. Due to multiple sources of randomness (e.g., demand volatility) embedded in the business environment, cash flow arising from any operational decision is risky and this complicates the decision making process. In principle, an analyst could construct a dynamic model to evaluate a decision that will reveal the corresponding cash flow profile, i.e., the probability distributions of the present values of the risky cash flow over time. The overarching challenges in a dynamic decision problem are to determine the appropriate decision criterion that translates the preference of the decision maker and also to find the "best" policy in which the criterion is optimized. Unfortunately, such a problem often suffers from the "curse of dimensionality" and its computational tractability depends critically on size of the underlying state-space, which may be aggravated by the choice of decision criterion. Even in the simplest setting where decision makers are risk neutral, finding the optimal policy that maximizes the expected net present value of the consumptions profile is #P-hard and possibly PSPACE-hard; see Dyer and Stougie (2006). Nevertheless, the optimal risk neutral policies of many important dynamic decision problems with smaller state space can be analyzed and solved via Bellman's (1957) dynamic programming.

Despite the technical attractiveness of risk neutral decision criteria, they neglect the risks involved in the operational cash flows and may not appeal to managers who are averse to potential losses along the planning horizon. They also ignore the fact that decision makers can be sensitive to the timing of the resolution of uncertainties. For example, suppose a firm will have a risky cash flow 10 years from now, knowing it today can be preferred than knowing it 10 years later because by having the knowledge of the cash flow today, managers can better plan (through borrowing and lending) corporate activities such as expansion and investment in new technology. Markowitz (1959), Matheson and Howard (1989) among others recognize this as the problem of "temporal risk." In order to resolve this problem, decision makers should be allowed to borrow or lend to smooth out *consumptions* over the planning horizon. In the corporate world, consumptions can refer to expenditures such as wages, dividend payments, and R&D costs. More details on "temporal risk problem" can be found in Smith (1998).

Recognizing the risks in the operational cash flow as well as the temporal risk, we study a firm's operational and financing decisions concurrently by modelling a finite horizon dynamic decision problem. We refer to *operational decisions* as those (e.g., production quantity and inventory planning) that would directly affect the operational cash flow in response to underlying uncertainties, and *financing decisions* as the amount of money the firm borrows or lends through financial markets. In every period except for the last, the total cash flow arising from the operational and financing decisions is the firm's consumption. To illustrate, if the firm needs to consume more than the operational cash flow, she'll borrow from the financial markets at a cost and if she consumes less, she will lend and earn the interest. The consumption in the final period is defined as the total wealth of the company in the consideration that a firm's ultimate goal is to grow the company. We define *consumptions profile* as the probability distributions of the present values of the risky consumptions over time.

One widely adopted criterion for evaluating the consumptions profile is expected utility, which captures decision maker's risk awareness and also has strong normative basis. Dynamic programming under the expected utility criterion has been proposed and studied by Howard and Matheson (1972), Porteus (1975) and Jaquette (1976) among others. Smith (1998) shows that under an additive-exponential utility function, joint operational and financing decisions can be made without overburdening the analysis. In the context of inventory management, Chen et al. (2007) show that this approach also preserves the structures of optimal joint inventory replenishment and pricing policies and hence, extends the result of Bouakiz and Sobel (1992).

However, from the descriptive point of view, preferences based on expected utility have been shown to contradict behavioral experiments, notably in the famous Allais' (1953) paradox. Prospect theory (Kahneman and Tversky (1979) and Tversky and Kahneman (1992)) corrects the inadequacies of expected utility by incorporating probability weighting and adopting the characteristic S-shaped reference-dependent value function, where we may interpret the inflection point in the S-shaped value function as a target that the decision maker would like to attain, see Heath et al. (1999).

While models like prospect theory could better account for behavioral preferences, they are generally not amiable to tractable analysis in dynamic decision problems. To bridge the gap between behavioral predictability and computational intractability, we propose a new dynamic decision criterion, *Consumptions Profile Riskiness Index (CPRI)*, that evaluates the prospects of a consumptions profile in achieving consumptions targets over time. This decision criterion is especially of practical meaning, as an important aspect of managers' decision making is attainment of predetermined targets (of consumption levels, profits, or company stock prices), which is a direct reflection of their performance. Simon (1955), who has coined the term *satisfice*, elucidates that most firms' goals are not maximizing profits but attaining their target profits. From the interviews of executives in large corporations, Lanzillotti (1958) concludes that managers are primarily concerned about target returns on investment. Likewise, Mao (1970) also concludes from his empirical study that managers perceived *risk* as the "prospect of not meeting some target rate of return". From the normative perspective, research interests in target-oriented utility can be traced to Borch (1968) and have rekindled in recent years (Bordley and LiCalzi, 2000; Bordley and Kirkwood, 2004; Castagnoli and LiCalzi, 1996 and Tsetlin and Winkler, 2007).

Our decision criterion CPRI is based on the extension of the riskiness index axiomatized by Aumann and Serrano (2008). Under this criterion, a joint operational and financing decision that returns the lowest CPRI is most preferred. We show that if there is no limit on the amount of borrowing and lending (*full financing*), all consumption targets, except the terminal one, are met with certainty. Moreover, the optimal policy of this criterion has the same control state as the optimal risk neutral policy and could be achieved with relatively modest computational effort. When financing is restricted, we show that for convex dynamic decision problems, the optimal policies correspond to those that optimize expected additive-exponential utilities. We also provide an algorithm to find the optimal policies.

Applying the CPRI decision framework to the joint inventory-pricing decision problem, we identify the optimal inventory and pricing policies for the case of with fixed ordering cost under full financing, and the policy structures for the case of zero fixed ordering cost under restricted financing. These results fill the void of inventory and inventory-pricing literature which has been mainly focused on risk neutral decision and expected utility to incorporate risk aversion. With our numerical studies for inventory control problems, we also report favorable computational results for using targets in regulating uncertain consumptions over time.

The rest of the paper is organized as follows. Section 2 introduces the decision criterion CPRI.

Section 3 discusses a general framework for joint operational and financing decisions under the CPRI. Optimal policies are provided for both the scenarios of full and restricted financing. Section 4 applies CPRI to a joint inventory and pricing decision problem and identifies the optimal inventory replenishment and pricing policies. Section 5 presents numerical results for inventory control problems. We conclude the paper in Section 6.

2 Consumptions profile riskiness index (CPRI)

In this section, we propose a new decision criterion for evaluating the prospects of a consumptions profile in achieving consumptions targets over time. Although the joint probability of achieving targets is a natural candidate for a target-oriented decision criterion, Diecidue and van de Ven (2008) argue against success probability as it tacitly assumes that the decision maker is indifferent to the magnitude of the losses when they occur. Our criterion is built upon the *riskiness index* recently axiomatized by Aumann and Serrano (2008).

We first introduce some general notations used in the paper. A vector such as $\boldsymbol{x}$ is represented as a boldface character and x^i denotes the ith component of the vector. In particular, $\mathbf{1}$ represents a vector of ones. We denote a random variable by a character with the tilde sign such as $\tilde{z}$, and z is its realization. A random variable is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω is the set of possible outcomes, $\mathcal{F}$ is a σ-algebra that describes the set of all possible events, and $\mathbb{P}$ is the probability measure function. Let $\mathcal{C}$ be a set of random variables on Ω in which $\tilde{c} \in \mathcal{C}$ denotes the present values of the uncertain consumptions that will be realized in future.

Definition 1 *The riskiness index is a function,* $\rho : \mathcal{C} \mapsto [0, \infty]$ *defined as follows*

$$\rho(\tilde{c}) = \min \left\{ \alpha \;\middle|\; \mathrm{C}_\alpha(\tilde{c}) \geq 0, \alpha > 0 \right\} \tag{1}$$

where $C_\alpha : \mathcal{C} \mapsto \Re$ *is the certainty equivalent function defined as*

$$\mathrm{C}_\alpha(\tilde{c}) = -\alpha \ln \mathbb{E} \left[\exp \left(-\tilde{c}/\alpha \right) \right].$$

By convention, we define $\inf \emptyset = \infty$.

Aumann and Serrano (2008) interpret the riskiness index as the reciprocal of the absolute risk aversion (ARA) of an individual with constant ARA who is indifferent between accepting and not accepting the uncertain consumptions. (While $\tilde{c}$ can refer to any random position such as investment return, we specifically call it consumption in the context of this paper.) Uncertain consumptions with lower riskiness index appeal to a greater subset of individuals who are willing to accept the uncertain consumptions over nothing. The riskiness index has the same unit as the underlying consumptions and has the following properties (Aumann and Serrano, 2008). It is *positively homogeneous*, i.e.,

$$\rho(k\tilde{c}) = k\rho(\tilde{c}) \qquad \forall k \geq 0,$$

and *subadditive*, i.e., for all $\tilde{c}_1, \tilde{c}_2 \in \mathcal{C}$,

$$\rho(\tilde{c}_1 + \tilde{c}_2) \leq \rho(\tilde{c}_1) + \rho(\tilde{c}_2).$$

The positively homogeneous property reflects upon the cardinal nature of riskiness such that $k\tilde{c}$ is k times as risky as $\tilde{c}$. Subadditivity implies that pooling of consumptions is less risky than the sum of the individual parts. These two properties imply that the riskiness index is also a convex function and hence consistent with convex preference. Furthermore, the riskiness index is monotone with second-order stochastic dominance and therefore it is a suitable decision making criterion for risk averse individuals.

Brown and Sim (2009) show that the riskiness index can be extended to a target-oriented decision criterion by evaluating the riskiness index of the consumption excesses over targets, $\tilde{c} - \tau$, where $\tau \in \Re$ is the target at present value. The target-oriented riskiness index embodies the property of *satisficing*, i.e., $\rho(\tilde{c} - \tau) = 0$ if and only if $\mathbb{P}(\tilde{c} \geq \tau) = 1$. Hence, similar to the probability measure, uncertain consumptions that can almost surely achieve the target will be most preferred and equally valued. However, unlike probability measure, this target-oriented criterion is diversification favoring and also rejects consumptions that fail to meet their targets in expectation, i.e., if $\mathbb{E}[\tilde{c}] < \tau$ then $\rho(\tilde{c} - \tau) = \infty$. We call this property *loss aversion awareness*, which captures the effect of loss aversion described by Kahneman and Tversky (1979) that, the disutility of losses below the reference point or target is far larger than the value derived from the same size of gains. The loss aversion awareness property is reflected in Payne et al. (1980, 1981), which studies that managers tend to eradicate investment possibilities that underperform against their targets. Further, Brown et al. (2011) show that the target-oriented riskiness index can resolve several well-known behavioral experiments that contradict the expected utility theory. (See Appendix for detailed illustration.)

Motivated by the normative and behavioral relevance of the target-oriented riskiness index, we generalize this approach and propose the new decision criterion CPRI for evaluating the prospect of a consumptions profile in achieving future targets. Let a vector of T random variables $\tilde{\boldsymbol{c}} = (\tilde{c}_1, \ldots, \tilde{c}_T)$ denote the decision maker's consumptions profile from period 1 to T, where $\tilde{c}_t \in \mathcal{C}$ is the present value of the uncertain consumption that will realize in period t. $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_T)$ is the vector of consumptions targets from 1 to T, with τ_t as the present value of the consumption target in period t. (Note that if the interest rate compounded at every period is β, then the future consumption and target in period t would be $(1+\beta)^t \tilde{c}_t$ and $(1+\beta)^t \tau_t$, respectively.) Also let $\mathcal{C}^{\dim(\boldsymbol{\tau})}$ be the set of random vectors that have the same dimension as $\boldsymbol{\tau}$.

Definition 2 *The consumptions profile riskiness index (CPRI) of a consumptions profile $\tilde{\boldsymbol{c}}$ with respect to the vector of targets $\boldsymbol{\tau}$ is a function, $\varphi_{\boldsymbol{\tau}} : \mathcal{C}^{\dim(\boldsymbol{\tau})} \mapsto [0, \infty]$ defined as follows:*

$$\varphi_{\boldsymbol{\tau}}(\tilde{\boldsymbol{c}}) = \sum_{t=1}^{\dim(\boldsymbol{\tau})} \rho(\tilde{c}_t - \tau_t). \tag{2}$$

Essentially, the CPRI criterion is the sum of riskiness indices of the present values of the consumption excesses over targets. The present values are used to resolve the time value incompatibility of riskiness over different periods. The evaluation of the uncertain consumptions in each period is compartmentalized and hence, if there exist a period t with disfavoring uncertain consumptions such that $\rho(\tilde{c}_t - \tau_t) = \infty$, then the consumptions profile will be least favored, i.e., $\varphi_{\boldsymbol{\tau}}(\tilde{\boldsymbol{c}}) = \infty$. On the other hand, the most favored consumptions profile, i.e., $\varphi_{\boldsymbol{\tau}}(\tilde{\boldsymbol{c}}) = 0$ will require the stringent condition that all future consumptions attain their targets almost surely.

