# Stochastic approaches for solving Rapid Transit Network Design models with random demand

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**Abstract**

We address rapid transit network design problems characterized by uncertainty in the input data. Network design has a determinant impact on the future effectiveness of the system. Design decisions are made with a great degree of uncertainty about the conditions under which the system will be required to operate. The demand is one of the main parameters which determines design decisions. We present two uncertainty rapid transit network model approaches to study the impact that the estimation of the future demand will have in the design of new rapid transit networks. Considering that the new topology is oriented to define public lines, to cover the demand by public transportation, a bounded budget and the modal competition between the old transportation system and the new rapid transit network will be included. Computational experiments are developed for these uncertainty approaches studying different network size problems under the main parameters of the model.

**Keywords**: rapid transit networks design, stochastic programming, fixed resource approach, joint probability constraints approach.
**AMS subject classification:** 90B06, 90C10, 90C35

# 1 Introduction

Transportation networks design is highly dependent on the future use of the system. One of the more important factors is the demand. The demand is increased as consequence of the development around urban areas, and as consequence of it, the traffic increases between the center and the surrounding areas. The result of this phenomenon is traffic congestion, which leads to build new rapid transit systems to cover trips by public transportation.
We design a new rapid transit network, which maximizes the number of travelers using it, such that the travel time on the new network is lower than the travel time in the existing network, *i.e.* the number of users using the new rapid transit network would be as high as possible.
Design decisions are considered at strategic level, but they must include the demand and its behavior in relation with mode choosing. The Rapid Transit Network Design (RTND) tries to maximize the demand coverage by the new network subject to design and budget constraints, but considering the user decisions to evaluate the network design alternatives. These decisions must consider two topology alternatives, the old and the new (to be constructed) network. The comparison of these alternatives is carried out by users choosing both travel path and mode.
One of the first papers in RTND are[Bruno et al(1998)], [Bruno et al(2002)]. In the first one the user travel cost is minimized, while in the other one the coverage of the demand by public network is maximized. [García and Marín(2001)], and [García and Marín(2002)], study the mode interchange location and parking network design problems using a Bilevel Programming model. They address the multimodal traffic assignment problem with combined modes at lower level.
[Laporte et al(2007)], extend the previous models by incorporating the station location problem, the alternative of several lines and defining the model using the maximum coverage of the public demand as an objective function and the budget constraints as side constraints. [Marín(2007)], studies the inclusion in the previous models of free but bounded number of lines, and the origin-destination of the lines is chosen in the rapid transit network design model. [Escudero and Muñoz(2009)] extend the RTND model to allow circular lines.
The network to design depends on the number of trips. The estimation of the future demand is based on the current mobility patterns for which a new infrastructure does not yet exist. Therefore, data obtained by samples or analytical models are uncertain. The model will include some uncertain coefficients, those related with the number of users. In this paper we are interested in solving the problem of designing an optimal RTN where the amount of users between every o/d pair is a random variable. This model will be

called rapid transit network design under uncertainty (RTNDU).
Problems with uncertain coefficients can be solved by stochastic programming techniques. Solutions approaches to stochastic programming have been studied in [Rockafellar and Wets(1991)] and [Kall and Mayer(2005)]. A difficulty in these approaches is to find a proper balance between the terms of the objective function. One must achieve a tradeoff between the mean and variance of the solution, and deviations from feasibility under all scenarios. Textbooks on stochastic programming are [Birge and Louveaux(1997)], [Prékopa(1995)] and [Kall and Stein(1994)], the last of which is the one that is most easily accessible. The first two give more technical insight and are more comprehensive. A research survey on various aspects of stochastic programming is given in [Ruszczynski and Shapiro(2003)].
Stochastic programming problems are optimization problems in which uncertainty about parameters is modeled as probability distributions over these parameters. Then, the optimization takes the randomness into account by either optimizing the expected value of an objective function or by requiring stochastic constraints to be satisfied with high probability. The latter is called Chance Constrained Programming. In the former, the objective function may, besides the expectation of the costs, also contain some deviation measure and it is called Fixed Recourse Programming.
The probability space is often modeled as a finite set of scenarios. Then, in principle, a linear optimization problem with stochastic input data, chance constraints and some stochastic objective function is a linear program itself. But the size of the original program is multiplied by the number of scenarios.
In the Fixed Recourse model, the time is modeled discretely by means of stages, corresponding to the available information. Each step decisions can be made after observing the realizations of the random problem parameters. The difference between those decisions lies in the cost: Early decisions, i.e., decisions under little information are cheaper than later, informed decisions. If all uncertainty is dissolved at the same moment, this is captured by a recourse model with two stages: present and future, and such models are known as two-stage stochastic programming models. The objective is to minimize the total expected cost.
The second main class of stochastic programming problems consists of probabilistic or chance-constrained problems, which model random constraints by requiring that they should be satisfied with some prescribed reliability. Clearly, such models are alternatives in the above RTN context.
A new research area is the computational complexity and polynomial time approximation of probabilistic constraint stochastic programming problems. Here, there are a lot of research opportunities. For large scale-networks most methods are in some sense sampling based. The basic idea, formalized in the Sample Average Approximation method, see [Kleywegt et al(2001)], is to substitute the original, vast set of scenarios by a small set

of samples drawn according to the present probability distribution. By means of the theory of limited deviations it is often possible to show that the probability increases exponentially with the sample size, that the results calculated with respect to the sample set are an approximation of the desired stochastic optimum. We have analyzed such rapid convergence for stochastic RTND problems. But we also point out that the rapid convergence is of no useless if the sample size is too small. In fact, for large-scale networks the minimum size for reasonable samples may already lie beyond practical tractability.
In this paper, the RTNDU model is formulated by first time, using Fixed Resource Approach and Chance Constrained Programming. For the second case, a disaggregated chance constraint approach is specifically defined. The numerical tests using the above approaches are compared for different networks with 4 and 9 nodes. In the computational tests different scenario set sizes and different random generation rules are used.
The paper is organized as follows. In the next section we will present the RTND model. Then, in Section 3, the stochastic solution approaches will be discussed: their advantages and disadvantages. Finally the last part, RTND models are solved by the different approaches in order to illustrate their computational behavior and compare their success.

# 2 Rapid transit network design model

In this section we are going to define the transit network design model. First we will establish the goals and the particularities of the model in the deterministic case and then we will present the stochastic version.

## 2.1 Deterministic model

In a traffic network, users move from a certain origin to a certain destination according to an optimality criterion. Grouping these demands, we can assume that there are $g_w$, $w = (o, d)$ users moving from centroid origin $o$ to centroid destination $d$, shortly denoted as o/d-pair. Let us consider $W$, set of o/d-pairs, we want to design a rapid transit network (RTN) such that the following criteria are fulfilled:

- The demand using RTN is maximized as first objective, but also
- minimizing the cost of the network design, and the routing costs, so the users choose the shortest path on a network composed by the RTN and the current network.

In our deterministic model for RTND the demand mobility patterns in a metropolitan area are assumed to be known. This implies that the number of potential passengers from

each origin to each destination is given. We also assume that the locations of the potential stations are given. There already exists a different current mode of transportation (for example, private cars or an alternative public transportation is already operating in the area) competing with the RTN. When deciding which mode each demand is allocated to, the comparison between the generalized costs of the travelers is used. The aim of the model is to design a network, i.e. to decide at which nodes to locate the stations and how to connect them covering as many trips as possible. Since resources are limited we also impose a budget constraint on construction costs.
Rapid transit network design (RTND) model is defined by the following data:

1. For key stations, the set of potential locations is $N = \{1, \ldots, I\}$. ¿From that, the set $E$ of feasible (bidirectional) edges linking the key stations $N$ is defined. Therefore, we have a potential network $(N, E)$ from which the optimum rapid transit network is selected. Let us denote by $N(i) = \{j : \exists a \in E, a = (i, j)\}$ the set of nodes adjacent to node $i$.

2. The node set is composed by centroids ($N_c$) and stations at RTN ($N_r$), the node set is then $N = N_c \cup N_r$. The edges represent: alignments in RTN ($A_r$), dummy links between origin centroids and any station ($A_o$), dummy links between stations and every destination centroid ($A_d$), and fictitious links between any origin-destination pair ($A_f$). The arc set is then $A = A_r \cup A_f \cup A_o \cup A_d$.

3. Each feasible edge $(i, j)$ has an associated length $d_{ij}$. The length of the edges usually correspond to the Euclidean distance between pairs of nodes $(i, j)$ if the system is underground and street distance if it is at ground. However, forbidden regions will increase the distance and $d$ can also be interpreted as the generalized cost of using the arc $(i, j)$.

4. The demand is given by the matrix $G = [G_{i,j}]$ where $G_{i,j} \in \mathbb{R}_+$ is the number of users of pair $w = (o_i, d_j)$, *i.e.* $G_{i,j} = g_{(o_i,d_j)} = g_w$.

5. Let $c_{ij}^l$ and $c_i^l$ be the costs of constructing an edge (alignment) $(i, j)$ of line $l$ and a node (station) $i$ of line $l$. The upper budget bound is $c_{\max}$.

