PARTIALLY AFFINE CONTROL PROBLEMS: SECOND ORDER CONDITIONS AND A WELL-POSED SHOOTING ALGORITHM[1,2]

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ABSTRACT. This paper deals with optimal control problems for systems that are affine in one part of the control variables and nonlinear in the rest of the control variables. We have finitely many equality and inequality constraints on the initial and final states. First we obtain second order necessary and sufficient conditions for weak optimality. Afterwards, we propose a shooting algorithm, and we show that the sufficient condition above-mentioned is also sufficient for the injectivity of the shooting function at the solution.

1. INTRODUCTION

In this article we investigate an optimal control that is affine in one part of the control variables and nonlinear in the rest of the control variables. We consider a finite quantity of initial-final state constraints. First we provide second order necessary and sufficient conditions for weak optimality. We do not assume uniqueness of the multipliers. Then we prove that the stated sufficient condition is also sufficient for the well-posedness of the shooting algorithm. Some of the techniques used in Aronna et al. [1, 3] are employed.

The investigation of this particular framework is motivated by some models encountered in practice. Among these we can mention: 1. an hydrothermal electricity production problem studied in Bortolossi et al. [6] and Aronna et al. [2], 2. the Goddard's problem in 3 dimensions introduced in Goddard [15] and analysed in Martinon et al. [4].

The literature on the subject of optimality conditions developped until this moment is limited to the works of Goh [17, 18], Dmitruk [12], Dmitruk-Shishov [13] and Maurer-Osmolovskii [23]. In the papers by Goh, second order necessary conditions are derived assuming uniqueness of the multiplier. They are consequence of the classical Legendre-Clebsch condition applied to a transformed problem, and are currently known as Goh-Legendre-Clebsch conditions. Dmitruk and Shishov [13] analysed the quadratic functional associated to the second variation of the Lagrangian function. They provided a set of necessary conditions for the non-negativity of this quadratic

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functional. Dmitruk [12] proposed, without proof, necessary and sufficient conditions for a problem having a particular structure: the affine control variables and the nonlinear one are not multiplied in the dynamics, i.e., they are uncoupled. This hypothesis is not used in the present work. All of these four articles use Goh's Transformation, first introduced in [16], to derive their conditions. We use this transformation as well. On the other hand, in [23] Maurer and Osmolovskii gave a sufficient condition for the case where the affine control is subject to bounds and it is bang-bang at the solution. This structure is not studied here since no control constraints are considered. Our affine control is suppose to be totally singular.

The article is organised as follows. In section 2 we present the problem, the basic definitions and properties. A necessary condition is established in section 3. In section 4 we introduce Goh's Transformation, that is an essential tool for the rest of the article. In Section 5 we show a new necessary condition, and in Section 6 we give a sufficient for weak optimality. A shooting algorithm is proposed in Section 7, and in Section 8 we give a sufficient condition for this algorithm to be well-posed.

2. Statement of the problem and assumptions

2.1. **Statement of the problem.** In this paper we investigate the optimal control problem (P) given by

$$J := \varphi_0(x_0, x_T) \to \min, \tag{1}$$

$$\dot{x}_t = \sum_{i=0}^{m} v_{i,t} f_i(x_t, u_t), \tag{2}$$

$$\eta_j(x_0, x_T) = 0, \text{ for } j = 1\dots, d_\eta, \tag{3}$$

$$\varphi_i(x_0, x_T) \leq 0, \text{ for } i = 1, \dots, d_\varphi, \tag{4}$$

where $f_i : \mathbb{R}^{n+l} \to \mathbb{R}^n$ for $i = 0, \dots, m$, $\varphi_i : \mathbb{R}^{2n} \to \mathbb{R}$ for $i = 0, \dots, d_\varphi$, $\eta_j : \mathbb{R}^{2n} \to \mathbb{R}$ for $j = 1, \dots, d_\eta$ and $v_0 \equiv 1$ (it is not a variable). The nonlinear control u belongs to $\mathcal{U} := L_\infty(0, T; \mathbb{R}^l)$, while by $\mathcal{V} := L_\infty(0, T; \mathbb{R}^m)$ we denote the space of affine controls, and $\mathcal{X} := W^1_\infty(0, T; \mathbb{R}^n)$ refers to the state space. When needed, put $w = (x, u, v)$ for a point in $\mathcal{W} := \mathcal{X} \times \mathcal{U} \times \mathcal{V}$. Assume that all data functions f_i are twice continuously differentiable.

