Optimization over the Efficient Set of a Bicriteria Convex Programming Problem *

Nguyen Thi Bach Kim[1] and Tran Ngoc Thang[2]
[1,2]*Faculty of Applied Mathematics and Informatics*
Hanoi University of Science and Technology
1 *Dai Co Viet, Hai Ba Trung, Hanoi, Vietnam.*
[1]kimntb-fami@mail.hut.edu.vn
[2]thangtn-fami@mail.hut.edu.vn

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Abstract

The problem of optimizing a real function over the efficient set of a multiple objective programming problem arises in a variety of applications. In this article, we propose an outer approximation algorithm for maximizing a function $h(x) = \varphi(f(x))$ over the efficient set X_E of the bi-criteria convex programming problem $\text{Vmin}\{f(x) = (f_1(x), f_2(x))^T | x \in X\}$, where φ is an increasing function on $f(X)$. The convergence of the algorithm is established. To illustrate the new algorithm, we apply it to the solution of the sample problem. Preliminary computational results with the proposed algorithm are reported.

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Key words. Global optimization, Optimization over the efficient set, Outcome set, Bicriteria convex programming, Outer approximation, Branch-and-reduce scheme

1 Introduction

We consider the bicriteria convex programming problem

$$\text{Vmin } f(x) \text{ s.t. } x \in X, \qquad (VP)$$

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where $X \subset \mathbb{R}^n$ is a nonempty compact convex set, $f(x) = (f_1(x), f_2(x))^T$ and for each $j = 1, 2$, the function $f_j : \mathbb{R}^n \to \mathbb{R}$ is finite, positive and convex on X.

A point x^* is said to be an *efficient solution* for Problem (VP) if there exists no point $x \in X$ such that $f(x^*) \geq f(x)$ and $f(x^*) \neq f(x)$. Here for any two vectors $a, b \in \mathbb{R}^2$, the notation $a \geq b$ mean $a - b \in \mathbb{R}^2_+ = \{y = (y_1, y_2)^T | y_1 \geq 0,\ y_2 \geq 0\}$. Let X_E denote the set of all efficient solutions for Problem (VP). Since X is a compact set, the efficient set X_E is not empty [10].

The central problem of interest in this paper is the problem of maximizing a real function over the efficient set X_E of Problem (VP). This problem, which we shall denote as Problem (P), is given by

$$\max h(x) \quad s.t. \quad x \in X_E, \tag{P}$$

where $h(x) = \varphi(f(x))$ with φ is an increasing function defined on $f(X) = \{y \in \mathbb{R}^2 \ |y = f(x) \text{ for some } x \in X\}$. As usual, the set $Y = f(X)$ is called the *outcome set* (or *image*) *of* X *under* f. By the definition, a function φ is increasing on $f(X)$ if for $y', y \in f(X)$, $y' \geq y$ and $y' \neq y$, we have $\varphi(y') > \varphi(y)$.

It is well known that X_E is, in general, a non-convex set, even in special case when X is a polyhedron and f_1, f_2 are linear function on $\mathbb{R}^n$ for each $i = 1, 2$. Hence, the problem of optimizing over the efficient set can be classified as a hard global optimization problem [13]. Because of its interesting mathematical aspects as well as its wide range of applications, this problem has attracted the attention of many authors (cf. [1], [2], [3], [5] [7] [8], [9], [11], [13] [14], [15] and references therein).

The outcome-space reformulation of Problem (P) is given by

$$\max \varphi(y) \quad s.t. \quad y \in f(X_E), \tag{OP}$$

where $f(X_E) = \{y \in \mathbb{R}^2 | y = f(x) \text{ for some } x \in X_E\}$.

Recall that for a given nonempty set $Q \subset \mathbb{R}^2$, a point q^* is said to be an *efficient point* of Q if there is no $q \in Q$ satisfying $q^* \geq q$ and $q^* \neq q$, i.e. $Q \cap (q^* - \mathbb{R}^2_+) = \{q^*\}$. Let Q_E be the set of all efficient points of Q. By definition, it can be verified that

$$Y_E = f(X_E) = \{y \in \mathbb{R}^2 | y = f(x) \text{ for some } x \in X_E\}. \tag{1}$$

Therefore, the set Y_E is also known as the *efficient outcome set* for Problem (VP). From definition, it is easily observed that if $y^* \in Y_E$ then any $x^* \in X$

such that $f(x^*) \leq y^*$ is an efficient solution to Problem (P), i.e. $x^* \in X_E$. Furthermore, if $y^* \in Y_E$ is a global optimal solution to Problem (OP) then any $x^* \in X$ such that $f(x^*) \leq y^*$ is a global optimal solution to Problem (P). For the sake of convenience, x^* is said to be *an efficient solution associated with the outcome efficient point* y^*.

