Characterization of local quadratic growth for strong minima in the optimal control of semi-linear elliptic equations*

Térence Bayen† J. Frédéric Bonnans‡ Francisco J. Silva§

October 14, 2011

Abstract

In this article we consider an optimal control problem of a semi-linear elliptic equation, with bound constraints on the control. Our aim is to characterize local quadratic growth for the cost function J in the sense of strong solutions. This means that the function J growths quadratically over all feasible controls whose associated state is close enough to the nominal one, in the uniform topology. The study of strong solutions, classical in the Calculus of Variations, seems to be new in the context of PDE optimization. Our analysis, based on a decomposition result for the variation of the cost, combines Pontryagin's principle and second order conditions. While these two ingredients are known, we use them in such a way that we do not need to assume that the Hessian of Lagrangian of the problem is a Legendre form, or that it is uniformly positive on an extended set of critical directions.

Keywords. Optimal control of PDE, strong minima, Pontryagin's minimum principle, second order optimality conditions, control constraints, local quadratic growth.

MSC. 35J61, 49J20, 49K20.

1 Introduction

Over the last decades important progress has been done in the field of optimal control of Partial Differential Equations (PDEs). In the case of a semi-linear elliptic equations, we have particularly in mind (i) the extensions of Pontryagin's minimum principle [23], the first papers being due to [24], and then [3, 4, 5], and (ii) the theory of second order optimality conditions for weak minima [2, 10, 11, 12, 13, 14, 15, 16, 26]. By weak minimum we mean that optimality is ensured in a L^∞-neighborhood in the *control space*.

To be more precise, regarding second order optimality conditions, the authors of this article are aware of two sufficient conditions that imply local quadratic growth for the cost function in the *weak sense*. This means that the cost growths quadratically, with respect to the L^2-norm, in a feasible L^∞- neighborhood of the nominal control. Both aforementioned conditions ask that a *weak* form of the Pontryagin's principle is satisfied, while they differ on the second order condition. In references [2, 10] it is supposed that the Hessian of Lagrangian of the problem is a Legendre form

*The support of the Chair "Modélisation Mathématique et Simulation Numérique", EADS-Polytechnique-INRIA and of the European Union under the "7th Framework Programme FP7-PEOPLE-2010-ITN Grant agreement number 264735-SADCO" are gratefully acknowledged.

†Université Montpellier 2, Case courrier 051, 34095 Montpellier cedex 5, France (tbayen@math.univ-montp2.fr).

‡INRIA-Saclay and CMAP, École Polytechnique, 91128 Palaiseau, France (Frederic.Bonnans@inria.fr).

§Dipartimento di Matematica Guido Castelnuovo, 00185 Rome, Italy (fsilva@mat.uniroma1.it).

and that it is uniformly positive over a cone of *critical directions*, which is exactly the cone provided by second order necessary conditions. On the other hand, in references [11, 12, 13, 14, 15, 16, 26] the Legendre form assumption is not needed, but it is required a uniform positivity condition of Hessian of Lagrangian over a slightly larger cone than the critical one.

This paper is devoted to the study of *strong solutions* for the optimal control problem of a semi-linear elliptic equation with Dirichlet boundary conditions, under bounds constraints on the controls. By strong solutions we mean, as in the classical calculus of variations, optimality in a neighborhood in the *state space* only. Thus, the neighborhoods are considered in the state space rather than in the control space.

We are particularly interested in providing a characterization of local quadratic growth for the cost function in the *strong sense*. This means that quadratic growth for the cost, with respect to the L^2-norm, holds over all feasible controls whose associated states are uniformly close to the nominal one. Our main result is theorem 4.23 which states that local quadratic growth for the cost holds in the strong sense at $\overline{u}$ if and only if the Hessian of Lagrangian of the problem is uniformly positive over the cone of critical directions, and the Hamiltonian satisfies a global quadratic growth property at $\overline{u}$. Since this is the first result in this direction for the PDE framework, we decided to present it in the rather simple framework of bounds constraints. Important extensions like non-local constraints over the control and state constraints will be addressed in future works.

The proof of theorem 4.23 relies on the combination of Pontryagin's minimum principle and second order conditions, which is possible thanks to the extension of a decomposition result in [6, theorem 2.14] to the elliptic framework. Roughly speaking, theorem 3.5 says that the variation of the cost function under a perturbation of the control can be expressed as the sum of two terms: the first one is the variation due to a *large* perturbation in the L^∞-norm but with support over a set of small measure, while the second one correspond to the variation due to a perturbation *small* in the L^∞-norm. The key elements in the proof of the decomposition result are the well known regularity estimates from the L^s-theory of linear elliptic equations (see e.g. [18]).

The article is organized as follows: in section 2 we set some useful notations, recall some basics facts about linear and semi-linear elliptic equations, define the optimal control problem $(\mathcal{CP})$ and set some standard assumptions over the data. Using the estimates stated in section 2, we are able in section 3 to give first and second order expansions for the state and cost functions. The latter are expressed in terms of an associated Hamiltonian. The novelty of this section is theorem 3.5, where the decomposition result is proved. Section 4 begins with the statement and proof of an extension of the standard Pontryagin's minimum principle. This result allows us to show that weak local solutions satisfy a *local Pontryagin's minimum principle*. Next, after recalling some well-known facts about necessary conditions for weak solutions, we prove in theorem 4.16 that if local quadratic growth for the cost holds at $\overline{u}$ in the strong sense, then $\overline{u}$ satisfies a strict Pontryagin inequality and the associated quadratic form, i.e. the Hessian of the Lagrangian of $(\mathcal{CP})$, is uniformly positive over the cone of critical directions. Regarding sufficient conditions, the decomposition result allows us in theorems 4.19 and 4.22 to extend to the strong sense, respectively, theorems [2, theorem 2.9] and [15, theorem 2] . Finally, in theorem 4.23 we characterize the local quadratic growth in the strong sense, i.e. we provide the converse implication of theorem 4.16. In order to prove the result, we adapt the technique of projection on the pointwise critical cone due to [6, theorem 5.5]. The article concludes with an appendix containing the proofs of some technical lemmas of sections 3 and 4.

2 Problem statement and preliminary results.

We first fix some useful notations. For a function $\psi : \mathbb{R}^d \to \mathbb{R}$ ($d \in \mathbb{N}$), a nominal $\bar{x} \in \mathbb{R}^d$ and a perturbation $z \in \mathbb{R}$, we set $\psi_{x_i}(\bar{x})z := D_{x_i}\psi(\bar{x})z$ for $i = 1, \ldots, d$. Analogously, for $z_1, z_2 \in \mathbb{R}$ we use the following convention

$$\psi_{x_ix_i}(\bar{x})z_1^2 := D^2_{x_ix_i}\psi(\bar{x})(z_1,z_1), \quad \psi_{x_ix_j}(\bar{x})z_1z_2 := D^2_{x_ix_j}\psi(\bar{x})(z_1,z_2),$$
$$\psi_{(x_i,x_j)^2}(\bar{x})(z_1,z_2)^2 := \psi_{x_ix_i}(\bar{x})z_1^2 + 2\psi_{x_ix_j}(\bar{x})z_1z_2 + \psi_{x_jx_j}(\bar{x})z_2^2.$$

From now on, we fix a non-empty bounded open set $\Omega \subseteq \mathbb{R}^n$ ($n \in \mathbb{N}$) with a $C^{1,1}$ boundary and for $s \in [1,\infty]$, $k \in \mathbb{N}$, we denote by $\|\cdot\|_s$ and $\|\cdot\|_{k,s}$ the standard norms in $L^s(\Omega)$ and $W^{k,s}(\Omega)$, respectively. For any Borel set $A \subseteq \mathbb{R}^n$ we denote by $|A|$ its Lebesgue measure and when a property holds for almost all $x \in \Omega$, with respect to the Lebesgue measure, we use the abbreviation "for a.a. $x \in \Omega$". For future reference we recall the following Sobolev embeddings (cf. [1], [17], [18]):

$$W^{m,s}(\Omega) \subseteq \begin{cases} L^{q_1}(\Omega) & \text{with } \frac{1}{q_1} = \frac{1}{s} - \frac{m}{n} \quad \text{if } s < \frac{n}{m}, \\ L^q(\Omega) & \text{with } q \in [1,+\infty) \quad \text{if } s = \frac{n}{m}, \\ C^{m-\lfloor \frac{n}{s} \rfloor - 1, \gamma(n,s)}(\overline{\Omega}) & \text{if } s > \frac{n}{m}, \end{cases} \tag{2.1}$$

where the injections are continuous and $\gamma(n,s)$ is defined as

$$\gamma(n,s) = \begin{cases} \lfloor \frac{n}{s} \rfloor - \frac{n}{s} + 1, & \text{if } \frac{n}{s} \notin \mathbb{Z}, \\ \text{any positive number} < 1 & \text{if } \frac{n}{s} \in \mathbb{Z}. \end{cases} \tag{2.2}$$

The next well known regularity result for *linear elliptic equations* will be very useful.

Theorem 2.1. *Let $R > 0$ be given. Consider the following Dirichlet problem*

$$\begin{cases} -\Delta z + \alpha(x) z &= f(x) \quad \text{in } \Omega, \\ z &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

where $\alpha \in L^\infty(\Omega)$ satisfies $\alpha(x) \geq 0$ for a.a. $x \in \Omega$, and $\|\alpha\|_\infty \leq R$. Then, for every $s \in (1,\infty)$ and $f \in L^s(\Omega)$, equation (2.3) admits a unique strong solution $z \in W^{2,s}(\Omega) \cap W_0^{1,s}(\Omega)$ and there exists $c_s = c_s(R) > 0$ such that for every $f \in L^s(\Omega)$,

$$\|z\|_{2,s} \leq c_s \|f\|_s. \tag{2.4}$$

Moreover, there exists $c_1 = c_1(R) > 0$ such that the following L^1-estimate holds true

$$\|z\|_1 \leq c_1 \|f\|_1. \tag{2.5}$$

Remark 2.2. The proof of (2.4) can be found in [18, Theorem 9.15 and Lemma 9.17] while (2.5) is a corollary of Stampacchia's results in [27] (see also [9, lemma 2.11] for a simple proof). By the Sobolev embeddings (2.1), inequality (2.4) implies that if $s > n/2$ ($s = 2$ if $n \leq 3$), then $z \in C(\overline{\Omega})$ and $\|z\|_\infty \leq c_s \|f\|_s$.

In this work, we are concerned with the following controlled semi-linear elliptic equation:

$$\begin{cases} -\Delta y + \varphi(x,y,u) &= 0 \quad \text{in } \Omega, \\ y &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2.6}$$

where $\varphi : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. For $a, b \in C(\overline{\Omega})$ with $a \leq b$ in $\overline{\Omega}$, we suppose that the controls u take values in the set $\mathcal{K}$ of *admissible controls*:

$$\mathcal{K} = \{u \in L^\infty(\Omega) \mid a(x) \leq u(x) \leq b(x), \text{ for a.a. } x \in \Omega\}. \tag{2.7}$$

Let us denote by $M := \max\{\|a\|_\infty, \|b\|_\infty\}$. We will assume the following assumption:

(H1) The function φ is continuous and satisfies:

(i) For all $x \in \overline{\Omega}$ we have that $\varphi(x,\cdot,\cdot)$ is C^1. Moreover, uniformly on $x \in \overline{\Omega}$ we have that (i.1) $D_{(y,u)}\varphi(x,0,0)$ is bounded; (i.2) $D_{(y,u)}\varphi(x,\cdot,\cdot)$ is locally Lipschitz.

(ii) For all $(x,y)\in\overline{\Omega}\times\mathbb{R}$ and $|u|\leq M$, we have $\varphi_y(x,y,u)\geq 0$.

Example 2.3. A typical example of φ satisfying **(H1)** is $\varphi(x,y,u)=g(y)+u+f(x)$, where $f\in C(\overline{\Omega})$, and $g\in C^2(\mathbb{R})$ and satisfies $g_y\geq 0$.

