On the connection of facially exposed, and nice cones

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Abstract

A closed convex cone K is called *nice,* if the set $K^* + F^\perp$ is closed for all F faces of K, where K^* is the dual cone of K, and $F^\perp$ is the orthogonal complement of the linear span of F. The niceness property plays a role in the facial reduction algorithm of Borwein and Wolkowicz, and the question whether the linear image of a nice cone is closed also has a simple answer.

We prove several characterizations of nice cones and show a strong connection with facial exposedness. We prove that a nice cone must be facially exposed; in reverse, facial exposedness with an added technical condition implies niceness.

We conjecture that nice, and facially exposed cones are actually the same, and give some supporting evidence.

Key words: duality; closedness of the linear image; badly behaved instances; excluded minor characterizations

1   Introduction

Closed convex cones appear in many areas of optimization. Conic linear programs – optimization problems with a linear objective function, and a feasible set expressed as the intersection of a closed convex cone with an affine subspace – were introduced by Duffin in [11]. They serve as a natural framework to study the duality theory of convex programs. The seminal interior-point framework of Nesterov and Nemirovskii [14] was also developed for conic LPs.

The properties of the underlying cone determine the easiness of a conic LP to a large extent. The nonnegative orthant is arguably the simplest cone useful in optimization. Second order, p-order, and semidefinite cones are more complex, but still admit efficient optimization algorithms (see e.g. [1], [18], [13]). Copositive, and completely positive cones lie at the other end of the spectrum: though they are very useful in optimization (see e.g. [7]), optimizing over them is more difficult, as is understanding their geometry ([9]).

The goal of this paper is to better understand the connection of two intriguing geometric properties of closed convex cones: facial exposedness and niceness. The niceness property is important for two reasons: it plays a role in the facial reduction algorithm of Borwein and Wolkowicz [6]. Also, the question whether the linear image of a nice cone is closed has

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a very simple characterization. Noting that for a convex set C the relative interior of C is denoted by $\operatorname{ri} C$, and if $x \in C$ then the set of feasible directions at x in C is defined as $\operatorname{dir}(x, C) = \{\, y \mid x + \epsilon y \in C \text{ for some } \epsilon > 0 \,\}$, we recall a simplified version of Theorem 1 in [15]:

Theorem 1. *Let M be a linear map, C a nice cone, and $x \in \operatorname{ri}(C \cap \mathcal{R}(M))$. Then*

(1) the set $M^* C^*$ is closed, if and only if $\mathcal{R}(M) \cap (\operatorname{cl} \operatorname{dir}(x, C) \setminus \operatorname{dir}(x, C)) = \emptyset$. $\qquad\square$

For better intuition, we can note that if C is a polyhedral cone, then $\operatorname{dir}(x, C)$ is closed for all $x \in C$, and that polyhedral cones are nice. So the sufficiency of the condition in (1) shows that $M^* C^*$ is closed for an arbitrary M map, as expected. Also, if x is in $\operatorname{ri} C$, then $\operatorname{dir}(x, C)$ is just a subspace, hence closed. Thus Theorem 1 unifies two seemingly unrelated, sufficient conditions for the closedness of $M^* C^*$.

Many cones appearing in the optimization literature, for instance polyhedral, second order, p-cones, and the semidefinite cone are both facially exposed, and nice: see for instance [15]. Here we find a close connection between these concepts: we prove that a nice cone must be facially exposed; in reverse, facial exposedness with an added technical condition implies niceness. We conjecture that the two classes of cones are actually the same, and we give supporting evidence.

The rest of the paper is structured as follows. In Section 2 we collect preliminaries. Section 3 has our main characterizations of nice cones, and describes the connection with facial exposedness. Section 4 states the conjecture, shows a supporting example, and shows that proving a seemingly weaker version would already suffice. In this section we also describe a reformulated version of our main characterization of nice cones (Theorem 3) and shows how it may lead to the proof of the main conjecture.

2   Preliminaries

For a set S we write $\operatorname{cl} S$ for its closure, $\operatorname{ri} S$ for its relative interior, $\operatorname{lin} S$ for its linear span and $S^\perp$ for the orthogonal complement of its linear span. For a one-element set $\{y\}$ we abbreviate $\{y\}^\perp$ by $y^\perp$.

