A semi-discrete in time approximation for a model 1^{st} order-finite horizon mean field game problem

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Abstract

In this article we consider a model first order mean field game problem, introduced by J.M. Lasry and P.L. Lions in [17]. Its solution (v, m) can be obtained as the limit of the solutions of the second order mean field game problems, when the *noise* parameter tends to zero (see [17]). We propose a semi-discrete in time approximation of the system and, under natural assumptions, we prove that it is well posed and that it converges to (v, m) when the discretization parameter tends to zero.

Keywords. First order mean field game, semi-discrete in time approximation.

1 Introduction

The aim of Mean Field Game theory (MFG), introduced by Lasry and Lions in [15, 16, 17], is to describe analytically when $N \uparrow \infty$ the behavior of the limit of Nash equilibria for N-players stochastic differential games. In the presence of stochastic dynamics of Itô type for the state of each player, the limit system is at least formally, of the following form (see [17]):

$$-\partial_t v(t,x) - \sigma^2 \Delta v(x,t) + H(x, Dv(x,t)) = F(x, m(t)), \text{ in } \mathbb{R}^d \times (0,T), \tag{1.1}$$

$$\partial_t m(t,x) - \sigma^2 \Delta m(x,t) - \operatorname{div}\big(m(t,x) \frac{\partial H}{\partial p}(x, Dv(t,x))\big) = 0, \quad \text{in } \mathbb{R}^d \times (0,T), \tag{1.2}$$

$$v(x,T) = G(x, m(T)) \quad \text{for } x \in \mathbb{R}^d \quad , \quad m(0) = m_0 \in \mathcal{P}_1. \tag{1.3}$$

In the notation above we have: the *noise* parameter $\sigma \in \mathbb{R}$, the Hamiltonian H, which is convex with respect to the second variable, the space $\mathcal{P}_1$ of probability measures on $\mathbb{R}^d$ and the functions $F, G : \mathbb{R}^d \times \mathcal{P}_1 \to \mathbb{R}$, which can be non local operators. It is shown in [17] that, under natural assumptions and in an appropriate sense, system (1.1)-(1.3) admits a unique solution (v_σ, m_σ). Let us point out that when the time and the space are discrete, existence and uniqueness also hold as well as exponential convergence, as $T \uparrow \infty$, to the equilibrium of the discrete version of (1.1)-(1.3) (see [12]).

Regarding numerical methods, a finite-difference scheme for (1.1)-(1.3) is proposed in [2], as well as the corresponding scheme for the stationary version. See also [1] where a Newton method is used to solve the discretized system in the context of the planning problem. A

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different approach is explored in [14], based on the interpretation of (1.1)-(1.3) as the conditions characterizing a saddle point of an associated optimization problem. Let us also mention [13] where, in the case of a quadratic Hamiltonian, a change of variable allows to rewrite system (1.1)-(1.3) in a simpler manner which in turns permits to design a monotone scheme.

In this work we focus our attention on the *vanishing viscosity* limit of (1.1)-(1.3) as $\sigma \downarrow 0$, when $H(x,p) = \frac{1}{2}|p|^2$. This is interpreted as that the noise σ for each player tends to disappear. The limit equations are the so-called first order mean field system (see [17])

$$\begin{array}{rcl} -\partial_t v(t,x) + \frac{1}{2}|Dv(t,x)|^2 & = & F(x,m(t)), \text{ in } \mathbb{R}^d \times (0,T), \\ \partial_t m(t,x) - \operatorname{div}\big(Dv(t,x)m(t,x)\big) & = & 0, \quad \text{in } \mathbb{R}^d \times (0,T), \\ v(x,T) = G(x,m(T)) \quad \text{for } x \in \mathbb{R}^d & , & m(0) = m_0 \in \mathcal{P}_1. \end{array} \tag{1.4}$$

It is well known (see [17])) that, under natural assumptions, system (1.4) has a unique solution (v,m) and that $(v_\sigma, m_\sigma) \to (v,m)$ in appropriate spaces.

In this work we study a time discretization of (1.4), which is precisely described in section 4. This is motivated by a forthcoming work in which a fully discrete (i.e. where space discretization is added) semi-lagrangian method is studied, together with some numerical experiments. Our main results is this article is that the resulting discretization is well-posed (see theorem 4.1) and that, as the discretization parameter goes to zero, we have the convergence to the solution (v,m) of (1.4). This is the first result concerning the approximation of a first order MFG system.

The article is organized as follows: After setting the notation and introducing the fundamental assumptions in section 2, we analyze in section 3 a discrete in time approximation of the first and second equations in (1.4), separately. This is fundamental in order to prove in section 4 the well-posedness of the semi-discrete scheme, as well as its convergence. Finally, we provide in the appendix a proof of a technical result stated in section 3.

2 Prerequisites

We denote by $\mathcal{P}_1$ the set of the Borel probability measures m such that $\int_{\mathbb{R}^d} |x| \mathrm{d}m(x) < \infty$. The set $\mathcal{P}_1$ is be endowed with the Kantorovich-Rubinstein distance

$$\bar{d}_1(\mu,\nu) = \sup\left\{ \int_{\mathbb{R}^d} \phi(x)\mathrm{d}[\mu-\nu](x) \ ; \ \phi : \mathbb{R}^d \to \mathbb{R} \ \text{ is 1-Lipschitz} \right\}. \tag{2.1}$$

Given a measure $\mu \in \mathcal{P}_1$ we denote by $\operatorname{supp}(\mu)$ its support. In what follows, in order to simplify the notation, the operator D (resp. D^2) will denote the derivative (resp. the second derivative) with respect to the space variable $x \in \mathbb{R}^d$. We suppose that the functions $F, G : \mathbb{R}^d \times \mathcal{P}_1 \to \mathbb{R}$ and $m_0 \in \mathcal{P}_1$, which are the data of system (1.4), satisfy the following assumptions:

(H1) F and G are continuous over $\mathbb{R}^d \times \mathcal{P}_1$.

(H2) There exists a constant $C > 0$ such that for any $m \in \mathcal{P}_1$

$$||F(\cdot,m)||_{C^2} + ||G(\cdot,m)||_{C^2} \leq C.$$

(H3) The following monotonicity conditions holds true

$$\int_{\mathbb{R}^d} [F(x,m_1) - F(x,m_2)]\, \mathrm{d}[m_1 - m_2](x) > 0 \quad \text{for all } m_1,\, m_2 \in \mathcal{P}_1, \ m_1 \neq m_2, \tag{2.2}$$

$$\int_{\mathbb{R}^d} [G(x,m_1) - G(x,m_2)]\, \mathrm{d}[m_1 - m_2](x) > 0 \quad \text{for all } m_1,\, m_2 \in \mathcal{P}_1,\, m_1 \neq m_2. \tag{2.3}$$

(H4) The measure m_0, appearing in the third equation of (1.4), is absolutely continuous with respect to the Lebesgue measure. Its density, still denoted as m_0, is essentially bounded and satisfies $\text{supp}(m_0) \subset B(0, C)$.

Remark 2.1 *Assumption* (2.3) *is slightly stronger than the corresponding assumption in [7].*

Now we define what we mean by a solution of (1.4).

Definition 2.1 *The pair* $(v, m) \in W^{1,\infty}_{loc}(\mathbb{R}^d \times [0,T]) \times L^1(\mathbb{R}^d \times (0,T))$ *is a solution of* (1.4) *if the first equation is satisfied in the viscosity sense, while the second one in sense of distributions.*

The following existence and uniqueness result is proved in [17, 18, 7]

Theorem 2.1 *Under* **(H1)-(H4)** *there exists a unique solution* (v, m) *to* (1.4)*.*

In the proof of the above theorem, as well that in the the proof of our main results, the concept of semi-concavity plays a crucial role. For a complete account of the theory and its applications to the solution of HJB equations, we refer the reader to [6].