In this paper, instead of solving Problem (OP), we construct an outcome-space outer approximation algorithm for solving globally a problem (OP_G) that is equivalent to Problem (OP). The algorithm is established based on the branch-and-reduce scheme that proposed in [4]. It is worth pointing out that when the algorithm terminates, we simultaneously get an optimal solution to Problem (OP) and an optimal solution to Problem (P). Since the number of variables n, in practice, is often much larger than 2, we expect potentially that considerable computational savings could be obtained.

The paper is organized as follows. In Section 2, theoretical prerequisites for the algorithm are given. The algorithm is presented in Section 3. Computational experiments are reported in Section 4.

2 Theoretical Prerequisites

We will assume henceforth that in Problem (VP), X is a nonempty compact, convex set given by

$$X := \{x \in \mathbb{R}^n | g_i(x) \leq 0, i = 1, 2, \ldots, m\},$$

where $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, m$, are the convex function. Now, define the set G by

$$G := Y + \mathbb{R}^2_+ = \{y \in \mathbb{R}^2 | y \geq f(x) \gg 0 \text{ for some } x \in X\}.$$

It is easy to show that $G \subset \text{int}\mathbb{R}^2_+$ is a nonempty, full-dimension closed convex set. The following fact is very useful in the sequel.

Proposition 2.1. (Theorem 3.2 in Yu (pg. 22 in [16]) *The efficient outcome set for Problem (VP) is equal to the set of all efficient points of G, i.e. $Y_E = G_E$.*

We invoke (1) and Proposition 2.1 to deduce that Problem (OP) is equivalent to the following problem

$$\max \varphi(y) \quad s.t. \quad y \in G_E. \qquad (OP_G)$$

Therefore, to globally solve Problem (OP), we instead globally solve Problem (OP_G).

In the next section, based on the structure of the efficient set G_E of the convex $G \subset \text{int}\mathbb{R}^2_+$, the outer approximation algorithm is developed for solving the problem (OP_G).

Now we present some more particular results that will be needed to develop the outer approximation algorithm. For each $i = 1, 2$, let

$$\alpha_i = \min\{y_i \ : \ y \in G\}. \qquad (SP_i)$$

Note that α_i is also the optimal value of the convex programming problem $\min\{f_i(x) \ : \ x \in X\}$, $i = 1, 2$. Since $G \subset \mathbb{R}^2$, the problem

$$\min\{y_2 \ : \ y \in G, \ y_1 = \alpha_1\}$$

has an unique optimal solution $\hat{y}^1$ and the problem

$$\min\{y_1 \ : \ y \in G, \ y_2 = \alpha_2\}$$

has an unique optimal solution $\hat{y}^2$. These solutions $\hat{y}^1, \hat{y}^2$ belong to G_E.

By definition, for each $i = 1, 2$, if $(\widetilde{x}^i, \widetilde{y}^i) \in \mathbb{R}^{n+p}$ is an optimal solution for the problem (P_i) given by

$$\begin{array}{ll} \min & y_k \\ \text{s.t.} & f_j(x) - y_j \leq 0, \quad j = 1, 2 \\ & g_i(x) \qquad \leq 0, \quad i = 1, \ldots, m \\ & y_i = \alpha_i \quad i \in \{1, 2\} \setminus \{k\} \end{array} \qquad (P_i)$$

then we have $\hat{y}^i = \widetilde{y}^i$ and the efficient solutions $\hat{x}^i = \widetilde{x}^i$ associated with the outcome efficient points $\hat{y}^i$.

Since $G \subset \mathbb{R}^2$ is a closed convex set, it is well known (see [10], [12]) that the efficient set G_E is homeomorphic to a nonempty closed interval of $\mathbb{R}^1$. If $\hat{y}^1 \equiv \hat{y}^2$, we have $G_E = \{\hat{y}^1\}$ and $\hat{y}^1$ is an unique optimal solution to problem (OP_G). Therefore, we assume henceforth that $\hat{y}^1 \neq \hat{y}^2$. Then, the efficient set $G_E \subset \partial G$ is a curve with starting point $\hat{y}^1$ and end point $\hat{y}^2$, where ∂G is the boundary of the set G (see Figure 1).

