Informatively optimal levels of confidence for measurement uncertainty

D. Kisets
National Physical Laboratory, Jerusalem, Israel

Abstract

The conception of dimensional perfection and based on principles of qualimetry and information theory the criterion of informational optimality have been used for analyzing modeling functions of measurement. By means of variances of uncertainty contributions, transformed into their relative weights, the possibility of determining informatively rational and optimal levels of confidence for expanded uncertainty has been demonstrated. Commonly used 95% level of confidence has been analyzed for establishing an extent of corresponding to the proposed new approach and proven as being optimally redundant. The author believes that this paper may be of interest and practical value for professionals engaged in improving metrological estimations.

Keywords: Information optimality; Perfect numerical ratios; Optimal redundancy; Measurement uncertainty; Level of confidence.

1. The problem

One of existing problems of applied metrology is the choosing of adequate level of confidence (LC) for expanded measurement uncertainty. The commonly used 95% level of confidence is well recognized in the Statistical community and then adopted in metrology for evaluating the expanded uncertainty of measurement as a historical artifact, and not as a strictly substantiated value. The increasing of LC (e.g. 99%) or its decreasing (e.g. 90% or even 68%) one may also meet in practice.

In some cases a practical task can predefine a degree of adequacy in choosing certain LC as, say, when are being dealt with so-called risks analysis. There is also an attempt of suggesting this approach in combination with an economic optimization for determining the uncertainty being declared by calibration laboratories in regard to the so-called calibration and measurement capability (CMC) [1]. However, for many standard measurement methods, measurement standards and measuring instruments the optimization reducing to such approaches most commonly is non-adequate. This is owing to the universalism in applying of above metrological objects demanding strict accuracy classification. The approaches can likely be useful as an additional tool if they are not violating main classification criteria of optimization.

Even when possess application universality, LC apparently should be informatively well-grounded or, preferably, informatively optimal as an intrinsic characteristics of measurement method used in calibration or testing. In particular, it follows that most commonly used 95% LC (or any others) should be subjected to informational analysis. Taking into account the natural evolutionary trend to optimization in human activity, one can obviously expect a comparative closeness of the scientifically proven optimal criteria on the one hand and the criteria that have been formed and commonly used as the result of persistent metrology practice (as 95% confidence) on the other hand. The degree of such closeness is of interest.

2. Conceptions and criteria

Although measurement uncertainty is of random nature, it is not a statistical value, because along with a contribution due to repeated measurements (Type A) its evaluation involves a number of non-statistical components. Thus, while using probabilistic calculation technique, the combined and expanded uncertainty should reasonably be explained on qualimetrical basis rather than on statistical. Importantly also that the common probabilistic idea of uncertainties enables to apply elements of information theory in formulating and substantiating needed quantitative criteria.

Beyond question a selection of informative uncertainty contributions and an estimation of LC relate to the quality of measurement result and its informational sufficiency. Then principles and criteria of doing this, borrowed from qualimetry [2] and information theory or developed on their base, imply that:

(a) the analysis of a modeling function of measurement involves the identification of uncertainty contributions (from their theoretically infinite variety) according to their influence upon the combined uncertainty which, in turn, characterizes the quality (in terms of accuracy) of measurement;

(b) the quality of measurement associated with measurement accuracy ought to be determined strictly in terms of classification, i.e. each method and procedure of measurement belong to certain accuracy grade, or class (even when such a gradation is not yet documentarily specified or somehow indicated);

(c) each grade, or class of accuracy is bounded regarding the composition of contributions of uncertainty to be taken into account, so that the informational sufficiency of each classification group is characterized by the adequate relation between minimum and maximum contribution. The providing of information sufficiency in such a way is the only logically justified possibility to classify measurement accuracy for traceability chains, for measurement standards and measuring instruments, and for many other purposes [3].

A standard modeling function of measurement is being analogous to a function of modeling a quantitative estimation of quality. The output estimate y of measurand as a function of input estimates $x_1, x_2, ..., x_N$ for N quantities is given by $y = f\,(x_1, x_2, ..., x_N)$. Any integral estimate that represents a function (or a set) of some N contributions, such as a modeling function of measurement, can be subjected to qualimetrical analysis. The analysis is always based on determining influences (weights) of contributions on the certain quality of the object undergoing consideration.

Through weights (K_j) each "j" contribution may be characterized by specific relative index ρ_j so that $1/\rho_j$ indicates to what extent the weight related to this contribution exceeds the weight related to the contribution possessing 50% confidence to be informatively redundant:

$$\rho_j = K_{\varphi o} / K_j, \qquad (1)$$

where $K_{\varphi o}$ is the weight of the lesser by value contribution amongst informatively optimal (necessary and sufficient) number φ_o of contributions ($\varphi_o \leq N$).