A set C is called a *cone,* if $\lambda x \in C$ holds for all $x \in C$, and $\lambda \geq 0$. For a set S the set of all nonnegative combinations of S is clearly a cone, which is called the cone generated by S. For a one-element set $\{y\}$ we abbreviate $\operatorname{cone}\{y\}$ by $\operatorname{cone} y$.

Good references on convex analysis in general are for instance [16, 5, 13]. References [4, 17, 2] cover more specifically the theory of cones. If C is a convex cone, its lineality space is defined as
$$\operatorname{lspace} C = C \cap -C,$$
and its dual cone as
$$C^* = \{\, y \mid \langle y, x \rangle \geq 0 \ \forall x \in C \}.$$

We say that C is pointed, if $\operatorname{lspace} C = \{0\}$. For convex cones $C, C_1,$ and C_2 we have

$$
\begin{aligned}
C^{**} &= \operatorname{cl} C, & (2.1) \\
(C_1 + C_2)^* &= C_1^* \cap C_2^*. & (2.2)
\end{aligned}
$$

Furthermore, if C_1 and C_2 are also closed, then

$$
(C_1 \cap C_2)^* = \operatorname{cl}(C_1^* + C_2^*). \qquad (2.3)
$$

Given a closed convex cone C, a convex subset E of C is called a *face* of C, if $x_1, x_2 \in C$, $x_1 + x_2 \in E$ imply that x_1 and x_2 both are in E. The cone C itself is clearly a face of C. For all $x \in C$ there is a unique face of C that contains x, namely the face containing x in its relative interior.

We write $E \trianglelefteq C$ to denote that E is a face of C, and $E \triangleleft C$ to abbreviate $E \trianglelefteq C$, $E \neq C$. The definition implies that the intersection of faces is again a face. Also, $\operatorname{lspace} C$ is the inclusionwise minimal face of C. We call an E face of C *proper,* if $\operatorname{lspace} C \triangleleft E \triangleleft C$. We call a proper face minimal (maximal) if it is inclusionwise minimal and distinct from $\operatorname{lspace} C$ (inclusionwise maximal and distinct from C). Minimal proper faces of pointed cones are called extreme rays.

A remark on notation: we will look at characterizations of the niceness of a closed convex cone, and will generally denote this cone by K. In collecting relevant results we usually reference a closed convex cone by C, since the role of C later on will be played sometimes by K, and sometimes by F^*, where F is a face of K.

The following proposition is easy to prove:

Proposition 1. *If C is a closed convex cone, and $E \trianglelefteq C$, then E is a cone in its own right, and*

$$
\begin{aligned}
\operatorname{lspace} E &= \operatorname{lspace} C \text{ and} & (2.4) \\
\operatorname{ri} E &= \operatorname{ri} E + \operatorname{lspace} C. & (2.5)
\end{aligned}
$$

hold.

$\square$

A subset E of C is called an *exposed face* of C, if it is the intersection of C with a supporting hyperplane, i.e.

$$
E = C \cap y^\perp
$$

for some y satisfying $\langle y, x \rangle \geq 0$ for all $x \in C$, i.e. y must be in C^*. We say that y exposes E. Also, if H is the smallest face of C^* that contains y, then $E = C \cap H^\perp$ holds for the above E. The set of all vectors exposing E is $C^* \cap E^\perp$.

An exposed face of C is always a face, but a face E_1 may not be exposed. This happens when every vector that exposes E_1 actually exposes a strictly larger face E_2, i.e. $C^* \cap E_1^\perp =$

$C^* \cap E_2^\perp$ holds for some $E_2 \trianglelefteq C$ with $E_1 \subsetneq E_2$. (Example 1 is an example of a cone, which has a nonexposed face.)

The intersection of exposed faces is again an exposed face, so the smallest exposed face containing a subset of C is well-defined. Hence a face E is not exposed, iff the smallest exposed face containing it is distinct from E.

We say that a closed convex cone C is facially exposed, if all of its faces are exposed. Based on the above argument, an equivalent definition is requiring

$$C^* \cap E_2^\perp \subsetneq C^* \cap E_1^\perp \tag{2.6}$$

for all E_1 and E_2 faces of C with $E_1 \subsetneq E_2$.

In the following proposition the 'if" part is Proposition 2.1 in [17] and the "only if" part is trivial.