Definition 2.2 *We say that* $w : \mathbb{R}^d \to \mathbb{R}$ *is semi-concave with constant* $C_{conc} > 0$ *if for every* $x_1, x_2 \in \mathbb{R}^d$, $\lambda \in (0,1)$ *we have*

$$w(\lambda x_1 + (1-\lambda)x_2) \geq \lambda w(x_1) + (1-\lambda) w(x_2) - \lambda(1-\lambda) C_{conc} |x_1 - x_2|^2. \tag{2.4}$$

A function $\hat{w}$ *is said to be semi-convex if* $-\hat{w}$ *is semi-concave.*

Recall that for $w : \mathbb{R}^d \to \mathbb{R}$ the super-differential $D^+ w(x)$ at $x \in \mathbb{R}^d$ is defined as

$$D^+ w(x) := \left\{ p \in \mathbb{R}^d \ ; \ \limsup_{y \to x} \frac{w(y) - w(x) - \langle p, y - x \rangle}{|y - x|} \leq 0 \right\}.$$

We collect in the following lemmas some useful properties of semi-concave functions (see [6]) .

Lemma 2.1 *For a function* $w : \mathbb{R}^d \to \mathbb{R}$, *the following assertions are equivalent:*

(i) *The function* w *is semi-concave, with constant* C_{conc}.

(ii) *For all* $x, y \in \mathbb{R}^d$, *we have*

$$w(x + y) + w(x - y) - 2w(x) \leq C_{conc} |y|^2.$$

(iii) *For all* x_1, $x_2 \in \mathbb{R}^d$ *and* $p \in D^+ w(x_1)$, $q \in D^+ w(x_2)$

$$\langle q - p, x_2 - x_1 \rangle \leq C_{conc} |x - y|^2. \tag{2.5}$$

(iv) *Setting* I_d *for the identity matrix, we have that* $D^2 w \leq C_{conc} I_d$ *in the sense of distributions.*

Lemma 2.2 *Let* $w : \mathbb{R}^d \to \mathbb{R}$ *be semi-concave. Then:*

(i) w *is locally Lipschitz.*

(ii) *If* w_n *is a sequence of semi-concave converging point-wisely to* w, *then the convergence is locally uniform and* $Dw_n(x) \to Dw(x)$ *for a.e.* $x \in \mathbb{R}^d$.

3 The semi-discrete scheme for the "separated" equations

In order to describe a discrete in time approximation for (1.4), we first analyze the discretization of both equations separately.

3.1 Approximation of the Hamilton-Jacobi equation

As we explained in the introduction, for a given $m \in C([0,T];\mathcal{P}_1)$ the first equation in (1.4), together with the corresponding boundary value at $t=T$, yields the *value function* of a model optimal control problem. In this subsection we recall some basic properties of this problem and some facts about its semi-discrete in time discretization (see [8, 10, 9] for the semi-discrete approximation and the forthcoming book [11] for a complete account of the theory including the fully discrete semi-lagrangian approximation).

For $t \in [0,T]$ set $\mathcal{A}(t) := L^2([t,T];\mathbb{R}^d)$ for the *set of admissible controls*. Given $\alpha \in \mathcal{A}(t)$ its associated *state* $X^{x,t}[\alpha]$ is defined as the unique solution of

$$\dot{X}(s) = -\alpha(s) \quad \text{for } s \in (t,T), \quad X(t) = x. \tag{3.1}$$

Now, let $f : \mathbb{R}^d \times [0,T] \to \mathbb{R}$, $g : \mathbb{R}^d \to \mathbb{R}$ be continuous, satisfying that

$$\|f(\cdot,t)\|_{C^2} \leq C \quad \forall t \in [0,T], \qquad \|g\|_{C^2} \leq C. \tag{3.2}$$

Consider the following cost function

$$J(\alpha; x,t) := \int_t^T \left[\tfrac{1}{2}|\alpha(s)|^2 + f(X^{x,t}[\alpha](s),s)\right] ds + g(X^{x,t}[\alpha](T)). \tag{3.3}$$

We define $w : \mathbb{R}^d \times [0,T] \to \mathbb{R}$ as the value function of the optimal control problem with dynamics (3.1) and cost (3.3), i.e.

$$w(x,t) = \inf_{\alpha \in \mathcal{A}(t)} J(\alpha; x,t) \quad \text{for all } (x,t) \in \mathbb{R}^d \times [0,T]. \tag{3.4}$$

We recall in next proposition some basic properties of the function w (see e.g. [7] for our model problem and [4, 6] for generalizations).

Proposition 3.1 *The following statement hold true:*
(i) *The value function is globally Lipschitz w.r.t. the pair* (x,t).
(ii) *The value function is semi-concave w.r.t.* x.

It is well known that w is the unique viscosity solution of (see e.g. [4])

$$\begin{cases} -\partial_t w(x,t) + \frac{1}{2}|Dw(x,t)|^2 &= f(x,t), \quad \text{in } \mathbb{R}^d \times (0,T), \\ w(x,T) &= g(x) \quad \text{in } \mathbb{R}^d. \end{cases} \tag{3.5}$$

We define the set of optimal controls $\mathcal{A}(x,t)$ as

$$\mathcal{A}(x,t) := \{\alpha \in \mathcal{A}(t)\ ;\ J(\alpha; x,t) = w(x,t)\}. \tag{3.6}$$

Under our assumptions, classical arguments show that for all $(x,t) \in \mathbb{R}^d \times (0,T)$ the set $\mathcal{A}(x,t)$ is non-empty. Moreover, the next lemma (proved in e.g. [7, Lemma 4.8 and Lemma 4.9]) shows that unicity in $\mathcal{A}(x,t)$ characterizes the existence of $Dw(x,t)$.

Lemma 3.1 *Let* $(x,t) \in \mathbb{R}^d \times [0,T]$. *Then,*
(i) *For any* $\alpha \in \mathcal{A}(x,t)$ *and* $s \in (t,T]$, *we have that* $\mathcal{A}(X^{x,t}[\alpha](s),s) = \left\{\alpha(\cdot)\big|_{(s,T]}\right\}$.
(ii) *The function* w *is differentiable with respect to* x *at* (x,t) *iff there exists* $\alpha \in \mathcal{A}(t)$ *such that* $\mathcal{A}(x,t) = \{\alpha\}$. *In this case,* $D_x w(x,t) = -\alpha(t)$.

(iii) *If* $\alpha \in \mathcal{A}(x,t)$ *then is* C^1 *in* $[t,T]$ *and satisfies*

$$\dot{\alpha}(t) = -Df(X^{x,t}[\alpha](s), s), \text{ for } s \in [t,T]; \quad \alpha(T) = Dg(X^{x,t}[\alpha](T)). \tag{3.7}$$

(iv) *For every* $(x,t) \in \mathbb{R}^d$, *any optimal trajectory* $X(\cdot) : [t,T] \to \mathbb{R}^d$ *for problem* (3.4) *satisfies*

$$\dot{X}(s) = -Dw(X(s), s) \text{ for } s \in [t,T], \quad X(t) = x, \tag{3.8}$$

i.e. every $\alpha(\cdot) \in \mathcal{A}(x,t)$ *admits a feedback representation as* $\alpha(s) = Dw(X(s), s)$ *where* X *is a solution of* (3.8). *Conversely, any solution* $X(\cdot) : [t,T] \to \mathbb{R}^d$ *of* (3.8) *is an optimal trajectory for problem* (3.4). *In particular, equation* (3.8) *admits a unique solution iff the optimal control problem for* $w(x,t)$ *has a unique minimum.*

Remark 3.1 *Proposition 3.1*(i) *and lemma 3.1*(ii) *imply that given* $t \in [0,T]$ *we have that* $\mathcal{A}(x,t)$ *is a singleton for a.e.* $x \in \mathbb{R}^d$.