In relation to the maximal weight (K_{max}) the theoretically stated index, symbolized as ρ_o, may be called *informatively optimal classification ratio*:

$$\rho_o = K_{\varphi o} / K_{max} = 1/2\pi \approx 0.159 \quad (2)$$

This ratio may be qualified as the fundamental informational constant, tightly bound with the mathematical constant π. Its substantiation is performed through the use of information entropy (H) regarding two boundary components with weights $K_{max}(\rho)$ and $K_{\varphi o}(\rho)$, provided they form the complete system, i.e. $K_{max}(\rho) + K_{\varphi o}(\rho) = 1$, as the solution of the following equations system:

$$\begin{cases} \rho_o = \arg[\varphi_o(\rho) = 1.5]; & (3) \\ \varphi_o = \exp(H) = \exp[-K_{max}(\rho)\ln K_{max}(\rho) - K_{\varphi o}(\rho)\ln K_{\varphi o}(\rho)]; & (4) \\ K_{max}(\rho) = 1/(1+\rho); & (5) \\ K_{\varphi o}(\rho) = \rho/(1+\rho), & (6) \end{cases}$$

where $\varphi_o(\rho) = 1.5$ is true for the most uncertain classification situation (50% confidence) about allowing or ignoring the lesser component [4];

$K_{max}(\rho)$, and $K_{\varphi o}(\rho)$ are considered as analogs of probabilities enabling the entropy usage.

Another way of substantiating φ_o, $K_{\varphi o}$ and ρ_o is the analysis of poly-component system that in general form is briefly dealt with in Appendix 1.

Indexes ρ_j and ρ_o have been demonstrated as being kernel [5] in solving measurement accuracy problems including LC.

Perfect numerical ratios

Since measurement is the process of assigning a number to a physical property, the identification of the properties' relations with some numerical ratios that constitute the system possessing conceptual formal-informational features is worthwhile. These features we can define as those related to conceptions of *dimensional perfection*.

Informational *optimality* expressed (2) by ρ_o is one of three perfect numerical ratios (PNR) related to conceptions of dimensional perfection which will be used as methodological instruments in the present study. Another two conceptions are: mathematical *harmony* $f_o = [(\sqrt{5} - 1)/2]^{-1} \approx 0.618$ (harmonious relation) – fundamental constant, otherwise known as *phi*, and *balance* $\lambda = 0.5$. By PNR we will imply the arithmetic ratio either of two parts of a whole or (depending on an objective) of one of parts to their sum being related to certain quality.

Virtually PNRs represent objectively existing numerical ratios as preferable in the evolution of nature and human practice; among them the commonly known is the harmony ratio. Provided that systemic connection between PNRs exists, they may be and will be jointly used as one more tool in solving declared problems.

Any PNR manifestations in nature and human practice always oscillate between ideal and approximations, i.e. are being characterized by deviations from a classification constant C_{inf} (i.e. ρ_o, f_o, *or* λ) that in many cases make their accepting problematic. The problem is successfully solved through the conception of informational optimality itself. The deviation equal to $\pm 0.5\rho_o$ =

±1/4π multiplied on C_{inf} can reasonably be attributed to the constant. This permissible deviation amounts to approximately ±8% of C_{inf} that is rigorously substantiated in Appendix 2.

While the harmony (f_o) is the widely known conception, until now manifestations of balance (λ) practically are not being draw attention. At the same time, both conceptions are fundamentally associated via Fibonacci numbers ratios, which start with pure balance, and very quickly approach the harmony. This feature is especially noteworthy, some details of which are reflected in the discussion section too. Interestingly, the next after λ = 0.5 ratio of adjacent numbers in Fibonacci succession, which among other ratios is being characterized by maximum deviation from f_o, equals 0.667 that meets the boundary permissible requirement, i.e. f_o (1 + 1/4π). Besides, in terms of C_{inf} permissible deviations there is the deep informatively optimal connection via ρ_o between the existing Fibonacci ratios, i.e. $r_\lambda = \lambda\ (1 \pm 1/4\pi)$ and $r_{fo} = f_o\ (1 \pm 1/4\pi)$. The proof consists in determining ratios r_λ = 0.52 and r_{fo} =0.614 as the solution of following equation: $(f_o - r_\lambda)/f_o = (r_{fo} - \lambda)/r_{fo} = \rho_o$.

Each PNR bears upon simplest case of dividing a whole onto two complementary parts or components that in terms of classification is called *classification dichotomy*. The question, whether information optimality, mathematical harmony and balance represent the complete set of dimensional perfection characteristics implies two approaches: firstly the informational one (informational non-redundancy) and secondly the geometrical one (completeness in various coordinates measures). The dimensional separability of the characteristics is their quality that allows reliably distinguishing between them, and also needs verification. These problems of systemic character are being discussed and solved in Appendix 3.

The found of informatively optimal connection between critical Fibonacci ratios, as well as above considered the systemic character of PNRs classification dichotomy enable to hypothesize that the harmony and balance rank directly among informational conceptions in terms of Shannon's information theory. The verification of this hypothesis is dealt with in Appendix 4.

Now, given the discussed basic conceptions and criteria generally, we will proceed to the main content of the present study demonstrating how PNRs allow solving the LC problem.

3. Critical and optimal levels of confidence

Dimensionally it is convenient to characterize an interval having certain percentage level of confidence by the following relative value:

$$\mathcal{E}_{LC} = 1/(1 - \mathrm{LC}/100), \tag{7}$$

which we name *dimensional factor of confidence* (DFC). There is one chance from the number equal to the factor that the value of the measurand lies outside the interval.