Proposition 2. *If C is a closed convex cone, then an E face of C is proper, iff $C^* \cap E^\perp$ is a proper face of C^*.* $\qquad\square$

The space of n by n symmetric, and the cone of n by n symmetric, positive semidefinite matrices are denoted by $\mathcal{S}^n$, and $\mathcal{S}^n_+$, respectively. The space $\mathcal{S}^n$ is equipped with the inner product

$$X \bullet Z \quad := \quad \sum_{i,j=1}^{n} x_{ij} z_{ij},$$

and it is a well-known fact, that $\mathcal{S}^n_+$ is self-dual with respect to it.

The faces of $\mathcal{S}^n_+$ have an attractive, and simple description. After applying a rotation $V^T(.)V$ by a full-rank matrix V, any face can be brought to the form

$$E \quad = \quad \left\{ \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \mid x \in \mathcal{S}^r_+ \right\}.$$

For a proof, see [3], [12, Appendix A] for a somewhat simpler one. For a face of this form we will use the shorthand

$$E = \begin{pmatrix} \oplus & 0 \\ 0 & 0 \end{pmatrix}, \ \operatorname{lin} E = \begin{pmatrix} \times & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.7}$$

when the size of the partition is clear from the context. The $\oplus$ sign denotes a positive semidefinite submatrix, and a $\times$ a submatrix with arbitrary elements.

Facially nonexposed cones can be constructed by taking sums of facially exposed ones, as Example 1 shows. The cross-section of the cone in Example 1 is illustrated on Figure 1. We give this example in detail, since we will return to it later.
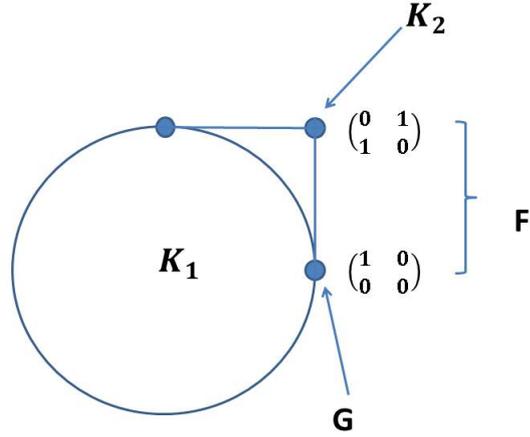
Figure 1: Cross section of a facially nonexposed cone

Example 1. Define the cone $K \subseteq \mathcal{S}^2$ as $K = K_1 + K_2$, with $K_1 = \mathcal{S}_+^2$, $K_2 = \text{cone} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Let

$$G = \text{cone} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.8}$$

$$F = \text{cone}\{G \cup K_2\}. \tag{2.9}$$

Clearly, G and F are both faces of K.

Also,
$$K^* = K_1^* \cap K_2^* = \{ X \in \mathcal{S}_+^2 \mid x_{12} \geq 0 \}. \tag{2.10}$$

Hence

$$K^* \cap G^\perp = K^* \cap F^\perp = \text{cone} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.11}$$

hence G is not exposed, and on the figure one can easily check that the smallest exposed face of K that contains G is F.

The main definition of the paper is

Definition 1. *A closed convex cone K is nice, if the set $K^* + F^\perp$ is closed for all $F \trianglelefteq K$.*

Remark 1. Since $F = K \cap \text{lin}\, F$ holds for all $F \trianglelefteq K$, by (2.3) the definition is equivalent to requiring

$$F^* = K^* + F^\perp \text{ for all } F \trianglelefteq K. \tag{2.12}$$

Also, (2.12) trivially holds for $F = \text{lspace}\, K,$ and $F = K,$ so it suffices to require it for proper faces in Definition 1.

The following proposition, which is based on the remarks in Section 18 in [16] collects properties of closed, convex, possibly nonpointed cones:

Proposition 3. *Let C be a closed convex cone, and $L = \text{lspace}\, C.$ Then the following hold:*

(1) $C = C_0 + L,$ *where* $C_0 = C \cap L^\perp,$ *and C_0 is pointed.*

(2) E *is a face of C iff* $E = E_0 + L,$ *with E_0 a face of $C_0.$*

(3) *For the above E and E_0 we have*

    (a) $E_0 = E \cap L^\perp.$

    (b) E *is a minimal proper face of C iff E_0 is an extreme ray of $C_0.$*

    (c) E *is exposed iff E_0 is.*

$\square$

In general it seems to be known that a nonpointed closed, convex cone is generated by the union of its minimal proper faces. Since we were not able to find a result stated precisely in this form, we state, and prove:

Proposition 4. *Let C be a closed convex cone. Then*

$$C = \text{cone}\,\{\, E \mid E \text{ is a minimal proper face of } E \,\}. \tag{2.13}$$

Proof Let $L = \text{lspace}\, C,$ and write $C = C_0 + L,$ with $C_0 = C \cap L^\perp.$ Then

$$
\begin{aligned}
C &= \text{cone}\,\{\, E_0 \mid E_0 \text{ is an extreme ray of } C_0 \,\} + L \\
&= \text{cone}\,\{\, E \cap L^\perp \mid E \text{ is a minimal proper face of } C \,\} + L \\
&= \text{cone}\,\{\, E \cap L^\perp + L \mid E \text{ is a minimal proper face of } C \,\} \\
&= \text{cone}\,\{\, E \mid E \text{ is a minimal proper face of } C \,\}.
\end{aligned}
$$

Here the first equation comes from the fact that C_0 is pointed, and Theorem 18.5 in [16], while the others follow from Proposition 3. $\square$

$\square$

3 Characterizations of nice cones, and connections to facial exposedness

Throughout this section we assume that

> K is a closed, convex cone.

In Theorem 2 we give several characterizations of the situation $F^* = K^* + F^\perp$ for a specific F face of K. In Theorem 3 we build on this to make the connection of the niceness of K to its facial exposedness.

We first informally discuss Theorem 2. If F is a face of K, then $\operatorname{lspace} F^* = F^\perp$ holds, hence a proper face H of F^* satisfies $H \supsetneq F^\perp$. Clearly,

$$K^* \cap H \supseteq K^* \cap F^\perp, \tag{3.14}$$

and by the definition of faces, and $K^* \subseteq F^*$ both sets in (3.14) are faces of K^*. However, they may be equal.

The equality of F^* and $K^* + F^\perp$ is characterized by strict containment holding in (3.14) for *all* H proper faces of F^*; in fact, it is enough to assume it for all *minimal* proper faces. These are conditions (3) in Theorem 2, and (3') in Remark 2, the latter is what we use in Theorem 3 to connect niceness to facial exposedness.

We need the following

Proposition 5. *Let K be a closed convex cone, $F \trianglelefteq K$, and H a proper face of F^*. Then*

$$\operatorname{lspace} H \;=\; F^\perp \tag{3.15}$$
$$\operatorname{ri} H \;=\; \operatorname{ri} H + F^\perp. \tag{3.16}$$

Proof Directly from Proposition 3. $\qquad\square$

Theorem 2. *Let $F \trianglelefteq K$. Then the following statements are equivalent:*

(1) $F^* = K^* + F^\perp$.

(2) $K^* \cap \operatorname{ri} H \neq \emptyset$ holds for all H proper faces of F^*.

(3) $K^* \cap H \supsetneq K^* \cap F^\perp$ holds for all H minimal proper faces of F^*.

Proof of $(1) \Rightarrow (2)$ Let H be a proper face of F^*, and $x \in \operatorname{ri} H$. Write $x = x_1 + x_2$, with $x_1 \in K^*$, $x_2 \in F^\perp$. Hence $x_1 = x - x_2 \in \operatorname{ri} H + F^\perp = \operatorname{ri} H$, where the last equation follows from Proposition 5.

The implication $(2) \Rightarrow (3)$ follows from the fact that if H is a minimal proper face of F^*, then its only proper face is $F^\perp$, hence $H = \operatorname{ri} H \cup F^\perp$.

Proof of $(3) \Rightarrow (1)$ Proposition 4 implies

$$F^* \;=\; \operatorname{cone} \{\, H \mid H \text{ is a minimal proper face of } F^* \,\}. \tag{3.17}$$

Let H be an arbitrary minimal face of F^*. Given (3.17), it suffices to show that if $K^* \cap H \supsetneq K^* \cap F^\perp$, then $H \subseteq K^* + F^\perp$.