Let us now introduce a time discretization of the evolutive equation (3.5). Fix $h > 0$, set $N = T/h$ (we assume N to be an integer) and for $n = 0, ..., N-1$ define the set of *discrete controls* $\mathcal{A}^n := \mathbb{R}^{d\times(N-n)}$. Given $\alpha = (\alpha_k)_{k=n}^{N-1} \in \mathcal{A}^n$, let us define, for $k = n, ..., N$, the *discrete states* $X_k^{x,t}[\alpha]$ as the solution of

$$\begin{cases} X_{k+1} &= X_k - h\alpha_k = x - h\sum_{i=n}^{k} \alpha_i \quad \text{for } k = n, \dots, N-1, \\ X_n &= x. \end{cases} \tag{3.9}$$

In order to simplify the notation, when the context is clear, we do not write the dependence of the discrete state on (x,n) and α and we will simply write X_k. The *discrete cost function* $J_h(\cdot; x, n) : \mathcal{A}^n \to \mathbb{R}$ is defined by

$$J_h(\alpha; x, n) = h \sum_{k=n}^{N-1} \left[\tfrac{1}{2}|\alpha_k|^2 + f(X_k, kh)\right] + g(X_N). \tag{3.10}$$

The value function w_h is given by

$$(x,n) \in \mathbb{R}^d \times \{0, ..., N-1\} \to w_h(x,n) := \inf_{\alpha \in \mathcal{A}^n} J(\alpha; x, n), \quad w_h(x,N) = g(x). \tag{3.11}$$

Lemma 3.2 *For* $h > 0$ *small enough, there exists a unique* $\alpha \in \mathcal{A}^n$ *such that*

$$w_h(x,n) = J_h(\alpha; x, n). \tag{3.12}$$

Proof. First note that assumption (3.2) and lemma 2.1(iv) imply that $f(\cdot,\cdot)$ is semi-convex with respect to its first variable. Since $\{X_k\}_{k=n}^{N}$ is affine with respect to $(\alpha_k)_{k=n}^{N-1}$, the proof follows from the following claim:
Claim: For a semi-convex function $\hat{f} : \mathbb{R}^{d'} \to \mathbb{R}$ ($d' \in \mathbb{N}$), the problem $\min_{\alpha \in \mathbb{R}^{d'}} \frac{1}{2}h|\alpha|^2 + \hat{f}(h\alpha)$, admits a unique solution, for h small enough.
Proof of the claim: Suppose that $\alpha_1 \neq \alpha_2$ are two solutions and denote by $\bar{a}$ the optimal value. Let C be the constant in (2.4) associated to $\hat{f}$ and for $\lambda \in (0,1)$ set $\alpha_\lambda := \lambda\alpha_1 + (1-\lambda)\alpha_2$. Then

$$\begin{aligned} \tfrac{1}{2}h|\alpha_\lambda|^2 + \hat{f}(h\alpha_\lambda) &= \lambda h\alpha_1 + (1-\lambda)h\alpha_2 - \tfrac{1}{2}h\lambda(1-\lambda)|\alpha_1 - \alpha_2|^2 + \hat{f}(\alpha_\lambda) \\ &\leq \bar{a} + \lambda(1-\lambda)|\alpha_1 - \alpha_2|^2(Ch^2 - \tfrac{1}{2}h) < \bar{a}, \end{aligned}$$

for h small enough. This yields the desired contradiction. ■

Remark 3.2 *In order to emphasize the dependence on x, h and t, we will denote by $\bar{\alpha}_h(x,n)$ the optimal control in* (3.12).

From (3.10), we easily obtain that w_h satisfies the following *dynamic programming equation*

$$\begin{cases} w_h(x,n) & = \inf_{\alpha\in\mathbb{R}^d}\left\{w_h(x-h\alpha,n+1)+\frac{1}{2}h|\alpha|^2\right\}+hf(x,nh), \ \ \forall x\in\mathbb{R}^d,\ n=1,\dots,N-1, \\ w_h(x,N) & = g(x) \quad \text{for } x\in\mathbb{R}^d. \end{cases} \tag{3.13}$$

The next lemma, proved in the appendix, provide useful properties of the discrete value function.

Lemma 3.3 *For every $h>0$ small enough, we have:*
(i) *The function w_h satisfies (recall that C is the constant in* (3.2)*):*

$$|w_h(x,n)|\leq C(1+T) \quad \textit{for } x\in\mathbb{R}^d,\ \ n=1,\dots,N-1. \tag{3.14}$$

(ii) *For all $x\in\mathbb{R}^d$ and $n=1,\dots,N-1$ the following problem has a unique solution*

$$\inf_{\alpha\in\mathbb{R}^d}\left\{w_h(x-h\alpha,n+1)+\tfrac{1}{2}h|\alpha|^2\right\}+hf(x,nh). \tag{3.15}$$

(iii) *There exists a constant $C_1>0$ independent of h, such that*

$$|w_h(x_1,n_1)-w_h(x_2,n_2)|\leq C_1\left(|x_1-x_2|+h|n_1-n_2|\right).$$

(iv) *$w_h(\cdot,n)$ is semi-concave with an associated constant C_2 independent on h.*

Remark 3.3 *As in remark 3.2, for the sake of clarity, we denote by $\alpha_h(x,n)$ for the solution of* (3.15)*. Setting $\bar{\alpha}^i_h(x,n)$ for the i-th coordinate of $\bar{\alpha}_h(x,n)$, lemma 3.3*(iii) *yields*

$$\bar{\alpha}^i_h(x,n)=\alpha_h(X_i^{x,n}[\bar{\alpha}_h(x,n)],i), \quad \textit{for } i=1,...,N-n. \tag{3.16}$$

In the next lemma we study some properties of $\alpha_h(\cdot,\cdot)$.

Lemma 3.4 *Let $h>0$ be small enough. Then, the following assertions hold true:*
(i) *The set $\{\alpha_h(x,n)\ ;\ x\in\mathbb{R}^d,\ \ n=1,...N-1\}$ is bounded by a constant independent of h.*
(ii) *The function $x\in\mathbb{R}^d\to\alpha_h(x,n)$ is continuous.*
(iii) *For every $n=1,...,N-1$ the function w_h is C^1 with respect to x, and we have:*

$$\begin{aligned} Dw_h(x,n) &= \alpha_h(x,n)+hDf(x,hn), \\ Dw_h(x-h\alpha_h(x,n),n+1) &= \alpha_h(x,n). \end{aligned} \tag{3.17}$$

Proof. For all $x\in\mathbb{R}^d$ and $n=1,...,N-1$, lema 3.3(iii) yields

$$\tfrac{1}{2}h|\alpha_h(x,n)|^2=w_h(x,n)-w_h(x-h\alpha_h(x,n),n+1)-hf(x,hn)\leq C_1h\left(|\alpha_h(x,n)|+1\right)+hC,$$

which easily implies (i). Let $x_j\to x$ as $j\to\infty$ and let $\bar{\alpha}$ be a limit point of $\alpha_h(x_j,n)$. Then, by continuity of $w_h(\cdot,n)$ we have that $w_h(x_j,n)\to w_h(x,n)$. Thus, up to subsequences,

$$w_h(x,n)=\tfrac{1}{2}h|\bar{\alpha}|^2+w_h(x-h\bar{\alpha})+hf(x,hn),$$

and we get that $\bar{\alpha}=\alpha_h(x,n)$ by lemma 3.3(ii), proving (ii). Now, by Danskin theorem (see e.g. [5, Theorem 4.13]) we have that $w_h(\cdot,N-1)$ is directionally differentiable, with

$$Dw_h(x,N-1)d'=Dg(x-h\alpha_h(x,N-1))d'+hDf(x,h(N-1))d' \ \text{ for all } d'\in\mathbb{R}^d.$$

On the other hand, the optimality of $\alpha_h(x, N-1)$ yields $\alpha_h(x, N-1) = Dg(x - h\alpha_h(x, N-1))$. This fact, together with (ii) gives the first identity in (iii) for $n = N-1$, and the general result follows by iterating. The second identity in (iii) is a direct consequence of lemma 3.3(ii). ■

Consider the function $\hat{w}_h : \mathbb{R}^d \times [0,T] \to \mathbb{R}$ defined by $\hat{w}_h(x,t) := w_h(x, [t/h])$. For notational convenience, when the context is clear we will write also w_h for $\hat{w}_h$.