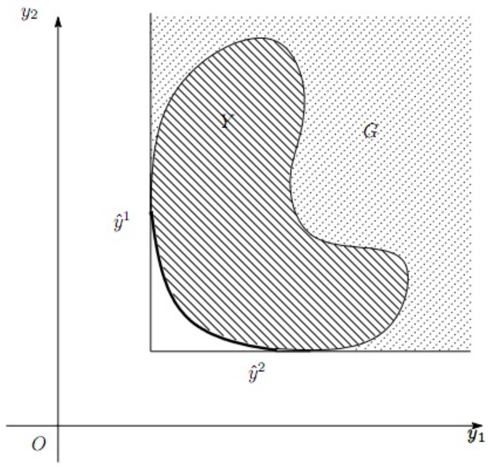

Figure 1

Let y^L and y^R be arbitrary points in G_E such that

$$y_1^L < y_1^R \quad \text{and} \quad y_2^R < y_2^L. \tag{2}$$

Denote by Γ the unique curve lying in G_E and connecting y^L and y^R. Let $y^O = (y_1^O, y_2^O)$ where

$$y_1^O = y_1^L > 0, \quad y_2^O = y_2^R > 0. \tag{3}$$

The points y^L, y^O, y^R can not belong to the same line. Therefore, the convex hull $\text{conv}\{y^L, y^O, y^R\}$ is a 2-simplex contained in the cone $(y^O + \mathbb{R}^2_+)$. We denote this simplex by S. Obviously, S contains the above efficient curve Γ,

$$\Gamma \subset S.$$

By definition, it can easily be seen that the ray starting at the origin 0 of the outcome space $\mathbb{R}^2$ and passing through the vertex y^O of the simplex intersects the boundary ∂G of the set G at an unique point $y^\omega \in \Gamma \subseteq G_E$ (see an illustration in Figure 2). Namely,

Proposition 2.2. *Let $y^L, y^R \in G_E$ which satisfy (2). Consider the 2-simplex $S = \text{conv}\{y^L, y^O, y^R\}$, where y^O determined by (3). Then, the ray starting at the origin O of the outcome space $\mathbb{R}^2$ and passing through the vertex y^O of the simplex intersects the boundary ∂G of the set G at an unique point $y^\omega \in G_E$. Furthermore,*

$$y_1^L < y_1^\omega < y_1^R \quad \textit{and} \quad y_2^R < y_2^\omega < y_2^L. \tag{4}$$

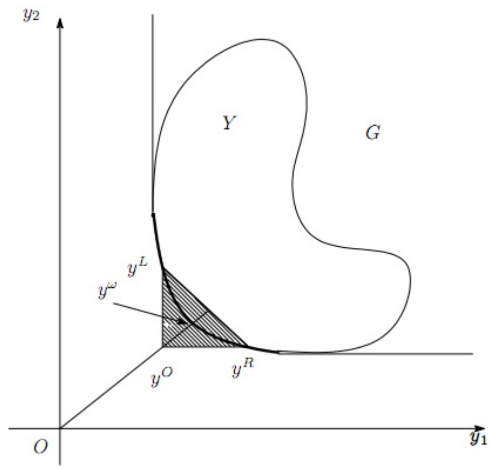

Figure 2

Remark 2.1. To determine this efficient outcome point y^ω described in Proposition 2.2, we have to find the unique value λ^* of $\lambda \geq 1$, such that

$$y^\omega = \lambda^* y^O \tag{5}$$

belongs to the boundary of G. It means that λ^* is the optimal value for the problem

$$\begin{aligned} &\min \ \lambda \\ &\text{s.t.} \ \ \lambda y^O \in G, \\ &\qquad \lambda \geq 1, \end{aligned}$$

i.e., λ^* is the optimal value for the following convex programming problem with linear objective function

$$\begin{aligned} &\min \ \lambda \\ &\text{s.t.} \ \ f(x) - \lambda y^O \leq 0, \\ &\qquad g_i(x) \leq 0, \quad i = 1, \cdots, m, \\ &\qquad \lambda \geq 1. \end{aligned} \tag{$T(y^O)$}$$

Let (x^*, λ^*) be an optimal solution to Problem $(T(y^O))$. Then we get x^* is an efficient solution associated with the outcome efficient point y^ω.

Definition. A 2-simplex S is said to be *a simplex generated by two points* y^L *and* y^R if $S = \text{conv}\{y^L, y^O, y^R\}$, where y^L, y^R satisfy (2) and y^O is determined by (3).