Let $x \in H$. If $x \in F^\perp$, then of course $x \in K^* + F^\perp$, so suppose $x \notin F^\perp$. By the assumption there is $y \in (K^* \cap H) \setminus F^\perp$. If $x = y$, then again $x \in K^* + F^\perp$. If $x \neq y$, then let us define the two half-lines

$$
\begin{aligned}
r_{x,y}^+ &= \{\, x + \lambda y \mid \lambda \geq 0 \,\} \text{ and} \\
r_{x,y}^- &= \{\, x - \lambda y \mid \lambda \geq 0 \,\}.
\end{aligned}
$$

Then $r_{x,y}^+ \subseteq H$, since H is a cone, and x and y are both in H. Hence $r_{x,y}^- \not\subseteq H$, since both $r_{x,y}^-$ and $r_{x,y}^-$ being in H would imply $y \in \mathrm{lspace}\, H = F^\perp$.

Let z denote the intersection of $r_{x,y}^-$ with the relative boundary of H. Since H is a minimal proper face, its relative boundary consists of $F^\perp$ alone, so $z \in F^\perp$. Also, x is a convex combination of y and z, hence we obtain $x \in K^* + F^\perp$.

$\square$

Remark 2. If F is as in Theorem 2, it is straightforward to see that two other conditions equivalent to $F^* = K^* + F^\perp$ are

(2') $K^* \cap \mathrm{ri}\, H \neq \emptyset$ holds for all H minimal proper faces of F^*.

(3') $K^* \cap H \supsetneq K^* \cap F^\perp$ holds for all H proper faces of F^*.

Also, for an H proper face of F^*, we have $\mathrm{ri}\, H = \mathrm{ri}\, H + F^\perp$. Hence

$$
K^* \cap \mathrm{ri}\, H \neq \emptyset \Leftrightarrow (K^* + F^\perp) \cap \mathrm{ri}\, H \neq \emptyset, \tag{3.18}
$$

so replacing K^* by $K^* + F^\perp$ in (2) in Theorem 2 and (2') yields equivalent conditions. Also, since $\mathrm{lspace}\, H = F^\perp$, we have that $K^* \cap H \supsetneq K^* \cap F^\perp$ iff $(K^* + F^\perp) \cap H \supsetneq (K^* + F^\perp) \cap F^\perp$ (and in fact it is easy to check that the last set is just $F^\perp$). Thus, replacing K^* by $K^* + F^\perp$ in (2) in Theorem 2 and (3') we also obtain equivalent conditions.

Theorem 3. *The following hold.*

(1) *If K is nice, then it is facially exposed.*

(2) *If K is facially exposed, and for all $F \trianglelefteq K$ all minimal proper faces of F^* are exposed, then K is nice.*

Proof Consider the statements

$$
K^* \cap H \supsetneq K^* \cap F^\perp, \tag{3.19}
$$

where $F \trianglelefteq K$, and H is a proper face of F^*, and

$$
K^* \cap G^\perp \supsetneq K^* \cap F^\perp, \tag{3.20}
$$

where F and G are faces of K satisfying $G \subsetneq F$.

Theorem 2 shows that K is nice, iff (3.19) holds for *all* $F \trianglelefteq K$, and *all* H proper faces of F^*, or equivalently for *all* $F \trianglelefteq K$, and *all* H minimal proper faces of F^*. Also, K is facially exposed, iff (3.20) holds for *all* F and G faces of K with $G \subsetneq F$.

To prove (1), let F and G be faces of K with $G \subsetneq F$, and set $H = F^* \cap G^\perp$. Since G is a proper face of F, Proposition 2 implies that H is a proper face of F^*. Since K is nice, (3.19) holds, and since

$$K^* \cap H = K^* \cap F^* \cap G^\perp = K^* \cap G^\perp, \qquad (3.21)$$

(3.20) follows.

To prove (2), let F be a face of K, and H a minimal proper face of F^*. By the assumption H is an exposed face, so $H = F^* \cap G^\perp$ for a G proper face of F. Then (3.21) holds. Since K is facially exposed, (3.20) holds as well, hence (3.19) follows.

$\square$

Given $F \trianglelefteq K$ the proofs of Theorem 2 and 3 show how to find points in $F^* \setminus (K^* + F^\perp)$, when this set is nonempty. In particular, we have

Corollary 1. *The following hold.*

(1) *If $F \trianglelefteq K$, and H is a proper face of F^* with $K^* \cap H = K^* \cap F^\perp$, then*

$$\mathrm{ri}\, H \subseteq F^* \setminus (K^* + F^\perp).$$

(2) *If G is a nonexposed face of K, and F the smallest exposed face of K that contains G, then*

$$\mathrm{ri}(F^* \cap G^\perp) \subseteq F^* \setminus (K^* + F^\perp).$$

Proof of (1) The containment $\mathrm{ri}\, H \subseteq F^*$ is obvious. Clearly, $K^* \cap \mathrm{ri}\, H = \emptyset$, and the equivalence (3.18) proves $(K^* + F^\perp) \cap \mathrm{ri}\, H \neq \emptyset$.