Proposition 1 *The CPRI criterion, $\varphi_{\boldsymbol{\tau}} : \mathcal{C}^{\dim(\boldsymbol{\tau})} \mapsto [0, \infty]$ has the following properties:*

1) Satisficing*: $\varphi_{\boldsymbol{\tau}}(\tilde{\boldsymbol{c}}) = 0$ if and only if $\mathbb{P}(\tilde{\boldsymbol{c}} \geq \boldsymbol{\tau})$.*

2) Loss aversion awareness*: If there exists a time period t such that $\mathbb{E}[\tilde{c}_t] < \tau_t$ then $\varphi_{\boldsymbol{\tau}}(\tilde{\boldsymbol{c}}) = \infty$.*

3) Convexity*: For all* $\tilde{\boldsymbol{c}}_1, \tilde{\boldsymbol{c}}_2 \in \mathcal{C}^{\dim(\boldsymbol{\mathcal{T}})}$,

$$\varphi_{\boldsymbol{\mathcal{T}}}(\lambda\tilde{\boldsymbol{c}}_1 + (1-\lambda)\tilde{\boldsymbol{c}}_2) \leq \lambda\varphi_{\boldsymbol{\mathcal{T}}}(\tilde{\boldsymbol{c}}_1) + (1-\lambda)\varphi_{\boldsymbol{\mathcal{T}}}(\tilde{\boldsymbol{c}}_2).$$

4) Subadditivity*: For all* $\tilde{\boldsymbol{c}} \in \mathcal{C}^{\dim(\boldsymbol{\mathcal{T}})}$,

$$\varphi_{\boldsymbol{\mathcal{T}}}(\tilde{\boldsymbol{c}}) \geq \varphi_{\mathbf{1}'\boldsymbol{\mathcal{T}}}\left(\mathbf{1}'\tilde{\boldsymbol{c}}\right),$$

Proof : The first two properties are trivial to show. The last two properties follow directly from the homogeneity and subadditivity of riskiness index. ■

We note that the convexity and subadditivity properties of the CPRI criterion have important ramifications that ensure tractable analysis in dynamic decision problems, which we will discuss in the next section.

3 Optimizing the CPRI criterion

We consider a firm making operational decisions (e.g., inventory planning, procurement) in the presence of uncertainties such as demand variability and supply volatility. The resulting cash flow from operational decisions are used for consumption as well as increasing firm's wealth (or value in other words). The firm has access to financial markets to borrow or lend at an interest rate of β in order to smooth out consumptions over the planning horizon T. With the objective of minimizing the CPRI, the firm needs to make operational as well as financing decisions in every period.

We have $\tilde{\boldsymbol{z}}_t : \Omega \mapsto \Re^n$, $t \in \{1, \ldots, T\}$ represent the vector of uncertainties in period t, which are independently distributed and resolved overtime. We further define the $n \times t$ vector $\boldsymbol{\zeta}_t = (\boldsymbol{z}_1, \ldots, \boldsymbol{z}_t)$, $t \in \{1, \ldots, T\}$ as the realizations of the uncertainties at the end of period t. We also define $\boldsymbol{\zeta}_0 = \{\}$. For convenience, we define the index sets, $\mathcal{T} = \{1, \ldots, T\}$ and $\mathcal{T}^- = \{1, \ldots, T-1\}$.

The sequence of events in any period t, $t \in \mathcal{T}^-$ is as follows: I) At the beginning of the period, the firm observes her state of wealth w_t and operational state of the system, $\boldsymbol{x}_t$, which is an element in $\Re^{s_t}$. II) The firm then administers an operational control $\boldsymbol{u}_t$, which takes values in a nonempty set $U_t(\boldsymbol{x}_t) \subseteq \Re^{v_t}$, i.e., $\boldsymbol{u}_t \in U_t(\boldsymbol{x}_t)$. III) Near the end of the period, the uncertainty $\tilde{\boldsymbol{z}}_t$ is resolved and takes a value of $\boldsymbol{z}_t$, which results in an operational cash flow of $r_t = g_t(\boldsymbol{x}_t, \boldsymbol{u}_t, \boldsymbol{z}_t)$. The operational state of the system is updated as $\boldsymbol{x}_{t+1} = \boldsymbol{f}_t(\boldsymbol{x}_t, \boldsymbol{u}_t, \boldsymbol{z}_t)$. IV) The firm then makes the financing decision to borrow b_t (or lend if $b_t < 0$). In the event that the financing is restricted, we note that it can be constrained by the firm's wealth level w_t and also depends on the system's updated operational state $\boldsymbol{x}_{t+1}$. So we assume $b_t \in F_t(\boldsymbol{x}_{t+1}, w_t)$. For the case of full financing, we have $F_t(\cdot, \cdot) = \Re$ and in the absence of financing, we have $F_t(\cdot, \cdot) = \{0\}$. V) Finally, the state of wealth is updated as $w_{t+1} = (1 + \beta)w_t - b_t$.

The sequence of events in the terminal period T is the same as that for earlier periods except that there is no financing decision involved, i.e., $b_T = 0$. This is because we require all outstanding borrowing and interest earning to be settled by the end of period T so that the firm does not have outstanding balances. Given the different events in the terminal period and other periods, the present values of the consumptions over the horizon are given as follows. For $t \in \mathcal{T}^-$, $c_t = (r_t + b_t)/(1 + \beta)^t$ and in the terminal period, $c_T = (r_T + (1 + \beta)w_T)/(1 + \beta)^T$.Taking a closer look at c_T, we can see that it represents the accumulated wealth gained from the operational cash flow subtracting the consumptions made in earlier periods.

Definition 3 *An operations control state $\boldsymbol{s}^o_t \in \Re^{l^o_t}$, $t \in \mathcal{T}$ is non-anticipative if it is only influenced by the resolved uncertainty of $\boldsymbol{\zeta}_{t-1}$ and uninfluenced by future uncertainty $\boldsymbol{z}_t, \ldots, \tilde{\boldsymbol{z}}_T$. Similarly, a financing control state $\boldsymbol{s}^f_t \in \Re^{l^f_t}$, $t \in \mathcal{T}^-$ is non-anticipative if it is only influenced by the resolved uncertainty of $\boldsymbol{\zeta}_t$ and uninfluenced by future uncertainty $\tilde{\boldsymbol{z}}_{t+1}, \ldots, \tilde{\boldsymbol{z}}_T$.*

We highlight that the *operational state* $\boldsymbol{x}_t$ refers to the state of the system in the current period t, and *operations (financing) control state* $\boldsymbol{s}^o_t$ ($\boldsymbol{s}^f_t$) governs the operational (financing) policies in period t, which can contain more or less information than x_t.

Definition 4 *An admissible operational policy is a sequence of T measurable functions given by $\boldsymbol{\Pi} = \{\boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_T\}$ where $\boldsymbol{\pi}_t : \Re^{l^o_t} \mapsto \Re^{v_t}$ maps from a non-anticipative operations control state $\boldsymbol{s}^o_t$ into operational control $\boldsymbol{u}_t = \boldsymbol{\pi}_t(\boldsymbol{s}^o_t)$ and is such that $\boldsymbol{\pi}_t(\boldsymbol{s}^o_t) \in U_t(\boldsymbol{x}_t)$ for all possible states $\boldsymbol{s}^o_t$. Likewise, an admissible financing policy is a sequence of $T - 1$ measurable functions given by $\boldsymbol{\Phi} = \{\phi_1, \ldots, \phi_{T-1}\}$ where $\phi_t : \Re^{l^f_t} \mapsto \Re$ maps from a non-anticipative financing control state $\boldsymbol{s}^f_t$ into financing decision $b_t = \phi_t(\boldsymbol{s}^f_t)$ and is such that $\phi_t(\boldsymbol{s}^f_t) \in F_t(\boldsymbol{x}_{t+1}, w_t)$ for all states $\boldsymbol{s}^f_t$.*

We let $\mathcal{P}$ be the set of all admissible operational and financing policies. Starting with an initial operational state $\boldsymbol{x}_1$, an initial wealth w_1, and an admissible policy, $\boldsymbol{\Psi} = (\boldsymbol{\Pi}, \boldsymbol{\Phi}) \in \mathcal{P}$, the operational states $\tilde{\boldsymbol{x}}_t$ and the wealth of the firm $\tilde{w}_t$ are random variables with distributions defined through the following system equations:

$$\tilde{\boldsymbol{x}}_{t+1} = \boldsymbol{f}_t(\tilde{\boldsymbol{x}}_t, \boldsymbol{\pi}_t(\tilde{\boldsymbol{s}}_t^o), \tilde{\boldsymbol{z}}_t)$$

and

$$\tilde{w}_{t+1} = (1+\beta)\tilde{w}_t - \phi_t(\tilde{\boldsymbol{s}}_t^f)$$

for all $t \in \mathcal{T}^-$. The consumptions at the end of period t is a random variable given by

$$\tilde{c}_t(\boldsymbol{\Psi}) = \begin{cases} (g_t(\tilde{\boldsymbol{x}}_t, \boldsymbol{\pi}_t(\tilde{s}_t^o), \tilde{\boldsymbol{z}}_t) + \phi_t(\tilde{\boldsymbol{s}}_t^f)/(1+\beta)^t & \text{if } t \in \mathcal{T}^-, \\ (g_T(\tilde{\boldsymbol{x}}_T, \boldsymbol{\pi}_T(\tilde{\boldsymbol{s}}_T^0), \tilde{\boldsymbol{z}}_T) + (1+\beta)\tilde{w}_T)/(1+\beta)^T & \text{if } t = T. \end{cases}$$

We define $\tilde{\boldsymbol{c}}(\boldsymbol{\Psi}) = (\tilde{c}_1(\boldsymbol{\Psi}), \ldots, \tilde{c}_T(\boldsymbol{\Psi}))$ to represent the consumptions profile as a function of the operational and financing policies, $\boldsymbol{\Psi} \in \mathcal{P}$.

Definition 5 *A history dependent policy is an admissible policy in which the control states are the history of resolved uncertainties, i.e.,*

$$\begin{aligned} \boldsymbol{s}_t^o &= \boldsymbol{\zeta}_{t-1} \quad t \in \mathcal{T}, \\ \boldsymbol{s}_t^f &= \boldsymbol{\zeta}_t \quad\quad t \in \mathcal{T}^-. \end{aligned}$$

We define $\mathcal{P}_H$ as the set of all admissible history dependent operational and financing policies.

Note that $\boldsymbol{s}_{t+1}^o = \boldsymbol{s}_t^f$. This is because by the sequence of events described earlier, there is one time period lag between the operations control states and the financing control states.

In classical dynamic programming in which expected reward is maximized, the optimal policy is usually dependent on system states instead of being superfluously dependent on its history. However, depending on the decision criterion, the system states may not necessarily be sufficient to describe the optimal policy. Nevertheless, although there exists an history dependent policy that is also optimal, such a policy can lead to the “curse of dimensionality” and are computationally intractable to implement in practice. In subsequent sections, we will show that in some interesting cases, optimal control policies may also be concisely dependent on system states, which has important ramifications on computations.

The consumptions profile that minimizes the target-oriented CPRI criterion and the associated optimal policy can be obtained by solving the following optimization problem:

$$\begin{aligned} \varphi^* = \min \quad & \varphi_{\boldsymbol{\tau}} \left(\tilde{\boldsymbol{c}}(\boldsymbol{\Psi}) \right) \\ \text{s.t.} \quad & \boldsymbol{\Psi} \in \mathcal{P}_H, \end{aligned} \tag{3}$$

where $\varphi_{\boldsymbol{\tau}}$ is defined as in (2).