It follows from the informational approach that $\mathcal{E}_{50} = 1/(1 - 50/100) = 2$, being the reference value, corresponds with 50% confidence. This value is of principle significance for determining the optimal level of confidence $\mathrm{LC_{jo}}$ for each uncertainty component (u_j) in a system of $N \geq 2$ components of a modeling function of measurement. In terms of informational optimality, that is when considering the ratio $\mathcal{E}_{50}/\mathcal{E}_{LCjo} = \rho_{jo}$, the further determining is carried out by means of the following equations system:

$$\begin{cases} 1 / (1 - \mathrm{LC_{jo}} / 100) = \mathcal{E}_{LCjo} & (8) \\ \mathcal{E}_{50}/\mathcal{E}_{LCjo} = \rho_{jo} & (9) \end{cases}$$

The solution of the system regarding LC_{jo} (when taking into account the above factor $\mathcal{E}_{50} = 2$) results in the general expression (10), as well as singly through uncertainty contributions, i.e. $LC_o(u_j)$, to the expression (11):

$$LC_{jo} = (1 - \rho_{jo}/\mathcal{E}_{50})*100\% = (1 - 0.5\,\rho_{jo})*100\% = $$
$$= (1 - 0.5\,K_{\varphi o}/K_j)*100\%, \quad (10)$$
$$LC_o(u_j) = (1 - 0.08\,u_{max}^2/\,u_j^2)*100\% \quad (11)$$

In case K_{max} = constant, for a non-redundant practical use the range of LC_{jo} is from 50% ($K_j = K_{\varphi o}$) up to 92% ($K_j = K_{max}$). This range characterizes informatively necessary and sufficient confidence to the estimation quality. The increasing of the quality above this optimum on account of information redundancy is limited by 96% level of confidence for the contribution with maximal weight. The decreasing of the quality reaches 0% of the level for K_j when $K_j = 0.5K_{\varphi o}$. A further redundancy increase ($K_j < 0.5K_{\varphi o}$) leads to the failure of classification criterion and formally results in a negative confidence level for at least one of contributing components. Thus, when applying formula (11), the most concrete magnitude for the common LC is 92%. Because LC = 95% is located within permissible redundancy, this confirms the practical possibility of its usage and will be singly treated further.

Fragmentarily the obtained results are illustrated on Fig. 1 as the interrelation $LC_j = 100[(96 - LC_\varphi)/(100 - LC_\varphi)]$ between the current confidence (LC_j) on the one hand and the classification-minimal confidence (LC_φ) on the other hand in the system of two critical classification components.

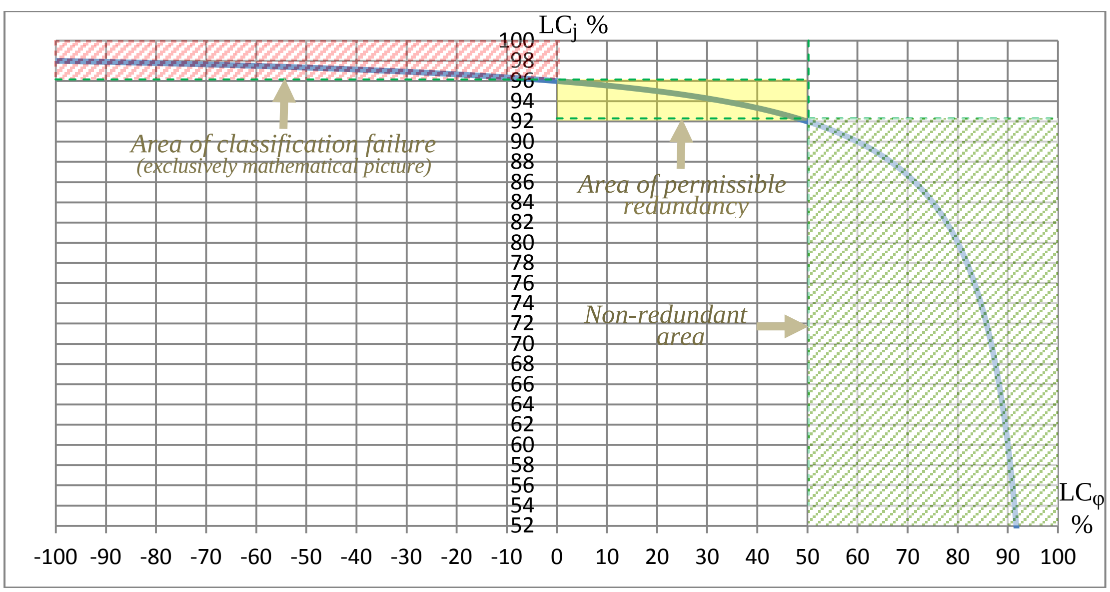

Fig.1: Functional interconnection between LC_j and LC_φ

We will analyze the obtained critical LC and DFC, to what extent they meet requirements of dimensional perfection expressed by PNR. Calculations of dimensional factors' ratios of confidence regarding above critical levels of confidence (including 95%) result in the data presented in Table 1.

Table 1: Results of analyzing relations of DFC for the critical LCs

LC (%)	$\mathcal{E}_{LOC}$	$\mathcal{E}_{50}/\mathcal{E}_{92}$	$\mathcal{E}_{92}/\mathcal{E}_{95}$	$\mathcal{E}_{92}/\mathcal{E}_{96}$	Identification to C_{inf}	Deviation from the constant (%)
50	2					
92	12.5	0.16			$\rho_o = 0.159$	0.6
95	20		0.625		$f_o = 0.618$	1.1
96	25			0.5	$\lambda = 0.5$	0

Table 1 convincingly demonstrates the informational perfection (in terms of PNRs) of the system composed by the critical levels of confidence being determined according to the proposed function (10). The additional useful argumentation in favor of formula (10) will be reflected in the further consideration when analyzing adequate critical coverage factors.