Remark 2.2. Assume that two points $y^L, y^R \in G_E$ satisfy (2) and $\Gamma \subseteq G_E$ is the efficient curve connecting y^L with y^R. Consider the 2-simplex S generated by y^L and y^R. Let y^ω be a point determined as Proposition 2.2. From (4), let S^1 be a simplex generated by y^L and $y^R = y^\omega$ and S^2 be a simplex generated by $y^L = y^\omega$ and y^R (see Figure 3). It is easy to see that

$$\Gamma \subset (S^1 \cup S^2) \subset S.$$

We refer to y^ω as *reduced bisection point* for the simplex S. It means that *the branch-and-reduce scheme is applied to* S. This interesting property will be used to construct the algorithm solving Problem (OP_G).

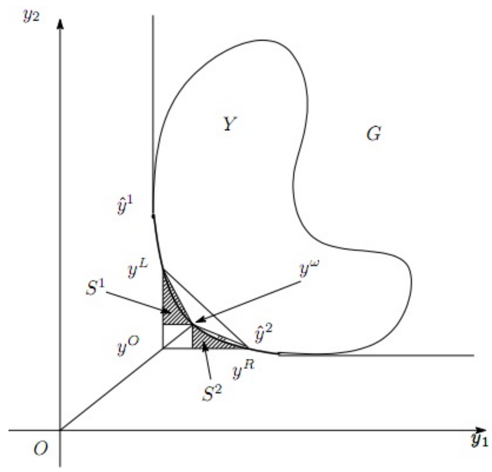

Figure 3

It is clearly that (2) is satisfied with the two point $y^L = \hat{y}^1$ and $y^R = \hat{y}^2$. Let S^0 be the 2-simplex generated by two points $\hat{y}^1$ and $\hat{y}^2$. We have

$$G_E \subset S^0.$$

3 The Outer Approximation Algorithm

Starting with the 2-simplex S^0, the algorithm will iteratively generate a two sequence $\{T^k = \{S^{k,1}, S^{k,2}, \ldots, S^{k,\eta_k}\}\}$ of the 2− simplices, where η_k is the number of elements of T^k, and $\{U^k = \bigcup_{S \in T^k} S\}$ satisfies

$$U^0 \supset U^1 \supset \cdots \supset U^k \supset U^{k+1} \supset \cdots \supset G_E.$$

Let $S^{0,1} = S^0$. At Step 0 of the algorithm for solving Problem (OP_G), set $T^0 := \{S^{0,1}\}$ and $U^0 := S^{0,1}$. Since $\hat{y}^1, \hat{y}^2 \in G_E$, let $\alpha_0 = \max\{\varphi(\hat{y}^1), \varphi(\hat{y}^2)\}$. Then α_0 is the *currently best lower bound* of Problem (OP_G) and the point $y^{best} \in \{\hat{y}^1, \hat{y}^2\}$ such that $\varphi(y^{best}) = \alpha_0$ is the *currently best feasible point.*

Let ε be a given sufficient small real number. A point $y^* \in G_E$ is said to be an $\varepsilon-$*optimal solution* to problem (OP_G) if there is a upper bound β_* for this problem such that $\beta_* - h(y^*) < \varepsilon$. Then, the efficient solution x^* associated with y^* is *an approximation optimal solution* to Problem (P).

At the beginning of a typical iteration $k \geq 0$, we have available from the previous iteration the lower bound α_k, the feasible point y^{best}, the efficient solution x^{best} associated with y^{best}, the set T^k of the $2-$ simplices, and U^k satisfies $G_E \subset U^k$. For each $\eta \in \{1, 2, \ldots, \eta_k\}$, the simplex $S^{k,\eta}$ is generated by two known points. In iteration k, the algorithm carries out:

i) Find the optimal value β_k of the problem $\max\{\varphi(y) | y \in U^k\}$; Since $G_E \subset U^k$, the optimal value β_k is an upper bound of Problem (OP_G);

ii) If $\beta_k - \alpha_k \leq \varepsilon$ then the algorithm terminates. As a result, y^{best} is an $\varepsilon-$ optimal solution to problem (OP_G) and x^{best} is an approximation optimal solution to Problem (P). Otherwise, the algorithm yields a new set T^{k+1} of $2-$simplices whose union U^{k+1} satisfies $U^k \supset U^{k+1} \supset G_E$.

For each $S \in T^k$, denote by $\beta(S)$ the optimal value of problem $\max\{\varphi(y) | y \in S\}$. Since $U^k = \bigcup_{S \in T^k} S$, we have $\beta_k = \max\{\beta(S) | S \in T^k\}$.