A *trajectory* is an element $w \in \mathcal{W}$ that satisfies the state equation (2). If in addition constraints (3) and (4) hold, say that w is a *feasible trajectory* of the problem (P). Denote by $\mathcal{A}$ the *set of feasible points.*

The following type of minimum is considered:

Definition 2.1. A feasible trajectory $\hat{w} = (\hat{x}, \hat{u}, \hat{v}) \in \mathcal{W}$ is said to be a *weak minimum* of (P) if there exists $\varepsilon > 0$ such that the cost function attains at $\hat{w}$ its minimum in the set of feasible trajectories satisfying

$$\|x - x^0\|_\infty < \varepsilon, \quad \|u - \hat{u}\|_\infty < \varepsilon, \quad \|v - \hat{v}\|_\infty < \varepsilon.$$

In the sequel, we study a nominal feasible trajectory $\hat{w} = (\hat{x}, \hat{u}, \hat{v}) \in \mathcal{W}$. An element $\delta w \in \mathcal{W}$ is termed *feasible variation* for $\hat{w}$ if $\hat{w} + \delta w \in \mathcal{A}$. We write $\mathbb{R}^{d,*}$ for the $d-$dimensional space of row vectors with real components. Take $\lambda = (\alpha, \beta, p)$ in $\mathbb{R}^{d_\varphi+1,*} \times \mathbb{R}^{d_\eta,*} \times W^1_\infty(0,T;\mathbb{R}^{n,*})$, i.e., p is a Lip-continuous function with values in $\mathbb{R}^{n,*}$. Define the *pre-Hamiltonian* function

$$H[\lambda](x,u,v,t) := p_t \sum_{i=0}^{m} v_i f_i(x,u),$$

the *terminal Lagrangian* function

$$\ell[\lambda](q) := \sum_{i=0}^{d_\varphi} \alpha_i \varphi_i(q) + \sum_{j=1}^{d_\eta} \beta_j \eta_j(q),$$

and the *Lagrangian* function

$$\Phi[\lambda](w) := \ell[\lambda](\hat{x}_0, \hat{x}_T) + \int_0^T p_t \left(\sum_{i=0}^{m} v_{i,t} f_i(x_T, u_t) - \dot{x}_T \right) \mathrm{d}t. \tag{5}$$

In the sequel, whenever some argument of f_i, H, ℓ, Φ or their derivatives is omitted, assume that they are evaluated over $\hat{w}$. Without loss of generality suppose that

$$\varphi_i(\hat{x}_0, \hat{x}_T) = 0, \text{ for all } i = 0,1,\ldots,d_\varphi. \tag{6}$$

2.2. **Lagrange and Pontryagin multipliers.**

Definition 2.2. An element $\lambda = (\alpha,\beta,p) \in \mathbb{R}^{d_\varphi+1,*} \times \mathbb{R}^{d_\eta,*} \times W^1_\infty(0,T;\mathbb{R}^{n,*})$ is a *Lagrange multiplier* associated to $\hat{w}$ if it satisfies the following conditions:

$$|\alpha| + |\beta| = 1, \tag{7}$$

$$\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{d_\varphi}) \geq 0, \tag{8}$$

the function p is solution of the *costate equation*

$$-\dot{p}_t = H_x[\lambda](\hat{x}_t, \hat{u}_t, \hat{v}_t, t), \tag{9}$$

and satisfies the *transversality conditions*

$$\begin{aligned} p_0 &= -D_{x_0}\ell[\lambda](\hat{x}_0,\hat{x}_T),\\ p_T &= D_{x_T}\ell[\lambda](\hat{x}_0,\hat{x}_T), \end{aligned} \tag{10}$$

and the following *stationarity* conditions hold

$$\begin{cases} H_u[\lambda](\hat{x}(t),\hat{u}(t),\hat{v}(t),t) = 0,\\ H_v[\lambda](\hat{x}(t),\hat{u}(t),\hat{v}(t),t) = 0, \end{cases} \quad \text{a.e. on } [0,T]. \tag{11}$$

We say that λ is a *Pontryagin multiplier* if it satisfies (7)-(10) and the following *minimum condition*

$$H[\lambda](\hat{x}_t,\hat{u}_t,\hat{v}_t,t) = \min_{(u,v)\in\mathbb{R}^{l+m}} H[\lambda](\hat{x}_t,u,v,t), \quad \text{a.e. on } [0,T]. \tag{12}$$

Denote by Λ_L and Λ_P the sets of Lagrange and Pontryagin multipliers, respectively.