The following convergence result is well known (see e.g. [4, Chapter V, theorem 1.1], for the stationary case and for more general assumptions).

Theorem 3.1 *As $h \downarrow 0$, w_h converges locally uniformly in $\mathbb{R}^d \times [0,T]$ to solution w of* (3.5).

3.2 Approximation of the continuity equation

We now discuss the approximation scheme for the continuity equation

$$\left\{ \begin{array}{rcl} \partial_t \mu(x,t) - \operatorname{div}(Dw(x,t)\mu(x,t)) &=& 0 \quad \text{for } (x,t) \in \mathbb{R}^d \times (0,T], \\ \mu(x,0) &=& m_0(x), \quad \text{for } x \in \mathbb{R}^d. \end{array} \right. \tag{3.18}$$

with w being the solution of (3.5). We recall that μ is said to be a solution of (3.18) if it satisfies the equation in the distribution sense. Let us first present some well-known results about the existence and uniqueness for a solution of (3.18).

Consider the multivalued mapping $x \in \mathbb{R}^d \to \mathcal{A}(x,0)$. Standard arguments (see e.g. [3]) easily show that it admits a measurable selection α. Thus, we can define a measurable map $x \in \mathbb{R}^d \to \Phi(x,\cdot) \in C([0,T];\mathbb{R}^d)$ by

$$\Phi(x,t) = x - \int_0^t \alpha(x,s)\mathrm{d}s \quad \text{for all } x \in \mathbb{R}^d. \tag{3.19}$$

Note that proposition 3.1(iv), implies that given $x \in \mathbb{R}^d$, we have

$$\partial_t \Phi(x,t) = -Dw(\Phi(x,t),t) \quad \text{for } t \in (0,T], \quad \Phi(x,0) = x. \tag{3.20}$$

Recall that for $\mathcal{T} : \mathbb{R}^d \to \mathbb{R}^d$ and $\mu \in \mathcal{P}_1$, the *push-forward* $\mathcal{T}\sharp\mu \in \mathcal{P}_1$ of μ is defined by $\mathcal{T}\sharp\mu(A) := \mu(\mathcal{T}^{-1}(A))$ for all $A \in \mathcal{B}(\mathbb{R}^d)$. We have (see e.g. [7, theorem 4.18 and lemma 4.14])

Theorem 3.2 *Given a solution w of* (3.4)*, the map $t \in [0,T] \to \mu(t) := \Phi(\cdot,t)\sharp m_0 \in \mathcal{P}_1$ is the unique solution of* (3.18)*. Moreover, the following assertions hold true:*

(i) *There exists $C_3 > 0$ such that for all $s \in [0,T]$, $\mu(s)$ is absolutely continuous (with density still denoted by $\mu(s)$), has a support in $B(0, C_3)$ and $||\mu(s)||_\infty \leq C_3$.*

(ii) *For all $t, t' \in [0,T]$, we have that*

$$\bar{d}_1(\mu(t), \mu(t')) \leq ||Dw(x,t)||_\infty |t - t'|.$$

Our aim now is to define a discrete in time approximation of $t \in [0,T] \to \mu(t) := \Phi(\cdot,t)\sharp m_0 \in \mathcal{P}_1$. Given $h > 0$ and $x \in \mathbb{R}^d$, the optimal flux $(\Phi_h(x,k))_{k=0}^N$ for (3.11) is defined by

$$\Phi_h(x,k+1) = \Phi_h(x,k) - h\alpha_h(\Phi_h(x,k),k) \;\text{ for } k = 0,\dots,N-1, \quad \Phi_h(x,0) = x, \tag{3.21}$$

where we recall that $\alpha_h(x,k)$ is defined in remark 3.2. Equivalently (see remark 3.3)

$$\Phi_h(x,k+1) = x - h\sum_{i=0}^{k} \bar{\alpha}_h^i(x,0). \tag{3.22}$$

Proposition 3.2 *There exists a constant $C_4 > 0$ (independent of small enough h) such that for all $k = 1, ..., N$ and $x, y \in \mathbb{R}^d$ we have*

$$C_4|\Phi_h(x,k) - \Phi_h(y,k)| \geq |x - y| \tag{3.23}$$

Thus, $x \mapsto \Phi_h(x,k)$ is invertible in $\Phi_h(\mathbb{R}^d, k)$ and the inverse $\Upsilon_h(\cdot, k)$ is Lipschitz continuous.

Proof. Let Φ_k, Ψ_k be the trajectories given by (3.21) with respectively $\Phi_0 = x$ and $\Psi_0 = y$. We have that

$$|\Phi_{k+1} - \Psi_{k+1}|^2 \geq |\Phi_k - \Psi_k|^2 + 2h\left[\alpha_h(\Phi_k, k) - \alpha_h(\Psi_k, k)\right] \cdot (\Phi_k - \Psi_k). \tag{3.24}$$

Now, lemma 3.4(iii) yields that

$$\alpha_h(\Phi_k, k) - \alpha_h(\Psi_k, k) = Dw_h(\Phi_k, k) - Dw_h(\Psi_k, k) - h\left[Df(\Phi_k, hk) - Df(\Psi_k, hk)\right].$$

Lemma 3.3(iv) and lemma 2.1 imply that

$$\left[\alpha_h(\Phi_k, k) - \alpha_h(\Psi_k, k)\right] \cdot (\Phi_k - \Psi_k) \leq (C_2 + Ch)\left|\Phi_k - \Psi_k\right|^2. \tag{3.25}$$

By (3.24) and (3.25), there is $\hat{C} > 0$ (independent of h small enough) such that

$$|\Phi_{k+1} - \Psi_{k+1}|^2 \geq (1 - h\hat{C})\left|\Phi_k - \Psi_k\right|^2.$$

Therefore, for every $k = 1, ..., N$, we get

$$|\Phi_{k+1} - \Psi_{k+1}|^2 \geq (1 - h\hat{C})^k\left|x - y\right|^2 \geq (1 - h\hat{C})^{[T/h]}\left|x - y\right|^2.$$

and the result follows from the convergence of $(1 - h\hat{C})^{[T/h]}$ to $\exp(-\hat{C}T)$ as $h \downarrow 0$. ■

We define $\mu_h : \{1, ..., N\} \to \mathcal{P}_1$ by

$$\mu_h(k) := \Phi_h(\cdot, k)\sharp m_0, \quad \text{for all } k = 1, ..., N. \tag{3.26}$$

Equivalently,

$$\int_{\mathbb{R}^d} \phi(x)\mathrm{d}\mu_h(k)(x) = \int_{\mathbb{R}^d} \phi(\Phi_h(x,k))m_0(x)\mathrm{d}x, \quad \text{for any } \phi \in C(\mathbb{R}^d). \tag{3.27}$$

We have the following lemma, whose proof follows the lines of lemma 4.14 in [7].