4. Optimally redundant level of confidence

The determination of *informatively optimal* LC_{opt} in the range of *permissible information redundancy* (PIR) can be carried out by using Benford's Law [6]. Since PIR ranges between $1/4\pi$ and $1/2\pi$, these boundaries correspond to the integers of classification character $R\{2\pi\} = 6$ and $R\{4\pi\} = 13$. The critical integer $z_o = R\{1/\rho_{or}\}$ of optimal redundancy can be determined by using Benford's probabilities.

The method of investigation based on the Benford's Law consists in the following. Any integer z greater than one can be used as the base of certain system of numbers; the system will employ z different digits. Benford's probability $P(d)$ of any number d from 1 to $(z-1)$ is calculated as follows:

$$P(d) = log_z (1 + 1/d) \qquad (12)$$

These probabilities form a complete group of independent events, i.e. their sum = 1, and a logarithmic sequence has obvious classification character. In so doing, the informatively sufficient number:

$$d_{\varphi o} = 1/\rho_{or} = \exp\left[-\sum_{i=1}^{(z-1)} P(d)_i \, ln\, P(d)\right], \qquad (13)$$

and thus for $z = R\{4\pi\} = 13$ the critical integer z_o is calculated as follows:

$$z_o = (R\{d_{\varphi o}\}+1) = 1 - R\{\exp \sum_{i=1}^{(z-1)} log_{13}[1+1/d] * ln\, (log_{13}[1+1/d])\} = 10 \qquad (14)$$

Incidentally, this result is of great original significance because it proves the informational optimality of decimal numbering system.

Another way of achieving this result [7] consists in applying the minimum $P_{min} = P\ [d = (z - 1)]$ and maximum Pmax = P (d = 1) Benford's probabilities that form the system of two components. For their ratio the optimization criterion may be expressed as follows:

$$(P_{min} / P_{max})_o = P\ [d = (z - 1)] \ / \ P\ (d = 1) = \rho_o = 1/2\pi \qquad (15)$$

The index of optimal classification z_{o1} = 10 (that is the logarithm base) is determined as follows:

$$z_{o1} = R\{arg\ min\ |\ log_z\ [1 + 1/(z - 1)] / \ log_z\ (1 + 1) - (1/2\pi)|\} = 10 \qquad (16)$$

Now, coming back to expression (13), using results (18) and (20), the optimally redundant level of confidence one may determine as follows:

$$LC_{opt} = [1 - 0.5\ (1/\ z_o)]*100\% = 95\% \qquad (17)$$

It is also noteworthy in the systems of components' pairs consisting of C_{inf} and (1- C_{inf}) regarding conceptions of harmony and balance the following are outcomes:

$\rho_{fo} = \rho_o\ f_o = 0.618/2\pi = 0.1$, and respectively $LC_{fo} = (1 - 0.5\ \rho_{fo})*100\% = 95\%$;

$\rho_{\lambda} = \rho_o\ \lambda = 0.5/2\pi = 0.08$, and respectively $LC_{fo} = (1 - 0.5\ \rho_{\lambda})*100\% = 96\%$.

Therefore, if using LC of the contributing component with K_{max} as the common one for the system, the range 92% ÷ 96% favors the commonly used 95% as:

(a) being informatively permissible, but redundant;
(b) possessing optimal information redundancy.

So the hypothesis of adequacy of commonly used one-third ratio and 95% level of confidence to the informational classification principle is true. It should be noted that LC = 95%, conforming with nearly average of permissible redundancy, in terms of generally accepted rule of providing reliability of informational estimates is even preferable. However, such a preference, while possessing heuristic value, explicitly causes additional estimation uncertainty.

5. Intrinsic and common coverage factors and confidence levels

Clearly, when dealing with an uncertainty budget, the obtained informatively optimal levels of confidence may be jointly used with the statistical instrument of determining coverage factors for calculating valid expanded uncertainty. Normal distribution or Student distribution might be used for this purpose. Theoretical statements and calculations will further be derived based on normal distribution.

The calculation may result in determining either of intrinsic (k_{in}) or of common (k_c) coverage factor. The usage of k_{in} is aimed at the precise estimation of expanded uncertainty that is the unique for certain measurement model, whereas k_c is convenient for traditional universalizing the uncertainty expansion. In case of normal distribution the calculations are performing as follows:

$$k_{in}= ([\sum_{j=1}^{N-m} (k_j u_j)^2]/[\sum_{j=1}^{N-m} u_j^2])^{1/2} \qquad (18)$$

$$k_c = 1.75 \text{ (that resembles } LC_{oj} = 92\% \text{ when } u_j = u_{max}), \qquad (19)$$

where k_j = arg [erf ($k/\sqrt{2}$)=LC_{oj}/100] = arg [erf($k/\sqrt{2}$) = (1– 0.08 u_{max}^2/u_j^2)], (20)
erf ($k/\sqrt{2}$) is the Gauss error function.

In so doing, the intrinsic level of confidence LC_{in} or the common one LC_c = 92% (or either in the range of permissible redundancy, i.e. from 92% to 96%) correspond according to the error function either to k_{in} or k_c respectively.

From the above consideration one may conclude that the feature of applying any LC_c is an incontestable overstating of expanded uncertainty, except the theoretical case when all selected uncertainty contributions are equal. At the same time this traditional way provides a habitual unified estimation norm. Clearly the passing from LC_c to LC_{in} requires changing the philosophy of uncertainty expansion.