It easily follows from the previous definitions that

$$\Lambda_P \subset \Lambda_L. \tag{13}$$

and

$$H_{v_j}[\lambda] = p f_j(\hat{x}, \hat{u}) \equiv 0, \quad \text{for } j = 1, \dots, m. \tag{14}$$

Recall the following well-known result:

Theorem 2.3. *The set Λ_L and Λ_P are non empty and compact.*

Proof. Regarding the existence of a Pontryagin multiplier, the reader is referred to Ioffe-Tihomirov [20].

In order to prove the compactness, observe that p may be expressed as a linear continuous mapping of (α, β). Thus, since the normalization (7) holds, both Λ_L and Λ_P are finite-dimensional compact sets. □

In view of previous result, note that Λ_L and Λ_P can be identified with compact subsets of $\mathbb{R}^s$, where $s := d_\varphi + d_\eta + 1$.

Given a square symmetric real matrix X, we write $X \succeq 0$ to indicate that it is positive semidefinite and $X \succ 0$ when it is positive definite. The minimum condition (12) yields following set of properties:

Lemma 2.4. *For every Pontryagin multiplier $\lambda \in \Lambda_P$,*

(i)

$$H_{uu}[\lambda] = p \sum_{i=0}^{m} \hat{v}_i D^2_{uu} f_i(\hat{x}, \hat{u}) \succeq 0, \tag{15}$$

(ii)

$$\begin{gathered} H_{uv}[\lambda] \equiv 0, \\ p D_u f_j(\hat{x}, \hat{u}) \equiv 0, \quad \text{for } j = 0, \dots, m. \end{gathered} \tag{16}$$

Proof. Observe that (12) implies that the matrix

$$\begin{pmatrix} H_{uu}[\lambda] & H_{vu}[\lambda]^\top \\ H_{vu}[\lambda] & H_{vv}[\lambda] \end{pmatrix} \tag{17}$$

is positive semidefinite. Since $H_{vv}[\lambda] \equiv 0$, the positive semidefiniteness in (i) and the first identity in (ii) follow. The latter equation implies the second identity in (ii) for $j = 1, \dots, m$. Finally, use (11) to get the analogous condition for $j = 0$. □

For $(\bar{x}_0, \bar{u}, \bar{v}) \in \mathbb{R}^n \times \mathcal{U} \times \mathcal{V}$, consider the *linearized state equation:*

$$\dot{\bar{x}}_t = A_t \bar{x}_t + D_t \bar{u}_t + B_t \bar{v}_t, \quad \text{for } t \in (0, T), \tag{18}$$

$$\bar{x}(0) = \bar{x}_0, \tag{19}$$

where

$$A_t := \sum_{i=0}^{m} \hat{v}_i f_{i,x}(\hat{x}, \hat{u}), \quad D_t := \sum_{i=0}^{m} \hat{v}_i f_{i,u}(\hat{x}, \hat{u}), \tag{20}$$

and $B:[0,T]\to\mathcal{M}_{n\times m}(\mathbb{R})^*$ such that for every $v\in\mathbb{R}^m$,

$$B_t v := \sum_{i=0}^{m} v_i f_i(\hat{x}_t,\hat{u}_t), \tag{21}$$

i.e., the ith. column of B is $f_i(\hat{x},\hat{u})$. The variable $\bar{x}$ in (18)-(19) is called *linearized state variable.*