By the expressions of $\tilde{c}_t$ introduced in the earlier part of Section 3, we know that $\tilde{c}_t$, $t \in \mathcal{T}^-$ refers to the consumption and $\tilde{c}_T$ refers to the firm's wealth increase after consumptions in the previous periods. The target-oriented CPRI framework is thus consistent with such a scenario that when managers make decisions for a planning horizon, while meeting day-to-day corporate consumptions such as labor cost is important, at the end of the horizon, what's more crucial is whether achieving a pre-determined target profit, which often plays a significant role in managers' performance evaluation.

In this model, we say a policy $\boldsymbol{\Psi} \in \mathcal{P}_H$ is feasible if $\varphi_{\boldsymbol{\tau}} \left(\tilde{\boldsymbol{c}}(\boldsymbol{\Psi}) \right) < \infty$. Hence, we assume that the decision maker can aptly set her targets, $\boldsymbol{\tau}$ so that $\varphi^* \in (0, \infty)$.

There are several criticisms of Model (3) which are related to the setting of targets. If there exist some policies such that their corresponding consumptions profiles achieve the targets almost surely, then $\varphi^* = 0$ and the CPRI criterion cannot distinguish among any of these profiles. In the other extreme, if there does not exist a policy that yields a consumptions profile with finite CPRI, then $\varphi^* = \infty$ and this framework would fail to obtain a feasible policy. Nevertheless, it is not unreasonable to assume that targets are set based on the perceived economic outlook and one could argue that the decision maker is not overly pessimistic or optimistic in setting targets. Simon (1955) provides an example of selling a house and the agent's target is determined after she learns about the climate of the housing market. Similarly, we may also argue that one may conjure her targets after examining the consumptions profiles of some policies such as the risk neutral optimal policy.

Another related criticism is the lack of time consistency such that an optimal policy perceived in one time period may not be perceived as optimal in another. While the decision maker may change her targets as uncertainty resolves over time, time inconsistency may occur even when her targets remain fixated. For instance, it is plausible that when the economic outlook is bad, the targets imposed at earlier periods may no longer yield a feasible policy unless the decision maker is willing to lower her targets. Time inconsistency also occurs in dynamic decision problems with non-exponential discounting factors. Nevertheless, while time consistency is a desirable feature in dynamic decision making, it is violated in behavioral experiments even in the absence of uncertainty; see for instance Thaler (1981), Frederick et al. (2002) and Loch and Wu (2007). Loewenstein (1988) shows experimentally that shifts of reference points, or targets in our language, could better account for behavioral time consistency than discounted utility models. In some situations, we may circumvent the issue of time inconsistency by simply adhering to the original policy announced in the first period, which may be applicable when there are heavy penalties against deviations from original plans. In other situations, it may be reasonable to implement a rolling (or folding) horizon approach to dynamic decision making in which only the "here-and-now" solution is obtained and implemented in every period without the need to announce future "wait-and-see" policy. This is achieved by solving the optimization problem based on a fresh set of targets and using the latest information available whenever we need to make and implement the decisions.

Despite these valid criticisms, we will next show that in some interesting cases, optimizing over consumptions profiles under the CPRI criterion can be made almost as easy as solving the underlying dynamic programming model. We will also show numerically on an inventory control problem that by using this approach we can better regulate consumptions over time.

3.1 Optimal policy under full financing

We first analyze the case in which the decision maker has unrestricted access to borrowing and lending to finance consumptions in periods $t \in \mathcal{T}^-$, which is similar to the assumptions made in Smith (1998) and Chen et al. (2007) to obtain tractable analysis. In the absence of financing restrictions, we will show that there exists an optimal financing policy in which the consumptions at periods $t \in \mathcal{T}^-$ are exactly at their targets. We call this a *financing-at-target (FAT)* policy.

Definition 6 *Given an admissible operational policy* $\mathbf{\Pi}$*, the financing-at-target (FAT) policy is a financing policy* $\mathbf{\Phi} = \{\phi_1, \ldots, \phi_{T-1}\}$*,* $\phi_t : \Re \mapsto \Re$ *such that for all* $t \in \mathcal{T}^-$*,*

$$\phi_t(r_t) = (1+\beta)^t \tau_t - r_t,$$

where $r_t = g_t(\boldsymbol{x}_t, \boldsymbol{u}_t, \boldsymbol{z}_t)$ is the realized operational cash flow in period t under policy $\mathbf{\Pi}$.

Theorem 1 *Under full financing, there exists an optimal FAT policy that minimizes the CPRI criterion.*

Proof : Let $(\mathbf{\Pi}^*, \mathbf{\Phi}^*) \in \mathcal{P}_H$ be an optimal admissible history dependent policy to Model (3) in which $\mathbf{\Phi}^*$ may not be a FAT policy. For given operational policy $\mathbf{\Pi}^*$, we show that the corresponding FAT policy, $\hat{\mathbf{\Phi}}$ would not yield a consumptions profile with worse CPRI. The consumptions under the FAT policy is

$$\tilde{c}_t = \begin{cases} \tau_t & \text{for } t \in \mathcal{T}^-, \\ \sum_{k \in \mathcal{T}} \dfrac{\tilde{r}_k}{(1+\beta)^k} + w_1 - \sum_{t \in \mathcal{T}^-} \tau_k & \text{for } t = T. \end{cases}$$

where $\tilde{r}_t$, $t \in \mathcal{T}$ is the uncertain operational cash flows under the operational policy $\mathbf{\Pi}^*$. Let $\tilde{\boldsymbol{c}}^* = (\tilde{c}_1^*, \ldots, \tilde{c}_T^*)$ denote the consumptions profile under the optimal policy $(\mathbf{\Pi}^*, \mathbf{\Psi}^*)$ and $\tilde{b}_t$, $t \in \mathcal{T}^-$ denotes the corresponding financing cash flows. Therefore,

$$\tilde{c}_t^* = \begin{cases} \dfrac{\tilde{r}_t + \tilde{b}_t}{(1+\beta)^t} & \text{for } t \in \mathcal{T}^-, \\ w_1 + \dfrac{\tilde{r}_T}{(1+\beta)^T} - \sum_{k \in \mathcal{T}^-} \dfrac{\tilde{b}_k}{(1+\beta)^k} & \text{for } t = T. \end{cases}$$

Hence, by subadditivity property of the CPRI criterion, we have

$$\begin{aligned} \varphi_{\boldsymbol{\tau}}(\tilde{\boldsymbol{c}}^*) &\geq \varphi_{\mathbf{1}'\boldsymbol{\tau}}(\mathbf{1}'\tilde{\boldsymbol{c}}^*) \\ &= \varphi_{\mathbf{1}'\boldsymbol{\tau}}\left(\sum_{t \in \mathcal{T}^-} \left(\frac{\tilde{r}_t + \tilde{b}_t}{(1+\beta)^t} \right) + w_1 + \frac{\tilde{r}_T}{(1+\beta)^T} - \sum_{t \in \mathcal{T}^-} \frac{\tilde{b}_t}{(1+\beta)^t} \right) \\ &= \rho \left(\sum_{t \in \mathcal{T}} \frac{\tilde{r}_t}{(1+\beta)^t} + w_1 - \sum_{t \in \mathcal{T}} \tau_t \right) \\ &= \sum_{t \in \mathcal{T}^-} \rho(\tilde{c}_t - \tau_t) + \rho(\tilde{c}_T - \tau_T) \\ &= \varphi_{\boldsymbol{\tau}}(\tilde{\boldsymbol{c}}). \end{aligned}$$

■

The optimality of the FAT policy implies that as long as Model (3) is feasible, one can always attain the desired consumption targets through financing, with the exception of the last target.

Hence, we may perfectly regulate consumptions in periods $t \in \mathcal{T}^-$ by minimizing the CPRI and in doing so, relegate consumption uncertainty to the last period. We observe that the corresponding CPRI of the consumptions profile under the FAT policy can be expressed as follows:

$$\rho\left(\sum_{t\in\mathcal{T}} \frac{\tilde{r}_t}{(1+\beta)^t} + w_1 - \sum_{t\in\mathcal{T}} \tau_t\right) = \inf\left\{\alpha > 0 \;\middle|\; \mathrm{C}_\alpha\left(\sum_{t\in\mathcal{T}} \frac{\tilde{r}_t}{(1+\beta)^t}\right) \geq \sum_{t\in\mathcal{T}} \tau_t - w_1\right\},$$

where $\tilde{r}_t$, $t \in \mathcal{T}$ are the uncertain operational cash flows. Therefore, we can formulate the optimization problem to minimize CPRI as follows:

$$\begin{aligned} \varphi^* \quad = \quad & \min \alpha \\ \text{s.t.} \quad & \max_{\mathbf{\Pi}\in\mathcal{Q}} \mathrm{C}_\alpha\left(\sum_{t\in\mathcal{T}} \frac{\tilde{r}_t(\mathbf{\Pi})}{(1+\beta)^t}\right) \geq \sum_{t\in\mathcal{T}} \tau_t - w_1 \\ & \alpha \geq 0, \end{aligned} \tag{4}$$

where $\mathcal{Q}$ is the set of all admissible operational policies and $\tilde{r}_t(\mathbf{\Pi})$ denotes the uncertain operational cash flows under policy $\mathbf{\Pi} \in \mathcal{Q}$. We next present the optimal operational policy.

Theorem 2 *Under full financing, there exists an optimal operational state dependent policy that can be obtained by solving the dynamic programming given by*

$$\boldsymbol{\pi}_t(\boldsymbol{x}_t) = \begin{cases} \arg\max_{\boldsymbol{u}_T\in U_T(\boldsymbol{x}_T)} \mathrm{C}_{\varphi^*}\left(\dfrac{g_T(\boldsymbol{x}_T, \boldsymbol{u}_T, \tilde{\boldsymbol{z}}_T)}{(1+\beta)^T}\right) & \text{for } t = T, \\ \arg\max_{\boldsymbol{u}_t\in U_t(\boldsymbol{x}_t)} \mathrm{C}_{\varphi^*}\left(\dfrac{g_t(\boldsymbol{x}_t, \boldsymbol{u}_t, \tilde{\boldsymbol{z}}_t)}{(1+\beta)^t} + L^{\varphi^*}_{t+1}(f_t(\boldsymbol{x}_t, \boldsymbol{u}_t, \tilde{\boldsymbol{z}}_t))\right) & \text{for } t \in \mathcal{T}^-, \end{cases}$$

where

$$L^\alpha_t(\boldsymbol{x}_t) = \begin{cases} \max_{\boldsymbol{u}_T\in U_T(\boldsymbol{x}_T)} \mathrm{C}_\alpha\left(\dfrac{g_T(\boldsymbol{x}_T, \boldsymbol{u}_T, \tilde{\boldsymbol{z}}_T)}{(1+\beta)^T}\right) & \text{for } t = T, \\ \max_{\boldsymbol{u}_t\in U_t(\boldsymbol{x}_t)} \mathrm{C}_\alpha\left(\dfrac{g_t(\boldsymbol{x}_t, \boldsymbol{u}_t, \tilde{\boldsymbol{z}}_t)}{(1+\beta)^t} + L^\alpha_{t+1}(f_t(\boldsymbol{x}_t, \boldsymbol{u}_t, \tilde{\boldsymbol{z}}_t))\right) & \text{for } t \in \mathcal{T}^-, \end{cases}$$

defined for $\alpha > 0$ *and*

$$\varphi^* = \min\left\{\alpha > 0 \;\middle|\; L^\alpha_1(\boldsymbol{x}_1) \geq \sum_{t\in\mathcal{T}} \tau_t - w_1\right\}.$$

Moreover, $L^\alpha_1(\boldsymbol{x}_1)$ *is an nondecreasing function of* α *and hence,* φ^* *can be found by a standard binary search on* α*.*

Proof : In solving Problem (4), we observe the certainty equivalent function C_α is nondecreasing in $\alpha \in (0,\infty)$ and hence, we can obtain φ^* by standard binary search on α. For a given parameter, α, the subproblem to maximize certainty equivalent of total operational cash flows under exponential utility function can be solved by modifying standard dynamic programming (see Bertsekas 2005, pages 53-54) so that

$$L_1^\alpha(\boldsymbol{x}_1) = \max_{\boldsymbol{\Pi}\in\mathcal{Q}} \mathrm{C}_\alpha\left(\sum_{t\in\mathcal{T}} \frac{\tilde{r}_t(\boldsymbol{\Pi})}{(1+\beta)^t}\right).$$

Since this is a standard approach, we omit the proof for brevity. ■

3.2 Optimal policy for convex dynamic decision problems

We now consider the general case when financing is restricted. Since a FAT policy may not necessarily be admissible, it would not be always possible to obtain the optimal policy by solving a small collection of dynamic programming problems as we have done in Theorem 2. Nevertheless, we will focus on a special class of convex dynamic decision problems in which the structure of the optimal policies under the CPRI criterion can be analyzed. Analogous to a convex maximization problem, the feasible policies of a convex dynamic decision problem is convex and the consumptions are concave functions with respect to the policies. The first step is to ensure that the feasible set of the optimization problem is closed, which is necessary for our results to hold. Hence, we analyze the policy of an ϵ-closure of Model (3) defined as follows

$$\begin{array}{rll} \varphi_\epsilon^* = \min & \sum_{t\in\mathcal{T}} \alpha_t & \\ \text{s.t.} & \mathrm{C}_{\alpha_t}(\tilde{c}_t(\boldsymbol{\Psi})) \geq \tau_t, & t\in\mathcal{T} \\ & \boldsymbol{\Psi}\in\mathcal{P}_H & \\ & \boldsymbol{\alpha}\geq \mathbf{1}\epsilon, & \end{array} \tag{5}$$

where $\epsilon > 0$ is a small number. Since $\mathrm{C}_\alpha(\cdot)$ is nondecreasing in α, we can establish that

$$\varphi^* \leq \varphi_\epsilon^* \leq \varphi^* + T\epsilon,$$

and hence, the optimal policy of Model (5) can be made arbitrarily close to that of Model (3). Therefore, with an abuse of terminology, we refer to an optimal policy of Model (5) as one that also minimizes the CPRI criterion.