6. Dimensional perfection of the system of critical coverage factors

In accordance with the proposed approach permissible coverage factors range within boundary critical values k_{min} = 0.67 (for 50% confidence) and k_{max} = 2.05 (for 96% confidence). Another critical value in this range is the optimal coverage factor k_o = 1.75 (for 92% confidence) that divides the range onto two parts so that the sub-range from k_o to k_{max} is being characterized by permissible redundancy. We will analyze these results, to what extent they meet requirements of dimensional perfection expressed by perfect numerical ratios. The results of analysis that prove the dimensional perfection are presented in Table 2.

Table 2: Data proving the dimensional perfection of permissible coverage factors

#	Name of relative value (V_r)	V_r expression	V_r numerical	C_{inf}	Deviation V_r from C_{inf}
1	Maximum range of permissible redundancy	$(k_{max} - k_o)/k_o$	0.171	ρ_o	-7.8%
2	Mean range of permissible redundancy	$2(k_{max} - k_o)/(k_{max}+ k_o)$	0.158	ρ_o	-0.6%
3	Minimum range of permissible redundancy	$(k_{max} - k_o)/k_{max}$	0.146	ρ_o	+8%
4	Minimum range of permissible sufficiency	$k_{min}/(k_o - k_{min})$	0.620	f_o	+0.3%
5	Maximum range of permissible sufficiency	$k_{min}/(k_{max}– k_{min})$	0.486	λ	-2.8%

Over obtained permissible coverage factors Table 2 demonstrates:

(a) the nearly ideal correspondence between the range (from 92% to 96%) of LC_c that characterizes necessary and sufficient information on the accuracy of measurement on the one hand, and of boundaries of permissible deviation of relative values of coverage factor that match this range and meet the requirement of permissible (±8%) deviations for ρ_o on the other hand;

(b) the very good adequacy of minimum and maximum range of permissible sufficiency (from 50% to 92%, and from 50% to 96% LC) to the harmony and balance respectively.

This outcome allows stating the LC determination by formula (11) is definitely adequate to conceptions of dimensional perfection, and this is the significant argument in favor of its practical usage.

Table 3 indicates also onto the existence of interrelation of PNRs. This is of concern both to understanding PNRs themselves and as being significant also in aspects of measurement; more about the interrelation is presented in Appendix 3.

Interestingly, when considering relations between $k_\sigma = 1$ (for 68% confidence), related to one standard deviation for normal distribution, and the above critical coverage factors, the conditions of dimensional perfection are also satisfied (Table 3). Data of Table 3 show the surprising existence of harmony and balance between the above parameters of normal distribution.

Table 3: Relation of k_σ with k_{min} , k_{max} and k_o

V_r expression	V_r numerical	C_{inf}	Deviation V_r from C_{inf}
k_{min}/k_σ	0.67	f_o	+8%
k_σ/k_o	0.571	f_o	-7.6%
k_σ/k_{max}	0.488	λ	-2.4%

7. Concluding remarks

7.1 The proposed approach has allowed optimazing levels of confidence for the evaluation of measurement uncertainty. It follows from the present discussion that along with 68% confidence, related to one standard deviation of normal distribution, there are critical levels of confidence for the weightiest component of a classification group: approximately 92%, 95%, and 96%. These LCs are strictly associated with conceptions of information optimality, of harmony, and of balance respectively and are characterized by the following classification qualities regarding informational redundancy:

92% level of confidence is related to the boundary of redundancy absence;
95% level of confidence is related to the optimal redundancy;
96% level of confidence is related to the maximum permissible redundancy.

7.2 The analysis of coverage factors for normal distribution by means of the proposed method of LCs optimization has revealed the conformity of the factors with theoretically stated informatively permissible ranges and redundancy, and the dimensional perfection of critical coverage factors' ratios in terms of optimality, harmony and balance expressed by perfect numerical ratios.

7.3 The closeness of the confidence, related to the optimal redundancy, to commonly used LC = 95% has demonstrated that the last one is the result of natural evolutionary trend to optimization.

Appendix 1: Sufficient number of components in a classification group

The substantiation of the criterion of determining φ_o and ρ_o for poly-component system (N >2) traces to the following. Any system of components, which is being analyzed by their weights (significances for certain usage) possesses redundancy (except when all components are of the same weight). The dividing of components onto necessary and sufficient ones on the one hand and redundant on the other hand is the typical act of classification. The criterion of such classifying is the most important problem that can be easily solved using the entropy's approach in terms of theory of information.

In a system of N components the normalized weights are considered as analogs of probabilities that enables using the entropy's approach; and the system's entropy is

$H_s = -\sum_{i=1}^{N} K_j \ln K_j$. For the selected by weights optimal (necessary and sufficient) number of components φ_o all these components belong to the same classification group, i.e. they are equilibrated by the appertaining: each the component possesses equal *group weight* = $1/\varphi_o$ (otherwise the group could contain inadmissible classification redundancy, i.e. non-optimal selection). Thus, the *classification entrop*y of selected group is $H_c = \ln \varphi_o$.