Lemma 3.5 *There exists $C_5 > 0$ (independent of h) such that:*
(i) *For all $k_1, k_2 \in \{1, ..., N\}$, we have that*

$$\bar{d}_1(\mu_h(k_1), \mu_h(k_2)) \leq C_5 h|k_1 - k_2|. \tag{3.28}$$

(ii) *For all $k = 1, ..., N$, $\mu_h(k)$ is absolutely continuous (with density still denoted by $\mu_h(k)$), has a support in $B(0, C_5)$ and $||\mu_h(k)||_\infty \leq C_5$.*

Proof. Lemma 3.4(i) implies the existence of $C_6 > 0$ such that

$$|\Phi_h(x, k_1) - \Phi_h(x, k_2)| \leq C_6|k_1 - k_2|h. \tag{3.29}$$

For any 1-Lipschitz function $\phi : \mathbb{R}^d \to \mathbb{R}$, by (3.27)

$$\int_{\mathbb{R}^d} \phi(x)\, d[\mu_h(k_1) - \mu_h(k_2)](x) \leq \int_{\mathbb{R}^d} |\Phi_h(x,k_1) - \Phi_h(x,k_2)| dm_0 \leq C_6 h |k_1 - k_2|,$$

from which (3.33) follows. Since $\text{supp}(m_0) \subset B(0,C)$, then $\text{supp}(\mu_h(k)) \subset B(0, C + NC_6 h)$ for any $k = 1, \dots, N$. For any Borel set A and $k = 1, ..., N$, lemma 3.2 yields

$$\mu_h(k)(A) = m_0(\Upsilon_h(A,k)) \leq \|m_0\|_\infty \mathcal{L}^d(\Upsilon_h(A,k)) \leq \|m_0\|_\infty C_4 \mathcal{L}^d(A).$$

Thus, $\mu_h(k)$ is absolutely continuous and its density, still denoted by $\mu_h(k)$, satisfies $\|\mu_h(k)\|_\infty \leq C_4 \|m_0\|_\infty$, proving (ii). ■

Now, we consider the time continuous version for μ_h. Denote by $\hat{\Phi}_h(x,\cdot)$ the unique solution of

$$\partial_t \hat{\Phi}_h(x,t) = \bar{\alpha}_h(x,[t/h]), \quad \text{for } t \in [0,T], \quad \hat{\Phi}_h(x,0) = x, \tag{3.30}$$

which is nothing else than the linear interpolation of $\{\Phi_h(x,[t/h])\ ;\ t \in [0,T]\}$. Thus, when no confusion is allowed, we write still $\Phi_h(x,\cdot)$ for $\hat{\Phi}_h(x,\cdot)$. Analogously, define $\hat{\mu} : [0,T] \to \mathcal{P}_1$ as

$$\hat{\mu_h}(t) := \Phi_h(\cdot,t)\sharp m_0, \tag{3.31}$$

and, when no confusion is possible, we still write $\mu_h(\cdot)$ for $\hat{\mu}_h(\cdot)$. We summary in one lemma the extension of proposition 3.2 and lemma 3.5. The proofs being analogous are omitted.

Lemma 3.6 *There exists a constant $C_7 > 0$ (independent of small enough h) such that:*
(i) *For every $t \in [0,T]$ and $x, y \in \mathbb{R}^d$,*

$$C_7 |\Phi_h(x,t) - \Phi_h(y,t)| \geq |x - y|. \tag{3.32}$$

(i) *For all $t_1, t_2 \in [0,T]$, we have that*

$$\bar{d}_1(\mu_h(t_1), \mu_h(t_2)) \leq C_7 |t_1 - t_2|. \tag{3.33}$$

(ii) *For all $t \in [0,T]$, $\mu_h(t)$ is absolutely continuous (with density still denoted by $\mu_h(t)$), has a support in $B(0,C_7)$ and $||\mu_h(t)||_\infty \leq C_7$.*

We have the following convergence result.

Proposition 3.3 *As $h \downarrow 0$ we have that $\mu_h \to \mu$ in $C([0,T];\mathcal{P}_1)$.*

We do not provide the proof of the above result since it is a slight variation of the second part of the proof of theorem 4.2 in the next section.

4 The semi-discrete scheme for the deterministic MFG system

In this section we combine the analysis done in the previous one and we study an semi-discrete in time approximation of (1.4). Note that, in view of theorem 3.2, we seek for a pair (v,m) satisfying

$$\begin{array}{rcl} -\partial_t v(t,x) + \frac{1}{2}|Dv(t,x)|^2 &=& F(x,m(t)), \text{ in } \mathbb{R}^d \times (0,T), \\ m(t) &=& \Phi(\cdot,t)\sharp m_0 \quad \text{for } t \in [0,T], \\ v(x,T) = G(x,m(T)) & \text{for} & x \in \mathbb{R}^d, \quad m(0) = m_0 \in \mathcal{P}_1, \end{array} \tag{4.1}$$

where $\Phi(x,\cdot)$ is the given by (3.19), with $\alpha(x,\cdot)$ being a solution for the optimal control associated with $v(x,0)$. This reflects the coupled nature of (4.1). We set

$$\mathcal{K}_h = \left\{ (m(i))_{i=0}^N : \; m(i) \in \mathcal{P}_1 \quad \text{for all } i = 0, \dots, N \right\},$$

and, for given $h > 0$, consider the following discretization of (4.1)

$$\left\{ \begin{array}{rl} v_h(x,k) & = \inf_{\alpha \in \mathbb{R}^d} \left\{ v_h(x - h\alpha, k+1) + \frac{1}{2}h|\alpha|^2 \right\} + hF(x, m_h(k)); \; x \in \mathbb{R}^d, k = 0, \dots, N-1, \\ m_h(k) & = \Phi_h(\cdot,k)\sharp m_0 \quad \text{for } k = 1, \dots, N, \\ m_h(0) & = m_0 \in \mathcal{P}_1, \quad v_h(x,N) = G(x, m_h(N)) \quad \text{for } x \in \mathbb{R}^d. \end{array} \right. \tag{4.2}$$

where $\Phi_h(\cdot,k)$ is defined by (3.21) with discrete control being the solution of the r.h.s. in the first equation of (4.2). Our aim in this section is to analyze the existence and uniqueness of (v_h, m_h) satisfying (4.2), as well as its convergence to the solution (v,m) of (4.1). Let us first fix some notations that will be useful to prove a stability result (see lemma 4.1 below). For $m \in \mathcal{K}_h$ set $(w_h[m](\cdot,k))_{k=0}^N$ for the solution of

$$\left\{ \begin{array}{rl} w(x,k) & = \inf_{\alpha \in \mathbb{R}^d} \left\{ w(x - h\alpha, k+1) + \frac{1}{2}h|\alpha|^2 \right\} + F(x, m(k)); \quad x \in \mathbb{R}^d, \quad k = 0, \dots, N-1, \\ w(x,N) & = G(x, m(N)) \quad \text{for } x \in \mathbb{R}^d. \end{array} \right. \tag{4.3}$$

Next, we set $\alpha_h[m](x,k)$ for the unique solution of the r.h.s. of the first equation of (4.3) and we let $\mu_h[m](k) := \Phi_h[m](\cdot,k)\sharp m(0)$, where, for $x \in \mathbb{R}^d$, $\Phi_h[m](x,k)$ is given by the solution of

$$\left\{ \begin{array}{rcl} \Phi_{k+1} & = & \Phi_k - h\alpha[m](\Phi_k, k) \;\; \text{for } k = 0, \dots, N-1 \\ \Phi_0 & = & x. \end{array} \right. \tag{4.4}$$

Note that the existence of a solution for (4.2) is equivalent to find a fixed point of $m \in \mathcal{K}_h \to \mu_h[m] \in \mathcal{K}_h$. The following stability result holds true:

Lemma 4.1 *Let $h > 0$ be fixed and m^j, m in $\mathcal{K}_h$ such that $m^j(k) \to m(k)$ in $\mathbb{P}_1$ as $j \uparrow \infty$ for every $k = 0, \dots, N$. Then, the following assertions hold true:*
(i) *For $k = 0, \dots, N$, we have that $w_h[m^j](\cdot,k)$ converges locally uniformly to $w_h[m](\cdot,k)$.*
(ii) *For $k = 0, \dots, N-1$, the controls $\alpha_h[m^j](\cdot,k)$ converge locally uniformly to $\alpha_h[m](\cdot,k)$.*
(iii) *For $k = 1, \dots, N$, the flows $\Phi_h[m^j](\cdot,k)$ converge locally uniformly to $\Phi_h[m](\cdot,k)$*
(iv) *For $k = 1, \dots, N$, we have that $\mu_h[m^j](k)$ converges to $\mu_h[m](k)$ in $\mathcal{P}_1$.*

Proof. For notational convenience we denote $w^j(\cdot,k) := w_h[m^j](\cdot,k)$, with a similar convention for $\alpha_h[m^j](\cdot,k)$, $\Phi_h[m^j](\cdot,k)$, $\mu_h[m^j](k)$. For the limits we simply write $w(\cdot,k)$, $\alpha(\cdot,k)$, etc.
Proof of (i): Assumption **(H1)** for G imply easily (i) for $k = n$. Now, assuming the result for $k = n+1$, we have

$$|w^j(x,n) - w(x,n)| \leq \sup_{\alpha \in \mathbb{R}^d} \left\{ |w^j(x - h\alpha, n+1) - w(x - h\alpha, n+1)| \right\} + |F(x, m^j(n)) - F(x, m(n))|$$

and the result for $k = n$ follows from assumption **(H1)** for F and lemma 3.4(i).
Proof of (ii): Let us define the *semi-relaxed limits*

$$\alpha^*(x,k) := \limsup_{y \to x, \; j \uparrow \infty} \alpha^j(y,k), \quad \alpha_*(x,k) := \liminf_{y \to x, \; j \uparrow \infty} \alpha^j(y,k).$$

In the notation above, the limits have to be understood coordinate by coordinate. By definition,

$$w^j(y,k) = w^j(y - h\alpha^j(y,k), k+1) + \tfrac{1}{2}h|\alpha^j(y,k)|^2 + hF(y, m^j(k)). \tag{4.5}$$

Using (i) and passing to the limit along subsequences, lemma 3.3(iii) implies that

$$\alpha^*(x,k) = \alpha_*(x,k) = \alpha(x,k)$$

which gives the desired convergence.
Proof of (iii): Follows directly from the definition (4.4) and assertion (ii).
Proof of (iv):Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be 1- Lipschitz. By definition

$$\int_{\mathbb{R}^d} \phi(x) \mathrm{d}[\mu^j(k) - \mu(k)](x) \leq \sum_{i=0}^{k-1} \int_{\mathbb{R}^d} \left[\alpha^j(\Phi^j(x,i),i) - \alpha(\Phi(x,i),i)\right] \mathrm{d}m_0(x).$$

Using (ii) and (iii) we can pass to the limit in the above equation, which yields the result. ■

Theorem 4.1 *Given $h > 0$ small enough, there exists a unique solution for system* (4.2).

Proof.
Existence: Define the convex set convex set $\mathcal{C} = \{m \in \mathcal{K}_h : m(0) = m_0\}$, and consider the map $m \in \mathcal{C} \to \mu_h[m] \in \mathcal{C}$, which is a continuous map in view of lemma 4.1. Moreover, lemma 3.5 implies that $\hat{\mathcal{C}} := \{\mu[m] \; ; \; m \in \mathcal{C}\}$ is equi-continuous and $\mathrm{supp}\{\mu_h[m]\} \subseteq B(0, C_5)$. Therefore, by Ascoli theorem, $\hat{\mathcal{C}}$ is pre-compact and the existence results from Schauder fix point theorem.
Uniqueness: Let (v^1, m^1) and (v^2, m^2) be solutions of (4.2). We denote respectively by $\bar{\alpha}^1(\cdot)$, $\bar{\alpha}^2 \in \mathcal{A}^0(\cdot)$ the optimal controls for $v^1(\cdot, 0)$ and $v^2(\cdot, 0)$ and by $\Phi^1(\cdot,\cdot)$, $\Phi^2(\cdot,\cdot)$ the optimal flows. First note that for every continuous $\varphi : \mathbb{R}^d \times \mathcal{P}_1 \to \mathbb{R}$ and $k = 1, ..., N$, we have the identity

$$\int_{\mathbb{R}^d} \left[\varphi(x, m^1(k)) - \varphi(x, m^2(k))\right] \mathrm{d}\left[m^1 - m^2\right](k) = I_1(\varphi) + I_2(\varphi), \tag{4.6}$$

where

$$\begin{array}{rcl} I_1(\varphi) & := & \int_{\mathbb{R}^d} \left[\varphi(\Phi^1(x,k), m^1(k)) - \varphi(\Phi^2(x,k), m^1(k))\right] m_0(x)\mathrm{d}x, \\ I_2(\varphi) & := & \int_{\mathbb{R}^d} \left[\varphi(\Phi^2(x,k), m^2(k)) - \varphi(\Phi^1(x,k), m^2(k))\right] m_0(x)\mathrm{d}x. \end{array} \tag{4.7}$$

Now, let us set the following notations,

$$\begin{array}{r} \delta_1 F(k) := F(\Phi^1(x,k), m^1(k)) - F(\Phi^2(x,k), m^1(k)), \\ \delta_2 F(k) := F(\Phi^2(x,k), m^2(k)) - F(\Phi^1(x,k), m^2(k)), \\ \delta_1 G(N) := G(\Phi^1(x,N), m^1(N)) - G(\Phi^2(x,N), m^1(N)), \\ \delta_2 G(N) := G(\Phi^2(x,N), m^2(N)) - G(\Phi^1(x,N), m^2(N)). \end{array}$$

Since for $x \in \mathbb{R}^d$, $\bar{\alpha}^1(x,\cdot)$ is the minimum for $v^1(x,0)$, we have

$$\begin{array}{rcl} v^1(x,0) & = & \sum_{k=0}^{N-1} \left(\frac{1}{2}h|\bar{\alpha}^1(x,k)|^2 + hF(\Phi^1(x,k), m^1(k))\right) + G(\Phi^1(x,N), m^1(N)), \\ & \leq & \sum_{k=0}^{N-1} \left(\frac{1}{2}h|\bar{\alpha}^2(x,k)|^2 + hF(\Phi^2(x,k), m^1(k))\right) + G(\Phi^2(x,N), m^1(N)). \end{array} \tag{4.8}$$

Therefore, we have the following inequality

$$h \sum_{k=0}^{N-1} \delta_1 F(k) + \delta_1 G(N) \leq \tfrac{1}{2}h \sum_{k=0}^{N-1} \left(|\bar{\alpha}^2(x,k)|^2 - |\bar{\alpha}^1(x,k)|^2\right). \tag{4.9}$$

Analogously, since $\bar{\alpha}^2(x,\cdot)$ is the minimum for $v^2(x,0)$, we have

$$h \sum_{k=0}^{N-1} \delta_2 F(k) + \delta_2 G(N) \leq \tfrac{1}{2}h \sum_{k=0}^{N-1} \left(|\bar{\alpha}^1(x,k)|^2 - |\bar{\alpha}^2(x,k)|^2\right). \tag{4.10}$$