From here forward, we assume finite discrete distributions, i.e., $\Omega = \{\omega_1, \ldots, \omega_K\}$ and that $\mathbb{P}\{\omega_k\} > 0$. While this assumption aims to simplify the analysis, it is not practically limiting in most applications of dynamic decision problems. Under the assumption of finite sample space, at any point in time, there are only finitely many possible histories or states that will influence control decisions. Therefore, the history dependent policies can be perceived as vectors representing concatenation of controls corresponding to all the possible control states. Hence, Model (5) can be expressed as a finite dimensional optimization problem. Despite the case, there could be exponentially large number of decision variables and computationally prohibitive to solve Model (5) directly as a mathematical optimization problem. Instead, we propose to address the problem by solving a sequence of dynamic optimization problems, which may enable us to exploit the structures of their optimal policies for efficient computations. We make further assumptions on the problem.

Assumption 1 *The set of admissible policies $\mathcal{P}_H$ is closed, bounded, convex, i.e. for all $\mathbf{\Psi}^1, \mathbf{\Psi}^2 \in \mathcal{P}_H$,*

$$\lambda \mathbf{\Psi}^1 + (1-\lambda)\mathbf{\Psi}^2 \in \mathcal{P}_H \qquad \forall \lambda \in [0,1]$$

and strictly feasible, i.e. $\exists \mathbf{\Psi} \in \operatorname{int}\mathcal{P}_H$, where $\operatorname{int}\mathcal{P}_H$ refers to the interior of $\mathcal{P}_H$, such that $\varphi(\tilde{\mathbf{c}}(\mathbf{\Psi})) \in [0,\infty)$. The consumptions are concave with respect to the policy, i.e, for all $t \in \mathcal{T}$,

$$\tilde{c}_t(\lambda \mathbf{\Psi}^1 + (1-\lambda)\mathbf{\Psi}^2, \omega) \geq \lambda \tilde{c}_t(\mathbf{\Psi}^1, \omega) + (1-\lambda)\tilde{c}_t(\mathbf{\Psi}^2, \omega) \qquad \forall \lambda \in [0,1], \omega \in \Omega.$$

Theorem 3 *Under Assumption 1, the optimal policy under the CPRI criterion is one that maximizes the expected value of an additive-exponential utility function as follows:*

$$\begin{aligned} \max \quad & \mathbb{E}\left[\sum_{t=1}^{T} -\delta_t \exp\left(-\frac{\tilde{c}_t(\mathbf{\Psi})}{\alpha_t}\right)\right] \\ s.t. \quad & \mathbf{\Psi} \in \mathcal{P}_H, \end{aligned} \tag{6}$$

where $\boldsymbol{\alpha} > \mathbf{0}$ and $\boldsymbol{\delta} \geq \mathbf{0}$ take some specific values.

Proof : Note for any $\tilde{c} \in \mathcal{C}$, $\alpha > 0$, $\mathrm{C}_\alpha(\tilde{c}) \geq 0$ if and only if $\alpha \mathbb{E}[\exp(-\tilde{c}/\alpha)] - \alpha \leq 0$. We can therefore formulate Model (5) equivalently as follows:

$$
\begin{array}{ll}
\min & \sum_{t\in\mathcal{T}} \alpha_t \\
\text{s.t.} & \alpha_t \mathbb{E}\left[\exp\left(-(\tilde{c}_t(\boldsymbol{\Psi}) - \tau_t)/\alpha_t\right)\right] \le \alpha_t, \quad t \in \mathcal{T} \\
& \boldsymbol{\Psi} \in \mathcal{P}_H, \\
& \boldsymbol{\alpha} \ge \mathbf{1}\epsilon.
\end{array} \tag{7}
$$

We claim that the function $\alpha\mathbb{E}\left[\exp\left(-(\tilde{c}_t(\boldsymbol{\Psi}) - \tau)/\alpha\right)\right]$ is jointly convex in $(\alpha, \boldsymbol{\Psi})$, $\alpha > 0$. Indeed, given $\alpha^1, \alpha^2 > 0$, $\boldsymbol{\Psi}^1, \boldsymbol{\Psi}^2 \in \mathcal{P}_H$, let $\alpha^\lambda = \lambda\alpha^1 + (1-\lambda)\alpha^2$ and $\boldsymbol{\Psi}^\lambda = \lambda\boldsymbol{\Psi}^1 + (1-\lambda)\boldsymbol{\Psi}^2$ for $\lambda \in (0, 1)$. Observe that

$$
\begin{aligned}
& \alpha^\lambda \mathbb{E}\left[\exp\left(-(\tilde{c}_t(\boldsymbol{\Psi}^\lambda) - \tau)/\alpha^\lambda\right)\right] \\
\le\ & \alpha^\lambda \mathbb{E}\left[\exp\left(-(\lambda\tilde{c}_t(\boldsymbol{\Psi}^1) + (1-\lambda)\tilde{c}_t(\boldsymbol{\Psi}^2) - \tau)/\alpha^\lambda\right)\right] \\
=\ & \alpha^\lambda \mathbb{E}\left[\exp\left(-\tfrac{\lambda\alpha^1}{\lambda\alpha^1+(1-\lambda)\alpha^2}(\tilde{c}_t\left(\boldsymbol{\Psi}^2\right) - \tau)/\alpha^1 - \tfrac{(1-\lambda)\alpha^2}{\lambda\alpha^1+(1-\lambda)\alpha^2}(\tilde{c}_t\left(\boldsymbol{\Psi}^2\right) - \tau)/\alpha^2\right)\right] \\
\le\ & \lambda\alpha^1 \mathbb{E}\left[\exp\left(-(\tilde{c}_t\left(\boldsymbol{\Psi}^2\right) - \tau)/\alpha^1\right)\right] + (1-\lambda)\alpha^2 \mathbb{E}\left[\exp\left(-(\tilde{c}_t\left(\boldsymbol{\Psi}^2\right) - \tau)/\alpha^2\right)\right]
\end{aligned}
$$

where the first inequality holds for the convexity of the dynamic decision problem, the second inequality follows from the convexity of the exponential function. Hence, Problem (7) is a convex optimization problem with finite number of decision variables and it is strictly feasible by assumption. We let this be the primal problem and the Lagrange dual problem follows:

$$
\begin{array}{ll}
\max & g(\boldsymbol{\lambda}) \\
\text{s.t.} & \boldsymbol{\lambda} \ge \mathbf{0},
\end{array} \tag{8}
$$

where

$$
g(\boldsymbol{\lambda}) = \min_{\boldsymbol{\alpha}\ge\mathbf{1}\epsilon,\, \boldsymbol{\Psi}\in\mathcal{P}_H} L(\boldsymbol{\alpha}, \boldsymbol{\Psi}, \boldsymbol{\lambda}),
$$

$$
L(\boldsymbol{\alpha}, \boldsymbol{\Psi}, \boldsymbol{\lambda}) = \sum_{t\in\mathcal{T}} \left(\alpha_t + \lambda_t\left(\alpha_t\mathbb{E}\left[\exp\left(-(\tilde{c}_t(\boldsymbol{\Psi}) - \tau_t)/\alpha_t\right)\right] - \alpha_t\right)\right).
$$

Since the primal problem has finite objective, is convex and strictly feasible, strong duality holds and the dual variables $\boldsymbol{\lambda}$ are attainable. (See for instance, Boyd and Vandenberghe 2004, section 5.2.3). Let $(\boldsymbol{\alpha}^*, \boldsymbol{\Psi}^*)$ be any optimal solution to the primal problem (7), and $\boldsymbol{\lambda}^*$ be any optimal solution to the dual problem (8). We have

$$
L(\boldsymbol{\alpha}^*, \boldsymbol{\Psi}^*, \boldsymbol{\lambda}^*) = \sum_{t\in\mathcal{T}} \left(\alpha_t^* + \lambda_t^*\left(\alpha_t^*\mathbb{E}\left[\exp\left(-(\tilde{c}_t(\boldsymbol{\Psi}^*) - \tau_t)/\alpha_t^*\right)\right] - \alpha_t^*\right)\right) = \sum_{t\in\mathcal{T}} \alpha_t^* = g(\boldsymbol{\lambda}^*).
$$

Therefore, $L(\boldsymbol{\alpha}^*, \boldsymbol{\Psi}^*, \boldsymbol{\lambda}^*) = \min_{\boldsymbol{\Psi}\in\mathcal{P}_H} L(\boldsymbol{\alpha}^*, \boldsymbol{\Psi}, \boldsymbol{\lambda}^*)$, and $\boldsymbol{\Psi}^*$ is an optimal solution to the problem given by

$$\max_{\boldsymbol{\Psi}\in\mathcal{P}_H} \mathbb{E}\left[\sum_{t\in\mathcal{T}} -\delta_t^* \exp(-\tilde{c}_t(\boldsymbol{\Psi})/\alpha_t^*)\right],$$

where $\delta_t^* = \lambda_t^* \alpha_t^* \exp(\tau_t/\alpha_t^*)$. ■

Proposition 2 *The optimal policy that maximizes an expected additive-exponential utility can be obtained by solving the dynamic programming algorithm given by*

$$\boldsymbol{\pi}_t(\boldsymbol{x}_t, w_t) = \begin{cases} \arg\max\limits_{\boldsymbol{u}_T\in U_T(\boldsymbol{x}_T)} \mathbb{E}_{\tilde{\boldsymbol{z}}_T}\left[-\delta_T \exp\left(-\dfrac{(1+\beta)w_T + g_T(\boldsymbol{x}_T, \boldsymbol{u}_T, \tilde{\boldsymbol{z}}_T)}{(1+\beta)^T \alpha_T}\right)\right] & t = T, \\ \arg\max\limits_{\boldsymbol{u}_t\in U_t(\boldsymbol{x}_t)} \mathbb{E}_{\tilde{\boldsymbol{z}}_t}\left[V_t^f\left(\boldsymbol{f}_t(\boldsymbol{x}_t, \boldsymbol{u}_t, \tilde{\boldsymbol{z}}_t), w_t, g_t(\boldsymbol{x}_t, \boldsymbol{u}_t, \tilde{\boldsymbol{z}}_t)\right)\right] & t \in \mathcal{T}^-, \end{cases}$$

$$\boldsymbol{\phi}_t(\boldsymbol{x}_{t+1}, w_t, r_t) = \arg\max_{b_t\in F_t(\boldsymbol{x}_{t+1}, w_t)} \left\{-\delta_t \exp\left(-\frac{r_t + b_t}{(1+\beta)^t \alpha_t}\right) + V_{t+1}^o\left(\boldsymbol{x}_{t+1}, (1+\beta)w_t - b_t\right)\right\},$$

where

$$V_t^o(\boldsymbol{x}_t, w_t) = \begin{cases} \max\limits_{\boldsymbol{u}_T\in U_T(\boldsymbol{x}_T)} \mathbb{E}_{\tilde{\boldsymbol{z}}_T}\left[-\delta_T \exp\left(-\dfrac{(1+\beta)w_T + g_T(\boldsymbol{x}_T, \boldsymbol{u}_T, \tilde{\boldsymbol{z}}_T)}{(1+\beta)^T \alpha_T}\right)\right] & t = T \\ \max\limits_{\boldsymbol{u}_t\in U_t(\boldsymbol{x}_t)} \mathbb{E}_{\tilde{\boldsymbol{z}}_t}\left[V_t^f\left(\boldsymbol{f}_t(\boldsymbol{x}_t, \boldsymbol{u}_t, \tilde{\boldsymbol{z}}_t), w_t, g_t(\boldsymbol{x}_t, \boldsymbol{u}_t, \tilde{\boldsymbol{z}}_t)\right)\right] & t \in \mathcal{T}^- \end{cases}$$

$$V_t^f(\boldsymbol{x}_{t+1}, w_t, r_t) = \max_{b_t\in F_t(\boldsymbol{x}_{t+1}, w_t)} \left\{-\delta_t \exp\left(-\frac{r_t + b_t}{(1+\beta)^t \alpha_t}\right) + V_{t+1}^o\left(\boldsymbol{x}_{t+1}, (1+\beta)w_t - b_t\right)\right\}.$$