Theoretically the criterion to be formulated is based on the equivalence of the entropy of initial system and the classification entropy, i.e. $H_s = H_c$ that results in the expression:

$$\varphi_o = \exp\left(-\sum_{i=1}^{N} K_j \ln K_j\right) \qquad (21)$$

Then the optimal classification coefficient to be used as the criterion in terms of PIC is being expressed as follows:

$$\rho_o = 1 - (\varphi_o/N)_L = 0.16 \approx 1/2\pi,$$

where: $(\varphi_o/N)_L = 0.5[min(\varphi_o/N)_L + max(\varphi_o/N)_L] = 0.840,$

$$min(\varphi_o/N)_L = \lim_{N\to\infty} [(1/N) \exp(-\sum_{i=1}^{N} K_j \ln K_j)] = 0.824;$$

$$max(\varphi_o/N)_L = min(\varphi_o/N)_L + \lim_{N\to\infty} \sum_{i=1}^{N} K_j - min(\varphi_o/N)_L]^{j+1} = 0.856;$$

$K_{Lj} = \dfrac{2(N+1-j)}{N(N+1)}$ is the weight in a linear diagram of weights, i.e. when

$$\Delta K_j = K_j - K_{j+1} = \text{constant}.$$

Optimal classification ratio is limited between (1 - $max(\varphi_o/N)_L$) and (1 - $min(\varphi_o/N)_L$), i.e. possesses the range $\rho_o \pm 0.1\ \rho_o$ (that is $0.5/\pi \pm 0.05/\pi$ or 0.159 ± 0.0159). In practice the usage of more universal criterion is convenient, according to which φ_o is determined as the number of components with weights $K_j \geq K_\varphi$, where $K_\varphi = \rho_o K_1$. This became possible after proving the linear diagram of weights as the proper equivalent of using the criterion for the infinite variety of weights diagrams [8].

Appendix 2: Permissible PNR deviations

If $\pm 0.5\delta$ is the estimation error in determining a classification constant C_{inf}, one may prove that this allows determining whether any practical manifestations conform to the constant within the optimal range δ_o that is defined as $1/2\pi$ of the basic constant, i.e. $\delta_o/C_{inf} = 1/2\pi$. For this purpose we by analogy with the proposed above way of analysis, expressions (3) ÷ (6) shall proceed to the above method and consider the following equations system:

$$\begin{cases} K_1 = (C_{inf} - 0.5\delta) / (C_{inf} + 0.5\delta); & (22) \\ K_2 = \delta / (C_{inf} + 0.5\delta); & (23) \\ \varphi_o = exp\,(-K_1 \ln K_1 - K_2 \ln K_2) = 1.5 & (24) \end{cases}$$

The solution of these equations system regarding the sought permissible range of estimation error, i.e. $\delta_o = arg\,[\varphi_o(\delta) = 1.5]$, results in the following: $\delta_o = 0.026$ for $C_{inf} = \rho_o$; $\delta_o = 0.098$ for $C_{inf} = f_o$; $\delta_o = 0.080$ for $C_{inf} = \lambda$, and correspondingly with high estimation accuracy the proportion $\delta_o/C_{inf} = 1/2\pi$ is true. Thus, classification constants with their permissible deviations (approximately ±8% of any the constant) are determined as follows:

$\rho_o \pm \rho_o/4\pi = 0.159 \pm 0.013$,

$f_o \pm f_o/4\pi = 0.618 \pm 0.049$,

$\lambda \pm \lambda/4\pi = 0.5 \pm 0.04$

These values can be taken as tolerated ones by the criterion of acceptability in determining the conformity to the requirements of the optimality, harmony and balance. A deflection from the tolerances demonstrates how much an analyzed object does not meet respective PNR, and is the stimulus for more deep investigation of the object or existing estimation criteria and method used.

Appendix 3: Sufficiency, interrelations and separability of PNRs' system

The dimensional perfection as the system consisting of three independent components with respective weights (W) can be represented as a complex quality. Then the condition of non-redundancy, i.e. of including all n components into informative ones can be expressed as $(3 - \varphi) \leq 0.5$, where: $\varphi = - exp\ (W_1\ ln\ W_1 + W_2\ ln\ W_2 + W_3\ ln\ W_3)$, and $W_j = C_j /(C_1 + C_2 + C_3)$.

The equality in the condition matches the most uncertain classification situation (50% confidence) about allowing or ignoring the system's component with lesser weight. Thus, satisfying the condition, a system under consideration does not contain inadmissible redundancy, in other words the system possesses informational sufficiency.

There is the duality in analyzing the dimensional perfection in such a way that consists in the necessity in considering four separate systems ($s_1 \div s_4$); each one contain n = 3 independent components C_1, C_2 and C_3 so that C_1 equals either ρ_o or $(1 - \rho_o)$, C_2 equals either f_o or $(1 - f_o)$, and $C_3 = \lambda = (1 - \lambda)$. In so doing, results of calculation, embracing all combinations due to the duality, are presented in Table 4.

Table 4: Weights and sufficiency of dimensional perfection system

s_i	$C_1 = \rho_o$	$C_1 = 1-\rho_o$	$C_2 = f_o$	$C_2 = 1-f_o$	$C_3 = \lambda$	W_1	W_2	W_3	φ	$n - \varphi$
1	0.159		0.618		0.5	0.125	0.484	0.391	2.61	0.39
2	0.159			0.382	0.5	0.153	0.367	0.48	2.74	0.26
3		0.841	0.618		0.5	0.429	0.316	0.255	2.93	0.07
4		0.841		0.382	0.5	0.488	0.222	0.29	2.84	0.16

In so far as the above condition is satisfied for all the systems in question, i.e. there are no redundant components, the optimality, harmony and balance represent the complete set of dimensional ratios, i.e. wholly possess informational sufficiency.