The following result is well known (see e.g. [21, chapter 5, proposition 1.1])

Proposition 2.4. *Under* **(H1)**, *for every* $u\in\mathcal{K}$ *and* $s\in(n/2,\infty)$, *equation* (2.6) *has a unique strong solution* $y_u\in W_0^{1,s}(\Omega)\cap W^{2,s}(\Omega)$. *In particular, we have that* y_u *is continuous. Moreover, there exists a constant* C_s *depending only on* s, *such that*

$$\|y_u\|_\infty+\|y_u\|_{2,s}\leq C_s,\quad \textit{for all } u\in\mathcal{K}. \tag{2.8}$$

Consider a function $\ell:\overline{\Omega}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and assume that:

(H2) The function ℓ satisfies the same assumptions for φ in **(H1)** except for (ii).

Let us define the *cost function* $J:L^\infty(\Omega)\to\mathbb{R}$ by

$$J(u):=\int_\Omega \ell(x,y_u(x),u(x))\mathrm{d}x. \tag{2.9}$$

In this work we are concerned with the following *optimal control problem*:

$$\min J(u)\quad \text{subject to } u\in\mathcal{K}. \tag{$\mathcal{CP}$}$$

As we will see in the next section, assumptions **(H1)-(H2)** will allow us to obtain well known first order expansions for the state and the cost functions. However, in order to provide second order expansions we will need the following assumption:

(H3) For all $x\in\overline{\Omega}$ and $\psi=\varphi,\ell$, we have that $\psi(x,\cdot,\cdot)$ is C^2. Moreover, uniformly on $x\in\overline{\Omega}$ we have that

(i) $D^2_{(y,u)^2}\psi(x,0,0)$ is bounded; (ii) $D^2_{(y,u)^2}\psi(x,\cdot,\cdot)$ is locally Lipschitz.

See example 4.20 for a typical problem satisfying **(H1)-(H3)**. We end this section by recalling the notion of weak and strong local solutions and the so-called local quadratic growth conditions for J.

Definition 2.5. For a fixed $\overline{u}\in\mathcal{K}$ and $s\in(1,\infty)$, we say that:
(i) $\overline{u}$ is a L^s*-weak local minimum* (*weak local minimum if* $s=\infty$) of J on $\mathcal{K}$ if there exists $\varepsilon>0$ such that

$$J(u)\geq J(\overline{u})\quad \text{for all } u\in\mathcal{K} \text{ with } \|u-\overline{u}\|_s\leq\varepsilon. \tag{2.10}$$

In this case we also speak of L^s*-weak local solution* (*weak local solution if* $s=\infty$) of ($\mathcal{CP}$).
(ii) $\overline{u}$ is a *strong local minimum* of J on $\mathcal{K}$ if there exists $\varepsilon>0$ such that

$$J(u)\geq J(\overline{u})\quad \text{for all } u\in\mathcal{K} \text{ with } \|y_u-y_{\overline{u}}\|_\infty\leq\varepsilon. \tag{2.11}$$

In this case we also speak of *strong local solution* of ($\mathcal{CP}$).

The following remarks exploit the fact that the set $\mathcal{K}$ is bounded in $L^\infty(\Omega)$.

Remark 2.6. (i) Let $1 \leq p < q < \infty$. We claim that $L^p(\Omega)$ and $L^q(\Omega)$ have the same topology (open sets) on $\mathcal{K}$. Indeed, since Ω is bounded it is known that $L^q(\Omega) \subset L^p(\Omega)$ with continuous injection, proving that each (relatively) open set of $L^p(\Omega)$ is open in $L^q(\Omega)$. On the other hand, for $M = \max(\|a\|_\infty, \|b\|_\infty)$ and $u \in \mathcal{K}$, we have $\|u\|_p^p \leq M^{p-q}\|u\|_q^q$, proving that each open set of $L^q(\Omega) \cap \mathcal{K}$ is open in $L^p(\Omega) \cap \mathcal{K}$.
(ii) It follows that the notion of L^s-weak local solution is equivalent for all $s \in [1, \infty)$. In this case we simply speak of L^1-weak local solution. Obviously, any L^1-weak local solution is a weak local solution.
(iii) In lemma 3.1, stated in next section, we check that $\|y_u - y_{\overline{u}}\|_\infty = O(\|u - \overline{u}\|_s)$ for all $\overline{u}, u \in \mathcal{K}$ and $s \in (n/2, \infty)$. Therefore, every strong local solution is a L^s-weak local solution and, in view of point (ii), is also a L^1-weak local solution.

We now define the corresponding types of local quadratic growth for J at $\overline{u} \in \mathcal{K}$.

Definition 2.7. Given $\overline{u} \in \mathcal{K}$ and $s \in (1, \infty)$, we say that:
(i) J has local quadratic growth on $\mathcal{K}$ at $\overline{u}$ in the *L^s-weak sense* (in the *weak sense* if $s = \infty$) if there exists $\alpha, \varepsilon > 0$ such that

$$J(u) \geq J(\overline{u}) + \alpha\|u - \overline{u}\|_2^2 \quad \text{for all } u \in \mathcal{K} \text{ with } \|u - \overline{u}\|_s \leq \varepsilon. \tag{2.12}$$

(ii) J has local quadratic growth on $\mathcal{K}$ at $\overline{u}$ in the *strong sense* if there exists $\alpha, \varepsilon > 0$ such that

$$J(u) \geq J(\overline{u}) + \alpha\|u - \overline{u}\|_2^2 \quad \text{for all } u \in \mathcal{K} \text{ with } \|y_u - y_{\overline{u}}\|_\infty \leq \varepsilon. \tag{2.13}$$

Remark 2.8. In view of remark 2.6, local quadratic growth in the L^s-weak sense is equivalent to the local quadratic growth in the L^1-weak sense. The latter implies local quadratic growth in the weak sense, and it is implied by local quadratic growth in the strong sense.

3 Expansions for the state and the cost functions

In this section, we establish first and second order expansions for the state and the cost functions. The novelty is theorem 3.5 where a decomposition result for the variation of the cost is provided. In the entire section, we fix some $\overline{u} \in \mathcal{K}$ and we set $\bar{y} := y_{\overline{u}}$ for its associated state. For notational convenience, we often omit the dependence on x of certain functions such as $y_u(\cdot)$ and $u(\cdot)$.

3.1 First order expansions

We define the Hamiltonian $H : \Omega \times \mathbb{R}^3 \to \mathbb{R}$ associated to $(\mathcal{CP})$ by

$$H(x, y, p, u) = \ell(x, y, u) - p\varphi(x, y, u). \tag{3.1}$$

The *adjoint state* $\bar{p}$, associated to $\overline{u}$, is defined as the unique solution in $H_0^1(\Omega) \cap C(\overline{\Omega})$ of

$$\begin{cases} -\Delta\overline{p} = H_y(x, \overline{y}, \overline{p}, \overline{u}) & \text{in } \Omega, \\ \overline{p} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Let us now fix some useful notation. For $\psi = \ell, \varphi$, when there is no ambiguity, we write $\psi(x)$, $\psi_y(x)$, $\psi_u(x)$, $\psi_{yy}(x)$, $\psi_{yu}(x)$ for the value of ψ, ψ_y, ψ_u, ψ_{yy}, ψ_{yu} on $(x, \bar{y}(x), \overline{u}(x))$, respectively.

Similar notations are used for H and its derivatives evaluated at $(x, \overline{y}(x), \overline{p}(x), \overline{u}(x))$, for example $H(x) := H(x, \overline{y}(x), \overline{p}(x), \overline{u}(x))$. Moreover, for a fixed $u \in \mathcal{K}$ we set

$$\begin{cases} \delta\psi(x) = \psi(x, \overline{y}(x), u(x)) - \psi(x), \quad \delta\psi_y(x) = \psi_y(x, \overline{y}(x), u(x)) - \psi_y(x), \\ \delta H(x) = H(x, \overline{y}(x), \overline{p}(x), u(x)) - H(x), \quad \delta H_y(x) = H_y(x, \overline{y}(x), \overline{p}(x), u(x)) - H_y(x). \end{cases} \tag{3.3}$$

The *first order Pontryagin linearization* $z_1[u]$ of $u \in \mathcal{K} \to y_u \in H_0^1(\Omega) \cap C(\overline{\Omega})$ in the direction $u - \overline{u}$ is defined as the unique solution in $H_0^1(\Omega) \cap C(\overline{\Omega})$ of

$$\begin{cases} -\Delta z_1 + \varphi_y(x) z_1 + \delta\varphi(x) &= 0 \quad \text{on } \Omega, \\ z_1 &= 0 \quad \text{in } \partial\Omega. \end{cases} \tag{3.4}$$

For a fixed $u \in \mathcal{K}$, set

$$\delta u := u - \overline{u}, \quad \delta y[u] = y_u - \overline{y}, \quad d_1[u] = \delta y - z_1[u]. \tag{3.5}$$

When the context is clear, we will write $z_1 = z_1[u]$, $\delta y = \delta y[u]$, $d_1 = d_1[u]$. The next lemma is proved in the appendix.

Lemma 3.1. *Under **(H1)-(H2)**, for every $s \in (n/2, \infty)$ ($s = 2$ if $n \le 3$) we have:*

$$\begin{cases} \|\delta y\|_1 = O(\|\delta u\|_1), \ \ \|\delta y\|_2 = O(\|\delta u\|_2), \ \ \|\delta y\|_\infty = O(\|\delta u\|_s), \\ \|z_1\|_1 = O(\|\delta u\|_1), \ \ \|z_1\|_2 = O(\|\delta u\|_2), \ \ \|z_1\|_\infty = O(\|\delta u\|_s), \\ \|d_1\|_1 = O(\|\delta u\|_1 \|\delta u\|_s), \ \ \|d_1\|_2 = O(\|\delta y\|_\infty \|\delta u\|_2). \end{cases} \tag{3.6}$$

We now provide a first order expansion of the cost function. For future reference, we note that multiplying (3.4) by $\overline{p}$, integrating by parts and using the adjoint equation, we get

$$\int_\Omega \ell_y(x, \overline{y}, \overline{u}) z_1 \mathrm{d}x + \int_\Omega \delta\varphi(x) \overline{p} \ \mathrm{d}x = 0. \tag{3.7}$$

Lemma 3.2. *Under the assumptions of lemma 3.1 we have*

$$J(u) - J(\overline{u}) = \int_\Omega \delta H(x) \mathrm{d}x + O(\|\delta y\|_\infty \|\delta u\|_2), \tag{3.8}$$

$$J(u) - J(\overline{u}) = \int_\Omega \delta H(x) \mathrm{d}x + O(\|\delta u\|_1 \|\delta u\|_s). \tag{3.9}$$

Proof. By doing a Taylor expansion for ℓ, we get the following equalities:

$$\begin{aligned} J(u) - J(\overline{u}) &= \int_\Omega [\ell(x, y, u) - \ell(x, \overline{y}, \overline{u})] \, \mathrm{d}x = \int_\Omega [\ell(x, y, u) - \ell(x, \overline{y}, u) + \delta\ell(x)] \, \mathrm{d}x, \\ &= \int_\Omega [\ell_y(x, \overline{y}, u)\delta y + \delta\ell(x)] \, \mathrm{d}x + O\left(\int_\Omega |\delta y|^2 \mathrm{d}x \right), \\ &= \int_\Omega [\ell_y(x, \overline{y}, \overline{u})\delta y + \delta\ell_y(x)\delta y + \delta\ell(x)] \, \mathrm{d}x + O\left(\int_\Omega |\delta y|^2 \mathrm{d}x \right). \end{aligned}$$

Since ℓ_y is uniformly Lipschitz, we have $|\delta\ell_y(x)| = O(|\delta u|)$, hence, introducing z_1 leads to:

$$J(u) - J(\overline{u}) = \int_\Omega [\ell_y(x, \overline{y}, \overline{u}) z_1 + \delta\ell(x)] \, \mathrm{d}x + O\left(\int_\Omega \left[|\delta y|^2 + |\delta u \delta y| + |d_1| \right] \mathrm{d}x \right).$$

Using (3.7), we get:

$$J(u) - J(\overline{u}) = \int_\Omega \delta H(x)\mathrm{d}x + O\left(\int_\Omega \left[|\delta y|^2 + |\delta u \delta y| + |d_1|\right] \mathrm{d}x\right). \tag{3.10}$$

Using that $\|d_1\|_1 \leq |\Omega|^{1/2}\|d_1\|_2$, lemma 3.1 implies that

$$\int_\Omega \left(|\delta y|^2 + |\delta u \delta y| + |d_1|\right) \mathrm{d}x = O\left(\|\delta u\|_1 \|\delta y\|_\infty + \|\delta u\|_2 \|\delta y\|_\infty\right) = O\left(\|\delta u\|_2 \|\delta y\|_\infty\right),$$

which proves (3.8). Similarly, combining (3.10) and lemma 3.1 yields to (3.9). □

3.2 Second order expansions for the state and the cost function

The *second order Pontryagin linearization* $z_2[u]$ of $u \in \mathcal{K} \to y_u \in H_0^1(\Omega) \cap C(\overline{\Omega})$ in the direction $u - \overline{u}$ is defined as the unique solution in $H_0^1(\Omega) \cap C(\overline{\Omega})$ of

$$\begin{aligned} -\Delta z_2 + \varphi_y(x) z_2 + \tfrac{1}{2}\varphi_{yy}(x)(z_1[u])^2 + \delta\varphi_y(x) z_1[u] &= 0 \text{ in } \Omega, \\ z_2 &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{3.11}$$

where $z_1[u]$ is defined by (3.4). Let us define (recall (3.5))

$$d_2[u] := \delta y[u] - (z_1[u] + z_2[u]) = d_1[u] - z_2[u]. \tag{3.12}$$

When the context is clear, we will write $z_2 = z_2[u]$ and $d_2 = d_2[u]$. The next lemma is proved in the appendix.