Proof : Following standard dynamic programming procedure, let $V_t^o(\boldsymbol{x}_t, w_t)$ be the maximal value of $\sum_{i=t}^T \mathbb{E}[-\delta_i \exp(-\tilde{c}_i/\alpha_i)]$ given $(\boldsymbol{x}_t, w_t)$ at the beginning of period t, $t \in \mathcal{T}$. Similarly, let $V_t^f(\boldsymbol{x}_{t+1}, w_t, r_t)$ be the maximal value of $\sum_{i=t}^T \mathbb{E}[-\delta_i \exp(-\tilde{c}_i/\alpha_i)]$ given w_t, r_t and $\boldsymbol{x}_{t+1}$ at the end of period t. We observe that

$$V_T^o(\boldsymbol{x}_t, w_t) \;=\; \max_{\boldsymbol{u}_T\in U_T(\boldsymbol{x}_T)} \mathbb{E}_{\tilde{\boldsymbol{z}}_T}\left[-\delta_T \exp\left(-\frac{(1+\beta)w_T + g_T(\boldsymbol{x}_T, \boldsymbol{u}_T, \tilde{\boldsymbol{z}}_T)}{(1+\beta)^T \alpha_T}\right)\right],$$

and we can obtain $V_t^f(\boldsymbol{x}_{t+1}, w_t, r_t)$ and $V_t^o(\boldsymbol{x}_t, w_t)$ by standard backward induction. ■

To obtain the optimal policy of Model (5), it remains to find the appropriate parameters $\boldsymbol{\delta}$ and $\boldsymbol{\alpha}$. We next present an algorithm for obtaining the optimal policy as follows.

Algorithm Main

Input: Initial multipliers $\boldsymbol{\lambda} \geq \mathbf{0}$ and parameters $\boldsymbol{\alpha} \geq \mathbf{0}$, $i = 1$.

repeat

Run **Algorithm Coordinate Descent** with input $(\boldsymbol{\lambda}, \boldsymbol{\alpha})$ and output $(\boldsymbol{h}, \boldsymbol{\alpha}, \boldsymbol{\Psi})$.

Update multipliers, $\boldsymbol{\lambda} := \max\left(\boldsymbol{\lambda} + d_i \boldsymbol{h}, 0\right)$

Update index $i := i + 1$.

until stopping criterion is met.

Output: $\boldsymbol{\Psi}$

Here $d_i > 0$ is a decreasing sequence of step sizes satisfying $\lim_{i \to \infty} d_i = 0$ and $\sum_{i=1}^{\infty} d_i = \infty$.

Algorithm Coordinate Descent

Input: Multipliers $\boldsymbol{\lambda} \geq \mathbf{0}$ and parameters $\boldsymbol{\alpha} \geq \mathbf{0}$.

repeat

Update $\boldsymbol{\Psi} := \arg\min_{\boldsymbol{\Psi} \in \mathcal{P}_H} L\left(\boldsymbol{\alpha}, \boldsymbol{\Psi}, \boldsymbol{\lambda}\right)$.

Update $\boldsymbol{\alpha} := \arg\min_{\boldsymbol{\alpha} \geq \mathbf{1}\epsilon} L\left(\boldsymbol{\alpha}, \boldsymbol{\Psi}, \boldsymbol{\lambda}\right)$.

until stopping criterion is met.

Output: $(\boldsymbol{h}, \boldsymbol{\alpha}, \boldsymbol{\Psi})$, where $h_t := \left(\alpha_t \mathbb{E}\left[\exp\left(-(\tilde{c}_t(\boldsymbol{\Psi}) - \tau_t)/\alpha_t\right)\right] - \alpha_t\right), t \in \mathcal{T}$.

Here the Lagrangian $L(\boldsymbol{\alpha}, \boldsymbol{\Psi}, \boldsymbol{\lambda})$ is defined as follows:

$$L(\boldsymbol{\alpha}, \boldsymbol{\Psi}, \boldsymbol{\lambda}) = \sum_{t \in \mathcal{T}} \left(\alpha_t + \lambda_t \left(\alpha_t \mathbb{E}\left[\exp\left(-(\tilde{c}_t(\boldsymbol{\Psi}) - \tau_t)/\alpha_t\right)\right] - \alpha_t\right)\right).$$

Observe that from Proposition 2, we can use dynamic programming to obtain the optimal policy $\boldsymbol{\Psi}$ that minimizes the Lagrangian for given $(\boldsymbol{\alpha}, \boldsymbol{\lambda})$. To obtain the optimal solution $\boldsymbol{\alpha}$ that minimizes the Lagrangian for given $(\boldsymbol{\lambda}, \boldsymbol{\Psi})$, we can do so by solving the univariate convex optimization problem,

$$\alpha_t = \arg\min \left\{ \alpha + \lambda_t \left(\alpha \mathbb{E}\left[\exp\left(-(\tilde{c}_t(\boldsymbol{\Psi}) - \tau_t)/\alpha\right)\right] - \alpha\right) \, \middle| \alpha \geq \epsilon \right\},$$

via bisection search techniques such as the Golden search methods (Kiefer 1953). To ensure that the algorithm converges to the optimal policy, we require a differentiability assumption on the Lagrangian, which is implied in our next assumption.

Assumption 2 *Given* $\boldsymbol{\alpha} > \mathbf{0}$ *and* $\boldsymbol{\lambda} \geq 0$*, the Lagrangian* $L(\boldsymbol{\alpha}, \boldsymbol{\Psi}, \boldsymbol{\lambda})$ *is differentiable with respect to* $\boldsymbol{\Psi}$ *for all* $\boldsymbol{\Psi} \in \mathcal{P}_H$*.*

Theorem 4 *Under Assumptions 1 and 2, Algorithm Main returns an optimal policy that minimizes the CPRI criterion.*

Proof : Algorithm Main is a standard subgradient optimization routine for obtaining the optimal multiplier solution, $\boldsymbol{\lambda}$ in the nondifferential dual function (see for instance, Bertsekas 1999, section 6.3). It calls upon Algorithm Coordinate Descent to obtain the subgradient $\boldsymbol{h}$. We first show for any input $\boldsymbol{\lambda}$, the limit point of the sequence $\{(\boldsymbol{\alpha}^{(i)}, \boldsymbol{\Psi}^{(i)})\}$, which is generated by Algorithm Coordinate Descent, minimizes $L(\boldsymbol{\alpha}, \boldsymbol{\Psi}, \boldsymbol{\lambda})$. We write $f(\boldsymbol{\alpha}, \boldsymbol{\Psi}) = L(\boldsymbol{\alpha}, \boldsymbol{\Psi}, \boldsymbol{\lambda})$ for ease of notation.

We first observe that $\boldsymbol{\alpha}^{(i)}$ is bounded below by $\mathbf{1}\epsilon$ and that $\boldsymbol{\Psi}^{(i)}$ is bounded, since we assume that the policy set is bounded. We will next show that the sequence $\{\boldsymbol{\alpha}^{(i)}\}$ is bounded above by $\hat{\boldsymbol{\alpha}}$, which is defined by

$$\hat{\alpha}_t = \begin{cases} \epsilon + 2 + 2\lambda_t \left(\sup\limits_{\boldsymbol{\Psi} \in \mathcal{P}_H} \mathbb{E}\left[\exp(-\tilde{c}_t(\boldsymbol{\Psi}) + \tau_t)\right] - 1 \right) & \text{if } \lambda_t \leq 0.5, \\ \epsilon + \max\left\{ 2 + 2\lambda_t \left(\sup\limits_{\boldsymbol{\Psi} \in \mathcal{P}_H} \mathbb{E}\left[\exp(-\tilde{c}_t(\boldsymbol{\Psi}) + \tau_t)\right] - 1 \right), -\dfrac{\bar{c}_t}{\ln((\lambda_t - 0.5)/\lambda_t)} \right\} & \text{if } \lambda_t > 0.5, \end{cases}$$

where $\bar{c}_t = \sup_{\boldsymbol{\Psi} \in \mathcal{P}_H} \inf\{v \mid \mathbb{P}(\tilde{c}_t(\boldsymbol{\Psi}) - \tau_t \leq v) = 1\}$ represents the maximal consumption premium in period t. Let us assume $\epsilon \in (0, 1)$. For a given feasible policy $\boldsymbol{\Psi} \in \mathcal{P}_H$, suppose $\boldsymbol{\alpha} \geq \mathbf{1}\epsilon$ such that $\alpha_t > \hat{\alpha}_t$ for some $t \in \mathcal{T}$. We will show that $\boldsymbol{\alpha}$ is not the optimal solution that minimizes the Lagrangian for given $(\boldsymbol{\lambda}, \boldsymbol{\Psi})$. Indeed, we have:

$$\begin{aligned} &\alpha_t + \lambda_t \left(\alpha_t \exp\left(-(\tilde{c}_t(\boldsymbol{\Psi}) - \tau_t)/\alpha_t\right) - \alpha_t\right) \\ &\geq \alpha_t \left(1 + \lambda_t \exp\left(-\bar{c}_t/\alpha_t\right) - \lambda_t\right) = \upsilon_t. \end{aligned} \tag{9}$$

If $\lambda_t \leq 0.5$, then $\upsilon_t \geq 0.5\alpha_t$. If $\lambda_t > 0.5$ and $\bar{c}_t \leq 0$, then $\upsilon_t \geq \alpha_t(1 + \lambda_t - \lambda_t) = \alpha_t$. Finally, if $\lambda_t > 0.5$ and $\bar{c}_t > 0$, then $-\bar{c}_t/\alpha_t \geq -\bar{c}_t/\hat{\alpha}_t \geq \ln\left((\lambda_t - 0.5)/\lambda_t\right)$ and $\upsilon_t \geq \alpha_t(1 + \lambda_t \times (\lambda_t - 0.5)/\lambda_t - \lambda_t) = 0.5\alpha_t$. We have:

$$\begin{aligned} &\alpha_t + \lambda_t \left(\alpha_t \exp\left(-(\tilde{c}_t(\boldsymbol{\Psi}) - \tau_t)/\alpha_t\right) - \alpha_t\right) \\ &\geq 0.5\alpha_t > 0.5\hat{\alpha}_t \geq 1 + \lambda_t \left(\sup\nolimits_{\boldsymbol{\Psi} \in \mathcal{P}_H} \mathbb{E}\left[\exp(-\tilde{c}_t(\boldsymbol{\Psi}) + \tau_t)\right] - 1\right). \end{aligned}$$

Therefore, we can lower the value of the Lagrangian, $L(\boldsymbol{\alpha}, \boldsymbol{\Psi}, \boldsymbol{\lambda})$ by changing α_t to 1 and hence, $\boldsymbol{\alpha}^{(i+1)}$ must be bounded above by $\max\{\mathbf{1}, \hat{\boldsymbol{\alpha}}\}$. Since, $(\boldsymbol{\alpha}^{(i)}, \boldsymbol{\Psi}^{(i)})$ is a bounded sequence and there must exist at least one limit point.