The universality of informational optimality is of significance not only in determining permissible ranges for PNRs, but also when considering the hypothesis of their interrelations that we will discuss briefly both for one-dimensional and poly-dimensional models. One-dimensional model can be analyzed using Fig. 14 3.

Fig. 3: One-dimensional model for analyzing PNRs

Applying to ratios of parts of the straight line 0–a_3, one can easily prove the explicit quantitative interrelation between all three conceptions of dimensional perfection.
If the mean ratio $(a_3 - a_2)/0.5(a_3 + a_2)$ equals ρ_o and the following equations are true

$$(a_3 - a_2)/a_2 = \rho_o\ (1 + 1/4\pi) = 0.172; \quad (25)$$
$$(a_3 - a_2)/a_3 = \rho_o\ (1 - 1/4\pi) = 0.146, \quad (26)$$

then such a dimension a_1 (approximately $a_1 = a_3/3$) exists that satisfies to following conditions:

$$(a_2 - a_1)/a_2 = f_o\,(1 \pm 1/4\pi) \approx 0.618 \pm 0.049; \qquad (27)$$
$$(a_2 - a_1)/a_3 = \lambda\,(1 \pm 1/4\pi) \approx 0.5 \pm 0.04 \qquad (28)$$

Clearly these expressions match permissible PNR deviations. Thus one can assert about the dimensional connection of relative indexes of harmony and balance on the one hand and boundary values of relative index of informational optimality on the other hand. Recently published results of statistically analyzing critical temperatures of the Elements [9] are one of examples of the manifestation of PNRs and of their interrelations in nature. The revealed in the present study PNRs interrelations regarding levels of confidence is another example illustrating one-dimensional model.

As for poly-dimensional systems, any n-dimensional model is being mathematically described as a power of exponent (n). In our habitual world the numerical ratios geometrically are associating with one-, two- or three-dimensional model, i.e. with a line (n = 1) or a section (n = 2) or a volume (n = 3) respectively (Fig. 4). We shall deal with simplest unitary models: a straight line by length $x = 1$; a square by area $x^2 = 1$; a cube by volume $x^3 = 1$. The optimal ratio for each model may be presented as $\rho_o = y^n/x^n = y^n$, where $0 \le y \le 1$.Therefore, through classification duality one may consider the functions: $y = \rho_o^{1/n}$ or else $y' = 1 - \rho_o^{1/n}$. Corresponding results of calculations are presented in Table 5.

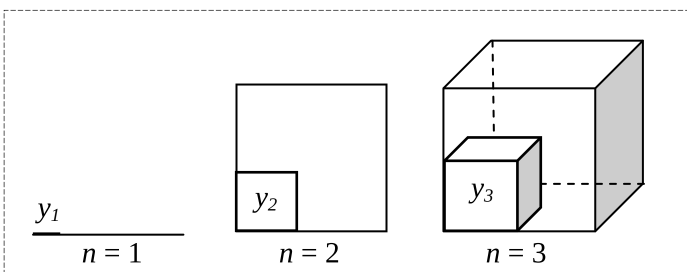

Fig. 4: Unitary dimensional models

Table 5: y and y' values

n	$y = \rho_o^{1/n}$	$y' = 1 - \rho_o^{1/n}$
1	0.159	0.841
2	0.399	0.601
3	0.54	0.46

The following conclusions can be drawn when using such a geometric simulation:

• The obvious connection exists between informational optimality on the one hand and the harmony and balance on the other hand.

• Informational optimality for two-dimensional model (n = 2) results in $y = 0.399$ and $y' = 0.601$ that is within the range $f_o \pm f_o/4\pi$ and illustrates mathematical harmony (golden section) adequate to flat models.

• Informational optimality for three-dimensional model (n = 3) results in $y = 0.54$ and $y' = 0.46$ that is within the range $\lambda \pm \lambda/4\pi$ and illustrates the balance inherent for volumetric models.

• Thus, depending on the type of normalized geometric model, informational optimality represents the optimality itself for one-dimensional model, and is adequate to the harmony for two-dimensional, and the balance for three-dimensional model.

PNRs dimensional separability

The system of PNRs itself can be considered as a one-dimensional classification and, therefore, should match the so-called first rule of classification: to avoid *cross-classification* [10]. One can prove the system possesses this important quality that, taking into account obtained PNRs' tolerances, can be called *dimensional separability*.

An expected characteristic feature of basic and indirect information constants is that in the ranges of their permissible values ($C_{inf} \pm \Delta C_{inf}$), accompanied by permissible estimation errors ($\pm \rho_o \Delta C_{inf}$), they do not have zones of mutual overlap on the axis of the values. If such feature exists, this excludes estimation uncertainty in determining what dimensional perfection the analyzed ratio belongs to. In the set of basic and indirect information constants the feature bears upon adjacent components $C_{inf} = C_{a1}$ and $C_{inf} = C_{a2}$, where $C_{ia2} > C_{a1}$. In so doing, the following condition regarding the difference (D), connected with adjacent components, indicates onto the dimensional separability:

$$D = (C_{a2} - \Delta C_{a2} - \rho_o \Delta C_{a2}) - (C_{a1} + \Delta C_{a1} + \rho_o \Delta C_{a1}) > 0 \quad (29)$$

Data and calculation results proving the reliable dimensional separability are set out in Table 6.