Proof of (2) We have $K^* \cap F^\perp = K^* \cap G^\perp$. Setting $H = F^* \cap G^\perp$, it clearly holds that H is a proper face of F^*, and $K^* \cap H = K^* \cap F^\perp$, so part (1) implies our claim.

$\square$

Example 1 continued With G a nonexposed face in K, and F the smallest exposed face containing it, we have

$$
\begin{aligned}
F^* &= \{\, X \in \mathcal{S}^2 \mid x_{11} \geq 0,\ x_{12} \geq 0 \,\} \\
F^* \cap G^\perp &= \{\, X \in \mathcal{S}^2 \mid x_{11} = 0,\ x_{12} \geq 0 \,\} \\
F^\perp &= \{\, X \in \mathcal{S}^2 \mid x_{11} = 0,\ x_{12} = 0 \,\}.
\end{aligned}
$$

Let

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then clearly $X \in \mathrm{ri}(F^* \cap G^\perp)$, hence Corollary 1 implies $X \notin K^* + F^\perp$. Given the description of K^* in (2.11) one can indeed easily verify this fact.

4 Are facially exposed, and nice cones the same?

The main conjecture of the paper is:

Conjecture 1. *A closed convex cone is nice if and only if it is facially exposed.* □

Proving Conjecture 1 would be very interesting, since facial exposedness and niceness are both fundamental, and at first sight unrelated geometric properties of cones.

Theorem 3 already finds a strong connection: niceness implies facial exposedness, and facial exposedness with an added extra, technical condition implies niceness. In support of Conjecture 1, we first present an example to show that the extra, technical conditon in general is not necessary. Next, we show that proving a weaker version of Conjecture 1 would already be sufficient. Finally, we give a reformulation of Theorem 2, and outline how this may lead to a proof of Conjecture 1.

We need the following

Proposition 6. *Suppose that K_1 and K_2 are nice cones. Then $K_1 \cap K_2$ is also nice.*

Proof Obviously, $K_1 \times K_2$ is nice. The proof is then straightforward from Proposition 2.2 of Chua and Tuncel [8], which shows that if C is a nice cone, and M a linear map, then $M^{-1}C := \{ x \,|\, Mx \in C \}$ is also nice: we need to apply it with $M = (I, I)$, and $C = K_1 \times K_2$. □

Precisely, Example 2 shows a closed, convex, facially exposed cone K, which is nice, however, there is a face F of K such that an H minimal proper face of F^* is not exposed. We first informally describe Example 2. We construct K as $K = K_1 \cap K_2$, where K_1 is a semidefinite cone, and K_2 is a halfspace. By Proposition 6 we have that $K_1 \cap K_2$ is nice.

Then we choose suitable faces F_1 of K_1, and F_2 of K_2. The definition of faces implies that $F := F_1 \cap F_2$ is a face of K (in fact, a theorem of Dubins in [10] shows that all faces of K arise in this manner). Also, if $\mathrm{ri}\, F_1 \cap \mathrm{ri}\, F_2 \neq \emptyset$, then $F^* = F_1^* + F_2^*$ (see Section 5 in [15]). As F^* is the sum of two simple, facially exposed closed, convex cones, one can expect it to have nonexposed faces, like K does in Example 1, and we can rigorously show that there is indeed such a face, which is minimal, and proper.

Example 2. Let

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and define $K = K_1 \cap K_2$, with

$$K_1 = \mathcal{S}_+^3, \; K_2 = \{ X \in \mathcal{S}^3 \,|\, M \bullet X \geq 0 \}. \tag{4.22}$$

Also define $F = F_1 \cap F_2$, with

$$F_1 = \begin{pmatrix} \oplus & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; F_2 = K_2.$$

(Recall the notation for the faces of the semidefinite cone from Section 2.)