Lemma 3.3. *Under **(H1)-(H3)** we have* $\|d_2\|_1 = O(\|\delta y\|_\infty \|\delta u\|_2^2)$.

We now prove the following second order expansion of the cost.

Proposition 3.4. *Suppose that **(H1)-(H3)** hold true, then for every* $u \in \mathcal{K}$ *we have*

$$J(u) - J(\overline{u}) = \int_\Omega \left[\delta H(x) + \delta H_y(x) z_1 + \tfrac{1}{2} H_{yy}(x) z_1^2\right] \mathrm{d}x + o(\|\delta u\|_2^2). \tag{3.13}$$

Proof. Writing $\ell(x, y, u) - \ell(x, \overline{y}, \overline{u}) = \ell(x, y, u) - \ell(x, \overline{y}, u) + \delta\ell(x)$, a Taylor expansion gives

$$J(u) - J(\overline{u}) = \int_\Omega \left[\delta\ell(x) + \ell_y(x, \overline{y}, u)\delta y + \tfrac{1}{2}\ell_{yy}(x, \overline{y}, u)(\delta y)^2\right] \mathrm{d}x + O\left(\int_\Omega |\delta y|^3 \mathrm{d}x\right).$$

Using that $\ell_{yy}(x, \cdot, \cdot)$ is locally Lipschitz, we get

$$J(u) - J(\overline{u}) = \int_\Omega \left[\delta\ell(x) + \delta\ell_y(x)\delta y + \ell_y(x, \overline{y}, \overline{u})\delta y + \tfrac{1}{2}\ell_{yy}(x, \overline{y}, \overline{u})(\delta y)^2\right] \mathrm{d}x + O\left(\int_\Omega [|\delta y|^2 |\delta u| + |\delta y|^3] \mathrm{d}x\right).$$

Introducing z_1, z_2 and using that $\ell_y(x, \cdot, \cdot)$ is locally Lipschitz gives:

$$\begin{cases} \int_\Omega \delta\ell_y(x)\delta y \ \mathrm{d}x &= \int_\Omega \delta\ell_y(x) z_1 \ \mathrm{d}x + O\left(\int_\Omega |\delta u||d_1| \ \mathrm{d}x\right), \\ \int_\Omega \ell_y(x, \overline{y}, \overline{u})\delta y \ \mathrm{d}x &= \int_\Omega \ell_y(x, \overline{y}, \overline{u})(z_1 + z_2) \ \mathrm{d}x + O(\int_\Omega |d_2| \ \mathrm{d}x), \\ \int_\Omega \ell_{yy}(x, \overline{y}, \overline{u})(\delta y)^2 \ \mathrm{d}x &= \int_\Omega \ell_{yy}(x, \overline{y}, \overline{u})(z_1)^2 \ \mathrm{d}x + O\left(\int_\Omega |d_1(\delta y + z_1)| \ \mathrm{d}x\right). \end{cases}$$

Lemmas 3.1 and 3.3 yield:

$$\int_\Omega \left(|\delta u||d_1| + |d_2| + |d_1||\delta y| + |d_1||z_1| + |\delta y|^2|\delta u| + |\delta y|^3\right) \mathrm{d}x = O(\|\delta y\|_\infty \|\delta u\|_2^2).$$

On the other hand, by remark 2.6, $\|\delta y\|_\infty \to 0$ when $\|\delta u\|_2^2 \to 0$, so that

$$J(u) - J(\overline{u}) = \int_\Omega \left[\delta\ell(x) + \delta\ell_y(x)z_1 + \ell_y(x,\overline{y},\overline{u})(z_1+z_2) + \tfrac{1}{2}\ell_{yy}(x,\overline{y},\overline{u})(z_1)^2\right] \mathrm{d}x + o(\|\delta u\|_2^2). \quad (3.14)$$

Multiplying (3.11) by $\overline{p}$ and integrating by parts gives

$$\int_\Omega \ell_y(x,\overline{y},\overline{u})z_2 \, \mathrm{d}x + \int_\Omega \tfrac{1}{2}\varphi_{yy}(x,\overline{y},\overline{u})(z_1)^2\overline{p} \, \mathrm{d}x + \int_\Omega \delta\varphi_y(x)z_1\overline{p} \, \mathrm{d}x = 0. \quad (3.15)$$

We conclude by combining (3.7) and (3.15) with (3.14). □

3.3 A Decomposition result

Let u_k be a sequence in $\mathcal{K}$, set $\delta_k u = u_k - \overline{u}$ and suppose that $\|\delta_k u\|_2 \to 0$ as $k \uparrow \infty$. Let us define $y_k := y_{u_k}$, $\delta_k y = y_k - \overline{y}$ and $z_1^k := z_1[u_k]$ (recall (3.4)). Next, consider a sequence of measurable sets A_k and B_k such that

$$|A_k \cup B_k| = |\Omega|, \;\; |A_k \cap B_k| = 0 \;\text{ and }\; |B_k| \to 0. \quad (3.16)$$

We decompose the sequence u_k into u_{A_k} and u_{B_k} defined by :

$$\begin{cases} u_{A_k} = u_k & \text{on } A_k, \quad u_{A_k} = \overline{u} \quad \text{on } B_k, \\ u_{B_k} = \overline{u} & \text{on } A_k, \quad u_{B_k} = u_k \quad \text{on } B_k. \end{cases}$$

We set

$$\delta_{A_k}u := u_{A_k} - \overline{u}, \qquad \delta_{B_k}u := u_{B_k} - \overline{u} \quad \text{and hence} \quad \delta_k u = \delta_{A_k}u + \delta_{B_k}u. \quad (3.17)$$

Let us set $z_{A_k} := z_1[u_{A_k}]$ and $z_{B_k} := z_1[u_{B_k}]$. Since $|A_k \cap B_k| = 0$ we have, by uniqueness of the Dirichlet problem, that $z_1^k = z_{A_k} + z_{B_k}$. From (2.4), we obtain

$$\|z_{A_k}\|_{2,s} \le c_s\|\delta_{A_k}u\|_s, \;\; \|z_{B_k}\|_{2,s} \le c_s\|\delta_{B_k}u\|_s \;\text{ for all } s \in (1,\infty). \quad (3.18)$$

Finally, let us note $\delta H^k(x)$ and $\delta H_y^k(x)$ the value of $\delta H(x)$ and $\delta H_y(x)$ with u_k in place of u in (3.3). We have the following *decomposition result*:

Theorem 3.5. *Suppose that **(H1)**-**(H3)** hold true and let $\overline{u} \in \mathcal{K}$. Let $u_k \in \mathcal{K}$ be such that $\|\delta_k u\|_2 \to 0$, and A_k, B_k and $\delta_{A_k}u$, $\delta_{B_k}u$ be as in* (3.16) *and* (3.17)*, respectively. If $\|\delta_{A_k}u\|_\infty \to 0$, then we have that*

$$J(u_k) - J(\overline{u}) = \int_{B_k} \delta H^k(x)\mathrm{d}x + \int_{A_k} H_u(x)\delta_{A_k}u \; \mathrm{d}x$$
$$+ \tfrac{1}{2}\int_\Omega \left[H_{uu}(x)(\delta_{A_k}u)^2 + 2H_{yu}(x)z_{A_k}\delta_{A_k}u + H_{yy}(x)(z_{A_k})^2\right] \mathrm{d}x + o(\|\delta_k u\|_2^2). \quad (3.19)$$

Proof. Given $p \in (1,\infty)$ we will denote by p^* its conjugate, i.e. $p^* = p/(p-1)$. Proposition 3.4, with u_k in place of u, yields

$$J(u_k) - J(\overline{u}) = \int_\Omega \left[\delta H^k(x) + \delta H_y^k(x)z_1^k + \tfrac{1}{2}H_{yy}(x)(z_1^k)^2\right] \mathrm{d}x + o(\|\delta_k u\|_2^2). \quad (3.20)$$

a) We first prove that

$$\int_\Omega \left[\delta H_y^k(x)z_1^k + \tfrac{1}{2}H_{yy}(x)(z_1^k)^2\right] \mathrm{d}x = \int_{A_k} \left[\delta H_y^k(x)z_{A_k} + \tfrac{1}{2}H_{yy}(x)(z_{A_k})^2\right] \mathrm{d}x + o(\|\delta_k u\|_2^2). \quad (3.21)$$

By (3.18) and (2.1), there exists $q_1 \in (1,2)$ such that $\|z_{B_k}\|_2 = O(\|\delta_{B_k} u\|_{q_1})$. Thus, by Hölder inequality and the fact that $|B_k| \to 0$, we get

$$\|z_{B_k}\|_2 = O(\|\delta_{B_k} u\|_{q_1}) = o(\|\delta_k u\|_2). \tag{3.22}$$

Henceforth, an easy computation, using the Cauchy-Schwarz inequality and (3.22), implies that

$$\int_\Omega \left[\delta H_y^k(x) z_1^k + \tfrac{1}{2} H_{yy}(x)(z_1^k)^2\right] \mathrm{d}x = \int_\Omega \left[\delta H_y^k(x) z_{A_k} + \tfrac{1}{2} H_{yy}(x)(z_{A_k})^2\right] \mathrm{d}x + o(\|\delta_k u\|_2^2). \tag{3.23}$$

On the other hand, using (3.18) and (2.1), there exists $q_2 \in (2,\infty)$ such that

$$\|z_{A_k}\|_{q_2} = O(\|\delta_{A_k} u\|_2). \tag{3.24}$$

Since $q_2^* \in (1,2)$, as in estimate (3.22) we get $\|\delta_{B_k} u\|_{q_2^*} = o(\|\delta_k u\|_2)$. Therefore, (3.24) yields

$$\int_{B_k} |z_{A_k}||\delta_k u| \mathrm{d}x \le \|z_{A_k}\|_{q_2} \|\delta_{B_k} u\|_{q_2^*} = o(\|\delta_k u\|_2^2), \tag{3.25}$$

showing that $\int_{B_k} \delta H_y^k(x) z_{A_k} \mathrm{d}x = o(\|\delta_k u\|_2^2)$. Moreover, using (3.24) and Hölder inequality we obtain

$$\int_{B_k} |z_{A_k}|^2 \mathrm{d}x \le |B_k|^{\frac{1}{(q_2/2)^*}} \left(\int_{B_k} |z_{A_k}|^{q_2} \mathrm{d}x\right)^{\frac{2}{q_2}} = o\left(\|\delta_k u\|_2^2\right),$$

which, together with (3.23), implies expression (3.21).

b) We now prove that

$$\int_{A_k} \delta H_y^k(x) z_{A_k} \mathrm{d}x = \int_{A_k} H_{yu}(x) z_{A_k} \delta_{A_k} u \ \mathrm{d}x + o(\|\delta_k u\|_2^2). \tag{3.26}$$

By a Taylor expansion, we have:

$$\int_{A_k} \delta H_y^k(x) z_{A_k} \mathrm{d}x = \int_{A_k} H_{yu}(x) z_{A_k} \delta_{A_k} u \ \mathrm{d}x + O\left(\int_{A_k} |z_{A_k}||\delta_k u|^2 \mathrm{d}x\right). \tag{3.27}$$

Hölder inequality, estimates (3.24) and $\|\delta_k u\|_{q_2^*} = O(\|\delta_k u\|_2)$ and the fact that $\|\delta_{A_k} u\|_\infty \to 0$, give

$$\int_{A_k} |z_{A_k}||\delta_k u|^2 \mathrm{d}x \le \|z_{A_k}\|_{q_2} \left(\int_{A_k} |\delta_k u|^{2q_2^*} \mathrm{d}x\right)^{\frac{1}{q_2^*}} = O(\|\delta_{A_k} u\|_\infty \|\delta_k u\|_2^2) = o(\|\delta_k u\|_2^2), \tag{3.28}$$

from which (3.26) follows.

c) Finally, a Taylor expansion and the fact that $\|\delta_{A_k} u\|_\infty \to 0$ imply that

$$\int_{A_k} \delta H^k(x) \mathrm{d}x = \int_{A_k} \left[H_u(x)\delta_{A_k} u + \tfrac{1}{2} H_{uu}(x)(\delta_{A_k} u)^2\right] \mathrm{d}x + o(\|\delta_k u\|_2^2). \tag{3.29}$$

The conclusion follows from (3.20) and expressions (3.21), (3.26) and (3.29). □

The following corollary is a straightforward consequence of Theorem 3.5.