Table 6: Data and results of calculating the differences of adjacent components

C_{a1}	C_{a2}	ΔC_{a1}	ΔC_{a2}	D
$\rho_o = 0.159$	$\lambda = 0.5$	0.013	0.04	0.280
$\rho_o = 0.159$	$(1 - f_o) = 0.382$	0.013	0.049	0.151
$(1 - f_o) = 0.382$	$\lambda = 0.5$	0.049	0.04	0.015
$\lambda = 0.5$	$f_o = 0.618$	0.04	0.049	0.015
$\lambda = 0.5$	$(1 - \rho_o) = 0.841$	0.04	0.013	0.280
$f_o = 0.618$	$(1 - \rho_o) = 0.841$	0.049	0.013	0.151

One can note the materially small difference (D = 0.015) for adjacent components of harmony and balance. Apparently this corresponds with a perception closeness of these conceptions.

Appendix 4: Harmony and balance as informational conceptions

In order to verify the hypothesis that the harmony and balance rank among the informational conceptions in terms of Shannon's theory one can proceed to entropies regarding C_{inf}, which can be determined as $H(C_{inf}) = - K_1(C_{inf}) \ln [K_1(C_{inf})] - K_2(C_{inf}) \ln [K_2(C_{inf})]$, where $K_1(C_{inf}) = C_{inf}/(1 + C_{inf})$, and $K_2(C_{inf}) = 1/(1 + C_{inf})$. Then the quantitative criteria of the verification is whether or not 1) all quantities of information $I(C_{inf}) = H_{\max} - H(C_{inf}) = \ln 2 - H(C_{inf})$ belong to the same informational class, and 2) the ratios of entropy $H(\rho_o)$ to $H(f_o)$ and to $H(\lambda)$ are in mathematically harmonious relation.

Graphical illustration of respective calculations' results that prove the satisfaction to the first criterion is presented in Fig. 5, where in coordinates of a ratio (r) as a decimal fraction and a quantity of information $I(r)$ one can find locations of $I(\rho_o)$, $I(\lambda)$ and $I(f_o)$ in relation to the following boundary levels: $I(r_b)_S = I(\rho_o)/2\pi$ – the maximum permissible loss of information, and $I(r_b)_R = I(\rho_o)/4\pi$ – the maximum permissible redundancy of information about the quantity $I(\rho_o)$.

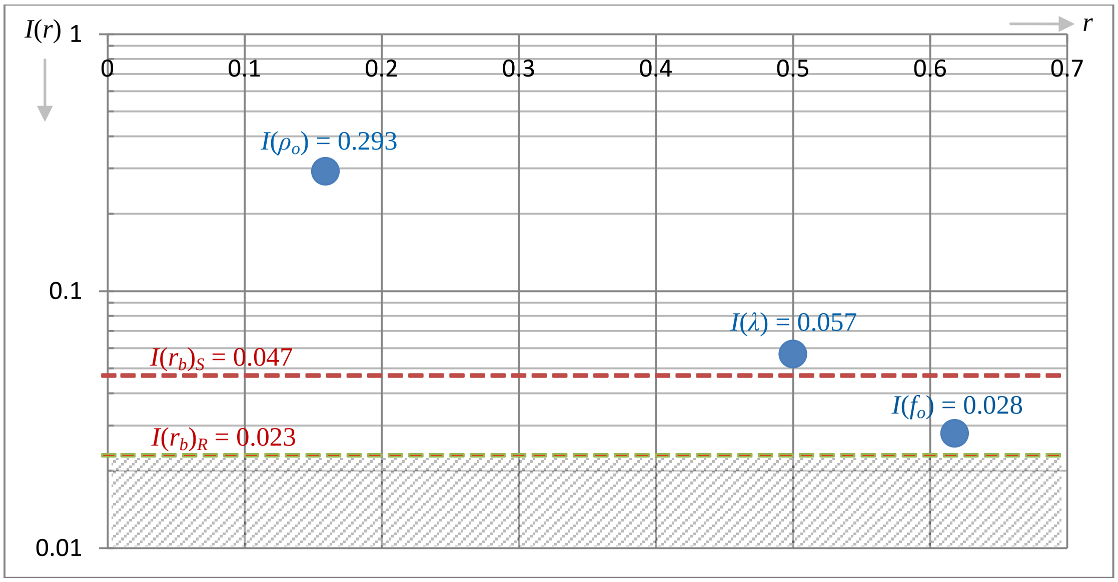

Fig.5: Graphical illustration of informational character of mathematical harmony and balance by their belonging to the same class with informational optimality. The shaded part illustrates the non-admissible redundancy. Any $I(r)$ within this area is not being in frame of the considered class of information quantities

Here are results of calculation as regards the second criterion: $H(\rho_o)/H(f_o) = 0.600$, and $H(\rho_o)/H(\lambda) = 0.627$; they demonstrate the full satisfaction of the condition that the entropy ratios are within the permissible range of harmony = f_o $(1 \pm 1/4\pi)$. Besides, the ratio $I(f_o)/I(\lambda) = 0.49$ demonstrates a balance of these two quantities within the permissible range λ $(1 \pm 1/4\pi)$.

Thus, along with ρ_o the mathematical harmony f_o and the balance λ ipso facto of their deep interrelations over informational properties can be considered as specific informational constants, useful for various quantitative.

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