Let $(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}})$ be a limit point of the sequence $\{(\boldsymbol{\alpha}^{(i)}, \boldsymbol{\Psi}^{(i)})\}$. Algorithm Coordinate Descent ensures that the following inequalities holds:

$$f(\boldsymbol{\alpha}^{(i)}, \boldsymbol{\Psi}^{(i)}) \geq f(\boldsymbol{\alpha}^{(i)}, \boldsymbol{\Psi}^{(i+1)}) \geq f(\boldsymbol{\alpha}^{(i+1)}, \boldsymbol{\Psi}^{(i+1)}), \tag{10}$$

for all i. Therefore, since $(\alpha^{(i)}, \boldsymbol{\Psi}^{(i)})$ is bounded, which implies $f(\boldsymbol{\alpha}^{(i)}, \boldsymbol{\Psi}^{(i)}) > -\infty$, the sequence $\{f(\boldsymbol{\alpha}^{(i)}, \boldsymbol{\Psi}^{(i)})\}$ is nonincreasing and converges to the limit point, $f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}})$. It remains to prove that $(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}})$ minimizes f, which we will show by contradiction. Let $\{(\boldsymbol{\alpha}^{(i_j)}, \boldsymbol{\Psi}^{(i_j)}) \mid j = 0, 1, \cdots\}$ be a subsequence of $\{(\boldsymbol{\alpha}^{(i)}, \boldsymbol{\Psi}^{(i)})\}$ that converges to $(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}})$. Suppose there exists $\boldsymbol{\Psi}^o \in \mathcal{P}_H$ such that $\Delta = f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}}) - f(\bar{\boldsymbol{\alpha}}, \boldsymbol{\Psi}^o) > 0$. Under Assumption 1, the function $f(\boldsymbol{\alpha}, \boldsymbol{\Psi})$ is differentiable with respect to $\boldsymbol{\alpha} > \mathbf{0}$ and $\boldsymbol{\Psi} \in \mathcal{P}_H$. Since continuity is implied by differentiability, we can find $\delta > 0$ such that $|f(\boldsymbol{\alpha}, \boldsymbol{\Psi}^o) - f(\bar{\boldsymbol{\alpha}}, \boldsymbol{\Psi}^o)| < \Delta/2$ for all $\boldsymbol{\alpha}$ with $\|\boldsymbol{\alpha} - \bar{\boldsymbol{\alpha}}\| \leq \delta$. Since $\{\boldsymbol{\alpha}^{(i_j)}\}$ converges to $\bar{\boldsymbol{\alpha}}$, we can find M_1 such that for all $j > M_1$, $||\boldsymbol{\alpha}^{(i_j)} - \bar{\boldsymbol{\alpha}}|| \leq \delta$ and therefore $|f(\boldsymbol{\alpha}^{(i_j)}, \boldsymbol{\Psi}^o) - f(\bar{\boldsymbol{\alpha}}, \boldsymbol{\Psi}^o)| < \Delta/2$. Moreover, since $\{f(\boldsymbol{\alpha}^{(i_j)}, \boldsymbol{\Psi}^{(i_j+1)})\}$ converges to $f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}})$, there exists M_2 such that for all $j > M_2$, $|f(\boldsymbol{\alpha}^{(i_j)}, \boldsymbol{\Psi}^{(i_j+1)}) - f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}})| < \Delta/2$. Therefore, for $j > \max\{M_1, M_2\}$, we have:

$$f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}}) < f(\boldsymbol{\alpha}^{(i_j)}, \boldsymbol{\Psi}^{(i_j+1)}) + \Delta/2 \leq f(\boldsymbol{\alpha}^{(i_j)}, \boldsymbol{\Psi}^o) + \Delta/2 < f(\bar{\boldsymbol{\alpha}}, \boldsymbol{\Psi}^o) + \Delta,$$

which contradicts the assumption that $f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}}) = f(\bar{\boldsymbol{\alpha}}, \boldsymbol{\Psi}^o) + \Delta$. Hence, we conclude that $f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}}) \leq f(\bar{\boldsymbol{\alpha}}, \boldsymbol{\Psi})$, $\forall \boldsymbol{\Psi} \in \mathcal{P}_H$. Therefore, $\bar{\boldsymbol{\Psi}}$ minimizes $f(\bar{\boldsymbol{\alpha}}, \cdot)$. Under the assumption of differentiability and convexity, the optimality condition is equivalent to $\nabla_{\boldsymbol{\Psi}} f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}})'(\boldsymbol{\Psi} - \bar{\boldsymbol{\Psi}}) \geq 0$ for all $\boldsymbol{\Psi} \in \mathcal{P}_H$ (see for instance, Bertsekas 1999, Proposition 2.1.2).

Similarly, suppose there exists $\boldsymbol{\alpha}^o \geq \mathbf{1}\epsilon$ such that $\Delta = f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}}) - f(\boldsymbol{\alpha}^o, \bar{\boldsymbol{\Psi}}) > 0$. By continuity argument, we can find $\delta > 0$ such that $|f(\boldsymbol{\alpha}^o, \boldsymbol{\Psi}) - f(\boldsymbol{\alpha}^o, \bar{\boldsymbol{\Psi}})| < \Delta/2$ for all $\boldsymbol{\Psi}$ with $\|\boldsymbol{\Psi} - \bar{\boldsymbol{\Psi}}\| \leq \delta$. Since $\{\boldsymbol{\Psi}^{(i_j)}\}$ converges to $\bar{\boldsymbol{\Psi}}$, we can find M_1 such that for all $j > M_1$, $||\boldsymbol{\Psi}^{(i_j)} - \bar{\boldsymbol{\Psi}}|| \leq \delta$ and therefore $|f(\boldsymbol{\alpha}^o, \boldsymbol{\Psi}^{(i_j)}) - f(\boldsymbol{\alpha}^o, \bar{\boldsymbol{\Psi}})| < \Delta/2$. Moreover, since $\{f(\boldsymbol{\alpha}^{(i_j)}, \boldsymbol{\Psi}^{(i_j)})\}$ converges to $f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}})$, there exists M_2 such that for all $j > M_2$, $|f(\boldsymbol{\alpha}^{(i_j)}, \boldsymbol{\Psi}^{(i_j)}) - f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}})| < \Delta/2$. Therefore, for $j > \max\{M_1, M_2\}$, we have:

$$f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}}) < f(\boldsymbol{\alpha}^{(i_j)}, \boldsymbol{\Psi}^{(i_j)}) + \Delta/2 \leq f(\boldsymbol{\alpha}^o, \boldsymbol{\Psi}^{(i_j)}) + \Delta/2 < f(\boldsymbol{\alpha}^o, \bar{\boldsymbol{\Psi}}) + \Delta,$$

which contradicts the assumption that $f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}}) = f(\boldsymbol{\alpha}^o, \bar{\boldsymbol{\Psi}}) + \Delta$. Hence, we conclude that $f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}}) \leq f(\boldsymbol{\alpha}, \bar{\boldsymbol{\Psi}})$, $\forall \boldsymbol{\alpha} \geq \mathbf{1}\epsilon$. Since $f(\boldsymbol{\alpha}, \boldsymbol{\Psi})$ is differentiable and convex in $\boldsymbol{\alpha}$, it implies by the optimality condition that $\nabla_{\boldsymbol{\alpha}} f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}})'(\boldsymbol{\alpha} - \bar{\boldsymbol{\alpha}}) \geq 0$ for all $\boldsymbol{\alpha} \geq \mathbf{1}\epsilon$.

Combining the two above results, we have:

$$\nabla f(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}})' \left((\boldsymbol{\alpha}, \boldsymbol{\Psi}) - (\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}})\right) \geq 0, \qquad \forall \boldsymbol{\alpha} \geq \mathbf{1}\epsilon, \boldsymbol{\Psi} \in \mathcal{P}_H.$$

Since $f(\boldsymbol{\alpha}, \boldsymbol{\Psi})$ is also jointly convex in $(\boldsymbol{\alpha}, \boldsymbol{\Psi})$, this implies that $(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}})$ minimizes $f(\cdot, \cdot)$.

Finally we show that the output of Algorithm Coordinate Descent, $\boldsymbol{h}$ is indeed a subgradient of $g(\cdot)$ at the input $\boldsymbol{\lambda}$. We observe $g(\cdot)$ is a pointwise minimum of a family of affine functions and hence is concave. Moreover, for any $\boldsymbol{\lambda}^o \geq \mathbf{0}$, we have:

$$\begin{aligned} g(\boldsymbol{\lambda}^o) - g(\boldsymbol{\lambda}) &= \min_{\boldsymbol{\alpha} \geq \mathbf{1}\epsilon, \boldsymbol{\Psi} \in \mathcal{P}_H} L(\boldsymbol{\alpha}, \boldsymbol{\Psi}, \boldsymbol{\lambda}^o) - L(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}}, \boldsymbol{\lambda}) \\ &\leq L(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}}, \boldsymbol{\lambda}^o) - L(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\Psi}}, \boldsymbol{\lambda}) \\ &= (\boldsymbol{\lambda}^o - \boldsymbol{\lambda})' \boldsymbol{h}. \end{aligned}$$

Therefore, $\boldsymbol{h}$ is a subgradient of $g(\cdot)$ at $\boldsymbol{\lambda}$. ■

4 Target-oriented inventory management

In this section, we study the joint inventory-pricing decision problem with financing control under the target-oriented CPRI decision criterion. The setup is similar to that in Chen et al. (2007), where a firm needs to determine the optimal replenishment and pricing policy spanning T periods. At the beginning of any period $t \in \mathcal{T}$, the inventory level x_t and the state of wealth w_t are observed. The decision maker then determines the selling price $p_t \in [\underline{p}_t, \bar{p}_t]$ and replenishes the inventory to the level $y_t \geq x_t$. The ordering cost, which consists of unit variable cost q_t and fixed ordering cost K_t, will be paid at the end of the period. The random demand is bounded and affected by the pricing decision p_t such that

$$\tilde{d}_t = D_t(p_t, \tilde{\boldsymbol{z}}_t) = \tilde{z}_t^1 - \tilde{z}_t^2 p_t,$$

where $\tilde{\boldsymbol{z}}_t = (\tilde{z}_t^1, \tilde{z}_t^2)$ are independently distributed nonnegative random variables satisfying $\mathbb{E}[\tilde{\boldsymbol{z}}_t] > \mathbf{0}$. The demand is realized near the end of the period and unsatisfied demand is backlogged. The inventory level is then updated as $x_{t+1} = y_t - d_t$ and the inventory cost $h_t(x_{t+1})$ is tabulated. The cost function, $h_t(x)$ is convex in x, represents holding cost if $x \geq 0$ and shortage cost otherwise. At the end of the planning horizon, $h_T(x)$ can be modified to include salvage values of unsold inventories.

Similar to Federgruen and Heching (1999), we assume

$$\lim_{x\to\infty} ((q_t - q_{t+1}/(1+\beta))x + h_t(x)) = \lim_{x\to-\infty} (q_t x + h_t(x)) = \infty \qquad t \in \mathcal{T},$$

with $q_{T+1} = 0$ for simplicity. The uncertain income at the end of the period $t \in \mathcal{T}$, is hence given by

$$\tilde{r}_t = p_t D_t(p_t, \tilde{\boldsymbol{z}}_t) - h_t(y_t - D_t(p_t, \tilde{\boldsymbol{z}}_t)) - K_t \gamma(y_t - x_t) - q_t(y_t - x_t),$$

where

$$\gamma(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Similar to the general dynamic decision model discussed in Section 3, once the income r_t is realized, the firm determines the financing level $b_t \in F_t(x_{t+1}, w_t)$ in period $t \in \mathcal{T}^-$ to fulfill the desired level of consumption c_t and the state of wealth w_{t+1} is updated accordingly. At any point in time, we define the *asset value* as the total wealth including the value of inventories on hand. Specifically, at the beginning of period $t \in \mathcal{T}$, the asset value is $a_t = w_t + q_t x_t/(1+\beta)$ and near the end of the period just after the demand is resolved and before the financing decision if $t \in \mathcal{T}^-$, the asset value is $\bar{a}_t = (1+\beta)w_t + q_{t+1}x_{t+1}/(1+\beta)$. Note here we apply a discount factor to the inventory value because the unit variable cost q_t is taken at the end of the period.