Corollary 3.6. *Under the assumptions of theorem 3.5 we have that*

$$J(u_k) - J(\overline{u}) = J(u_{B_k}) - J(\overline{u}) + J(u_{A_k}) - J(\overline{u}) + o(\|\delta_k u\|_2^2). \tag{3.30}$$

We now set some useful notation. Let us first define the linear map $Q_1[\overline{u}] : L^2(\Omega) \to \mathbb{R}$ as

$$Q_1[\overline{u}]v := \int_\Omega H_u(x)v(x)\mathrm{d}x. \tag{3.31}$$

Secondly, we define the quadratic form $Q_2[\overline{u}] : L^2(\Omega) \to \mathbb{R}$ as

$$Q_2[\overline{u}](v) = \int_\Omega \left[H_{yy}(x)(\zeta[v])^2 + 2H_{yu}(x)\zeta[v]v + H_{uu}(x)v^2\right] \mathrm{d}x, \tag{3.32}$$

where $\zeta[v]$ is defined as the unique solution in $H^2(\Omega) \cap H_0^1(\Omega)$ of the linearized state equation

$$\begin{cases} -\Delta\zeta + \varphi_y(x)\zeta + \varphi_u(x)v &= 0, \quad \text{in } \Omega, \\ \zeta &= 0, \quad \text{on } \partial\Omega. \end{cases} \tag{3.33}$$

It is easy to check, using theorem 2.1, that $\zeta[v]$ satisfies the same estimates as $z_1[v]$ in lemma 3.1. Therefore, we have that $Q_2[\overline{u}](v) = O(\|v\|_2^2)$, and thus $Q_2[\overline{u}]$ is a continuous quadratic form on $L^2(\Omega)$. For future reference, we denote by $\hat{Q}_2[\overline{u}] : L^2(\Omega) \times L^2(\Omega) \to \mathbb{R}$ for the associated continuous symmetric bilinear form, i.e.

$$\hat{Q}_2[\overline{u}](v_1, v_2) := \tfrac{1}{2}\left[Q_2[\overline{u}](v_1 + v_2) - Q_2[\overline{u}](v_1) - Q_2[\overline{u}](v_2)\right]. \tag{3.34}$$

Using the previous definitions we have

Theorem 3.7. *Under the assumptions of theorem 3.5 we have that*

$$J(u_k) - J(\overline{u}) = \int_{B_k} \delta H^k(x)\mathrm{d}x + Q_1[\overline{u}]\delta_{A_k}u + \tfrac{1}{2}Q_2[\overline{u}](\delta_{A_k}u) + o(\|\delta_k u\|_2^2). \tag{3.35}$$

Proof. Let ζ_k be the solution of (3.33) with $\delta_{A_k}u$ in place of v. By a Taylor expansion, the function $w_k := z_{A_k} - \zeta_k$ satisfies on Ω:

$$-\Delta w_k + \varphi_y(x)w_k = O(|\delta_{A_k}u|^2),$$

with Dirichlet boundary condition. Hence, we have $\|w_k\|_2 = O(\|\delta_{A_k}u\|_2\|\delta_{A_k}u\|_\infty) = o(\|\delta_{A_k}u\|_2)$. Using this estimate it is straightforward to prove that we can replace z_{A_k} by ζ_k in (3.19) up to an error $o(\|\delta_{A_k}u\|_2^2)$, from which the result follows. □

4 Optimality conditions

The purpose in this section is to provide some new results concerning optimality conditions for $(\mathcal{CP})$. More precisely, we first provide a general first order result that yields to the well-known Pontryagin's minimum principle for L^1-weak local solutions. Moreover, it also implies that weak solutions satisfy a *local* Pontryagin's minimum principle. Next, we study second order conditions and we extend to the strong sense two second order sufficient conditions for local quadratic growth. Finally, we characterize local quadratic growth in the strong sense.

4.1 Pontryagin's minimun principle for semi-linear elliptic equations

While Pontryagin's minimum principle for semi-linear elliptic equations is well-known, we will obtain it as a particular case of a more general statement; the latter will allow us to obtain a version of Pontryagin's minimum principle for weak local solutions. Let us set

$$\mathcal{K}(x) := [a(x), b(x)] \quad \text{for any } x \in \Omega. \tag{4.1}$$

Definition 4.1. We say that the multiplication $U : \Omega \to 2^{\mathbb{R}}$ (denoted by $U : \Omega \rightrightarrows \mathbb{R}$) is a measurable multifunction if, for any closed set $C \subset \mathbb{R}$, we have that $U^{-1}(C)$ is measurable. We say that it is feasible if $U(x) \subseteq \mathcal{K}(x)$ for a.a. $x \in \Omega$, and closed-valued if $U(x)$ is closed for a.a. $x \in \Omega$. If $u : \Omega \to \mathbb{R}$ is a measurable function, with $u(x) \in U(x)$ for a.a. $x \in \Omega$, we say that u is a *measurable selection* of U. We denote by $\text{select}(U)$ the set of measurable selections.

Example 4.2. Given $\overline{u} \in \mathcal{K}$, we will use the two following examples:

(i) For $\varepsilon > 0$, we define $U^\varepsilon(x) = [a(x), b(x)] \cap \overline{B}(\overline{u}(x), \varepsilon)$ for a.a. $x \in \Omega$. This multifunction will be useful in the study of weak local solutions.

(ii) Given $\overline{x} \in \Omega$ and $\varepsilon > 0$, the following multifunction will be useful for the study of L^1-weak local solutions

$$U^{\overline{x},\varepsilon}(x) := \begin{cases} \mathcal{K}(x) & \text{if } x \in B(\overline{x}, \varepsilon) \cap \Omega, \\ \{\overline{u}(x)\} & \text{if } x \in \Omega \setminus B(\overline{x}, \varepsilon). \end{cases} \tag{4.2}$$

Definition 4.3. Let $U : \Omega \rightrightarrows \mathbb{R}$ be a feasible, closed valued measurable multifunction. We say that $\overline{u} \in \mathcal{K}$ is a *Pontryagin extremal in integral form with respect to* U if

$$\int_\Omega H(x, \overline{y}(x), \overline{p}(x), \overline{u}(x)) \mathrm{d}x \leq \int_\Omega H(x, \overline{y}(x), \overline{p}(x), v(x)) \mathrm{d}x, \quad \text{for all } v \in \text{select}(U), \tag{4.3}$$

and that it is a *Pontryagin extremal with respect to* U if for a.a. $x \in \Omega$ we have

$$H(x, \overline{y}(x), \overline{p}(x), \overline{u}(x)) \leq H(x, \overline{y}(x), \overline{p}(x), v) \ \text{ for all } v \in U(x). \tag{4.4}$$

If $U(x) = \mathcal{K}(x)$ for a.a. $x \in \Omega$, a Pontryagin extremal with respect to U will be simply called *Pontryagin extremal.* On the other hand, if there exists $\varepsilon > 0$ such that $\overline{u}$ is a Pontryagin extremal with respect to U^ε, defined in example 4.2(i), we say that $\overline{u}$ is a *weak Pontryagin extremal.*

Obviously, a Pontryagin extremal is a Pontryagin extremal in integral form. Setting $F(x, u) := H(x, \overline{y}(x), \overline{p}(x), u)$ in the proposition below, whose proof can be found in [25, theorem 3A], we have that the converse is also true.

Proposition 4.4. *Let* $U : \Omega \rightrightarrows \mathbb{R}$ *be a feasible, closed valued measurable multifunction,* $\overline{u} \in \text{select}(U)$*, and* $F : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ *be a continuous fonction, such that* $\overline{u}$ *is solution of the problem*

$$\operatorname*{Min}_u \int_\Omega F(x, u(x)) \mathrm{d}x \quad u \in \text{select}(U). \tag{4.5}$$

Then, for a.a. $x \in \Omega$*,*

$$F(x, \overline{u}(x)) \leq F(x, v), \quad \text{for all } \ v \in U(x). \tag{4.6}$$

As a consequence we have:

Proposition 4.5. *Let* $U : \Omega \rightrightarrows \mathbb{R}$ *be a feasible, closed valued measurable multifunction. Suppose that* $\overline{u} \in \mathcal{K}$ *satisfies*

$$J(\overline{u}) \leq J(u), \quad \text{for all } \ u \in \text{select}(U). \tag{4.7}$$

Then $\overline{u}$ *is a Pontryagin extremal with respect to* U*.*

Proof. In view of proposition 4.4 it is enough to show that $\overline{u}$ is a Pontryagin extremal in integral form with respect to U. Set $F(x, u) := H(x, \overline{y}(x), \overline{p}(x), u)$ and $\mathcal{F}(u) := \int_\Omega F(x, u(x)) \mathrm{d}x$. By contradiction, suppose that there exists $\overline{v} \in \text{select}(U)$ such that $\mathcal{F}(\overline{v}) < \mathcal{F}(\overline{u})$. Consequently, for some $\varepsilon > 0$, there exists a measurable subset Ω_ε of Ω such that

$$H(x, \overline{y}(x), \overline{p}(x), \overline{v}(x)) + \varepsilon \leq H(x, \overline{y}(x), \overline{p}(x), \overline{u}(x)), \quad \text{for a.a. } x \in \Omega_\varepsilon. \tag{4.8}$$

Let Ω_k be a sequence of measurable subsets of Ω_ε such that $|\Omega_k| = 1/k$ and define $v_k : \Omega \to \mathbb{R}$ as

$$v_k(x) = \overline{v}(x) \text{ if } x \in \Omega_k, \qquad v_k(x) = \overline{u}(x) \text{ otherwise.} \tag{4.9}$$

Then $v_k \in \text{select}(U)$ and $\|v_k - \overline{u}\|_1 = O(1/k)$. Thus, $\|v_k - \overline{u}\|_1 \|v_k - \overline{u}\|_s = o(1/k)$ for $s > n/2$. Therefore, expression (3.9) in lemma 3.2, inequality (4.8) and the optimality condition (4.7) imply

$$0 \leq J(v_k) - J(\bar{u}) \leq -\varepsilon/k + o(1/k),$$

which gives the desired contradiction. □

As a corollary we obtain the well known Pontryagin's minimum principle for semi-linear elliptic problems, e.g. [3, 4, 5, 24], as well as a version for weak local solutions.

Theorem 4.6. *Let $\overline{u}$ be a L^1-weak (respectively weak) local solution of $(\mathcal{CP})$. Then $\overline{u}$ is a Pontryagin extremal (respectively weak Pontryagin extremal).*

Proof. The result is a straightforward consequence of proposition 4.5 applied to the multifunctions of example 4.2. □

Definition 4.7. Let $\overline{u}$ be a Pontryagin extremal with respect to the feasible, closed valued measurable multifunction U. We can change U and $\overline{u}$ over a negligible set, so that $U(x)$ is compact for all x, and $\overline{u}(x)$ minimizes the Hamiltonian for all $x \in \overline{\Omega}$. In that case we say that $\overline{u}$ is a *Pontryagin representative* (of the equivalence class of functions a.e. equal to $\overline{u}$). Note that a Pontryagin representative is also defined on $\partial\Omega$. In the sequel we will identify Pontryagin extremals with one of them which is a Pontryagin representative.