6. The generalized cost satisfying the demand of pair $w$ through the current network is $u_{cur}^w$. By the triangular inequality, at the optimum this demand will use the arc connecting its o/d pair in $A_f$. So, we can assume that $u_{cur}^w$ does not depend on the value of the demand, hence it is independent of scenario.

The variables are:

- $h_l = 1$, if line $l$ has at least a link, $h_l = 0$, otherwise.
- $y_i^l = 1$, if line $l$ is located using node $i$, $y_i^l = 0$, otherwise.
- $x_{ij}^l = 1$, if line $l$ is located using edge $i, j$, $x_{ij}^l = 0$, otherwise.
- $f_{ij}^w = 1$, if demand $w$ uses edge $(i, j)$ in the current network, $f_{ij}^w = 0$, otherwise. $f_{cur}^w$ denotes the use of the fictitious arc $(o, d) = w \in A_f$ by the demand $w$ (*i.e.* $f_{cur}^w = 1$ if and only if demand $g_w$ uses the current network).

RTND model is defined as the maximization of the public trip covering (equivalently to minimize the number of travelers using the current network), minimizing the location and routing costs. Hence we are dealing with a multiobjective model. As usual, we minimize a positive, linear combination of them:

$$z = \eta z_{cur} + \frac{3(1-\eta)}{4} z_{loc} + \frac{(1-\eta)}{4} z_{route},$$

where

$$z_{cur} = \sum_{w} g_w f_{cur}^w,$$

$$z_{loc} = \sum_{l \in L} \left( \sum_{(i,j) \in A_r, i<j} c_{ij} x_{ij}^l + \sum_{i \in N_r} c_i y_i^l \right)$$

and

$$z_{route} = \sum_{w \in W} \left( \sum_{(i,j) \in A_r \cup A_d \cup A_o} \left( d_{ij} f_{ij}^w \right) + u_{cur}^w f_{cur}^w \right).$$

The public trip covering is the main component of the objective function, but the location cost must be minimized to avoid the construction of inoperative parts of network, and minimizing the demand routing cost, so the users choose the shortest path. The weight between the first main term and the second and third secondary terms $\eta$ may be around .9, because the main goal is to minimize the amount of demand using the private network. The designer of the network has the following budget constraints:

$$z_{loc} \leq c_{\max}, \tag{1}$$

Line location constraints are included in order to ensure that the links are not located if their origin and destination nodes are not previously located, and to change the undirected link location variables to directed ones.

$$x_{ij}^l \leq y_i^l,\ \forall (i,j) \in A_r, i<j,\ \forall l \in L, \tag{2}$$

$$x_{ij}^l \leq y_j^l,\ \forall (i,j) \in A_r, i<j,\ \forall l \in L, \tag{3}$$

$$x_{ij}^l = x_{ji}^l,\ \forall (i,j) \in A_r, i<j,\ \forall l \in L, \tag{4}$$

We have to guarantee that the lines follow a path, without cycles.

$$\sum_{j \in N_r(i),\, i<j} x_{ij}^l + \sum_{j \in N_r(i),\, i<j} x_{ji}^l \leq 2,\ \forall i \in N_r, \forall l \in L, \tag{5}$$

$$\sum_{(i,j) \in B\ i<j} x_{ij}^l \leq |B|-1,\ \forall l \in L,\ \forall B \subset N_r, |B| \geq 2. \tag{6}$$

Now it should hold that $h_l = 1$ if there is at least a node associated to line $l$ and that no extra stations are located

$$\sum_{j \in A_r,\, i<j} x_{ij}^l \leq M h_l,\ \forall l \in L, \tag{7}$$

$$\sum_{j \in A_r,\, i<j} x_{ij}^l \geq h_l,\ \forall l \in L, \tag{8}$$

$$h_l + \sum_{j \in A_r,\, i<j} x_{ij}^l = \sum_{i \in N_r} y_i^l,\ \forall l \in L. \tag{9}$$

Here $M \geq \frac{|A_r|}{2}$.

With respect to the users from the o/d pair $w$, they will use a certain path on the RTN or use the current network according to the following constraints

$$\sum_{k \in N(i)} f_{ki}^w - \sum_{j \in N(i)} f_{ij}^w = \begin{cases} -1, & \text{if } i=o,\ w=(o,d), \\ 1, & \text{if } i=d\ w=(o,d), \\ 0, & \text{otherwise}, \end{cases} \tag{10}$$

$$f_{ij}^w \leq \sum_{l \in L} x_{ij}^l,\ \forall (i,j) \in A_r, \tag{11}$$

$$f_{o(w)j}^w \leq \sum_{l \in L} y_j^l,\ \forall j \in N_r, \tag{12}$$

$$f_{id(w)}^w \leq \sum_{l \in L} y_i^l,\ \forall i \in N_r, \tag{13}$$

$$\sum_{i,j \in A_r \cup A_d \cup A_o} d_{ij} f_{ij}^w \leq u_{cur}^w (1 - f_{cur}^w),\ \forall w \in W. \tag{14}$$

Equation (10) is the multi-commodity flow conservation at each node. Equation (11) guarantees that for each O/D pair, demand is routed on a link only if this edge has been constructed in the RTN. Equations (12, 13) guarantee that for each O/D pair, demand is routed from a centroid to a station, or from a station to a centroid only if the station has been constructed in the RTN. Equation (14) is used together with the objective function to obtain that, for each $o/d$ pair users traveling from $o$ to $d$ use the best path between the shortest way on the RTN and the current network. Indeed, if $f^w_{cur} = 0$, then there is a path $C_w$ on the RTN shorter than or equal to $u^w_{cur}$. As the distance is minimized, at the solution, $C_w$ coincides with the shortest path on the RTN. On the other hand if $f^w_{cur} = 1$, then $f^w_{i,j} = 0$, for all arc $(i, j) \in A_r \cup A_d \cup A_o$. So, users traveling from $o$ to $d$ are going to use the current network. If there was a shortest path on the RTN ($C_w$), then this solution is not optimal, because a smaller value of the objective function is possible taking $f^w_{cur} = 0$ and $f^w_{i,j} = 1$ if and only if $(i, j) \in C_w$.
This means that the RTND model is:

$$\begin{array}{c} \min \eta \sum_w g_w f^w_{cur} + \frac{3(1-\eta)}{4} z_{loc} + \frac{(1-\eta)}{4} z_{route} \\ \text{s.t. Constraints (1)-(14).} \end{array} \tag{15}$$

**Remark 1** *We want to point out that given $x^l_{ij}$, $y^l_i$ the values of $h_l$ and $f^w_{ij}$ are fixed.*

## 2.2 RTND with uncertain demands

Now we will present the rapid transit network design with uncertain demands(RTNDU). It is the case of model (15) where $G$ is a random matrix with finite many possible values. Different values of the matrix $G$ will lead us to a different scenario. We will denote by $R$ the set of scenarios. As a consequence of the characteristics of $g_w$, $R$ is a finite set and $R \subset \mathbb{N}$. Scenario $r \in R$ appears with probability $p_r$. The demand matrix $G_r = [g^r_w]$ denotes the demand of the o/d pair $w$ at scenario $r$.
In general, many scenarios may appear and its analysis may be complicated. The set $R$ can be reduced if we cluster the possible demands. Roughly speaking, we group the matrices of demand $G^r$ into a maximum of $R_1$ clusters $(C_1, \dots, C_{R_1})$ according to a minimal distance criterion. Now the problem will have $R_1$ scenarios, and scenario $r_i$ will appear with probability equal to the $\sum_{r:G_r \in C_{r_i}} p_r$ and demand equal to the centroid of cluster $C_{r_i}$. With these remarks, we can assume $|R|$ is not too big with respect to the number of o/d pairs.
We can expect that different scenarios will have different optimal solutions. That is why it is important to have a criterion that would determine which network is better, because what is good for a certain scenario, could be bad for the other. Of course, this

comparative criterion should take into account the characteristics of the scenarios and their probabilities. An option is the stochastic approach. Let us present the solution of RTNDU from this viewpoint.