2.3. **Critical cones.** Set $\mathcal{X}_2 := W_2^1(0,T;\mathbb{R}^n)$, $\mathcal{U}_2 := L_2(0,T;\mathbb{R}^l)$ and $\mathcal{V}_2 := L_2(0,T;\mathbb{R}^m)$. Put $\mathcal{W}_2 := \mathcal{X}_2\times\mathcal{U}_2\times\mathcal{V}_2$ for the corresponding product space. Define,

$$\mathcal{H}_2 := \{\bar{w}\in\mathcal{W}_2 : (18)\text{-}(19)\text{ hold}\},\ \mathcal{H}_\infty := \{\bar{w}\in\mathcal{W}_\infty : (18)\text{-}(19)\text{ hold}\}. \tag{22}$$

Given $\bar{w}\in\mathcal{H}_p$, consider the *linearization of the initial-final constraints and cost function:*

$$D\eta_j(\hat{x}_0,\hat{x}_T)(\bar{x}_0,\bar{x}_T) = 0,\ \text{for } j=1,\dots,d_\eta, \tag{23}$$

$$D\varphi_i(\hat{x}_0,\hat{x}_T)(\bar{x}_0,\bar{x}_T) \leq 0,\ \text{for } i=0,\dots,d_\varphi, \tag{24}$$

Define the L_p-*critical cone*

$$\mathcal{C}_p := \{\bar{w}\in\mathcal{H}_p :\ (23)-(24)\text{ hold}\}. \tag{25}$$

Lemma 2.5. *The critical cone $\mathcal{C}_\infty$ is a dense subset of $\mathcal{C}_2$.*

In order to prove previous lemma, recall the following technical result, (see e.g. Dmitruk [11] for a proof):

Lemma 2.6 (on density)**.** *Consider a locally convex topological space X, a finite-faced cone $C\subset X$, and a linear manifold L dense in X. Then the cone $C\cap L$ is dense in C.*

Proof. [of Lemma 2.5] Set $X:=\mathcal{H}_2$, $L:=\mathcal{H}_\infty$, $C:=\mathcal{C}_2$ and apply Lemma 2.6. □

3. Second order analysis

We begin this section by presenting the second variation of the Lagrangian function the we denote by Ω. All the second order conditions in this paper are established in terms of either Ω or some transformed form of Ω. The main result of the current section is the second order necessary condition provided by Theorem 3.6.

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*$\mathcal{M}_{n\times m}(\mathbb{R})$: the space of $n\times m-$real matrices

3.1. **Second variation.** Let us consider the quadratic mapping:

(26)
$$\begin{aligned}\Omega[\lambda](\bar{x},\bar{u},\bar{v}) :=\ & \tfrac{1}{2}\ell''[\lambda](\hat{x}_0,\hat{x}_T)(\bar{x}_0,\bar{x}_T)^2 \\ & + \int_0^T [\tfrac{1}{2}\bar{x}^\top Q[\lambda]\bar{x} + \bar{u}^\top E[\lambda]\bar{x} + \bar{v}^\top C[\lambda]\bar{x} + \tfrac{1}{2}\bar{u}^\top R_0[\lambda]\bar{u} + \bar{v}^\top K[\lambda]\bar{u}]\mathrm{d}t,\end{aligned}$$

where the involved matrices are, omitting arguments,

(27) $$Q := H_{xx}, \quad E := H_{ux}, \quad C := H_{vx}, \quad R_0 := H_{uu}, \quad K := H_{vu}.$$

Lemma 3.1 (Lagrangian expansion)**.** *Let $w \in \mathcal{W}$ be a solution of* (2)*, and set $\delta w := w - \hat{w}$. Then for every multiplier $\lambda \in \Lambda_L$,*

(28)
$$\begin{aligned}\Phi[\lambda](w) = \Phi[\lambda](\hat{w}) + \Omega[\lambda](\delta x,\delta u,\delta v) + \int_0^T & [H_{vxx}[\lambda](\delta x,\delta x,\delta v) \\ & + 2H_{vux}[\lambda](\delta x,\delta u,\delta v) + H_{vuu}[\lambda](\delta u,\delta u,\delta v)]\,\mathrm{d}t \\ & + \mathcal{O}(|(\delta x_0,\delta x_T)|^3) + (1+\|v\|_1)\|(\delta x,\delta u)\|_\infty \mathcal{O}(\|(\delta x,\delta u)\|_2^2),\end{aligned}$$

where the time variable was omitted for the sake of simplicity.