Corollary 4.8. *Let $\overline{u}$ be a weak local solution of $(\mathcal{CP})$. Let $x \in \Omega$ be such that $a(x) < b(x)$. Then*

$$H_u(x) \geq 0 \text{ if } \overline{u}(x) = a(x), \qquad H_u(x) \leq 0 \text{ if } \overline{u}(x) = b(x), \qquad \text{and} \qquad H_u(x) = 0 \text{ otherwise.} \tag{4.10}$$

Moreover, we have that

$$H_{uu}(x) \geq 0 \quad \text{if} \quad H_u(x) = 0. \tag{4.11}$$

Proof. This is a straightforward consequence of theorem 4.6. □

Definition 4.9. Let $\overline{u}$ be a Pontryagin extremal. We say that the *strict Pontryagin inequality* holds at $\overline{u} \in \mathcal{K}$ if *for all* $x \in \overline{\Omega}$,

$$H(x, \overline{y}(x), \overline{p}(x), \overline{u}(x)) < H(x, \overline{y}(x), \overline{p}(x), v), \quad \text{when } v \neq \overline{u}(x), \quad \text{for all } v \in \mathcal{K}(x). \tag{4.12}$$

Lemma 4.10. *A Pontryagin extremal that satisfies the strict Pontryagin inequality belongs to $C(\overline{\Omega})$.*

Proof. Let $\overline{u}$ satisfy the hypothesis of the lemma, and let $x_k \to \bar{x}$ in $\overline{\Omega}$. Extracting if necessary a subsequence, since a and b are continous, we may assume that $\overline{u}(x_k) \to \tilde{u} \in \mathcal{K}(\bar{x})$. By (4.12), we obtain by letting $k \uparrow \infty$:

$$\begin{aligned} H(\bar{x}, \overline{y}(\bar{x}), \overline{p}(\bar{x}), \tilde{u}) &= \lim_k H(x_k, \overline{y}(x_k), \overline{p}(x_k), \overline{u}(x_k)) \\ &\leq \lim_k H(x_k, \overline{y}(x_k), \overline{p}(x_k), \overline{u}(\bar{x})) \\ &= H(\bar{x}, \overline{y}(\bar{x}), \overline{p}(\bar{x}), \overline{u}(\bar{x})). \end{aligned} \tag{4.13}$$

By (4.12), $\tilde{u} = \overline{u}(\bar{x})$, implying that $\overline{u}$ is continuous at $\bar{x}$. The conclusion follows. □

4.2 Second order necessary conditions

Now, we establish second order necessary conditions. The novelty in this subsection is a second order necessary condition for local quadratic growth in L^1-sense (so in particular in the strong sense). Let us start with some standard definitions and results.

Consider a Banach space $(X, \|\cdot\|_X)$ and a non-empty closed convex set $K \subseteq X$. For $x, x' \in X$ define the *segment* $[x, x'] := \{x + \lambda(x' - x) \mid \lambda \in [0,1]\}$ and for $A \subseteq X$ set $\text{clo}_X(A)$ for its closure. In order to simplify the notation, when $X = L^p(\Omega)$ for some $p \in [1, \infty]$ we will write $\text{clo}_p(A) := \text{clo}_{L^p(\Omega)}(A)$ for $A \subseteq L^\infty(\Omega)$. The radial, the tangent and the normal cone to K at $\bar{x}$ are defined respectively by

$$\begin{array}{rcl} R_K(\bar{x}) & := & \{h \in X \mid \exists\, \sigma > 0 \quad \text{such that} \quad [\bar{x}, \bar{x} + \sigma h] \subseteq K\}, \\ T_K(\bar{x}) & := & \{h \in X \mid \exists\, x(\sigma) = \bar{x} + \sigma h + o(\sigma) \in K,\ \sigma > 0,\ \|o(\sigma)/\sigma\|_X \to 0, \text{ as } \sigma \downarrow 0\}, \\ N_K(\bar{x}) & := & \{x^* \in X^* \mid \langle x^*, x - \bar{x}\rangle_{X^*,X} \leq 0, \text{ for all } x \in K\}, \end{array} \tag{4.14}$$

where X^* is the dual topological space of X and $\langle \cdot, \cdot \rangle_{X^*,X}$ is the duality product. Recall that, since K is a closed convex set, we have $T_K(\bar{x}) = \text{clo}_X(R_K(\bar{x}))$ and $N_K(\bar{x})$ is the polar cone of $T_K(\bar{x})$, i.e.,

$$N_K(\bar{x}) = \{x^* \in X^* \mid \langle x^*, h\rangle_{X^*,X} \leq 0, \text{ for all } h \in T_K(\bar{x})\}. \tag{4.15}$$

In what follows we will consider the set $\mathcal{K}$, defined in (2.7), as a subset of $L^2(\Omega)$ rather than a subset of $L^\infty(\Omega)$. This will allow us to give explicit expressions for the tangent and normal cone. Note that, since $a, b \in L^\infty(\Omega)$, for all $\overline{u} \in \mathcal{K}$ we have $R_\mathcal{K}(\overline{u}) = R_\mathcal{K}(\overline{u}) \cap L^\infty(\Omega)$. The next lemma is standard (at least when $\min(b - a) > 0$). We provide its proof in the appendix for the sake of completeness of the paper.

Lemma 4.11. *Let $\mathcal{K}$ be defined by* (2.7) *and $\overline{u} \in \mathcal{K}$. Then the following assertions hold true:*
(i) *The tangent cone to $\mathcal{K}$ at $\overline{u}$ is given by*

$$\begin{array}{rcl} T_\mathcal{K}(\overline{u}) & = & \left\{v \in L^2(\Omega) \mid v(x) \in T_{\mathcal{K}(x)}(\overline{u}(x)) \quad \text{for a.a. } x \in \Omega\right\}, \\ & = & \left\{v \in L^2(\Omega) \mid v(x) \geq 0 \ \text{ if } \ \overline{u}(x) = a(x),\ v(x) \leq 0 \ \text{ if } \ \overline{u}(x) = b(x) \quad \text{for a.a. } x \in \Omega\right\}. \end{array} \tag{4.16}$$

(ii) *The normal cone to $\mathcal{K}$ at $\overline{u}$ is given by*

$$N_\mathcal{K}(\overline{u}) = \left\{v \in L^2(\Omega) \mid v(x) \in N_{\mathcal{K}(x)}(\overline{u}(x)) \text{ for a.a. } x \in \Omega\right\}. \tag{4.17}$$

(iii) *For every $q^* \in N_\mathcal{K}(\overline{u})$ we have that*

$$\text{clo}_2\left(R_\mathcal{K}(\overline{u}) \cap (q^*)^\perp\right) = T_\mathcal{K}(\overline{u}) \cap (q^*)^\perp = \{v \in T_\mathcal{K}(\overline{u}) \mid v(x) q^*(x) = 0 \text{ for a.a. } x \in \Omega\}. \tag{4.18}$$

Remark 4.12. Assertions (i), (ii) say that $T_\mathcal{K}(\overline{u})$ and $N_\mathcal{K}(\overline{u})$ can be *localized*, while the first equality in (iii) means that $\mathcal{K}$ is polyhedric in the sense of [19, 22].

The *critical cone* to $\mathcal{K}$ at a local Pontryagin extremal $\overline{u}$ is defined as

$$C_\mathcal{K}(\overline{u}) := T_\mathcal{K}(\overline{u}) \cap (Q_1[\overline{u}])^\perp, \tag{4.19}$$

while the *pointwise* critical cone is defined by

$$C_x := \left\{v \in T_{\mathcal{K}(x)}(\overline{u}(x)) \mid H_u(x) v = 0\right\} \quad \text{for a.a. } x \in \Omega. \tag{4.20}$$

Since $\overline{u}$ is a local Pontryagin extremal $\overline{u}$, we have that $-Q_1[\overline{u}] \in N_\mathcal{K}(\overline{u})$. Thus, lemma 4.11(iii) implies that

$$C_\mathcal{K}(\overline{u}) = \left\{v \in L^2(\Omega) \mid v(x) \in C_x \text{ for a.a. } x \in \Omega\right\}. \tag{4.21}$$

We have the following second order necessary condition (see e.g. [2, 12, 13] for more general settings).

Theorem 4.13. *Let $\overline{u}$ be a weak local solution of $(\mathcal{CP})$. Then, recalling* (3.32)*, we have*

$$Q_2[\overline{u}](v) \geq 0 \quad \text{for all } v \in C_{\mathcal{K}}(\overline{u}), \tag{4.22}$$

Proof. Let $v \in R_{\mathcal{K}}(\overline{u}) \cap (Q_1[\overline{u}])^{\perp}$ and set $u_k := \overline{u} + \frac{1}{k}v \in \mathcal{K}$, for k large. The local optimality of $\overline{u}$ and theorem 3.7 with $A_k \equiv \Omega$, imply that

$$0 \leq J(u_k) - J(\overline{u}) = \frac{1}{k^2} Q_2[\overline{u}](v) + o\left(1/k^2\right).$$

Multiplying by k^2 and letting $k \uparrow \infty$ yields (4.22) for all $v \in R_{\mathcal{K}}(\overline{u}) \cap (Q_1[\overline{u}])^{\perp}$. Using that $Q_1[\overline{u}] \in N_{\mathcal{K}}(\overline{u})$, relation (4.22) follows from the continuity of Q_2 in $L^2(\Omega)$ and the first equality in lemma 4.11(iii). □

Now we begin the study of necessary conditions for local quadratic growth on $\mathcal{K}$.

Definition 4.14. For $\overline{u} \in \mathcal{K}$, we say that

(i) The Hamiltonian satisfies the *a.e. local quadratic growth property* at $\overline{u}$ if and only if there exist $\alpha, \varepsilon > 0$ such that *a.e.* in Ω we have

$$H(x, \overline{y}(x), \overline{p}(x), \overline{u}(x)) + \alpha|v - \overline{u}(x)|^2 \leq H(x, \overline{y}(x), \overline{p}(x), v) \quad \text{for all } v \in \mathcal{K}(x) \text{ with } |v - \overline{u}(x)| \leq \varepsilon. \tag{4.23}$$

(ii) The Hamiltonian satisfies the *global quadratic growth* property at Pontryagin representative of $\overline{u}$ if and only if there exists $\alpha > 0$ such that *for all* $x \in \overline{\Omega}$ we have

$$H(x, \overline{y}(x), \overline{p}(x), \overline{u}(x)) + \alpha|v - \overline{u}(x)|^2 \leq H(x, \overline{y}(x), \overline{p}(x), v) \quad \text{for all } v \in \mathcal{K}(x). \tag{4.24}$$

Adapting the techniques of [7] to our framework we can characterize the global quadratic growth for the Hamiltonian.

Lemma 4.15. *Let $\overline{u} \in \mathcal{K}$. Then $\overline{u}$ is a Pontryagin extremal that satisfies the global quadratic growth property for the Hamiltonian iff both the strict Pontryagin inequality* (4.12) *and the a.e. local quadratic growth property for the Hamiltonian hold.*

Proof. It is enough to show that if (4.12) holds everywhere and (4.23) holds a.e. then (4.24) holds everywhere. First note that by lemma 4.10 we have that (4.23) holds everywhere. Now, let

$$\beta := \min_{x,v} \left\{ H(x, \overline{y}(x), \overline{p}(x), v) - H(x, \overline{y}(x), \overline{p}(x), \overline{u}(x)) \mid x \in \overline{\Omega},\ v \in \mathcal{K}(x),\ |v - \overline{u}(x)| \geq \varepsilon \right\}.$$

By (4.12) and since $\overline{u}$ is continuous, we get that $\beta > 0$. For $v \in \mathcal{K}(x)$ with $|v - \overline{u}(x)| \geq \varepsilon$, we have

$$H(x, \overline{y}(x), \overline{p}(x), v) - H(x, \overline{y}(x), \overline{p}(x), \overline{u}(x)) \geq \beta \geq \frac{\beta}{4M^2}|v - \overline{u}(x)|^2,$$

where we recall that $M = \max\{\|a\|_\infty, \|b\|_\infty\}$. Hence, by (4.23) we get that (4.24) is satisfied with $\min\left(\alpha, \frac{\beta}{4M^2}\right)$, which concludes the proof. □

We now provide second order necessary conditions for local quadratic growth on $\mathcal{K}$.