# 3 Deterministic approaches to RTNDU

In this part we are going to present two deterministic models for solving RTNDU.
In the first case, we consider the equivalent formulation of RTNDU

$$\begin{array}{c} \min \eta z_{cur} + \frac{3(1-\eta)}{4} z_{loc} + \frac{(1-\eta)}{4} z_{route} \\ \text{s.t. Constraints (1)-(14)} \\ z_{cur} = \sum_w g_w f^w_{cur}. \end{array}$$

The Fix Resource Approach (FRA),see [Kall and Mayer(2005)] lead us to the classical approach where the random variable is substituted by its expected value:

$$\begin{array}{c} (FRA) \quad \min \eta \sum_{w \in W} E(g_w) f^w_{cur} + 3\frac{1-\eta}{4} z_{loc} \frac{(1-\eta)}{4} z_{route} \\ \text{s.t. Constraints (1)-(14).} \end{array} \tag{16}$$

For the second approach, we analyze the following model

$$\begin{array}{c} \min \eta \sum_{w \in W} z^w_{cur} + \frac{3(1-\eta)}{4} z_{loc} + \frac{(1-\eta)}{4} z_{route} \\ \text{s.t. Constraints (1)-(14)} \\ g_w f^w_{cur} \leq z^w_{cur} \text{ for all } w. \end{array}$$

and solve it by the Joint Probability Constraints Approach (JPCA), where constraint $g_w f^w_{cur} \leq z^w_{cur}$ is substituted by

$$P(g_w f^w_{cur} - z^w_{cur} \leq 0) \geq p, \text{ for all } w \in W. \tag{17}$$

In order to linearize the model, we recall the percentile definition

**Definition 3.1** *Given a random variable $y$, $\alpha(p)$ is the percentile of order $p$ if $\alpha(p) = \inf\{\alpha : P(y \leq \alpha) \geq p\}$.*

Now we take into account that $f^w_{cur}$ can attain the following values:

- If $f^w_{cur} = 0$ we have $P(-z^w_{cur} \leq 0) \geq p$. As $z^w_{cur}$ is not random a variable, it holds that $P(g_w f^w_{cur} - z^w_{cur} \leq 0) = \begin{cases} 1, \text{ if } z^w_{cur} \geq 0, \\ 0, \text{ otherwise.} \end{cases}$

- If $f^w_{cur} = 1$ then $P(g_w f^w_{cur} - z^w_{cur} \leq 0) = P(g_w - z^w_{cur} \leq 0) = P(g_w \leq z^w_{cur}) \geq p$, but(recall Definition 3.1) this means $z^w_{cur} \geq \alpha^w(p)$ where $\alpha^w(p)$ is the order $p$ percentile of the random variable $g_w$.

So,

$$P(g_w f^w_{cur} - z^w_{cur} \leq 0) \geq p \Leftrightarrow z^w_{cur} \geq \alpha^w_p f^w_{cur}, \text{ for all } w \in W.$$

Using the particular structure of the model (*i.e.* $z^w_{cur}$ only appears in one constraint and $\sum_w z^w_{cur}$ is minimized), we obtain that at the solution the equality holds. Hence, we have the following result

**Proposition 1** *The model corresponding to the Joint Probability Constraints Approach can be linearized as*

$$\begin{array}{ll} \textit{(JPCA)} & \min \eta \sum_{w\in W} \alpha^w_p f^w_{cur} + 3\frac{1-\eta}{4} z_{loc} + \frac{(1-\eta)}{4} z_{route} \\ & \quad \textit{s.t. Constraints (1)-(14).} \end{array} \tag{18}$$

**Remark 2** *This model provides upper bounds to the best RTND optimal solution.*

A third approach is obtained if we consider

$$\begin{array}{c} \min \eta z_{cur} + \frac{3(1-\eta)}{4} z_{loc} + \frac{(1-\eta)}{4} z_{route} \\ \text{s.t. Constraints (1)-(14)} \\ \sum_{w\in W} g_w f^w_{cur} \leq z_{cur}. \end{array}$$

JPCA lead us to substitute constraint $\sum_{w\in W} g_w f^w_{cur} \leq z_{cur}$ by

$$P(\sum_{w\in W} g_w f^w_{cur} - z_{cur} \leq 0) \geq p. \tag{19}$$

This constraint can be linearized if we use the following system

$$\sum_{w\in W} g_w f^w_w \leq z_{cur} + M(1-\delta_r), \ \forall r \in R \tag{20}$$

$$\sum_{r\in R} \delta_r p_r \geq p. \tag{21}$$

where $\delta_r$ is a binary variable $r \in R$ and $M$ is an upper bound of $\sum_{w\in W} g^w_r f^w_{cur}$ for all $r$, for instance $M = \sum_{w\in W} \max_r\{g^w_r\}$.
So, the model corresponding to Joint Probability approach for aggregated constraints is

$$\begin{array}{c}\min \eta z_{cur} + 3\frac{1-\eta}{4} z_{loc} + \frac{(1-\eta)}{4} z_{route} \\ \text{s.t. Constraints (1)-(14),} \\ \sum_{w\in W} g_r^w f_{cur}^w \leq z_{cur} + M(1-\delta_r),\ \forall r \in R \\ \sum_{r\in R} \delta_r p_r \geq p.\end{array} \tag{22}$$

Note that FRA and JPCA approaches lead to solve a MIP problem of the same dimension of the original RTNDU model. This means that the dimension of the model will not increase as in general stochastic models. However the whole random variable is concentrated at one point. In the case of model (22), we use more information, but the model is larger.

# 4 Computational tests

In this section, we are going to illustrate the behavior of the solution approaches. We solve the MIP models (16) and (18) with CPLEX in the context of GAMS 22.2. The solutions, denoted by $N_{FRA}$ and $N_{JPCA}$ respectively, will be compared taking into account the value of the objective function at the scenarios. Let us define the involved parameters.

## 4.1 Parameters and experiment design

We assume that $c_i = c_i^l,\ c_{ij} = c_{ij}^l$ for all $l = 1, \ldots, L$. These costs and the distance between the nodes of the network are depicted in the graphs, where the number at node $i$ represents the value of $c_i$ and the ordered pair at arc $(i, j)$, the values $(c_{ij}, d_{ij})$.
Let $\text{RTND}_r$ be the RTND model with demand equal to the demand at scenario $r$. That is,

$$\text{RTND}_r \quad \min z_r = \eta \sum_w g_w^r f_{cur}^w + \frac{3(1-\eta)}{4} z_{loc} + \frac{(1-\eta)}{4} z_{route},$$

$$\text{s.t. Constraints (1)-(14).}$$

We denote by $N_r = (x_{ij}^l, y_i^l)_r$ the network corresponding to the optimal solution of $\text{RTND}_r$. Taking into account Remark 1, we can compute $z_r(N_r)$, the (optimal) value of the objective function using only $N_r$. We will compare $z_r(N_{FRA})$ and $z_r(N_{JPCA})$ with $z_r(N_r)$, because values closed to $z_r(N_r)$ means that the network is a good solution for scenario $r$.
Define $\xi(N) \in \mathbb{R}^{|R|}$ as $\xi_r = \frac{z_r(N) - z_r(N_r)}{z_r(N_r)}$, $r \in R$, we can say that $N$ is a good network if $\xi(N) \approx 0$. So, $\xi(N)$ denotes the values of a random variable which attains $\xi_r(N)$ with

probability $p_r$. Using this fact, we can say that $\xi \approx 0$ if the expected value is smaller than the desired tolerance $\varepsilon_1$, that is

$$\text{Expected value criterion } E(\xi) = \sum_{r \in R} p_r \xi_r \leq \varepsilon_1$$

or if $\xi$ is smaller than the desired tolerance $\varepsilon$ with large probability (larger than $\tilde{p}$):

$$\text{Probability criterion: } P_\xi = P(\{r : \xi \leq \varepsilon\}) \geq \tilde{p}$$

Although we will present both criteria, as small values of the $E(\xi)$ can mean large values of $\xi_r$ if the standard deviation is large, we will prefer networks with larger probability $P_\xi$. As already remarked, the quality of the solution may depend on the number of scenarios and the standard deviation on the involved random variables. So, we will consider 4 cases corresponding to the combinations of two possible number of scenarios and two different standard deviation of the distribution function which generate the demand. In order to determine the best approach for a certain combination from a statistical viewpoint, we generate 40 RTNDU models. Then we calculate the sample values $\xi_i(N)$ for the networks computed by the approaches and test if the previous two criteria can be assumed statistically.
Using the media test, see [Mood and Graybill(1974)], we will determine if $E(\xi) \leq \varepsilon_1$ can be accepted with probability .95. That is if

$$\frac{\sqrt{40}(\overline{\xi} - \varepsilon_1)}{S} \leq 1.684 \tag{23}$$

where $\overline{\xi} = \frac{\sum_i \xi_i}{40}$ and $S = \sqrt{\frac{\sum_i (\xi_i - \overline{\xi})^2}{39}}$. An approach will be good with respect to this criterium if the hypothesis is accepted. We take $\varepsilon = 0.001$
An analogous analysis will be done with $P_\xi \geq \tilde{p}$: we accept that the approach is good if

$$\sqrt{40}\frac{\overline{x} - \tilde{p}}{\sqrt{\overline{x}(1 - \overline{x})}} > -1.68 \tag{24}$$

where $\overline{x} = \frac{\text{total of cases where } P_\xi \geq \tilde{p}}{40}$. Here $\tilde{p} = .5$
For the experiment, we will take $|R| = 3$ or $|R| = 30$ as the two possible number of scenarios. The probability of scenario $r$ for $R = 3$, will be $p_1 = .25$, $p_2 = .5$ and $p_3 = .25$. In the 30 scenarios case, we assume $p_r = .025$ if $r \leq 10$ or $r \geq 21$ and $p_r = .05$ otherwise. It is reported in [Swan(2002)], that the demands distribute according to a gamma law if the expected value is small and to a normal distribution with expected value and variance

equals to $\mu$, otherwise. As the only available information is that the demand lies in a certain interval, we will focus our analysis in the case where the demand is generated by the uniform distribution function in $(a, b)$. However, at the end of this section, we will present some RTNDU problems where the demand $g_w^r$ is generated by the normal law.
We generate $G_r = g_w^r \in \mathbb{R}^{|W|}$, the demand matrix at scenario $r$, by means of distribution functions whose standard deviation (SD) is (relatively) small and (relatively) large and depend on a (fixed) matrix $G^0 \in R^{|W|}$.
We use the following uniform distribution functions:

$$unif\left(0, 2[G_w]^0\right) \qquad \text{(uniform rule, low SD)} \tag{25}$$

$$unif\left(0, 200[G_w]^0\right) \qquad \text{(uniform rule, high SD)} \tag{26}$$

Once the matrices are computed $G_r,\ \forall r \in R$, the expected value and the percentile needed for FRA and JPCA approaches are the mean value and the percentile in the sample. That is $E(g_w) = \sum_{r=1}^{R} p_r g_w^r$ and $[\alpha_w]_p = \min\left\{g_w^{r*} \text{ s.a. } \sum_{r\in R\ g_w^r < g_w^{r*}} p_r \geq p\right\}$.
We begin with a small test network.