Before presenting our results under the CPRI criterion, we first summarize in Table 1 the replenishment policies under various decision criteria based on risk sensitive additive utility functions. Under the umbrella of risk neutral models, in the absence of pricing decision, i.e., $\bar{p}_t = \underline{p}_t$, Scarf (1960) establishes the well-known optimality of (s, S) inventory policy in the presence of fixed ordering cost. The inventory replenishment strategy in period t is characterized by two parameters (s_t, S_t) such that if the inventory level x_t is below s_t, an order of size $S_t - x_t$ is made. Otherwise, no order is placed. A special case of this policy is the base-stock policy, in which $s_t = S_t$ is the base-stock level, which is optimal when $K_t = 0$. If pricing decision is available, Chen and Simchi-Levi's (2004) show that (s, S, A, p) joint inventory-pricing policy is optimal. The inventory strategy in period t is characterized by two parameters (s_t, S_t) and a set $A_t \subseteq [s_t, (s_t + S_t)/2]$, which can be empty depending on the problem instance. Whenever $x_t < s_t$ or $x_t \in A_t$, an order of size $S_t - x_t$ is placed. Otherwise, no order is placed. The price depends on the initial inventory level at the beginning of the period. Taking into account of decision maker's risk aversion, Chen et al. (2007) show that under full financing, the optimal policy under an additive-exponential utility has the same structure as the classical risk neutral counterpart. Chen et al. (2007) also present the structural results for general additive concave utility function and under full financing and $K_t = 0$. They show that the optimal policy is one of base-stock in which the base-stock level depends solely on the asset value a_t at the beginning of period $t \in \mathcal{T}$.

Table 1: Summary of results under additive utility decision criteria.

	Price Not a Decision		*Price is a Decision*	
	$K_t = 0$	$K_t > 0$	$K_t = 0$	$K_t > 0$
Risk-neutral model	Base-stock	(s, S)	Base-stock list price	(s, S, A, p)
Exponential utility under full financing	Base-stock	(s, S)	Base-stock	(s, S, A, p)
Increasing & concave utility under full financing	Asset dependent base-stock	?	Asset dependent base-stock	?

We now present our results for the joint inventory-pricing control problem under the CPRI criterion.

Theorem 5 *Under full financing, the optimal policies of the joint inventory-pricing control problem under the CPRI criterion are as follows:*

1) A base stock policy is optimal without fixed ordering cost.

2) An (s, S) policy is optimal in the absence of pricing control.

3) An (s, S, A, p) policy is optimal in the general case when pricing decision is allowed and the fixed ordering cost is positive.

Proof : We have established in Theorem 2 that the optimal policies of the inventory-pricing decision problems correspond to those that maximize an expected exponential utility of the total incomes. The structural results of these policies are derived by Chen et al (2004). ■

Unfortunately, structural results under constrained financing are limited and so we restrict ourselves to the case when where is no fixed ordering cost, which belongs to the class of convex dynamic decision problems. We consider a special type of financing restriction, which we call *asset constrained financing* where borrowing above the asset value is prohibited, that is,

$$F_t(x_{t+1}, w_t) = \{b \mid b \le \underbrace{(1+\beta)w_t + q_{t+1}x_{t+1}/(1+\beta)}_{=\bar{a}_t}\} \qquad t \in \mathcal{T}^-. \tag{11}$$

Under this restriction, borrowing is limited by the asset value, which comprises the wealth and inventory value.

Theorem 6 *Under asset constrained financing, asset dependent base-stock policies are optimal for inventory-pricing decision problems without fixed cost if the decision criterion is CPRI or an additive-exponential utility. In other words, the base-stock level, $S_t(a_t)$ depends solely on the asset value, $a_t = w_t + q_t x_t/(1+\beta)$ at the beginning of the period $t \in \mathcal{T}$.*

Proof : Observe that the uncertain consumptions are

$$\tilde{c}_t = \begin{cases} (1+\beta)a_T + l_T(y_T, p_t, \tilde{\boldsymbol{z}}_T) & t = T \\ (1+\beta)a_t + l_t(y_t, p_t, \tilde{\boldsymbol{z}}_t) - a_{t+1} & t \in \mathcal{T}^-, \end{cases}$$

where the function $l_t : \Re^4 \mapsto \Re$ is defined by:

$$l_t(y, p, z_1, z_2) = (q_{t+1}/(1+\beta) - q_t)\, y + (p - q_{t+1}/(1+\beta))\,(z_1 - z_2 p) - h_t(y - z_1 + z_2 p) \qquad t \in \mathcal{T},$$

and it is obvious that $l_t(y, p, z_1, z_2)$ is jointly concave in y and p for any $z_2 \ge 0$.

Given the state (x_t, a_t) at the beginning of period $t \in \mathcal{T}$, let $V_t(x_t, a_t)$ be the maximal value of $\mathbb{E}[\sum_{i=t}^{T} -\delta_i \exp(-\tilde{c}_i/\alpha_i)]$. We have

$$V_T(x_T, a_T) = \max_{\substack{y_T \ge x_T \\ p_T \in [\underline{p}_T, \bar{p}_T]}} \mathbb{E}\left[-\delta_T \exp\left(-\frac{(1+\beta)a_T + l_T(y_T, p_T, \tilde{\boldsymbol{z}}_T)}{(1+\beta)^T \alpha_T}\right)\right].$$

Since $\tilde{z}_T^2$ is always nonnegative, $l_T(y, p, \tilde{\boldsymbol{z}}_T)$ is concave in (y, p) for all realizations. Therefore, the objective function in the above equation is concave in (y_T, p_T, a_T). Hence, as shown in Chen et al. (2007), Proposition 4, $V_T(x_T, a_T)$ is concave in (x_T, a_T), and is nondecreasing in a_T. Moreover, since

$$V_T(x_T, a_T) = \exp\left(-\frac{a_T}{(1+\beta)^{T-1}\alpha_T}\right) \times \max_{\substack{y_T \geq x_T \\ p_T \in [\underline{p}_T, \bar{p}_T]}} \mathbb{E}\left[-\delta_T \exp\left(-\frac{l_T(y_T, p_T, \tilde{\boldsymbol{z}}_T)}{(1+\beta)^T \alpha_T}\right)\right],$$

the assumption on h_T implies that there exists a finite constant $S_T \in \Re$ such that the optimal order-up-to level is $y_T^* = \max\{x_T, S_T\}$.

Assume for some $t+1 \in \mathcal{T}$, $V_{t+1}(x_{t+1}, a_{t+1})$ is concave in (x_{t+1}, a_{t+1}) and nondecreasing in a_{t+1}. Given all information before the decision b_t and under asset constrained financing, the financing constraint $b_t \in F_t(x_{t+1}, w_t)$ can be represented by $a_{t+1} \geq 0$. Therefore, we have

$$V_t(x_t, a_t) = \max_{\substack{y_t \geq x_t \\ p_t \in [\underline{p}_t, \bar{p}_t]}} \mathbb{E}\left[\max_{a_{t+1} \geq 0} \left\{\mu_t\left((1+\beta)a_t + l_t(y_t, p_t, \tilde{\boldsymbol{z}}_t) - a_{t+1}\right) + V_{t+1}\left(y_t - \tilde{z}_t^1 + \tilde{z}_t^2 p_t, a_{t+1}\right)\right\}\right],$$

where $\mu_t : \Re \mapsto \Re$ is defined by:

$$\mu_t(c) = -\delta_t \exp\left(-\frac{c}{(1+\beta)^t \alpha_t}\right).$$

Since μ_t is concave and nondecreasing, and $l_t(y_t, p_t, \boldsymbol{z})$ is concave in (y_t, p_t) for all $z_t^2 \geq 0$, we know that $\mu_t\left((1+\beta)a_t + l_t(y_t, p_t, \boldsymbol{z}_t) - a_{t+1}\right)$ is concave in (y_t, p_t, a_t, a_{t+1}) for all $z_t^2 \geq 0$ and is nondecreasing in a_t. Further, since $V_{t+1}(\cdot, \cdot)$ is concave, it implies $V_{t+1}\left(y_t - z_t^1 + z_t^2 p_t, a_{t+1}\right)$ is concave in (y_t, p_t, a_{t+1}) and independent of a_t. Therefore, we conclude that

$$\mathbb{E}\left[\max_{a_{t+1} \geq 0} \left\{\mu_t\left((1+\beta)a_t + l_t(y_t, p_t, \tilde{\boldsymbol{z}}_t) - a_{t+1}\right) + V_{t+1}\left(y_t - \tilde{z}_t^1 + \tilde{z}_t^2 p_t, a_{t+1}\right)\right\}\right]$$

is concave in (y_t, p_t, a_t) and nondecreasing in a_t . Therefore, by the consumption on h_t, there exists finite $S_t \in \Re$, which depends on a_t but is independent from x_t, such that the optimal order-up-to level is $y_t^* = \max\{x_t, S_t\}$. Moreover, from Proposition 4 in Chen et al. (2007), $V_t(x_t, a_t)$ is concave in (x_t, a_t), and it is obviously nondecreasing in a_t.

By induction, we know for all $t \in \mathcal{T}$, $V_t(x_t, a_t)$ is concave in (x_t, a_t) and nondecreasing in a_t, and there exists an optimal S_t, which is independent from x_t but may depends on a_t, such that the optimal order up to level is $y_t^* = \max\{x_t, S_t\}$. Hence, an asset dependent base-stock policy is optimal to the additive-exponential utility criterion.

The result under the CPRI criterion follows from Theorem 3. With $K = 0$, the consumptions can be verified to be convex in the general history dependent policy. Therefore, Theorem 3 shows that the optimal policy can be solved by maximizing an expected additive-exponential utility, and hence the optimal policies are asset dependent base-stock policy. ■

We summarize our results in Table 2.

Table 2: Summary of new contributions.

	Price Not a Decision		*Price is a Decision*	
	$K_t = 0$	$K_t > 0$	$K_t = 0$	$K_t > 0$
CPRI under full financing	Base stock	(s, S)	Base stock	(s, S, A, p)
CPRI under asset constrained financing	Asset dependent base-stock	?	Asset dependent base-stock	?
Exponential utility under asset constrained financing	Asset dependent base-stock	?	Asset dependent base-stock	?

5 Computational study

In this section, we present a numerical study on an inventory control problem without pricing decisions. We consider a stylized five period problem, $T = 5$ and there is no fixed ordering cost, $K_t = 0$. The inventory cost functions are

$$h_t(x) = h_t^+ \max\{x, 0\} + h_t^- \max\{-x, 0\} \qquad t \in \mathcal{T},$$

where h_t^+ represents the unit inventory holding cost and h_t^- is the unit shortage cost. The values of the input parameters are presented in Table 3. We assume discrete demands uniformly distributed in $[0, 100]$ and independent across periods. The system starts with $x_1 = w_1 = 0$, and we assume unrestricted financing and zero interest rate.

With the above setting, we compare the consumptions profiles generated by the CPRI model against the risk neutral and additive-exponential utility models. In our computational studies,

Table 3: Input parameters of the inventory model.

unit ordering cost	$q_t = 3,\ t \in \mathcal{T}$
selling price	$p_t = 10,\ t \in \mathcal{T}$
unit holding cost	$h_t^+ = 1,\ t \in \mathcal{T}$
unit penalty cost	$h_t^- = 6,\ t \in \mathcal{T}$

we first obtain the optimal inventory policies under various decision criteria. The optimal policy for the additive-exponential utility model is solved using the dynamic programming recursion described in Chen et al. (2007). Next, we use Monte Carlo simulations with 100,000 independent trails to estimate the consumptions profiles. In Figure 1, we present the risky profile of the cash flows generated by the inventory system under the optimal risk neutral policy in the absence of financing. It is not surprising that the cash flows have large variability.