Theorem 4.16. *The following assertions hold true:*

(i) *If J has local quadratic growth on $\mathcal{K}$ at $\overline{u}$ in the weak sense, then the Hamiltonian satisfies the a.e. local quadratic growth property, and we have*

$$Q_2[\overline{u}](v) \geq \alpha\|v\|_2^2, \quad \text{for all } v \in C_{\mathcal{K}}(\overline{u}). \tag{4.25}$$

(ii) *If J has local quadratic growth on $\mathcal{K}$ at $\overline{u}$ in the L^1-weak sense, then the Hamiltonian satisfies the global quadratic growth property and* (4.25) *holds true.*

Proof. Let us define $J_\alpha : L^\infty(\Omega) \to \mathbb{R}$ by $J_\alpha(u) := J(u) - \alpha\|u - \overline{u}\|_2^2$ and consider the problem

$$\min J_\alpha(u) \quad \text{subject to } u \in \mathcal{K}. \tag{$\mathcal{CP}_\alpha$}$$

If J has local quadratic growth on $\mathcal{K}$ at $\overline{u}$ in the weak sense, then $\overline{u}$ is a weak local solution of $(\mathcal{CP}_\alpha)$. Thus, (4.25) follows from condition (4.22) in theorem 4.13. Theorem 4.6 implies that $\overline{u}$ is a local Pontryagin extremal for $(\mathcal{CP}_\alpha)$, which easily yields that the Hamiltonian satisfies the a.e. local quadratic growth property. On the other hand, if J has local quadratic growth on $\mathcal{K}$ at $\overline{u}$ in the L^1-weak sense, we have that $\overline{u}$ is a L^1-weak local solution of $(\mathcal{CP}_\alpha)$. Therefore, theorem 4.6 implies that $\overline{u}$ is a Pontryagin extremal for $(\mathcal{CP}_\alpha)$ which easily yields that (4.24) holds for a.a. $x \in \Omega$. By an elementary continuity argument, this implies that the strict Pontryagin inequality holds at $\overline{u}$. Therefore, lemma 4.15 gives that $\overline{u}$ is continuous and so (4.24) holds for all $x \in \Omega$. Assertion (ii) follows. □

4.3 Extension of standard second order sufficient conditions

In this subsection we extend to the strong sense two well known second order sufficient conditions for the local quadratic growth of J on $\mathcal{K}$ in the weak sense. The main tool for proving such extensions is the decomposition result in theorem 3.5.

We first consider the case studied in [2] which supposes that $Q_2[\overline{u}]$ is a Legendre form.

Definition 4.17. Given a Hilbert space X, a quadratic form $Q : X \to \mathbb{R}$ is said to be a Legendre form if it is sequentially weakly lower semicontinuous and that if h_k converges weakly to h in X and $Q(h_k) \to Q(h)$ then h_k converges strongly to h in X.

For the reader convenience, let us reproduce the sufficiency part of [2, theorem 2.9].

Theorem 4.18. *Suppose that **(H1)-(H3)** hold true and let $\overline{u} \in \mathcal{K}$. Assume that $H_u(x)v \geq 0$ for all $v \in T_{\mathcal{K}(x)}(\overline{u}(x))$ a.e. in Ω, that $Q_2[\overline{u}]$ is a Legendre form and that there exists $\alpha > 0$ such that the following second order condition holds true:*

$$Q_2[\overline{u}](v) \geq \alpha\|v\|_2^2, \quad \text{for all } v \in C_{\mathcal{K}}(\overline{u}). \tag{4.26}$$

Then J has local quadratic growth on $\mathcal{K}$ at $\overline{u}$ in the weak sense.

We recall that J has local quadratic growth on $\mathcal{K}$ at $\overline{u}$ in the strong sense if there exists $\alpha, \varepsilon > 0$ such that

$$J(u) \geq J(\overline{u}) + \alpha\|u - \overline{u}\|_2^2 \quad \text{for all } u \in \mathcal{K} \text{ with } \|y_u - y_{\overline{u}}\|_\infty \leq \varepsilon. \tag{4.27}$$

We have the following extension of theorem 4.18.

Theorem 4.19. *Suppose that **(H1)-(H3)** hold true and let $\overline{u} \in \mathcal{K}$. Assume that the strict Pontryagin inequality holds at $\overline{u}$, that $Q_2[\overline{u}]$ is a Legendre form and that there exists $\alpha > 0$ such that the following second order condition holds true:*

$$Q_2[\overline{u}](v) \geq \alpha\|v\|_2^2, \quad \text{for all } v \in C_{\mathcal{K}}(\overline{u}). \tag{4.28}$$

Then J has local quadratic growth on $\mathcal{K}$ at $\overline{u}$ in the strong sense.

Proof. a) Let assume that (4.27) does not hold. Then, there exists a sequence $u_k \in \mathcal{K}$ such that $\|y_k - \bar{y}\|_\infty \to 0$ as $k \uparrow \infty$ (we have denoted $y_k := y_{u_k}$) and

$$J(u_k) - J(\overline{u}) \leq o(\|\delta_k u\|_2^2), \tag{4.29}$$

where $\delta_k u := u_k - \overline{u}$. Theorem 4.18 implies that J has local quadratic growth on $\mathcal{K}$ at $\overline{u}$ in the weak sense. Therefore, theorem 4.16(i) yields that the Hamiltonian satisfies the a.e. local quadratic growth property. Using lemma 4.15, we obtain that the Hamiltonian satisfies the global quadratic growth property. Consequently, by expression (3.8) in lemma 3.2, inequality (4.29) implies that $\|\delta_k u\|_2 \to 0$. Let us define the measurable sets

$$A_k := \left\{ x \in \Omega \mid |u_k(x) - \overline{u}(x)| \leq \sqrt{\|u_k - \overline{u}\|_1} \right\} \quad \text{and} \quad B_k := \Omega \backslash A_k. \tag{4.30}$$

Chebyshev's inequality implies that $|B_k| \leq \sqrt{\|u_k - \overline{u}\|_1}$, hence $|B_k|$ goes to zero. Thus, introducing the notations of subsection 3.3 we clearly have that $\|\delta_{A_k} u\|_\infty \to 0$. Therefore, theorem 3.5 gives

$$\int_{B_k} \delta H^k(x) \mathrm{d}x + \int_{A_k} H_u(x) \delta_{A_k} u(x) \mathrm{d}x + \tfrac{1}{2} Q_2[\overline{u}](\delta_{A_k} u) \leq o(\|\delta_k u\|_2^2). \tag{4.31}$$

Now, set

$$\sigma_{A_k} := \|\delta_{A_k} u\|_2, \quad \sigma_{B_k} := \|\delta_{B_k} u\|_2 \quad \text{and hence } \|\delta_k u\|_2^2 = \sigma_{A_k}^2 + \sigma_{B_k}^2. \tag{4.32}$$

If $\sigma_{A_k} = o(\sigma_{B_k})$, using that $H_u(x)\delta_{A_k} u(x) \geq 0$ and that $Q_2[\overline{u}](\delta_{A_k} u) = O(\sigma_{A_k}^2) = o(\sigma_{B_k}^2)$, inequality (4.31) and (4.24) imply that $\sigma_{B_k}^2 \leq o(\sigma_{B_k}^2)$, which is impossible. Thus, let us assume, up to a subsequence, that $\sigma_{B_k} = O(\sigma_{A_k})$ and define $h_k := \delta_{A_k} u / \sigma_{A_k}$.

By (4.12) the first integral in (4.31) is nonnegative, and therefore, after minorizing it by 0,

$$\int_{A_k} H_u(x) h_k(x) \mathrm{d}x + \tfrac{1}{2} \sigma_{A_k} Q_2[\overline{u}](h_k) \leq o(\sigma_{A_k}). \tag{4.33}$$

It follows that

$$\int_{A_k} H_u(x) h_k(x) \mathrm{d}x \leq O(\sigma_{A_k}). \tag{4.34}$$

Also, minorizing the first integral in (4.33) by 0, we obtain that

$$Q_2[\overline{u}](h_k) \leq o(1). \tag{4.35}$$

b) Since $h_k \in T_{\mathcal{K}}(\overline{u})$ and $\|h_k\|_2 = 1$, up to a subsequence, it converges weakly in $L^2(\Omega)$ to some $\bar{h}$. Recalling that $T_{\mathcal{K}}(\overline{u})$ is weakly closed we get that $\bar{h} \in T_{\mathcal{K}}(\overline{u})$. Noting that $Q_2[\overline{u}](\delta_{A_k} u)/\sigma_k = o(1)$, condition (4.24) and equation (4.31) imply that

$$0 \leq \int_{A_k} H_u(x) h_k(x) \mathrm{d}x \leq o(1).$$

By passing to the limit in the above inequality, we get that $\bar{h} \in C_{\mathcal{K}}(\overline{u})$. On the other hand, since equation (4.31) implies that $Q_2[\overline{u}](h_k) \leq o(1)$, the lower semicontinuity of $Q_2[\overline{u}]$ and its positivity over $C_{\mathcal{K}}(\overline{u})$ give

$$0 \leq Q_2[\overline{u}](\bar{h}) \leq \liminf_{k \to \infty} Q_2[\overline{u}](h_k) \leq \limsup_{k \to \infty} Q_2[\overline{u}](h_k) \leq 0.$$

The above inequality and (4.28) imply that $\bar{h} = 0$ and that $Q_2[\overline{u}](h_k) \to Q_2[\overline{u}](\bar{h})$. Thus, since $Q_2[\overline{u}]$ is a Legendre form, we have that $h_k \to 0$ strongly in $L^2(\Omega)$ which contradicts that $\|h_k\|_2 = 1$. □

Example 4.20. Let us consider a slight variation of the problem treated in [2]. Let f, $y_d \in C(\overline{\Omega})$, $g \in C^2(\mathbb{R})$ with $g_y \geq 0$ and g_{yy} locally Lipschitz. Consider the following data for $(\mathcal{CP})$,

$$\ell(x, y, u) = \tfrac{1}{2}|u|^2 + \tfrac{1}{2}(y - y_d(x))^2, \qquad \varphi(x, y, u) = g(y) + u + f. \tag{4.36}$$

In this case it is easy to see (see e.g. [2, 8]) that for $\overline{u} \in \mathcal{K}$ the associated quadratic form $Q_2[\overline{u}]$ is a Legendre form. Therefore, since the Hamiltonian for this problem is strictly convex with respect to the control variable, we have that $H_u(x)v \geq 0$ for all $v \in T_{\mathcal{K}(x)}(\overline{u}(x))$ a.e. in Ω together with (4.28) are a sufficient condition for the local quadratic growth on $\mathcal{K}$ in the strong sense.

Our aim now is to extend to the strong sense the second order sufficient condition in [15], which is stated in terms of a larger cone than $C_{\mathcal{K}}(\overline{u})$ but the assumption for $Q_2[\overline{u}]$ of being a Legendre form is not needed. For $\tau > 0$ define the *strongly active set*

$$A^\tau(\overline{u}) := \{x \in \Omega \mid |H_u(x)| > \tau\}, \tag{4.37}$$

and the *τ-critical cone*

$$C^\tau_{\mathcal{K}}(\overline{u}) := \{v \in T_{\mathcal{K}}(\overline{u}) \mid v(x) = 0 \text{ for } x \in A^\tau(\overline{u})\}. \tag{4.38}$$

For the reader convenience, we reproduce [15, theorem 2] adapted to our setting. Let us remark that the result holds true under more general assumptions (more precisely, measurability conditions over the data, instead of our continuity assumptions in **(H1)-(H3)**). Moreover, the result in [15] is stated for Neumann boundary controls, but the technique is quite similar for distributed controls and Dirichlet boundary conditions.

Theorem 4.21. *Suppose that **(H1)-(H3)** hold true and let $\overline{u} \in \mathcal{K}$. Assume that $H_u(x)v \geq 0$ for all $v \in T_{\mathcal{K}(x)}(\overline{u}(x))$ a.e. in Ω and that there exist $\tau, \alpha > 0$ such that the following second order condition holds true:*

$$Q_2[\overline{u}](v) \geq \alpha\|v\|_2^2, \quad \text{for all } v \in C^\tau_{\mathcal{K}}(\overline{u}). \tag{4.39}$$

Then J has local quadratic growth on $\mathcal{K}$ at $\overline{u}$ in the weak sense.