## 4.2 4 nodes RTNDU network

In this part, we will study the RTNDU model corresponding to the following 4 nodes network, see Figure 1.

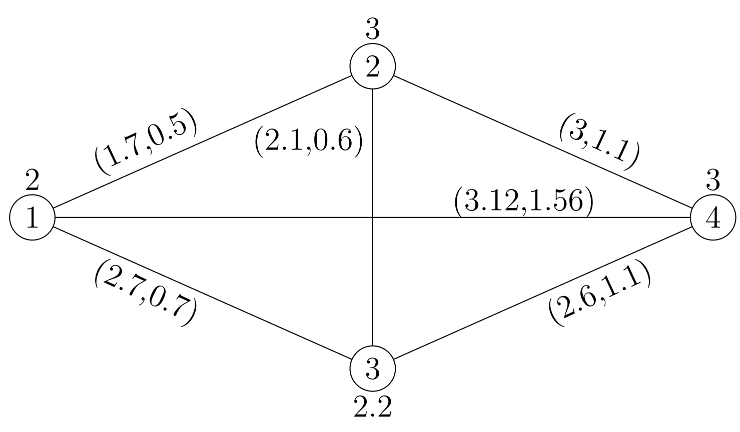



Figure 1: 4 nodes network

| | $o1$ | $o2$ | $o3$ | $o4$ | $d1$ | $d2$ | $d3$ | $d4$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 2.1 | 1.6 | 2 | 0 | 1.7 | .9 | 2.1 |
| 2 | 1.7 | 0 | 1.5 | 2.1 | 2.1 | 0 | 1 | 1.3 |
| 3 | .9 | 1 | 0 | 2 | 1.6 | 1.5 | 0 | 1.4 |
| 4 | 2.1 | 1.3 | 1.4 | 0 | 2 | 2.1 | 2 | 0 |

| | $d1$ | $d2$ | $d3$ | $d4$ |
|---|---|---|---|---|
| $o1$ | 0 | 1.6 | 0.8 | 2 |
| $o2$ | 1.5 | 0 | 0.9 | 1.2 |
| $o3$ | 2 | 1.4 | 0 | 1.3 |
| $o4$ | 1.9 | 2 | 1.9 | 0 |

Table 1: Distance between the centroids and the nodes of the network

The distances between nodes and centroids are in Table 1.
The maximal budget allowance $c_{\max} = 10$. The auxiliary matrix for computing the demand is

$$G^0 = \begin{pmatrix} 0 & 9 & 26 & 19 \\ 11 & 0 & 14 & 26 \\ 30 & 19 & 0 & 30 \\ 21 & 9 & 11 & 0 \end{pmatrix}$$

We will begin with an example and then we will present our statistical study.

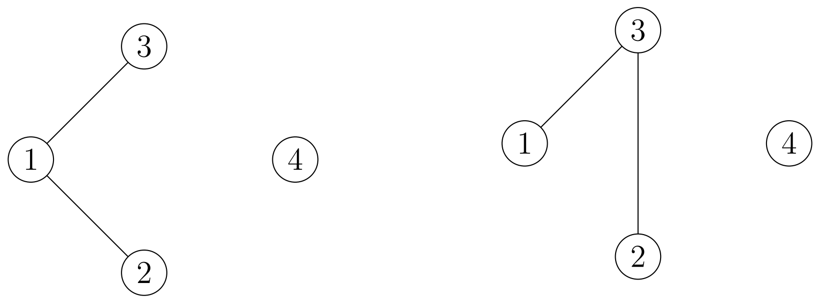
#### 4.2.1 An example

Let us present one example for the 3 scenarios case with demands $G_r$, $r = 1 \ldots 3$, generated by rule (25), equal to

$$G_1 = \begin{pmatrix} - & 3.9 & 39 & 35.3 \\ 0.2 & - & 27.3 & 41 \\ 34.4 & 33.4 & - & 8.8 \\ 7.7 & 0.3 & 4.1 & - \end{pmatrix} \quad G_2 = \begin{pmatrix} - & 10.3 & 19 & 3.2 \\ 15.3 & - & 10.2 & 8.6 \\ 44.1 & 30.2 & - & 24.22 \\ 1.91 & 6.76 & 1.93 & - \end{pmatrix}$$

$$G_3 = \begin{pmatrix} - & 12.9 & 12.9 & 11.6 \\ 16.5 & - & 21.9 & 1.3 \\ 19.6 & 20.7 & - & 1 \\ 18 & 12.5 & 6.5 & - \end{pmatrix}$$

The optimal networks for scenario $r = 1, 2, 3$ are depicted in Figure 2.
The network computed by FRA, $N_{FRA}$, coincided with $N_2$ and $N_3$, the optimal networks for scenario 2 and 3, while the JPCA approach, for percentile $p = .8$, calculated $N_1$, the optimal network for the first scenario. We observed that for FRA: $\xi_1 = 0.08$, $\xi_2 = \xi_3 = 0$ and for JPCA: $\xi_1 = 0$, $\xi_2 = 0.146$, $\xi_3 = 0.19$ With respect to the first comparison criterion, we obtained that the expected value of $\xi$ is clearly smaller for $N_{FRA}$. FRA



$N_1$ optimal network at Scenario 1 $\quad$ $N_2 = N_3$ optimal network at Scenario 2 and 3

Figure 2: Optimal networks for each scenario

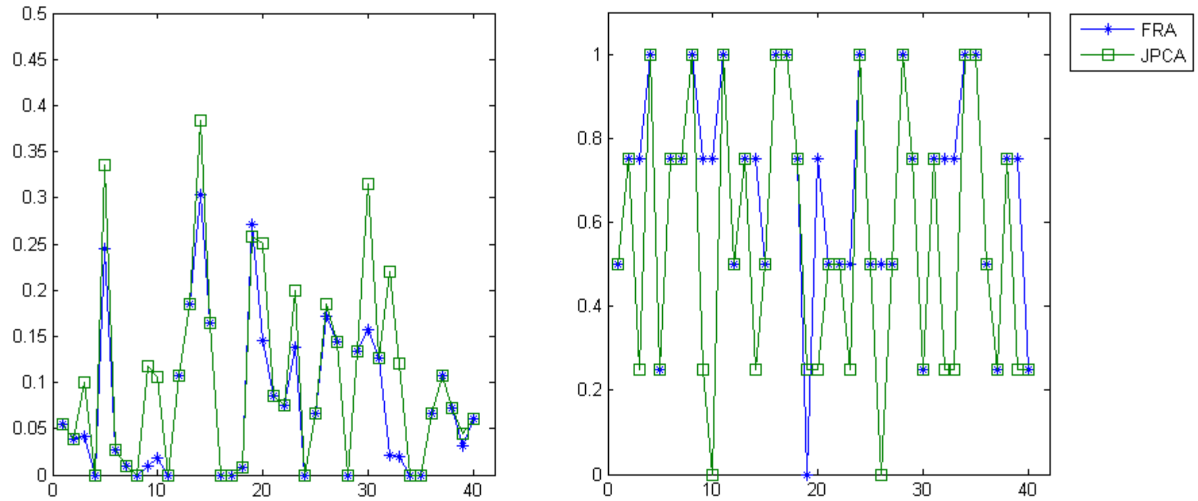
is also the best option with respect to the probability criterion, because $P(\{\xi(N_{FRA}) \leq \varepsilon\}) \geq P(\{\xi(N_{JPCA}) \leq \varepsilon\})$ for all $\varepsilon$.
The value of the objective function at $N_{JPCA}$ provides upper bounds of $z_r(N_r)$ for some scenarios $r \in R_1 \subset R$. Of course the set $R_1$ depends on the value of the percentile. For instance, in the previous example, when model (18) is solved for $p = .05i, i = 1, \ldots, 20$, we observe that for $p \leq .75$, network $N_2$ was the computed solution and the value of the objective function is 56.2, while for $p > .75$, $N_{JPCA} = N_1$ and the evaluation of the objective function 90.5. As the value of the objective function for the three scenarios is 90.5, 56,2, 63.8, respectively, it holds that:

$$R_1 = \begin{cases} \{2,3\} & p \geq .75 \\ \{1\} & \text{otherwise.} \end{cases}$$

In order to take a relative large percentile which bounds a relative large set, we choose p=.8 for our numerical experiments .
We also considered model (22). However it computed solutions with large values of $z_{cur}$. As the dimensions of the model are larger than the other two proposal, we will discard it.