In the step k, if the algorithm is not terminated, we choose the simplex $S = S^{k,\bar{\eta}} \in T^k$ satisfying $\beta_k = \beta(S^{k,\bar{\eta}})$ and S generated by two known points y^L and y^R. Let y^ω be the point that is determined in Proposition 2.2. Then the branch-and-reduce scheme is applied to S with reduced bisection point y^ω to obtain two simplices S^1 and S^2 as Remark 2.2. Let

$$T^{k+1} = \left(T^k \setminus \{S\}\right) \cup \{S^1, S^2\} \text{ and } U^{k+1} = \bigcup_{S \in T^{k+1}} S.$$

Then, from Remark 2.2 we have $G_E \subset U^{k+1} \subset U^k$.

Now, consider the following problem

$$\max \varphi(y) \text{ s.t. } y \in S, \qquad (P(S))$$

where $\varphi(y)$ is an increasing monotone on $\mathbb{R}^2_+$ and S is $2-$simplex generated by two points y^L, y^R.

Proposition 3.1. *Any global optimal solution to Problem* $(P(S))$ *must belong to the edge* $[y^L, y^R]$ *of the simplex* S.

Proof. Let y^* be a a global optimal solution to Problem $(P(S))$. Assume the contrary that $y^* \notin [y^L, y^R]$. By the definition of the simplex S, there is $\hat{y} \in [y^L, y^R]$ such that $y^* \leq \hat{y}$ and $y^* \neq \hat{y}$. Since $\varphi(y)$ is an increasing monotone, we have $\varphi(y^*) < \varphi(\hat{y})$. This contradicts the hypothesis that $y^* \in \text{Argmax}\{\varphi(y)|y \in S\}$ and the proof is complete. ■

Now, direct computation shows that the equation of the line through y^L and y^R is $\langle d, y\rangle = \alpha$, where

$$\alpha = \frac{y_1^L}{y_1^R - y_1^L} + \frac{y_2^L}{y_2^L - y_2^R}.$$

and the normal vector d defined by

$$d = \Big(\frac{1}{y_1^R - y_1^L}, \frac{1}{y_2^L - y_2^R}\Big).$$

It is clear that the simplex S generated by two point y^L and y^R is also given by the solution set of the system

$$\begin{cases} \langle d, y\rangle \leq \alpha \\ y_1 \geq y_1^L, y_2 \geq y_2^R. \end{cases}$$

By Proposition 3.1, Problem $(P(S)$ has an optimal solution lying on the edge $[y^L, y^R]$ of the simplex S. This edge is the solution set of the linear system

$$\begin{cases} \langle d, y\rangle = \alpha & (a) \\ y_1 \geq y_1^L, y_2 \geq y_2^R. & (b) \end{cases} \tag{6}$$

From $(6.a)$, we have

$$y_2 = \frac{\alpha}{d_2} - \frac{d_1}{d_2} y_1.$$

Therefore, Problem $(P(S))$ can be reformulated as a problem for maximizing of single-variable function on a closed line segment $(P^1(S))$

$$\max\{\theta(y_1)|y_1^L \leq y_1 \leq y_1^R\}, \tag{$P^1(S)$}$$

where

$$\theta(y_1) = \varphi\left(y_1, \frac{\alpha}{d_2} - \frac{d_1}{d_2} y_1\right).$$

Remark 3.1. By above argument, to solve Problem ($P(S)$), we instead solve the problem for maximizing of single-variable function ($P^1(S)$). Then, if y_1^{opt} is an optimal solution of Problem ($P^1(S)$) then $y^{opt} = (y_1^{opt}, y_2^{opt})$ is an optimal solution of Problem ($P(S)$), where

$$y_2^{opt} = \frac{\alpha}{d_2} - \frac{d_1}{d_2} y_1^{opt}.$$

The algorithm for solving Problem (OP_G) can be described as follows.

Outer Approximation Algorithm

Initialization Step.

Determine the optimal value α_i of Problem (SP_i), $i = 1, 2$.

Solve two problems (P_1) and (P_2) to obtain two efficient outcome points $\hat{y}^1, \hat{y}^2 \in G_E$ and two efficient solutions $\hat{x}^1, \hat{x}^2$ associated with $\hat{y}^1, \hat{y}^2$, respectively.

If $\hat{y}^1 \equiv \hat{y}^2$ **Then** STOP: $\hat{y}^1$ *is the optimal solution of Problem* (OP_G) *and* $\hat{x}^1$ *is the optimal solution of Problem* (P).