Then $K^* = K_1^* + K_2^*$, where

$$K_1^* = \mathcal{S}_+^3, \; K_2^* = \operatorname{cone} M.$$

It is easy to check that $\operatorname{ri} F_1 \cap \operatorname{ri} F_2 \neq \emptyset$, hence $F^* = F_1^* + F_2^*$, where

$$F_1^* = \begin{pmatrix} \oplus & & \times \\ & & \times \\ \times & \times & \times \end{pmatrix}, \; F_2^* = K_2^* = \operatorname{cone} M.$$

(More formally, F_1^* is the set of 3 by 3 symmetric matrices, whose upper left 2 by 2 block is positive semidefinite, and the rest of the components are arbitrary.) Now, let us define

$$H = \begin{pmatrix} 0 & 0 & \times \\ 0 & \oplus & \times \\ \times & \times & \times \end{pmatrix}.$$

(Again, more formally H is the set of 3 by 3 symmetric matrices X with $x_{11} = x_{12} = x_{21} = 0$, $x_{22} \geq 0$, and the rest of the components arbitrary.)

Proposition 7. *The set H is a minimal proper face of F^*, which is not exposed.*

Proof We first prove that H is a face. Let $X \in H$, and suppose $X = Y + Z$, where $Y, Z \in F^*$. We show that Y and Z are in F^*.

We can write $Y = S + T, Z = U + V$, where $S, U \in F_1^*, T, V \in F_2^*$. Let us write $s_{ij}, t_{ij}, u_{ij}, v_{ij}$ for the components of S, T, U, and V, respectively. Since $t_{11} = v_{11} = 0$, we have

$$x_{11} = s_{11} + u_{11}. \tag{4.23}$$

With $x_{11} = 0$, $s_{11} \geq 0$, $u_{11} \geq 0$, (4.23) implies

$$s_{11} = u_{11} = 0. \tag{4.24}$$

Next, since the upper left 2 by 2 corner of S and U are positive semidefinite, (4.24) implies that also $s_{12} = u_{12} = 0$ hold, so

$$x_{12} = t_{12} + v_{12}. \tag{4.25}$$

Finally, (4.25) with $x_{12} = 0$, $t_{12} \geq 0$, $v_{12} \geq 0$ implies $t_{12} = v_{12} = 0$, i.e. $T = V = 0$. Summarizing, $X = S + U$ with $S, U \in F_1^*$, $s_{11} = s_{12} = u_{11} = u_{12} = 0$, hence $Y = S$ and $Z = U$ are in H, as required.

Next we show that H is a minimal proper face: this comes from the easy-to-check fact that

$$F^{\perp} = \begin{pmatrix} 0 & 0 & \times \\ 0 & 0 & \times \\ \times & \times & \times \end{pmatrix}.$$

Finally, we prove that H is not exposed. Let

$$Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then clearly $Y \in \operatorname{ri} H$, and

$$F \cap Y^{\perp} = F \cap H^{\perp} = \begin{pmatrix} \oplus & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

However, any nonzero vector in $F \cap H^{\perp}$ exposes the larger face $H + \operatorname{cone} M$.

The following result shows that proving a weaker result suffices to prove Conjecture 1.

Theorem 4. *Suppose that*

$$K^* + F^{\perp} = F^* \tag{4.26}$$

holds whenever K is a closed, convex, facially exposed cone, and F is a maximal proper face of K. Then Conjecture 1 is true.

Proof   We show that if the assumption of the theorem is true, then (4.26) holds for all K closed, convex, facially exposed cones, and all F faces of K.

Let K be a closed, convex, facially exposed cone. We first prove that an F arbitrary face of K is facially exposed as a cone in its own right. Indeed, suppose that F is *not* facially exposed. Then there exist F_1 and F_2 faces of F with $F_1 \subsetneq F_2$, and $F^* \cap F_1^{\perp} = F^* \cap F_2^{\perp}$.

Intersecting both sides of this equation with K^* yields $K^* \cap F_1^\perp = K^* \cap F_2^\perp$. Since F_1 and F_2 are also faces of K, this means that K is not facially exposed, a contradiction.