#### 4.2.2 Statistical study of the criteria to validate the RTNDU approaches (4 nodes)

We begin analyzing 40 examples corresponding to 3 scenarios and demand generated by rules (25) and (26). Figures 3 and 4 show the values of $E(\xi)$ and $P_\xi$ for FRA and JPCA approaches, respectively.
$E(\xi)$ is very small for both approaches, but for FRA, it is smaller in most of the non




Expected value of $\xi$. Probability of $\{r : \xi \leq \varepsilon\}$.

Figure 3: 4 nodes RTNDU with 3 scenarios and demand generated by (25)

| Approaches | FRA | JPCA |
|---|---|---|
| Rule (25) | 0.96 | 1.02 |
| Rule (26) | 1.0285 | 0.9448 |

(a) $\sqrt{40}\ \frac{\overline{x}-0.01}{S}$

| Approaches | FRA | JPCA |
|---|---|---|
| Rule (25) | 7.17 | 0.61 |
| Rule (26) | 6.1993 | 0.9129 |

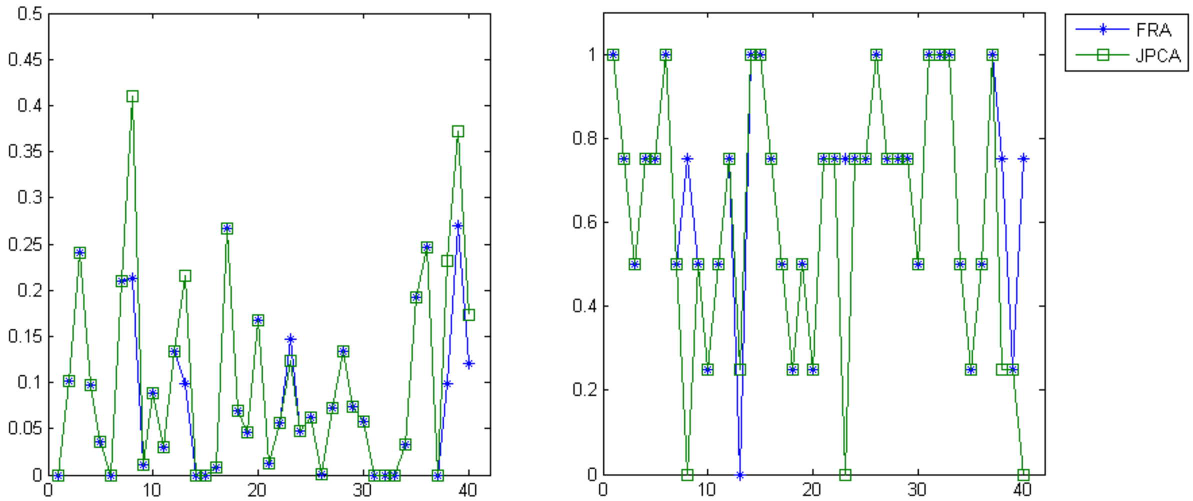
(b) $\sqrt{40}\frac{\overline{x}-\tilde{p}}{\sqrt{\overline{x}(1-\overline{x})}}$

Table 2: Statistical tests for 4 nodes RTNDU with 3 scenarios

coincident cases. Moreover, $\frac{\sqrt{40}(\overline{x}-0.01)}{S}$ is smaller than 1.68 in all cases, see Table 2(a). So, from a statistical viewpoint, the hypothesis $E(\xi) \leq 0.01$ with probability .95, can be accepted for both approaches.

For the probability criterion, we observe a similar result: $N_{FRA}$ was better than $N_{JPCA}$ in most of the non-coincident cases and we can accept that $P_\xi \geq .5$ with probability .95 for FRA and JPCA because $\sqrt{40}\frac{\overline{x}-\tilde{p}}{\sqrt{\overline{x}(1-\overline{x})}}$ is larger than -1.68 for all cases, see Table 2(b).

In the 30 scenarios instances, the solutions found are mainly networks $1-3-4$ and $1-3-2$, see Figure 2. Moreover $N_{FRA}$ and $N_{JPCA}$ coincided in many cases as it is shown




Expected value of $\xi$. Probability of $\{r : \xi \leq \varepsilon\}$.

Figure 4: 4 nodes RTNDU with 3 scenarios and demand generated by (26)

| Approaches | FRA | JPCA |
|---|---|---|
| Rule (25) | 1.2779 | 1.3606 |
| Rule (26) | 1.1698 | 1.4926 |

Value of $\frac{\sqrt{40}(\overline{x}-0.01)}{S}$

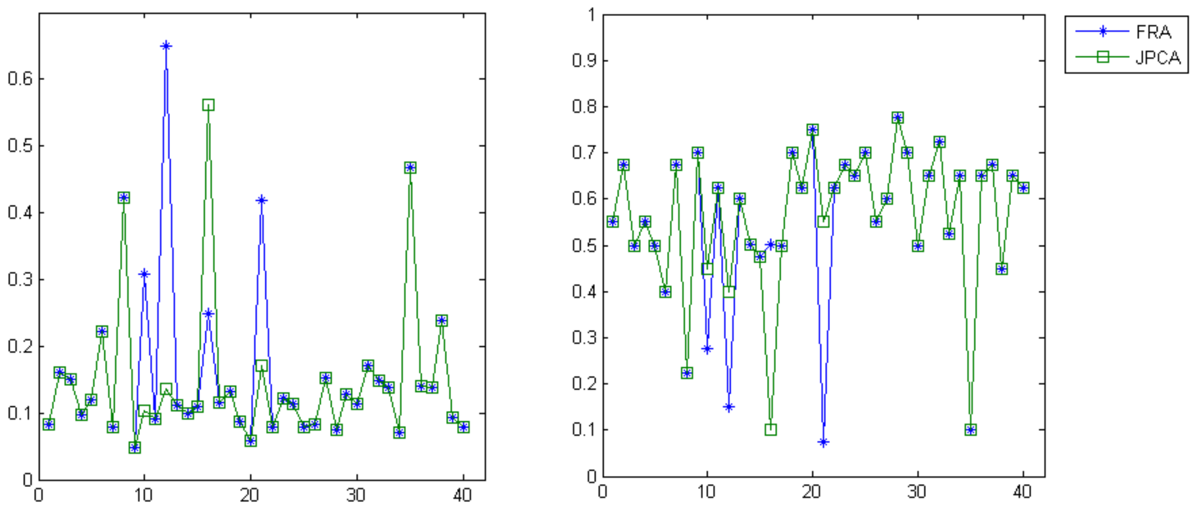
| Approaches | FRA | JPCA |
|---|---|---|
| Rule (25) | 4.7434 | 0.9487 |
| Rule (26) | 3.6515 | 0.9467 |

$\sqrt{40}\frac{\overline{x}-\tilde{p}}{\sqrt{\overline{x}(1-\overline{x})}}$

Table 3: Result of the statistical tests for 4 nodes RTNDU with 30 scenarios

in Figures 5 and 6. In fact only in 9 cases the networks computed by the approaches were different.

If the demand is generated by the low SD (rule (25)) JPCA have a better behavior for both comparison criteria in 3 of the 4 cases where $N_{FRA} \neq N_{JPCA}$. In the high SD case (demand generated by rule (26)), $N_{JPCA}$ was better in all the non-coincident cases. However both approaches behave good because the values of $E(\xi)$ are small, in fact smaller than .8, and $P_\xi \leq .4$ only in 7 cases. Again, see Table 3, we can accept that $E(\xi) \leq 0.001$ and $P_\xi \geq .5$.