Adding (4.9) with (4.10) and integrating with respect to m_0 gives

$$h\sum_{k=0}^{N-1}\int_{\mathbb{R}^d}[\delta_1 F(k)+\delta_2 F(k)]m_0(x)\mathrm{d}x+\int_{\mathbb{R}^d}[\delta_1 G(N)+\delta_2 G(N)]m_0(x)\mathrm{d}x\le 0,$$

which, by (4.6) and (4.7), implies that

$$h\sum_{k=0}^{N-1}\int_{\mathbb{R}^d}\left(F(x,m^1(k))-F(x,m^2(k))\right)\mathrm{d}[m^1-m^2](k)+\int_{\mathbb{R}^d}\left(G(x,m^1(N))-G(x,m^2(N))\right)\mathrm{d}[m^1-m^2](N)\le 0.$$

Therefore, **(H3)** gives $m^1(\cdot)=m^2(\cdot)$ which in turns yields $v^1(\cdot,\cdot)=v^2(\cdot,\cdot)$. ■

Let us define the function $\hat{v}_h(\cdot,t):=v_h(\cdot,[\frac{t}{h}])$ and $\bar{m}_h(t):=\Phi_h(\cdot,t)\sharp m_0$, where for $x\in\mathbb{R}^d$, $\Phi_h(x,t)$ is defined as in (3.30) with $\bar{\alpha}(x,\cdot)\in\mathcal{A}^0$ being the unique solution associated to $v_h(x,0)$. As before, we write still v_h, m_h for $\hat{v}_h$ and $\hat{m}_h$ respectively. We now prove a convergence result

Theorem 4.2 *As $h\downarrow 0$, v_h converges locally uniformly to v and m_h converge to m in $C([0,T];\mathcal{P}_1)$.*

Proof. Lemma 3.6 and Ascoli theorem implies that m_h converges in $C([0,T];\mathcal{P}_1)$, up to some subsequence, to some measure $\bar{m}$. Now, let us define

$$v^*(x,t):=\limsup_{(x',t')\to(x,t),h\downarrow 0}v_h(x',t'),\quad v_*(x,t):=\liminf_{(x',t')\to(x,t),h\downarrow 0}v_h(x',t').$$

We prove now that v^* is a viscosity subsolution of

$$\begin{array}{rcll}-\partial_t v(x,t)+\frac{1}{2}|Dv(x,t)|^2&=&F(x,\bar{m}(t))&\text{for }(x,t)\in\mathbb{R}^d\times[0,T],\\ v(x,T)&=&G(x,\bar{m}(t))&\text{for }x\in\mathbb{R}^d.\end{array}\tag{4.11}$$

Let $(\bar{x},\bar{t})\in\mathbb{R}^d\times(0,T)$, $\bar{B}$ a ball of radius $r>0$ centered at $(\bar{x},\bar{t})$ and $\phi\in C^1(\mathbb{R}^d\times(0,T))$ such that $(v^*-\phi)(\bar{x},\bar{t})=\max_{(x,t)\in\bar{B}}(v^*-\phi)(x,t)$. Classical arguments imply the existence of (x_h,t_h), with $(x_h,t_h)\to(\bar{x},\bar{t})$, as $h\downarrow 0$ such that

$$(v_h-\phi)(x_h,t_h)=\max_{(x,t)\in\bar{B}}(v_h-\phi)(x,t).$$

Setting $n_h=[t_h/h]$, for h small enough we have

$$v_h(x_h-h\alpha_h(x_h,n_h),t_h+h)-\phi(x_h-h\alpha_h(x_h,n_h),t_h+h)\le v_h(x_h,t_h)-\phi(x_h,t_h)\tag{4.12}$$

On the other hand, setting $t'_h=h[t_h/h]$, lemma 3.4(iii) implies that

$$v_h(x_h,t_h)=\tfrac{1}{2}h\left|Dv_h(x_h,t'_h)-hDF(x_h,m_h(t'_h))\right|^2+v_h(x_h-h\alpha_h(x_h,n_h),t'_h+h)+hF(x_h,m_h(t'_h)).$$

Using that $Dv_h(x_h,t'_h)=D\phi(x_h,t'_h)$ and replacing the above expression in (4.12) yields

$$0\le\frac{1}{h}\left[\phi(x_h-h\alpha_h(x_h,n_h),t'_h+h)-\phi(x_h,t_h)\right]+\tfrac{1}{2}\left|D\phi(x_h,t'_h)-hDF(x_h,m_h(t'_h))\right|^2+F(x_h,m_h(t'_h)).$$

By passing to the limit in the above inequality, using lemma 3.4(ii) and the convergence of m_h to $\bar{m}$ in $C([0,T];\mathcal{P}_1)$ we have that

$$-\partial_t\phi(x,t)+\tfrac{1}{2}|D\phi(x,t)|^2\le F(x,\bar{m}(t)),$$

which means that v^* is a viscosity sub-solution of (4.11). Analogously, we can prove that v_* is a viscosity super-solution of (4.11). Therefore, by a classical comparison argument, v_h converges locally uniformly to the unique viscosity solution $v[\bar{m}]$ of (4.11).

In order to conclude the proof we have to show that $\bar{m} = \Phi(\cdot, 0)\sharp m_0 = m$. By remark 3.1 there exist a set $A \subseteq \mathbb{R}^d$, with $\mathcal{L}^d(\mathbb{R}^d \setminus A) = 0$, such that $\mathcal{A}(x, 0)$ is a singleton for all $x \in A$. Lemma 3.4(i) gives that the optimal controls $\bar{\alpha}_h(x, \cdot)$, associated with $v_h(x, 0)$, are bounded in $L^\infty([0, T]; \mathbb{R}^d)$ (in particular they are bounded in $L^2([0, T]; \mathbb{R}^d)$). Thus, up to subsequence, $\alpha_h(x, \cdot)$ converges weakly in L^2 to some $\bar{\alpha}$ and the flow $\Phi_h(x, \cdot)$ converge uniformly to some $\bar{\Phi}(x, \cdot)$. Now, since $J_h(x, 0) \to J(x, 0)$, by the uniqueness of the optimal control for $x \in A$, we obtain that $\bar{\alpha} = \alpha(x, 0)$ (the optimal control for $v(x, 0)$), and

$$\bar{\Phi}(x, \cdot) = \Phi(x, \cdot) = x - \int_0^{\cdot} \alpha(x, s)\mathrm{d}s, \quad \text{for all } x \in A.$$

Using that m_0 has compact support, by the dominated convergence theorem, the uniform convergence of Φ_h to $\bar{\Phi} = \Phi$ yields that for every 1-Lipschitz $\varphi : \mathbb{R}^d \to \mathbb{R}$ and $t \in [0, T]$,

$$\int_{\mathbb{R}^d} \varphi(x)\mathrm{d}[m_h(t) - m(t)](x) \leq \int_{\mathbb{R}^d} |\Phi_h(x, t) - \bar{\Phi}(x, t)| m_0(x)\mathrm{d}x \to 0 \quad \text{as } h \downarrow 0.$$

Therefore $\bar{m} = m$, which ends the proof. ■

A Appendix

Proof of lemma 3.3. *Proof of (i)* Since g is bounded by C by (3.2), we have that $g(x - h\alpha) + \frac{1}{2}h|\alpha|^2$ is bounded by the parabolas $-C + \frac{1}{2}h|\alpha|^2$ and $C + \frac{1}{2}h|\alpha|^2$. Thus, using that f is also bounded by C, equation (3.13) implies

$$w(x, N-1) \leq C(1 + h),$$

and by a recurrence argument

$$|w_h(x, n)| \leq C(1 + (N - n)h),$$

from which the result follows.
Proof of (ii) The existence of a solution follows easily using (i). Regarding the uniqueness, when $n = N - 1$, the result is given by lemma 3.2. Suppose that the result holds true for $n = k + 1$ and all $x \in \mathbb{R}^d$. We have, given $\alpha \in \mathbb{R}^d$ and denoting by $\bar{\alpha}(\alpha) \in \mathcal{A}^{k+1}$ the unique solution associated to $w_h(x - h\alpha, k + 1)$ in (3.11),

$$w_h(x, k) := \inf_{\alpha \in \mathbb{R}^d} \left\{ J(\bar{\alpha}(\alpha); x - \alpha h, k + 1) + \tfrac{1}{2}h|\alpha|^2 \right\} + hf(x, kh). \tag{A.1}$$