If $\varphi(\hat{y}^1) < \varphi(\hat{y}^2)$ **Then** $\alpha_0 = \varphi(\hat{y}^1)$ and $y^{best} = \hat{y}^1$, $x^{best} = \hat{x}^1$;
Else $\alpha_0 = \varphi(\hat{y}^2)$ and $y^{best} = \hat{y}^2$, $x^{best} = \hat{x}^2$;

(α_0 is *the best currently lower bound*, y^{best} is *the best currently feasible solution*, x^{best} is *an efficient solution associated with* y^{best}.)

Let $y^L = \hat{y}^1$ and $y^R = \hat{y}^2$. Let $S^{0,1} = S^0$, where S^0 be the 2- simplex generated by two points y^L and y^R. Let $T^0 := \{S^{0,1}\}$ and $U^0 := S^{0,1}$.

Find the optimal value $\beta(S^0)$ of Problem $P(S^0)$;

Iteration Step $k, k = 0, 1, 2, ...$ See Step $k1$ through $k3$ below.
Step $k1$.

Find $S^{k,\bar{\eta}} \in T^k$, where $S = S^{k,\bar{\eta}}$ is the simplex generated by two known points y^L and y^R, such that

$$\beta(S^{k,\bar{\eta}}) := \max\{\beta(S') \mid S' \in T^k\}.$$

Let $\beta_k := \beta(S^{k,\bar{\eta}})$ (*the best currently upper bound*)

If $\beta_k - \alpha_k \leq \varepsilon$ **Then** STOP (y^{best} is *the ε-optimal solution to Problem (OP_G) and x^{best} is approximation optimal solution of Problem (P)).*

Else Let $S^k = S^{k,\bar{\eta}}$ and go to Step $k2$.

Step $k2$.

Determine the vertex y^O of the simplex S^k by (3), where y^L and y^R are two points that generate S^k.

Solve Problem $(T(y^O))$ to find the optimal solution $(\lambda^*, x^*)^T$.

Let $y^{\omega_k} = \lambda^* y^O \in G_E$ (*the reduced bisection point for the simplex S^k*) and we have x^* is an efficient solution associated with y^{ω_k}.

If $\varphi(y^{\omega_k}) > \alpha_k$ **Then**

$\alpha_{k+1} = \varphi(y^{\omega_k})$ (*the best currently lower bound*)

$y^{best} = y^{\omega_k}$ (*the best currently feasible solution*)

$x^{best} = x^*$; (*the efficient solution associated with y^{best}*)

Else go to Step $k3$.

Step $k3$. (*Branching*)

Let S_1^k be the 2-simplex generated by two points y^L and $y^R = y^{\omega_k}$, and S_2^k be the 2-simplex generated by two points $y^L = y^{\omega_k}$ and y^R.

For each i, $i = 1, 2$, find the optimal value $\beta(S_i^k)$ of Problem $(P(S_i^k))$

Let $T^{k+1} = \big(T^k \setminus \{S^k\}\big) \cup \{S_1^k, S_2^k\}$.

Set $k := k + 1$ and go to Iteration Step k.

To prove the convergence of the algorithm we need the following lemma that can be considered as a special case of Lemma 4.2 in [4].

Lemma 3.1. *Assume that the algorithm is infinite and that $\{S^k\}$ is a sequence of 2−simplices generated by the algorithm where, for each k the branch-and-reduce scheme is applied to S^k. Then $\{S^k\}$ has a subsequence $\{S^{k_q}\}$ such that* $\lim_q S^{k_q} = y^* \in G_E$.

Theorem 3.1. *If the algorithm does not terminate, then the sequence $\{y^{\omega_k}\}$ has a cluster point that solves Problem (OP_G) globally.*

Proof. Let f_* denote the optimal value for Problem (OP_G). If the algorithm does not terminate, then it generates two infinite sequence $\{S^k\}$ of $2-$simplices, and $\{y^{opt_k} \in S^k\}$ of the optimal solutions to Problem $P(S^k)$, where for each k the branch-and-reduce scheme is applied to S^k. Furthermore, by Lemma 3.1, by taking subsequences if necessary we may assume that $\lim_k S^k = y^* \in G_E$. It implies that $y^{opt_k} \to y^*$. Since the sequence of upper bound $\{\beta_k\}$ is monotone and $\beta_k = \beta(S^k) = \varphi(y^{opt_k})$, we obtain in the limit that

$$\lim_k \beta_k = \lim_k \varphi(y^{opt_k}) = \varphi(y^*) \geq f_*.$$

Since $y^* \in G_E$ is feasible for Problem (OP_G)), we deduce that y^* is a global optimal solution to Problem (OP_G). ■