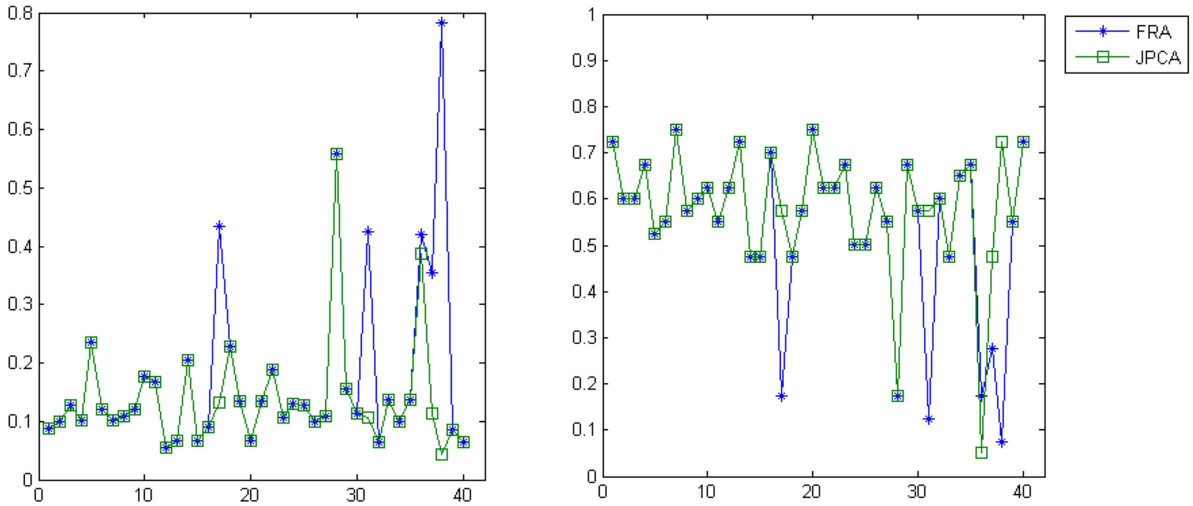

Expected value of $\xi$. Probability of $\{r : \xi \leq \varepsilon\}$.

Figure 5: 4 nodes RTNDU with 30 scenarios and demand generated by (25)

## 4.3 9 nodes network

In this section we present a RTNDU model corresponding to a 9 nodes network. It is a median size example, so the results are more similar to what can happen in applications. First we present the involved parameters. In Figure 7, the values of $c_i, c_{i,j}$ and $d_{ij}$ can be found. Here $c_{\max} = 15$. The auxiliary matrix for computing the origin-destination demands $G^0$ and the cost $u^w_{cur}$ for each demand pair $w \in W$ are given by the following matrices:

$$G^0 = \begin{pmatrix} 0 & 9 & 26 & 19 & 13 & 12 & 4 & 6 & 4 \\ 11 & 0 & 14 & 26 & 7 & 18 & 3 & 7 & 9 \\ 30 & 19 & 0 & 30 & 24 & 8 & 3 & 9 & 11 \\ 21 & 9 & 11 & 0 & 22 & 16 & 21 & 18 & 16 \\ 14 & 14 & 8 & 9 & 0 & 20 & 12 & 18 & 9 \\ 26 & 1 & 22 & 24 & 13 & 0 & 11 & 28 & 21 \\ 7 & 5 & 6 & 19 & 15 & 13 & 0 & 16 & 14 \\ 5 & 9 & 11 & 16 & 17 & 25 & 17 & 0 & 21 \\ 6 & 8 & 10 & 18 & 11 & 20 & 14 & 20 & 0 \end{pmatrix};$$




Expected value of $\xi$. Probability of $\{r : \xi \le \varepsilon\}$.

Figure 6: 4 nodes RTNDU with 30 scenarios and demand generated by (26)

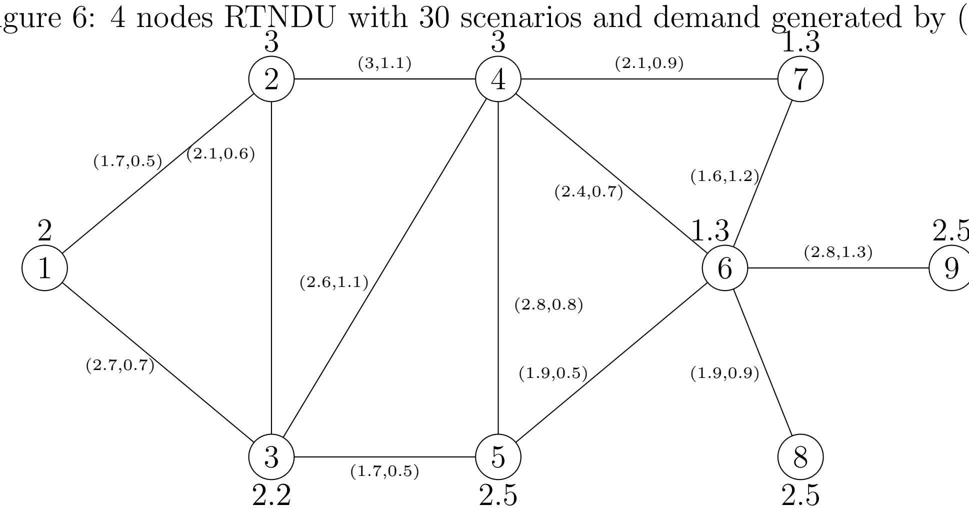



Figure 7: 9 nodes network.

$$
u^w_{cur} = \begin{pmatrix}
0 & 1.6 & 0.8 & 2 & 1.6 & 2.5 & 4 & 3.6 & 4.6 \\
2 & 0 & 0.9 & 1.2 & 1.5 & 2.5 & 3.2 & 3.5 & 4.5 \\
1.5 & 1.4 & 0 & 1.3 & 0.9 & 2 & 3.3 & 2.9 & 3.9 \\
1.9 & 2 & 1.9 & 0 & 1.8 & 2 & 2 & 3.8 & 4.1 \\
3 & 1.5 & 2 & 2 & 0 & 1.5 & 3 & 2 & 3 \\
2.1 & 2.7 & 2.2 & 1 & 1.5 & 0 & 2.5 & 3 & 2.5 \\
3.9 & 3.9 & 3.9 & 2 & 3 & 2.5 & 0 & 2.5 & 2.5 \\
5 & 3.5 & 4 & 4 & 2 & 3 & 2.5 & 0 & 2.5 \\
4.6 & 4.5 & 4 & 3.5 & 3 & 2.5 & 2.5 & 2.5 & 0
\end{pmatrix}.
$$

| Approaches | FRA | JPCA |
|---|---|---|
| Rule 25 | 0.9182 | 0.7439 |
| Rule 26 | 0.6589 | 0.6551 |

Value of $\frac{\sqrt{40}(\overline{x}-0.01)}{S}$

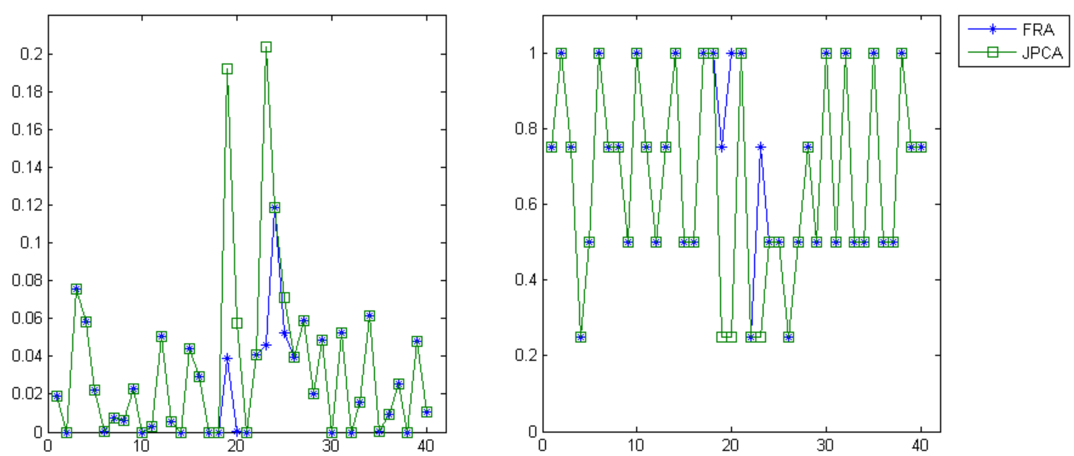
| Approaches | FRA | JPCA |
|---|---|---|
| Rule 25 | 10.2051 | 6.1993 |
| Rule 26 | 2.7603 | $\infty$ |