Now, let F again be an arbitrary face of K. To show that (4.26) holds for an arbitrary F face of K, define the chain of faces

$$F_0 = K,\, F_1,\, \ldots, F_{k-1},\, F_k = F, \tag{4.27}$$

where F_i is a maximal proper face of F_{i-1} for $i = 1, \ldots, k$. Since all the F_i are facially exposed, by the assumption we get

$$
\begin{aligned}
F_k^* &= F_{k-1}^* + F_k^\perp, \\
F_{k-1}^* &= F_{k-2}^* + F_{k-1}^\perp, \\
&\vdots \\
F_1^* &= F_0^* + F_1^\perp,
\end{aligned}
$$

hence

$$
\begin{aligned}
F_k^* &= F_{k-1}^* + F_k^\perp \\
&= F_{k-2}^* + F_{k-1}^\perp + F_k^\perp \\
&\vdots \\
&= F_0^* + F_1^\perp + \ldots F_k^\perp \\
&= F_0^* + F_k^\perp,
\end{aligned}
$$

as required. $\qquad\square$

Using Proposition 3, for a K closed convex cone, an F face of K, and an H minimal proper face of F^* we can define a vector $x_{F,H}$ as the unique vector with norm 1, and satisfying

$$\operatorname{cone} x_{F,H} = H \cap \operatorname{lin} F. \tag{4.28}$$

Also, for an F face of K we denote the orthogonal projection operator onto $\operatorname{lin} F$ by M_F.

We first rephrase a condition in Theorem 2.

Proposition 8. *Let K be a closed, convex cone, $F \trianglelefteq K$, and H a minimal proper face of F^*. Then $K^* \cap H \supsetneq K^* \cap F^\perp$ iff $x_{F,H} \in M_F K^*$.*

Proof   We have the following chain of equivalences:

$$
\begin{aligned}
K^* \cap H &\supsetneq K^* \cap F^\perp &\Leftrightarrow \\
(K^* \cap H) \setminus F^\perp &\neq \emptyset &\Leftrightarrow \\
(K^* \cap (\operatorname{cone} x_{F,H} + F^\perp)) \setminus F^\perp &\neq \emptyset &\Leftrightarrow \\
\exists \lambda \geq 0,\, f \in F^\perp : \lambda x_{F,H} + f &\in K^* + F^\perp &\Leftrightarrow \\
\exists \lambda > 0,\, f \in F^\perp : \lambda x_{F,H} + f &\in K^* + F^\perp &\Leftrightarrow \\
\exists f \in F^\perp : \lambda x_{F,H} + f &\in K^* + F^\perp &\Leftrightarrow \\
\exists f \in F^\perp : \lambda x_{F,H} + f &\in K^* &\Leftrightarrow \\
x_{F,H} &\in M_F K^*.
\end{aligned}
\tag{4.29}
$$

Here the second equivalence comes from (4.28), the fourth and sixth from $x_{F,H} \in \operatorname{lin} F$, and the others are trivial.

Corollary 2. *Let K be a closed convex cone, $F \trianglelefteq K$. Then $K^* + F^\perp = F^*$ iff $x_{F,H} \in M_F K^*$ for all H minimal proper faces of F^*.*

$\square$

We now outline a possible avenue of proving Conjecture 1. Let K and F be as before, and fix an H minimal proper face of F^*. We have that $\operatorname{cone} x_{F,H}$ is an extreme ray of $F^* \cap \operatorname{lin} F$. Hence applying Straszewicz's theorem (Theorem 18.6 in [16]) for cones we get

$$x_{F,H} = \lim_i x_i, \tag{4.30}$$

for some $x_i \in F^* \cap \operatorname{lin} F$ with $\| x_i \| = 1$, and $\operatorname{cone} x_i$ an exposed ray of $F^* \cap \operatorname{lin} F$ for all i. We have $x_i = x_{F,H_i}$ for some H_i minimal proper faces of F^* for all i. Also, since $\operatorname{cone} x_i$ is exposed, so is H_i, i.e. $H_i = F^* \cap G_i^\perp$ for some G_i proper faces of F for all i.

Let us assume that K is facially exposed. Then for the above G_i faces we have

$$K^* \cap H_i = K^* \cap F^* \cap G_i^\perp = K^* \cap G_i^\perp \supsetneq K^* \cap F^\perp$$

for all i. Using the equivalence of Lemma 8, we have $x_{F,H_i} \in M_F K^*$ for all i. Summarizing, the facial exposedness of K with the closedness of $M_F K^*$ implies $K^* + F^\perp = F^*$.

Unfortunately, as shown in , Proposition 2.1 in [15] the closedness of $M_F K^*$ is actually equivalent to $K^* + F^\perp = F^*$. Still, it would be sufficient, and perhaps possible to prove that $M_F K^*$ is "locally" closed, i.e. when a sequence of vectors from extreme rays of $F^* \cap \operatorname{lin} F$ is in this set, so is their limit.

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