4 Computational Experiments

Consider Problem (P) with

$$h(x) = (x_1 + x_2 - 0.4)(x_1 + 4x_2 + 0.2),$$

and X_E is the efficient solution set for the Problem (VP), where

$$\begin{aligned} f_1(x) &= x_1 + x_2, \\ f_2(x) &= x_1 + 4x_2 + 1, \\ g_1(x) &= (x_1 - 1)^2 + 4x_2^2 - 0.2, \\ g_2(x) &= 3x_1 - 8x_2 - 6. \end{aligned}$$

It is easily to see that $\varphi(y) = (y_1 - 0.4)(y_2 - 0.8)$. Now we solve Problem (OP) to find an $\varepsilon-$optimal solution with $\varepsilon = 0.0001$.

Initalization. Solving two convex problem (SP_1) and (SP_2) we obtain two optimal values $\alpha_1 = 0.500000, \alpha_2 = 1.000000$.
Solve Problem (P_1) and (P_2) obtaining the optimal solutions

$$\begin{aligned} (\hat{x}^1, \hat{y}^1) &= (0.599782, -0.099782, 0.500000, 1.998910), \\ (\hat{x}^2, \hat{y}^2) &= (0.799562, 0.199890, 0.999452, 1.000000), \end{aligned}$$

where

$$\hat{x}^1 = (0.599782, -0.099782), \hat{x}^2 = (0.799562, 0.199890),$$

are two efficient solutions associated with two efficient outcome points, respectively,

$$\hat{y}^1 = (0.500000, 1.998910), \hat{y}^2 = (0.999452, 1.000000).$$

Since $\varphi(\hat{y}^1) = 0.119891 > \varphi(\hat{y}^2) = 0.119890$, we take $\alpha_0 = \varphi(\hat{y}^2) = 0.119890$ and $y^{best} = \hat{y}^2, x^{best} = \hat{x}^2$.
Let $y^L = \hat{y}^1$, $y^R = \hat{y}^2$ and $S^{0,1} := S^0$ where S^0 generated by y^L and y^R. Let $T^0 := \{S^{0,1}\}$ and $U^0 := S^{0,1}$.
Solve Problem $P(S^0)$, we have the optimal value $\beta(S^0) = 0.244618$.

Iteration $k = 0$

Step 0.1. Since $T^0 = \{S^{0,1}\}$, $\beta_0 := \beta(S^{0,1}) = 0.244618$.
Since $\beta_0 - \alpha_0 = 0.124728 > \varepsilon = 0.0001$, set $S^0 = S^{0,1}$ and go to Step 0.2.

Step 0.2. From (3), determine the vertex $y^O = (y_1^L, y_2^R) = (0.500000, 1.000000)$.
Solving Problem $(T(y^O))$, we obtain the optimal solution

$$(\lambda^*, x^*) = (1.292892, 0.575735, 0.070711).$$

Let $y^{\omega_0} = \lambda^* y^O = (0.646446, 1.292892)$.
Since $\varphi(y^{\omega_0}) = 0.121471 > \alpha_0 = 0.119890$ then $\alpha_1 = 0.121471$ and $y^{best} = y^{\omega_0} = (0.646446, 1.292892)$, $x^{best} = x^* = (0.575735, 0.070711)$.

Step 0.3. Let S_1^0 be the 2-simplex generated by two points y^L and $y^R = y^{\omega_0}$, and S_2^0 be the 2-simplex generated by two points $y^L = y^{\omega_0}$ and y^R.
Solve $P(S_1^0)$ and $P(S_2^0)$ recieving the optimal values respectively

$$\beta(S_1^0) = 0.146536 \text{ and } \beta(S_2^0) = 0.146536.$$

Let $T^1 = \{S_1^0, S_2^0\} = \{S^{1,1}, S^{1,2}\}$, where $S^{1,1} = S_1^0$ and $S^{1,2} = S_2^0$.

Interation $k = 1$

Step 1.1. Since $\beta(S^{1,1}) = \max\{\beta(S') \mid S' \in T^1\}$, set $\beta_1 = \beta(S^{1,1}) = 0.146536$, where $S^{1,1}$ is generated by two points

$$y^L = (0.500000, 1.998910) \text{ and } y^R = (0.646446, 1.292892).$$

Since $\beta_0 - \alpha_0 = 0.025065 > \varepsilon = 0.0001$, set $S^1 = S^{1,1}$ and go to Step 1.2.