If $\hat{\alpha}$ is a solution of (3.15) then, denoting by $[\hat{\alpha}, \bar{\alpha}(\hat{\alpha})] \in \mathcal{A}^k$ the sequence obtained by the concatenation of $\hat{\alpha}$ with $\bar{\alpha}(\hat{\alpha})$, (A.1) implies that

$$\begin{aligned} w(x, k) &= J(\bar{\alpha}(\hat{\alpha}); x - h\hat{\alpha}, k + 1) + \tfrac{1}{2}h|\hat{\alpha}|^2 + hf(x, kh) \\ &= J([\hat{\alpha}, \bar{\alpha}(\hat{\alpha})]; x, k), \end{aligned}$$

and the result follows from lemma 3.2.
Proof of (iii) Using expression (3.11) and assumption (3.2) we obtain

$$\begin{aligned} |w_h(x, n) - w_h(y, n)| &\leq \sup_{\alpha \in \mathcal{A}^n} |J(\alpha; x, n) - J(\alpha; y, n)|, \\ &\leq \sup_{\alpha \in \mathcal{A}^n} \Big\{ h \textstyle\sum_{k=n}^{N-1} \left| f(X_k^{x,n}[\alpha], kh) - f(X_k^{y,n}[\alpha], kh) \right| \\ &\quad + \left| g(X_N^{x,n}[\alpha]) - g(X_N^{y,n}[\alpha]) \right| \Big\}, \\ &\leq C(Nh + 1)|x - y| = C(T + 1)|x - y|. \end{aligned}$$

On the other hand, let $x \in \mathbb{R}^d$ fixed and $n_1, n_2 \in \{1, ..., N\}$ with $n_1 < n_2$. Let $(\alpha)_{k=n_1}^{N-1} \in \mathcal{A}^{n_1}$ optimal for $J(\cdot; x, n_1)$ and let $(X_k)_{k=n_1}^{N}$ be its associated state. We have

$$\begin{aligned}
|w_h(x,n_1) - w_h(x,n_2)| &\leq |w_h(x,n_1) - w_h(X_{n_2},n_2)| + |w_h(X_{n_2},n_2) - w_h(x,n_2)|, \\
&\leq h \sum_{k=n_1}^{n_2-1} \left[\tfrac{1}{2}|\alpha_k|^2 + f(X_k,kh)\right] + C(T+1)|X_{n_2} - x|, \\
&= h \sum_{k=n_1}^{n_2-1} \left[\tfrac{1}{2}|\alpha_k|^2 + f(X_k,kh)\right] + C(T+1)h|\textstyle\sum_{k=n_1}^{n_2-1} \alpha_k|,
\end{aligned}$$

and the results follows the boundedness uniform on x in lemma 3.4(i).

Proof of (iv). By lemma 2.1(iv) $w_h(\cdot, N)$ is semi-concave. Now, suppose that the result is true for $n = k+1$ and call C_{k+1} the semi-concavity constant of $w_h(\cdot, k+1)$. We have, denoting by $\alpha_h(x,k)$ the optimal vector associated to $w_h(x,k)$,

$$\begin{aligned}
w_h(x,k) = w_h(x - h\alpha_h(x,k), k+1) + \tfrac{1}{2}h\,|\alpha_h(x,k)|^2 + hf(x,kh), \\
w_h(x \pm y, k) \leq w_h(x - h\alpha_h(x,n) \pm y, k+1) + \tfrac{1}{2}h\,|\alpha_h(x,k)|^2 + hf(x \pm y, kh).
\end{aligned}$$

Therefore, using the semi-concavity of $w_h(\cdot, k+1)$ and of $f(\cdot, kh)$, we easily obtain

$$w_h(x-y,k) + w_h(x+y,k) - 2w_h(x,k) \leq C_{k+1}|y|^2 + hC|y|^2,$$

where C is the constant in (3.2). By lemma 2.1(ii) this implies the result for $n = k$ with a semi-concavity constant $C_k \leq C_{k+1} + hC$. Thus, iterating we get that for every n, the function $w_k(\cdot, n)$ is semi-concave with a constant $C_n \leq C(1 + Nh) = C(T+1)$. ■

References

[1] Y. Achdou, F. Camilli, and I. Capuzzo Dolcetta. Mean field games: Numerical methods for the planning problem. *Preprint*, 2010.

[2] Y. Achdou and I. Capuzzo Dolcetta. Mean field games: Numerical methods. *SIAM Journal of Numerical Analysis*, 48-3:1136–1162, 2010.

[3] J.P. Aubin and H. Frankowska. *Set valued analysis.* Birkauser, Boston, 1990.

[4] M. Bardi and I. Capuzzo Dolcetta. *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations.* Birkauser, 1996.

[5] J.F. Bonnans and A. Shapiro. *Perturbation analysis of optimization problems.* Springer-Verlag, New York, 2000.

[6] P. Cannarsa and C. Sinestrari. *Semiconcave Functions, Hamilton-Jacobi Equations, and Optimal Control.* Progress in Nonlinear Differential Equations and Their Applications. Birkauser, 2004.

[7] P. Cardaliaguet. Notes on Mean Field Games: from P.-L. Lions' lectures at Collège de France. *Lecture Notes given at Tor Vergata*, 2010.

[8] I. Capuzzo Dolcetta. On a discrete approximation of the Hamilton-Jacobi equation of dynamic programming. *Appl. Math. Opt.*, 10:367–377, 1983.

[9] I. Capuzzo Dolcetta and M. Falcone. Discrete dynamic programming and viscosity solutions of the Bellman equation. *Annales de l'institut Henri Poincaré (C) Analyse non linéaire,*, S6:161–183, 1989.

[10] I. Capuzzo Dolcetta and H. Ishii. Approximate solutions of the Bellman equation of deterministic control theory. *Appl. Math. Opt.*, 11:161–181, 1984.

[11] M. Falcone and R. Ferretti. *Semi-Lagrangian Approximation Schemes for Linear and Hamilton-Jacobi Equations.* MOS-SIAM Series on Optimization, to appear.

[12] D.A. Gomes, J. Mohr, and R. Souza. Discrete time, finite state space mean field games,. *Journal de Mathématiques Pures et Appliquées*, 93:308–328, 2010.

[13] O. Guéant. Mean field games equations with quadratic hamiltonian: a specific approach. Submitted.

[14] A. Lachapelle, J. Salomon, and G. Turinici. Computation of mean field equilibria in economics. *Mathematical Models and Methods in Applied Sciences*, 20-4:567–588, 2010.

[15] J.-M. Lasry and P.-L. Lions. Jeux à champ moyen I. Le cas stationnaire. *C. R. Math. Acad. Sci. Paris*, 343:619–625, 2006.

[16] J.-M. Lasry and P.-L. Lions. Jeux à champ moyen II. Horizon fini et contrôle optimal. *C. R. Math. Acad. Sci. Paris*, 343:679–684, 2006.

[17] J.-M. Lasry and P.-L. Lions. Mean field games. *Jpn. J. Math.*, 2:229–260, 2007.

[18] P.-L. Lions. *Cours au Collège de France.* www.college-de-france.fr, 2007-2008.