We have the following extension of theorem 4.21 to the strong sense.

Theorem 4.22. *Suppose that **(H1)-(H3)** hold true and let $\overline{u} \in \mathcal{K}$. Assume that the strict Pontryagin inequality holds at $\overline{u}$ and that there exist $\tau, \alpha > 0$ such that the following second order condition holds true:*

$$Q_2[\overline{u}](v) \geq \alpha\|v\|_2^2, \quad \text{for all } v \in C^\tau_{\mathcal{K}}(\overline{u}). \tag{4.40}$$

Then J has local quadratic growth on $\mathcal{K}$ at $\overline{u}$ in the strong sense.

Proof. The beginning of the proof is exactly as in theorem 4.19. The rest of the proof is a slight modification of the proof in [28] for the weak case. Thus, by (4.31), we may assume that the following inequality holds true

$$\int_{A_k} H_u(x)\delta_{A_k}u(x)\mathrm{d}x + \tfrac{1}{2}Q_2[\overline{u}](\delta_{A_k}u) \leq o(\|\delta_{A_k}u\|_2^2). \tag{4.41}$$

Define $v_k^0(x) := \mathbf{1}_{\{A_k \setminus A^\tau(\overline{u})\}}(x)\delta_{A_k}u(x)$ (where $\mathbf{1}_A$ denotes the indicator function of A) and $v_k^1(x) := \delta_{A_k}u(x) - v_k^0(x)$. Obviously $v_k^0 \in C^\tau_{\mathcal{K}}(\overline{u})$. Using (3.34) and (4.40), we get

$$\tau\int_{A_k\cap A^\tau(\overline{u})} |v_k^1|\mathrm{d}x + \tfrac{1}{2}\alpha\|v_k^0\|_2^2 + \tfrac{1}{2}Q_2[\overline{u}](v_k^1) + \hat{Q}_2[\overline{u}](v_k^0, v_k^1) \leq o(\|\delta_{A_k}u\|_2^2). \tag{4.42}$$

There exists $c_1 > 0$ such that $\hat{Q}_2[\overline{u}](v_k^0, v_k^1) \geq -c_1\|v_k^0\|_2\|v_k^1\|_2$. With Young's inequality, we get

$$-Q_2[\overline{u}](v_k^0, v_k^1) \leq \frac{\alpha}{4}\|v_k^0\|_2^2 + c_2\|v_k^1\|_2^2 \leq \frac{\alpha}{4}\|v_k^0\|_2^2 + c_2\|v_k^1\|_\infty\|v_k^1\|_1, \tag{4.43}$$

for some $c_2 > 0$. On the other hand, there exists $c_3 > 0$ such that

$$\tfrac{1}{2}Q_2[\overline{u}](v_k^1) \geq -c_3\|v_k^1\|_2^2 \geq -c_3\|v_k^1\|_\infty\|v_k^1\|_1. \tag{4.44}$$

Recall that $\|\delta_{A_k}u\|_\infty \to 0$, so we can choose k large such that $\|v_k^1\|_\infty(c_2 + c_3) \leq \tau/2$. Therefore, combining inequalities (4.43)-(4.44) with (4.42) we easily get that

$$\min\left\{\frac{\tau}{2}, \frac{\alpha}{4}\right\}\|\delta_{A_k}u\|_2^2 \leq \frac{\tau}{2}\int_{A_k\cap A^\tau(\overline{u})} |v_k^1|^2\mathrm{d}x + \frac{\alpha}{4}\int_{A_k\setminus A^\tau(\overline{u})} |v_k^0|^2\mathrm{d}x \leq o(\|\delta_{A_k}u\|_2^2),$$

which gives the desired contradiction. □

4.4 Characterization of local quadratic growth in the strong sense

We now state the main result of the article, which characterize local quadratic growth in the strong sense. Note that in particular, the sufficient condition does not need the assumption that $Q_2[\overline{u}]$ is a Legendre form (as in theorem 4.19) or that it is uniformly positive on $C^\tau_{\mathcal{K}}(\overline{u})$ (as in theorem 4.22).

Theorem 4.23. *Suppose that **(H1)-(H3)** hold true and let $\overline{u} \in \mathcal{K}$. Then J has local quadratic growth on $\mathcal{K}$ at $\overline{u}$ in the strong sense if and only if the the Hamiltonian satisfies the global quadratic growth property at $\overline{u}$ and there exists $\alpha > 0$ such that the following second order condition holds true:*

$$Q_2[\overline{u}](v) \geq \alpha \|v\|_2^2, \quad \text{for all } v \in C_{\mathcal{K}}(\overline{u}). \tag{4.45}$$

The proof needs some preparation. Again, the main ingredient of the proof is the decomposition result in theorem 3.5. However, the choice of the sets A_k and B_k is slightly different from the one used in the preceding results. It takes into account the degeneracy of the so-called Hoffman constants associated with the pointwise critical cone C_x, defined in (4.20). By Hoffman's lemma [20] for each $x \in \Omega$ there exists a smallest possible (finite) nonnegative number κ_x, called the Hoffman constant, such that

$$\operatorname{dist}(v, C_x) \leq \kappa_x \left[|H_u(x)v| + \mathbf{1}_{\{\overline{u}(x)=a(x)\}}(-v)_+ + \mathbf{1}_{\{\overline{u}(x)=b(x)\}} v_+ \right], \tag{4.46}$$

where for $x \in \mathbb{R}$ we set $(x)_+ := \max\{x, 0\}$. Since these Hoffman constants play an important role in the analysis, we compute them. It is easily checked that

$$\begin{cases} C_x = T_{\mathcal{K}(x)}, & \kappa_x = 1 & \text{if } H_u(x) = 0, \\ C_x = \{0\}, & \kappa_x \leq \max\left(1, |H_u(x)|^{-1}\right) & \text{otherwise.} \end{cases} \tag{4.47}$$

Proof of theorem 4.23. Since by theorem 4.16(ii) the condition is necessary, we only need to prove that is also sufficient. In order to do this, we proceed as follows:
a) We first essentially repeat step (a) of the proof of theorem 4.19, with a slightly different choice of the sets A_k and B_k. If the conclusion does not hold, let $u_k \in \mathcal{K}$ be such that $\|y_k - \bar{y}\|_\infty \to 0$ and (4.29) holds. Setting $\delta_k u := u_k - \overline{u}$ we have that $\|\delta_k u\|_2 \to 0$. Remind that κ_x denotes the Hoffman constants of the pointwise cones C_x. For some sequence $\varepsilon_k \downarrow 0$ to be specified later, consider the measurable sets

$$B_k^1 := \left\{ x \in \Omega \ : \ |u_k(x) - \overline{u}(x)| \geq \sqrt{\|\delta_k u\|_2} \right\}, \quad B_k^2 := \{x \in \Omega \ : \kappa_x \geq 1/\varepsilon_k\}, \quad B_k := B_k^1 \cup B_k^2, \tag{4.48}$$

and $A_k := \Omega \setminus B_k$. Since $|B_k^i| \to 0$ for $i = 1, 2$, we have that $|B_k| \to 0$, and therefore, (4.31) holds. Fix σ_{A_k} and σ_{B_k} as in (4.32) and define $h_k := \delta_{A_k} u / \sigma_{A_k}$. If (for a subsequence) $\sigma_{A_k} = o(\sigma_{B_k})$, we obtain a contradiction in the same way. So we may assume that $\sigma_{B_k} = O(\sigma_{A_k})$, and we obtain that (4.34)-(4.35) hold.
b) We make the decomposition

$$h_k = \hat{h}_k + \tilde{h}_k, \text{ where } \hat{h}_k(x) := P_{C_x}(h_k(x)), \text{ for a.a. } x \in \Omega, \tag{4.49}$$

where $P_{C_x}(\cdot)$ denotes the projection on C_x. By the definition of A_k, and since u_k is feasible, we have that the contribution of the control constraints to the estimate of the distance to C_x in (4.46) is zero, and $H_u(x)h_k(x) \geq 0$, proving that

$$|\tilde{h}_k(x)| = \operatorname{dist}(h_k(x), C_x) \leq \frac{1}{\varepsilon_k} H_u(x) h_k(x). \tag{4.50}$$

With (4.34) we deduce that

$$\|\tilde{h}_k\|_1 = O\left(\frac{\sigma_{A_k}}{\varepsilon_k}\right). \tag{4.51}$$

Since a projection is non expansive, we also have $|\tilde{h}_k(x)| \leq |h_k(x)|$ for a.a. $x \in \Omega$. Therefore

$$\sigma_{A_k} \|\tilde{h}_k\|_\infty \leq \sigma_{A_k} \|h_k\|_\infty = \|\delta_{A_k} u\|_\infty. \tag{4.52}$$

We let $\varepsilon_k := \|\delta_k u\|_2^{1/4}$, that converges to 0 as required. Using that $\|\delta_{A_k} u\|_\infty \leq \sqrt{\|\delta_k u\|_2}$, we get

$$\|\tilde{h}_k\|_2^2 \leq \|\tilde{h}_k\|_\infty \|\tilde{h}_k\|_1 = O\left(\frac{\|\delta_{A_k} u\|_\infty}{\sigma_{A_k}} \frac{\sigma_{A_k}}{\varepsilon_k}\right) = O(\varepsilon_k). \tag{4.53}$$

By (4.49) and (4.53) we obtain that $\|h_k - \hat{h}_k\|_2 \to 0$ and so $\|\hat{h}_k\|_2 \to 1$. Combining with (4.35) and using (4.45) we deduce that $\alpha \|\hat{h}_k\|_2 \leq Q_2[\overline{u}](\hat{h}_k)^2 \leq o(1)$, which is impossible. ☐

Remark 4.24. If $\min(b - a) > 0$, an easy extension of the analysis in [7] to our framework, gives that (4.45) implies the a.e. local quadratic growth for the Hamiltonian. Thus, by lema 4.15, we have that the strict Pontryagin inequality at $\overline{u}$ together with (4.45) imply the global quadratic growth for the Hamiltonian.

5 Appendix

In this section we prove the technical lemmas in section 3.

Proof of lemma 3.1 : For convenience, we will omit the dependence of x in some parts of the proof. The equation satisfied by δy, with Dirichlet boundary condition, can be written as

$$-\Delta \delta y(x) + \varphi(x, y_u, u) - \varphi(x, \overline{y}, u) + \delta\varphi(x) = 0, \qquad \text{for } x \in \Omega.$$

Equivalently,

$$-\Delta \delta y + \left[\int_0^1 \varphi_y(x, \overline{y} + \theta \delta y, u) \mathrm{d}\theta\right] \delta y = O(|\delta u|), \qquad \text{for } x \in \Omega.$$

Thus, the estimates for δy, as well as those for z_1, follow directly from theorem 2.1. The equation satisfied by d_1, with Dirichlet boundary condition, becomes:

$$-\Delta d_1 + \varphi_y(x) d_1 + \varphi(x, y_u, u) - \varphi(x, \overline{y}, u) - \varphi_y(x) \delta y = 0. \tag{5.1}$$

Introducing $\varphi_y(x, \overline{y}, u)$ in $\varphi(x, y, u) - \varphi(x, \overline{y}, u) - \varphi_y(x)\delta y$, easily yields

$$\varphi(x, y, u) - \varphi(x, \overline{y}, u) - \varphi_y(x) \delta y = O\left(|\delta y|^2 + |\delta y \delta u|\right),$$

hence the equation satisfied by d_1 rewrites:

$$-\Delta d_1 + \varphi_y(x) d_1 = O\left(|\delta y|^2 + |\delta y \delta u|\right).$$

Theorem 2.1 implies that for $s \in (n/2, \infty)$

$$\|d_1\|_1 \leq O(\|(\delta y)^2\|_1 + \|\delta y \delta u\|_1) = O(\|\delta y\|_1 \|\delta y\|_\infty + \|\delta y\|_\infty \|\delta u\|_1) = O(\|\delta u\|_1 \|\delta u\|_s).$$

Finally, taking $s = 2$ in proposition 2.1 gives

$$\|d_1\|_2 = O(\|\delta y \delta u\|_2 + \|(\delta y)^2\|_2) = O(\|\delta y\|_\infty \|\delta u\|_2).$$