$\sqrt{40}\frac{\overline{x}-\tilde{p}}{\sqrt{\overline{x}(1-\overline{x})}}$

Table 4: Result of the statistical tests for 9 nodes RTNDU with 3 scenarios

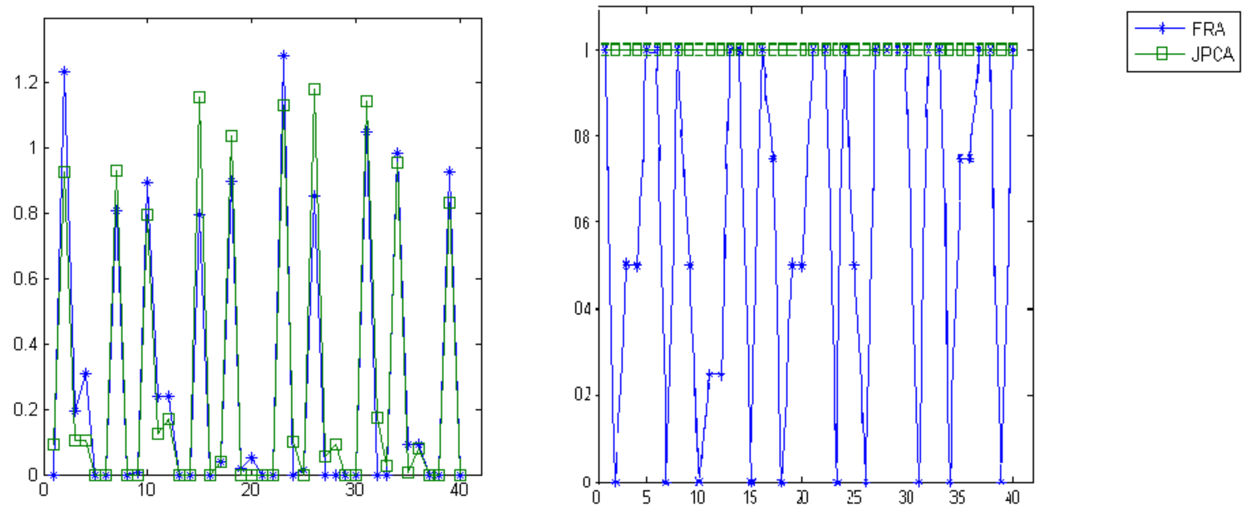
#### 4.3.1 Statistical study of the criteria to validate the RTNDU approaches (9 nodes)

Again we consider 40 examples for each combination of number of scenarios and distribution functions for generating the demand.
For the case of 3 scenarios, Figures 8 and 9 respectively, show the values of $E(\xi)$ and $P_\xi$ for FRA and JPCA approaches. We observe that if the demand is generated by (25) in the non-coincident cases, $E(\xi)$ was smaller and $P_\xi$, larger for $N_{FRA}$ in most cases. A similar analysis shows that JPCA is better if the demand is generated by (26). However we can accept the goodness of both approaches because inequalities (23) and (24) are fulfilled, see Table 4.




Expected value of $\xi$. Probability of $\{r : \xi \leq \varepsilon\}$.

Figure 8: 9 nodes RTNDU with 3 scenarios and demand generated by (25)




Expected value of $\xi$. Probability of $\{r : \xi \leq \varepsilon\}$.

Figure 9: 9 nodes RTNDU with 3 scenarios and demand generated by (26)

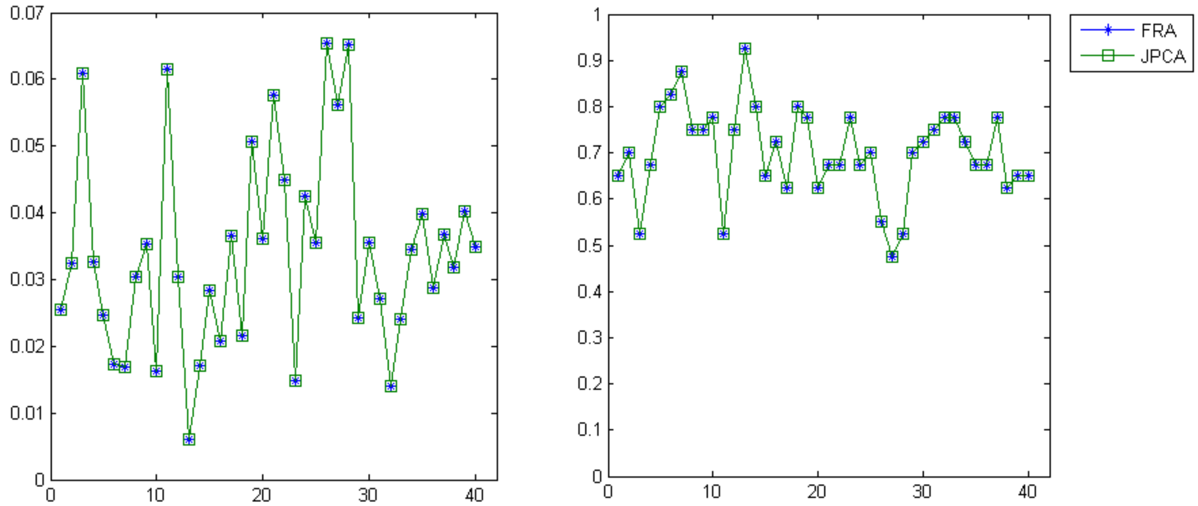
In the 30 scenarios case, independently of the generation of the demand, $N_{FRA} = N_{JPCA}$ at each instance, see Figures 10 and 11. Moreover, we can accept that $P_\xi \geq .5$, because for all the examples $P_\xi \geq .5$. However from a statistical viewpoint $E(\xi) > .01$ because as

$$1.68 < \frac{\sqrt{40}(\bar{\xi} - \varepsilon_1)}{S} = \begin{cases} 2.2437 & \text{if the demand is generated by (25)} \\ 2.6632 & \text{if the demand is generated by (26)} \end{cases},$$

inequality (23) fails. Nevertheless $E(\xi)$ is not too large because in all cases it is smaller than 0.1.

## 4.4 Solutions changing the budget

Another sensible parameter is the maximal allowance budget, $c_{\max}$. In this part we will present an illustrative example that shows that the $N_{FRA}$ and $N_{JCA}$ are highly dependent on this value. If the maximal budget allowance is close to 0 (for instance $c_{\max} < min\{c_i, c_{ij}\}$), the solution of RTND model is to build no network and hence all the demand will use the private network. On the other hand, for $c_{\max} > \sum_{i \in N_r} c_i + \sum_{i \in A_r} c_{ij}$, constraint $z_{loc} \leq c_{\max}$ is superfluous. So, the same network is optimal for all $c_{\max}$ and all the demand will use the public network. This result was observed numerically in [Laporte et al(2011)].




Expected value of $\xi$. Probability of $\{r : \xi \leq \varepsilon\}$.

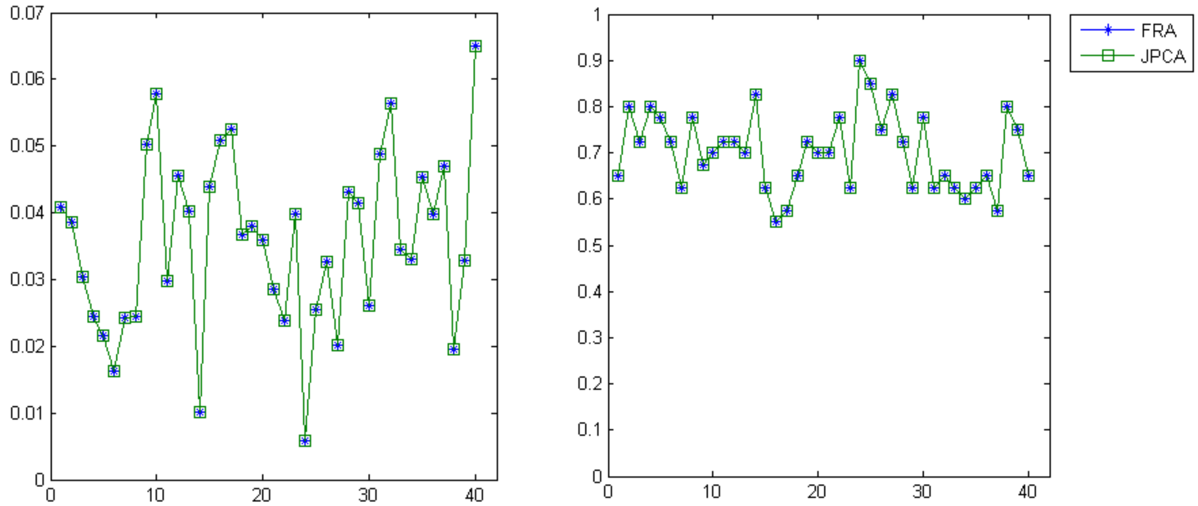
Figure 10: 9 nodes RTNDU with 30 scenarios and demand generated by (25)

In the stochastic model something similar happens. If $c_{\max} \to 0$ (again is enough to take $c_{\max} < min\{c_i, c_{ij}\}$) the solution of RTNDU$_r$ by FRA and JPCA is to build no network. Similarly for all $c_{\max} > \sum_{i \in N_r} c_i + \sum_{i \in A_r} c_{ij}$, the two approaches and the models RTNDU computed the same solution. So, we can conclude that for $c_{\max}$ large or small enough, $\xi = 0$.

We will consider the 4 nodes RTNDU model. In this case, we observed that for $c_{\max} < 5$ or $c_{\max} > 25$, the same network is feasible for all scenario and all approaches. So, we will compute $N_{FRA}$ y $N_{JPCA}$ for $c_{max} = 5 + i/2$, $i = 21, \ldots 20$. In the case of 3 scenarios and demand generated by rule (26), Figure 12 shows the values of the $E(\xi)$ and $P_\xi$ versus $c_{\max}$. As can be seen, both approaches computed good solutions, because the values of $E(\xi)$ are small and $P_\xi \geq .4$ in most of the cases. However, FRA computed better solutions in the non-coincident cases.