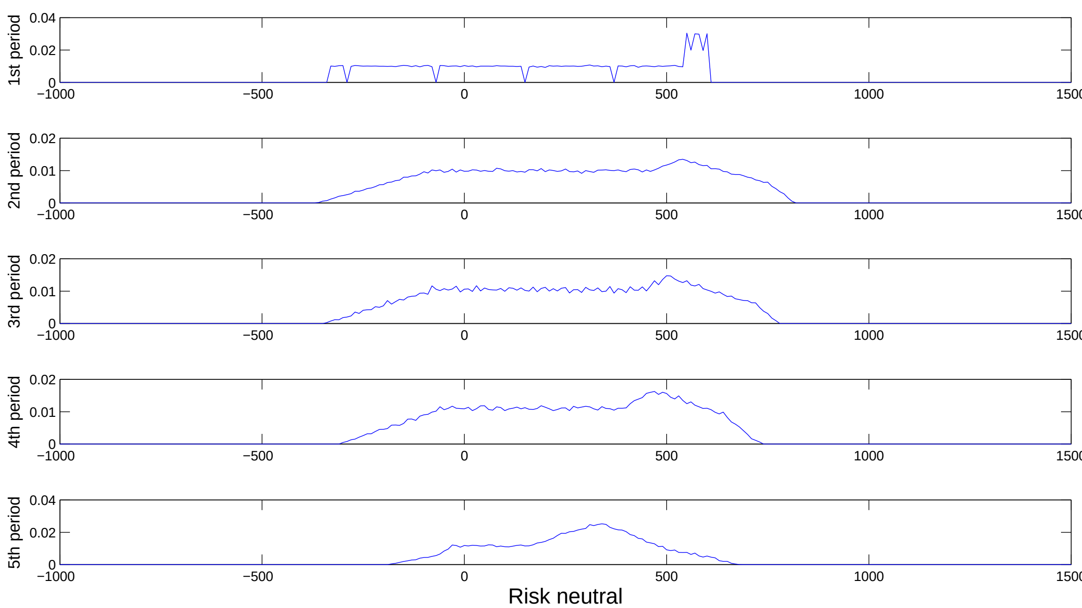

Figure 1: Cash flows profile under optimal risk neutral policy.

5.1 CPRI versus Risk Neutral Model

In the first study, we compare the results between the CPRI model and the risk neutral model. For the CPRI model, we assume consumption targets in periods $t \in \mathcal{T}^-$ are zeros and we vary only the last period target, τ_T. Under an optimal FAT policy, consumptions will only appear in the last period. For the risk neutral model, whose objective function is indifferent to any financing decisions, we assume all operational cash flows in $t \in \mathcal{T}^-$ are saved and are used to finance consumption at the last period. Hence, in both experiments, it suffices to compare the consumptions at the final period.

To provide a fair comparison, we apply seven performance measures. The first two are expectation and standard deviation of the final consumptions. The third is the probability that the final consumptions will achieve the given target, and we call it *attainment probability* (AP). The fourth is expected loss (EL) relative to the target, and by normalizing it with the probability of loss we get the conditional expected loss (CEL) as the fifth measure. Finally, we consider the value at risk (VaR), i.e., the threshold loss that the loss with respect to the target does not exceed with a specific probability, at 95% and 99%. While the first three measures are intuitive, the last four measures are also widely adopted in financial risk management (Embrechts et al. 1997, Jorion 2006).

Table 4 shows the performance from the risk neutral model against the CPRI model. Observe that while yielding the consumptions profile with lower risk, the CPRI model only reduces the expectation with a relatively small fraction. In particular, with $\tau_T = 900$, the standard deviation is reduced by 30% and the mean loss, conditional mean loss, VaR @ 95%, and VaR @99% are all reduced by nearly 25%. And the only sacrifice is a 9% reduction in the expectation. The attainment probabilities from these two models are almost identical. Moreover, we can see from Table 4 that as the targets increase, the difference between the two models becomes less significant. The reason is that with high targets, the CPRI model is less risk averse and hence the result is similar to the risk neutral model.

Table 4: Performance of CPRI and risk neutral models.

τ_T	**Criterion**	**Performance measures**						
		Expected consumptions	AP	Standard deviation	EL	CEL	VaR @ 95%	VaR @ 99%
1200	Risk Neutral	1254.98	56.0%	419.46	143.54	326.54	653.05	943.64
	CPRI	1253.25	56.2%	406.18	139.09	317.33	634.80	921.61
1150	Risk Neutral	1254.98	60.5%	419.46	122.68	310.92	603.05	893.64
	CPRI	1247.71	60.8%	391.35	114.86	292.62	567.92	845.34
1100	Risk Neutral	1254.98	64.8%	419.46	104.02	295.55	553.05	843.64
	CPRI	1237.99	65.0%	375.04	93.82	268.40	502.90	772.24
1050	Risk Neutral	1254.98	68.9%	419.46	87.46	281.05	503.05	793.64
	CPRI	1222.13	69.1%	355.64	75.65	244.63	438.12	698.26
1000	Risk Neutral	1254.98	72.8%	419.46	72.89	268.07	453.05	743.64
	CPRI	1202.59	73.0%	336.54	60.37	223.29	377.22	627.94
950	Risk Neutral	1254.98	76.4%	419.46	60.21	255.59	403.05	693.64
	CPRI	1175.99	76.4%	315.18	47.68	202.41	319.00	559.31
900	Risk Neutral	1254.98	79.8%	419.46	49.27	243.71	353.05	643.64
	CPRI	1143.52	79.7%	292.41	37.20	183.00	264.77	491.63

AP=Attainment probability; EL=Expected Loss;

CEL=Conditional Expected Loss; VaR=Value at Risk.

5.2 CPRI versus Additive-Exponential Utility Model

We next compare the CPRI model against the additive-exponential utility model that minimizes the following criterion $\mathbb{E}\left[\sum_{t=1}^{5}\exp(-\tilde{c}_t/\alpha)\right]$ for some $\alpha > 0$. For the CPRI model, we apply the same consumption targets, $\tau_t = \tau$ for all $t \in \mathcal{T}$. In Figure 2, we present the consumptions profiles under the additive-exponential utility model as we vary $\alpha \in \{10, 100, 400\}$. In contrast, we show in Figure 3 the consumptions profiles under the CPRI model as we vary $\tau \in \{100, 150, 200\}$. We can see that while the additive exponential utility model yields uncertain consumptions in each period, the CPRI model yield deterministic consumption in the first four periods and relegate uncertainty in consumption to the last target.

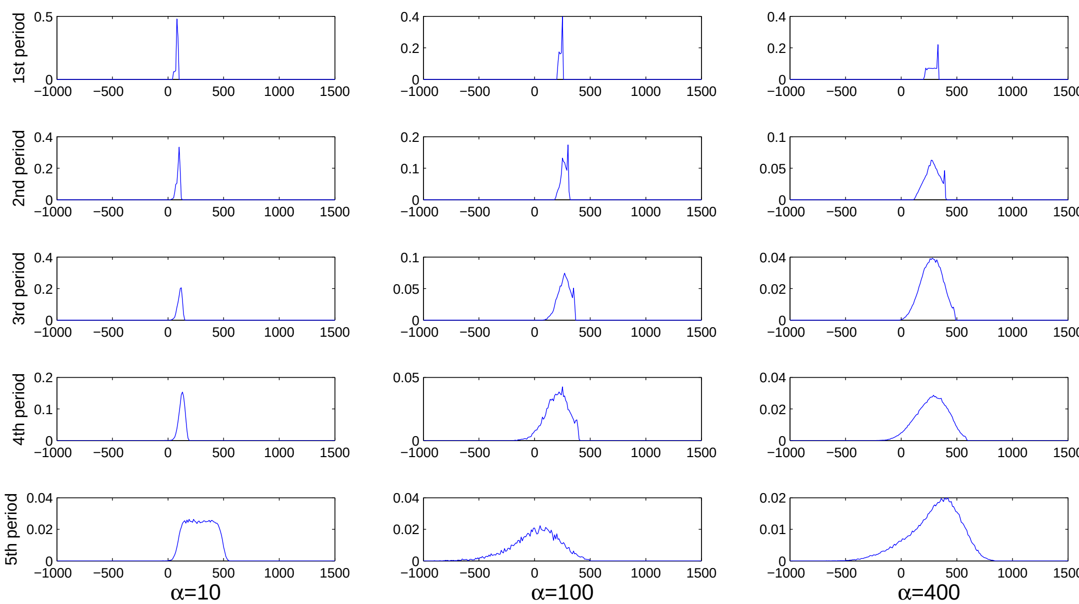

Figure 2: Consumptions profiles under the additive-exponential utility model as α varies.

6 Conclusion

In this paper, we propose a target-oriented decision model to help decision makers regulate their consumptions profile over time using some prescribed consumption targets. The model captures both the decision makers' risk aversion toward uncertain cash flow and their sensitive to the

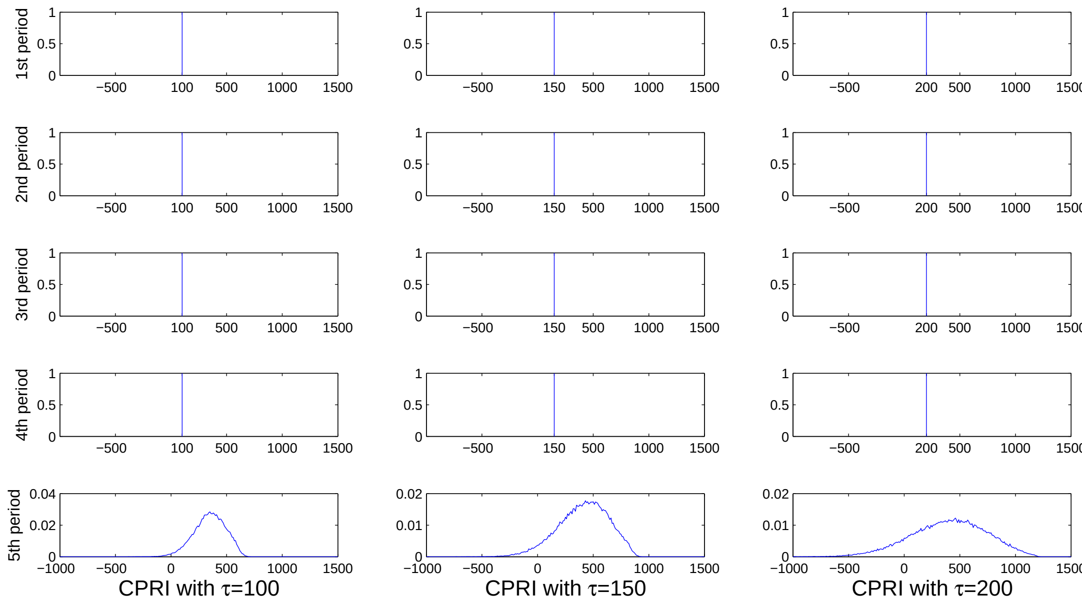

Figure 3: Consumptions profiles under the CPRI model as τ varies.

timing of the resolution of uncertainties. By taking into account of the prescribed targets, the model highlights managers' practical concern in planning corporate activities: not only they need to meet everyday corporate consumptions such as sending out pay checks to employees, they are also typically concerned about whether they can achieve a target profit level at the end of the planning horizon.

With the CPRI criterion, we show that under full financing a FAT policy is optimal and accordingly, we can obtain the optimal operational policy by solving a modest number of dynamic programming problems. When financing is restricted, we show that for convex dynamic decision problems, the optimal policies correspond to those that maximize expected additive-exponential utilities. We also provide an algorithm to find the optimal policies.

Applying the CPRI criterion to the joint inventory-pricing decision problem, we identify the optimal inventory and pricing policies for the case with fixed ordering cost under full financing, and the policy structures for the case of zero fixed ordering cost under restricted financing. We also provide the policy structures when the objective function is additive-exponential utility and the fixed cost is zero. Finally, our numerical studies suggest that our target-oriented dynamic decisions model provides an interesting alternative for regulating consummations over time.

Appendix

Brown et al. (2011) show that the target-oriented riskiness index can resolve several well-known behavioral experiments that contradict the expected utility theory. To illustrate this, we present the gambles of Allais' paradox in Table 5. Most subjects prefer Gamble A over Gamble B and Gamble C over Gamble D and this preference cannot be resolved by expected utility and also success probability, which can be perceived as a form of expected utility with step function. In contrast, this preference can be resolved via the target-oriented riskiness index. In particular, preferences for Gamble A over Gamble B and Gamble C over Gamble D are strict if $\tau \in (.023M, 0.25M]$. For $\tau \in (0.25M, 0.5M]$, Gamble A is strictly preferred over Gamble B, while both Gambles C and D are equally disliked since the expectation of the gambles are less than the target, τ[1].

Table 5: Gambles in Allais' paradox

Gamble A	Wins $0.5M for sure.
Gamble B	Wins $0M with 1% chance, $2.5M with 10% chance, and $0.5M with 89% chance.
Gamble C	Wins $0M with 90% chance, and $2.5M with 10% chance.
Gamble D	Wins $0M with 89% chance, and $0.5M with 11% chance.

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[1]In Brown et al. (2011) aspirations measure, which incorporates risk seeking behavior and hence generalizes Brown and Sim (2009) sacrificing measures, the strict preference can be extended to $\tau \in (.023M, 0.5M]$.

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