☐

Proof of lemma 3.3 : Since $d_2 = d_1 - z_2$, combining (5.1) with (3.11) yields

$$-\Delta d_2 + \varphi_y(x) d_2 + \varphi(x, y, u) - \varphi(x, \overline{y}, u) - \varphi_y(x, \overline{y}, u)\delta y + \delta\varphi_y(x)(\delta y - z_1) - \frac{1}{2}\varphi_{yy}(x) z_1^2 = 0.$$

Now, by a Taylor expansion, we have:

$$\varphi(x,y,u) - \varphi(x,\overline{y},u) - \varphi_y(x,\overline{y},u)\delta y = \frac{1}{2}\varphi_{yy}(x,\overline{y},u)(\delta y)^2 + O(|\delta y|^3),$$

and as φ_y is locally Lipschitz, $\delta\varphi_y(\delta y - z_1) = O(|\delta u\|\delta y - z_1|) = O(|\delta u||d_1|)$. Therefore,

$$-\Delta d_2 + \varphi_y(x)d_2 + \frac{1}{2}\varphi_{yy}(x,\overline{y},u)(\delta y)^2 - \frac{1}{2}\varphi_{yy}(x)z_1^2 = O(|\delta y|^3 + |\delta u||d_1|).$$

Now, we have $\varphi_{yy}(x,\overline{y},u)(\delta y)^2 = \varphi_{yy}(x)(\delta y)^2 + O(|\delta u||\delta y|^2)$. Therefore, we obtain:

$$-\Delta d_2 + \varphi_y(x)d_2 + \frac{1}{2}\varphi_{yy}(x)\left[(\delta y)^2 - z_1^2\right] = O(|\delta y|^3 + |\delta u||d_1| + |\delta u|\delta y^2).$$

As $d_1 = \delta y - z_1$, we get:

$$-\Delta d_2 + \varphi_y(x)d_2 = O(|\delta y|^3 + |\delta u||d_1| + |\delta u||\delta y|^2 + |d_1\|\delta y + z_1|).$$

From the theorem 2.1 we get the inequality:

$$\|d_2\|_1 = O(\|(\delta y)^3\|_1 + \|\delta u d_1\|_1 + \|\delta u(\delta y)^2\|_1 + \|d_1\delta y\|_1 + \|d_1 z_1\|_1).$$

Using the previous estimates, we easily obtain the result. □

Proof of lemma 4.11 (i): Set $\hat{T}(\overline{u}) := \left\{v \in L^2(\Omega) \mid v(x) \in T_{\mathcal{K}(x)}(\overline{u}(x)) \text{ for a.a. } x \in \Omega\right\}$ and let $v \in T_{\mathcal{K}}(\overline{u})$. By definition, there exists $r : \mathbb{R} \to L^2(\Omega)$, with $\|r(\sigma)\|_2/\sigma \to 0$ as $\sigma \downarrow 0$, such that for small σ

$$\overline{u}(x) + \sigma v(x) + r(\sigma)(x) \in \mathcal{K}(x), \quad \text{for a.a. } x \in \Omega. \tag{5.2}$$

Since, up to subsequence, $|r(\sigma)(x)|/\sigma \to 0$ for a.a. $x \in \Omega$, relation (5.2) implies that $v \in \hat{T}(\overline{u})$. Conversely, let $v \in \hat{T}(\overline{u})$ and for $\varepsilon > 0$ set

$$v_\varepsilon := \varepsilon^{-1}\left[P_{\mathcal{K}}(\overline{u} + \varepsilon v) - \overline{u}\right], \tag{5.3}$$

where $P_{\mathcal{K}}(\cdot)$ denotes the orthogonal projection in $L^2(\Omega)$ onto $\mathcal{K}$. Clearly, $v_\varepsilon \in R_{\mathcal{K}}(\overline{u})$ and classical results (see e.g. [25]) yields that v_ε is measurable and given by

$$v_\varepsilon(x) := \varepsilon^{-1}\left[P_{\mathcal{K}(x)}(\overline{u}(x) + \varepsilon v(x)) - \overline{u}(x)\right] \quad \text{for a.a. } x \in \Omega. \tag{5.4}$$

Clearly, $v_\varepsilon(x) \in R_{\mathcal{K}(x)}(\overline{u}(x))$ and using that $v(x) \in T_{\mathcal{K}(x)}(\overline{u}(x))$ we get $v_\varepsilon(x) \to v(x)$ for a.a. $x \in \Omega$. Finally, using that $|v_\varepsilon(x)| \le |v(x)|$, we obtain the convergence in $L^2(\Omega)$ and so $v \in T_{\mathcal{K}}(\overline{u})$.

Proof of (ii): Set $\hat{N}(\overline{u}) := \left\{v \in L^2(\Omega) \mid v(x) \in N_{\mathcal{K}(x)}(\overline{u}(x)) \text{ for a.a. } x \in \Omega\right\}$ and let $v^* \in \hat{N}(\overline{u})$. Using that $N_{\mathcal{K}(x)}(\overline{u}(x))$ is the polar cone of $T_{\mathcal{K}(x)}(\overline{u}(x))$, assertion (i) yields $v^* \in N_{\mathcal{K}}(\overline{u})$. Conversely, let $v^* \in N_{\mathcal{K}}(\overline{u})$ and let v_1, v_2 be such that

$$v^*(x) = v_1(x) + v_2(x), \quad v_1(x)v_2(x) = 0, \quad v_1(x) \in T_{\mathcal{K}(x)}(\overline{u}(x)),\ v_2(x) \in N_{\mathcal{K}(x)}(\overline{u}(x)) \text{ for a.a. } x \in \Omega.$$

Assertion (i) implies that $v_1 \in T_{\mathcal{K}}(\overline{u})$ and so $\|v_1\|_2^2 = \int_\Omega v^*(x)v_1(x)\mathrm{d}x \le 0$, which gives that $v^* = v_2$.

Proof of (iii): Since the second equality follows from (i) and (ii), we only prove the first one. The inclusion $\mathrm{clo}_2(R_{\mathcal{K}}(\overline{u}) \cap (q^*)^\perp) \subseteq T_{\mathcal{K}}(\overline{u}) \cap (q^*)^\perp$ being trivial, we prove the other one. Now, fix $q^* \in N_{\mathcal{K}}(\overline{u})$ and let $v \in T_{\mathcal{K}}(\overline{u}) \cap (q^*)^\perp$. For $\varepsilon > 0$ define

$$\widehat{v}_\varepsilon(x) := \begin{cases} v(x) & \text{if } \overline{u}(x) + \varepsilon v(x) \in \mathcal{K}(x), \\ 0 & \text{otherwise.} \end{cases} \tag{5.5}$$

Clearly, $\widehat{v}_\varepsilon \in \mathcal{R}_{\mathcal{K}}(\overline{u}) \cap (q^*)^\perp$ and, as $\varepsilon \downarrow 0$, we have $\widehat{v}_\varepsilon(x) \to v(x)$ for a.a. $x \in \Omega$. The dominated convergence theorem gives that $\widehat{v}_\varepsilon \to v$ in $L^2(\Omega)$ and the result follows. □

References

[1] R.A. Adams. *Sobolev spaces.* Academic Press, New York, 1975.

[2] J. F. Bonnans. Second order analysis for control constrained optimal control problems of semilinear elliptic systems. *Appl. Math. Optim.*, 38-3:303–325, 1998.

[3] J. F. Bonnans. Pontryagin's principle for the optimal control of semilinear elliptic systems with state constraints. *Proceedings of the 30th Conference on Decision and Control, Brigthon, England*, December 1991.

[4] J.F. Bonnans and E. Casas. Un principe de Pontryagin pour le contrôle des systèmes semilinéaires elliptiques. *Journal of Differential Equations*, 90:288–303, 1991.

[5] J.F. Bonnans and E. Casas. Maximum principles in the optimal control of semilinear elliptic systems. *5th IFAC Conference on Distributed Parameter Systems, A. El Jai ed. (Perpignan), Pergamon Press*, 33-1:274–298, 26-29 June 1989.

[6] J.F. Bonnans and N.P. Osmolovskiĭ. Second-order analysis of optimal control problems with control and initial-final state constraints. *Journal of Convex Analysis*, 17(3):885–913, 2010.

[7] J.F. Bonnans and N.P. Osmolovskiĭ. Characterization of a local quadratic growth of the hamiltonian for control constrained optimal control problems. *Dynamics of Continuous, Discrete and Impulsive Systems*, 2011. To appear.

[8] J.F. Bonnans and A. Shapiro. *Perturbation analysis of optimization problems.* Springer-Verlag, New York, 2000.

[9] J.F. Bonnans and F.J. Silva. Asymptotic expansion for the solutions of control constrained semilinear elliptic problems with interior penalties. *SIAM J. Control Optim., to appear*, 2011.

[10] J.F. Bonnans and H. Zidani. Optimal control problems with partially polyhedric constraints. *SIAM J. Control Optim.*, 37(6):1726–1741 (electronic), 1999.

[11] E. Casas, J.C. de Los Reyes, and F. Tröeltzsch. Sufficient second-order optimality conditions for semilinear control problems with pointwise state constraints. *SIAM J. Optim.*, 19-2:616–643, 2008.

[12] E. Casas and F. Tröeltzsch. Second order necessary optimality conditions for some state-constrained control problems of semilinear elliptic equations. *Appl. Math. Optim.*, 39:211–228, 1999.

[13] E. Casas and F. Tröeltzsch. Second order necessary and sufficient optimality conditions for optimization problems and applications to control theory. *SIAM J. Optim.*, 13:406–431, 2002.

[14] E. Casas and F. Tröeltzsch. Recent advances in the analysis of pointwise state-constrained elliptic optimal control problems. *ESAIM Cont. Optim. Calc. Var.*, 16-3:581–600, 2010.

[15] E. Casas, F. Tröeltzsch, and A. Unger. Second order sufficient optimality conditions for a nonlinear elliptic control problem. *Journal for Analysis and its Applications*, 15:687–707, 1996.

[16] E. Casas, F. Tröeltzsch, and A. Unger. Second order sufficient optimality conditions for some state-constrained control problems of semilinear elliptic equations. *SIAM J. Control Optim.*, 38:369–1391, 2000.

[17] L.C. Evans. *Partial differential equations.* Amer. Math Soc., Providence, RI, 1998. Graduate Studies in Mathematics 19.

[18] D. Gilbarg and N.S. Trudinger. *Elliptic partial differential equations of second order, 2nd edition.* Springer Verlag, Berlin, 1983.

[19] A. Haraux. How to differentiate the projection on a convex set in Hilbert space. some applications to variational inequalities. *J. Mathematical Society of Japan*, 29:615–631, 1977.

[20] A. Hoffman. On approximate solutions of systems of linear inequalities. *Journal of Research of the National Bureau of Standards, Section B, Mathematical Sciences*, 49:263–265, 1952.

[21] X. Li and J. Yong. *Optimal Control Theory for Infinite Dimensional Systems.* Birkhäuser, 1994. Systems and Control: Foundations and Applications.

[22] F. Mignot. Contrôle dans les inéquations variationnelles. *Journal of Functional Analysis*, 22:25–39, 1976.

[23] L. Pontriaguine, V. Boltianski, R. Gamkrelidze, and E. Michtchenko. *The Mathematical Theory of Optimal Processes.* Wiley Interscience, New York, 1962.

[24] U. E. Raitum. The maximum principle in optimal control problems for an elliptic equation. *Z. Anal. Anwendungen*, 5(4):291–306, 1986.

[25] R. T. Rockafellar. Integral functionals, normal integrands and measurable selections. In *Nonlinear operators and the calculus of variations (Summer School, Univ. Libre Bruxelles, Brussels, 1975)*, pages 157–207. Lecture Notes in Math., Vol. 543. Springer, Berlin, 1976.

[26] A. Rösch and F. Tröltzsch. Sufficient second-order optimality conditions for an elliptic optimal control problem with pointwise control-state constraints. *SIAM J. Optim.*, 17-3:776–794, 2006.

[27] G. Stampacchia. Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus. *Ann. Inst. Fourier (Grenoble)*, 15:189–258, 1965.

[28] F. Tröltzsch. *Optimal Control of Partial Differential Equations - Theory, Methods and Applications.* Graduate Studies in Mathematics, Vol. 112. American Mathematical Society, Providence, Rhode Island, 2010.