Step 1.2. From (3), determine the vertex $y^O = (y_1^L, y_2^R) = (0.500000, 1.292892)$. Solving Problem $(T(y^O))$, we obtain the optimal solution

$$(\lambda^*, x^*) = (1.145344, 0.554299, 0.018373).$$

Let $y^{\omega_1} = \lambda^* y^O = (0.572672, 1.480806)$.
Since $\varphi(y^{\omega_1}) = 0.117556 < \alpha_1 = 0.121471$ then the lower bound is not updated.

Step 1.3. Let S_1^1 be the 2-simplex generated by two points y^L and $y^R = y^{\omega_1}$, and S_2^1 be the 2-simplex generated by two points $y^L = y^{\omega_1}$ and y^R. Solve $P(S_1^1)$ and $P(S_2^1)$ recieving the optimal values respectively

$$\beta(S_1^1) = 0.128172 \text{ and } \beta(S_2^1) = 0.123256.$$

Let $T^2 = \left(T^1 \setminus \{S^{1,1}\}\right) \cup \{S_1^1, S_2^1\} = \{S^{1,2}, S_1^1, S_2^1\} = \{S^{2,1}, S^{2,2}, S^{2,3}\}$.

After 7 iterations, since $\beta_7 - \alpha_7 = 0.000089 < \varepsilon = 0.0001$, the algorithm terminates in step 7 with global $\varepsilon-$optimal solutions given by $y^{best} = (0.646446, 1.292892)$ and corresponding $x^{best} = (0.575735, 0.070711)$. We also obtain the optimal value of Problem (P) $\varphi(y^{best}) = 0.1214710$.

A set of randomly generated problems was used to test the above algorithm. The test was perform on a laptop HP Pavilion 1.8GHz, RAM 2G, using codes written in Matlab. We will test Problem (P) are given as the following type

$$\begin{aligned} &\max \varphi(f(x)) = (f_1(x) - \beta_1)(f_2(x) - \beta_2) \\ &\quad \text{s.t.}\ x \in X_E, \end{aligned}$$

with $\beta_i = \min\{f_i(x) \mid x \in X\}, i = 1, 2$ and X_E is the efficient solution set of the problem (VP), where

$$f_i(x) = \alpha^i x + x^T D^i x, \quad i = 1, 2,$$

$$X = \{x \in \mathbb{R}^n \mid \left(-2 + \sum_{j=1}^n \frac{x_j}{j}\right)^2 \le 100, Ax \le b, x \ge 0\}$$

and the parameters was defined as follows (as in [6])

- $\alpha^1, \alpha^2 \in \mathbb{R}^n$ are randomly generated vectors with all components belonging to $[0, 1]$.

- $A = (a_{ij}) \in \mathbb{R}^{m\times n}$ is a randomly generated matrix with elements belonging to $[-1, 1]$.
- $b = (b_1, b_2, ..., b_m)^T$ is randomly generated vector such that

$$b_i = \sum_{j=1}^{n} a_{ij} + 2b_0,$$

  with b_0 being a randomly generated real in $[0, 1]$
- $D^i \in \mathbb{R}^{n\times n}$ are diagonal matrices with diagonal elements d^i_j randomly generated in $[0, 1]$.

In all problems we find an ε-solution where $\varepsilon = 0.005$. The following table present the results of computational experiment. In the table, UB is upper bound, LB is lower bound, gap is defined as $\frac{UB-LB}{UB}$.

n	m	♯Inter	UB	LB	Gap
60	40	7	4.103116	4.096561	0.001598
70	50	8	1.865636	1.860822	0.002580
80	80	8	1.137924	1.136717	0.001061
100	60	8	5.153709	5.138920	0.002870
100	80	7	4.169557	4.166498	0.000731
120	120	5	15.94133	15.94088	0.000029
150	100	8	55.40159	55.29604	0.001905
150	120	6	7.587685	7.559456	0.003720

Table 1

From Table 1 we can see that, even in large scale setting, our algorithm works well. The computation time is small since the algorithm terminates after few iterations. Moreover, the quality of final solution obtained is much smaller than 0.005.

5 Conclusion

In the paper, we present an outcome space outer approximation algorithm for globally solving the problem $\max\{h(x) \mid x \in X_E\}$, where X_E is the efficient solution set to the bicriteria convex problem (VP) and $h(x) = \varphi(f(x))$ with φ is an increasing function on the outcome set $Y = f(X)$. In every step of the algorithm, the branch-and-reduce scheme [4] is used. We hope that the algorithm helps to reduce considerably the size of the problem when the number of decision variables n much larger than 2.

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