In the 9 nodes case, we observe differences between the optimal networks of the scenarios, $N_{FRA}$ and $N_{JPCA}$for $c_{\max} \in [14.5, 16]$, see Figure 13. In the non-coincident cases FRA approach computed the better solution because for all $r \in R$, it coincide with $N_r$ the optimal solution of scenario $r$.

As we have seen the stochastic approaches compute the same solution in most cases and FRA is better in the non coincident examples. From a statistical viewpoint both



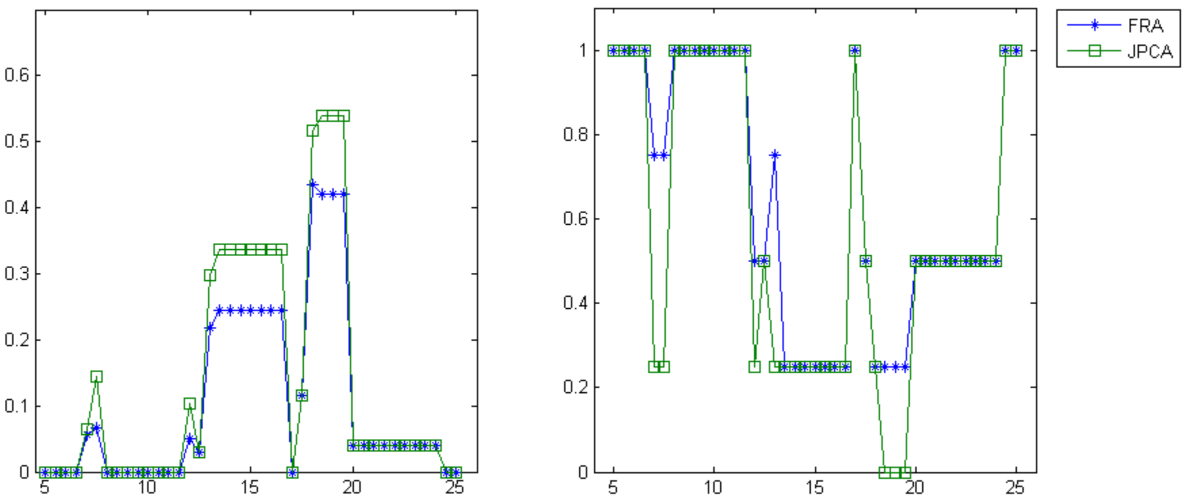

Expected value of $\xi$. Probability of $\{r : \xi \leq \varepsilon\}$.

Figure 11: 9 nodes RTNDU with 30 scenarios and demand generated by (26)

approaches are good, we can accept tha $E(\xi)$ is small and that $P_\xi$ is large enough.

### 4.5 Some experiments for demand generated according to a normal law

In this part we will present some examples of RTNDU where the expected value of the demand is large. As reported in [Swan(2002)], in this cases the demand follows a normal distribution with expected value and variance equal to $\mu$. We use $\mu = a^2[G_w]_0$, for $a = 100, 400, 10000$ in order to obtain a large variance distribution of $g_w$. Although $[g_w]_r$ must be non negative, $[g_w]_r \leq 0$ with probability $\int_{-\infty}^{-\sqrt{\mu}} e^{\frac{x^2}{2}} dx$, but as $\mu$ is a large value, this probability is small. For instance, if $\mu \geq 9$ the probability is smaller than 0.0014. Nevertheless, we generate $[G_w]_r$ by the distribution function $\max(normal(a^2 * [G_w]_0, a^2 * [G_w]_0), 0)$, where $normal(\mu, \sigma^2)$ is the normal distribution with expected value and variance equal to $\mu$ and $\sigma^2$ respectively.
We considered 40 examples for each value of $a$ for the RTNDU model of 4 and 9 nodes. We observed that, fixed the dimension, in all cases the same network is optimal for all criteria and for all scenario, see Figure 14.
This fact means that although the variance is large, as it is equal to the mean value, the




Expected value of $\xi$ vs. $c_{max}$. Probability of $\{r : \xi \leq \varepsilon\}$ vs. $c_{max}$.

Figure 12: 4 nodes RTNDU, 3 scenarios and demand generated by rule (25).

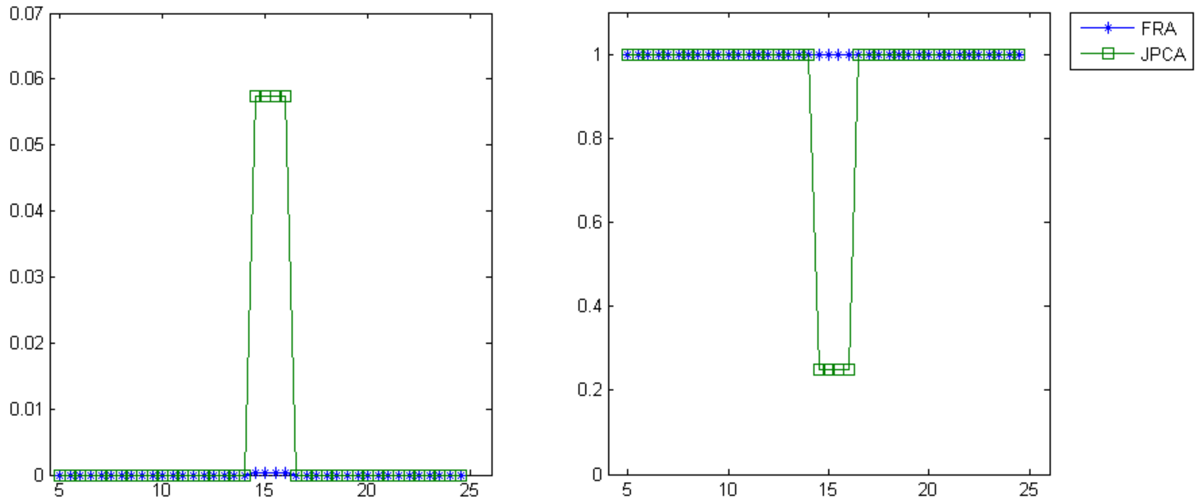
standard deviation is small and the generated values are very similar. So, we can not expect difference for the scenarios and hence similar solutions can be expected.

# 5 Conclusions

In this work we have presented two approaches for solving RTND problems with randomly generated demands, namely FRA and JPCA approaches. In both cases, their application to a RTNDU model corresponds to solve a RTND model with fixed demands: the expected value of the demand for FRA and percentiles for JPCA. However, in the last case we only have a bound of the value of the objective function that can be too large.

From a numerical point of view, we observe coincidences between the networks computed by the approaches. In fact, if the demand is generated by the normal law, the networks computed by the approaches coincided with the optimal solution for all scenarios.

For uniform generated demand, the solution of RTNDU model computed by FRA was better than JPCA in most of the non-coincident cases. Moreover, by means of statistical tests, we could accept that for the solutions computed by the approaches, $P_\xi$ is large. As only in one case (9 nodes networks with 30 scenarios and demands generated by the low SD. law) $E(\xi) < 0.01$ was rejected, we can say that FRA and JPCA behave well from




Expected value of $\xi$ vs. $c_{max}$. Probability of $\{r : \xi \leq \varepsilon\}$ vs. $c_{max}$.

Figure 13: 9 nodes RTNDU, 3 scenarios and demand generated by rule (25).

a statistical point of view. As FRA is more natural, computed better solutions and it is easier to calculate the expected value of a sample than the percentiles, we recommend this approach over JPCA.

Practical applications can be solved as follows, we collect a sample of demands and apply the approaches using the corresponding sample values. We expect that the solutions will be relative good options in most RTNDU cases.

The solution of RTNDU can change depending on parameters such as maximal budget and costs of construction of lines and stations. This is a topic for further research.

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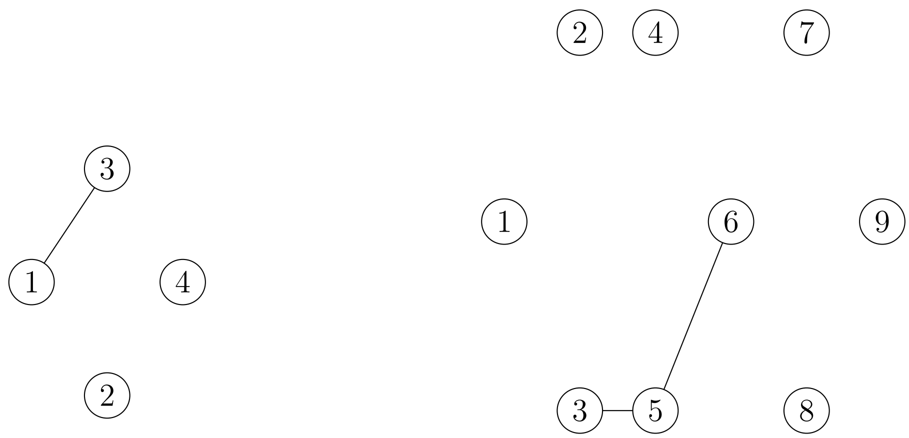


4 nodes case. 9 nodes case

Figure 14: Optimal networks for normal distributed demands

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