Proof. Omit the dependence on λ for the sake of simplicity. In order to achieve the expression (28) consider the second order Taylor representations below, written in a compact form,

(29) $$\ell(x_0,x_T) = \ell + D\ell(\delta x_0,\delta x_T) + \tfrac{1}{2}D^2\ell(\delta x_0,\delta x_T)^2 + \mathcal{O}(|(\delta x_0,\delta x_T)|^3),$$

(30) $$f_i(x_t,u_t) = f_{i,t} + Df_{i,t}(\delta x_t,\delta u_t) + \tfrac{1}{2}D^2 f_{i,t}(\delta x_t,\delta u_t)^2 + \mathcal{O}(\|(\delta x,\delta u)\|_3^3),$$

where, whenever the argument is missing the corresponding function is evaluated on the reference trajectory $\hat{w}$. Observe that the costate equation (9) and the transversality conditions (10) yield

(31) $$D\ell(\delta x_0,\delta x_T) = -p_0\delta x_0 + p_T\delta x_T = \int_0^T p\left[-\sum_{i=0}^m \hat{v}_i D_x f_i \delta x + \dot{\delta x}\right]\mathrm{d}t,$$

Recall the expression of the Lagrangian given in (5). Replacing $\ell(x_0,x_T)$ and $f_i(x,u)$ in (5) by their Taylor expansions (29)-(30) and using the identity (31) we get

$$\begin{aligned}\Phi(w) =\ & \Phi(\hat{w}) + \int_0^T [H_u\delta u + H_v\delta v]\mathrm{d}t + \Omega(\delta x,\delta u,\delta v) \\ & + \int_0^T [H_{vxx}(\delta x,\delta x,\delta v) + 2H_{vux}(\delta x,\delta u,\delta v) + H_{vuu}(\delta u,\delta u,\delta v)]\,\mathrm{d}t \\ & + \mathcal{O}(|(\delta x_0,\delta x_T)|^3) + \|(\delta x,\delta u)\|_\infty \int_0^T p\sum_{i=0}^m v_i\, \mathcal{O}(\|(\delta x,\delta u)\|_2^2)\mathrm{d}t.\end{aligned}$$

Finally, to obtain (28) use stationarity condition (11). □

Remark 3.2. The last lemma gives the identity:

(32) $$\Omega[\lambda](\bar{w}) = \tfrac{1}{2}D^2\Phi[\lambda](\hat{w})\bar{w}^2$$

3.2. **Second order necessary condition.** Recall the classical second order condition below, a proof of which can be found in [22].

Theorem 3.3 (Classical second order necessary condition)**.** *If $\hat{w}$ is a weak minimum of problem (P), then*

$$\max_{\lambda\in\Lambda_L} \Omega[\lambda](\bar{x},\bar{u},\bar{v}) \geq 0, \quad \text{on } \mathcal{C}_\infty. \tag{33}$$

In this paragraph we aim to strengthen previous necessary condition by proving that the maximum in (33) can be taken in a smaller set of multipliers. We shall first give a description of the subset of Lagrange multipliers we work with. Observe that:

Remark 3.4. Condition (33) can be extended to the cone $\mathcal{C}_2$ by the continuity of $\Omega[\lambda]$ and the compactness of Λ_L.

Recall the definition of the Hilbert space $\mathcal{H}_2$ introduced in (22), and consider the subset of Λ_L given by

$$\Lambda_L^{\#} := \{\lambda \in \Lambda_L : \Omega[\lambda] \text{ is weakly-l.s.c. on } \mathcal{H}_2\}. \tag{34}$$

The two results below are established in this section. Lemma 3.5 provides a characterization of $\Lambda_L^{\#}$ and Theorem 3.6 gives a strengthened second order necessary condition.

Lemma 3.5.

$$\Lambda_L^{\#} = \{\lambda \in \Lambda_L : R_0[\lambda] \succeq 0 \text{ and } K[\lambda] \equiv 0\}. \tag{35}$$

Theorem 3.6 (Second order necessary condition)**.** *If $\hat{w}$ is a weak minimum of problem (P), then*

$$\max_{\lambda\in\Lambda_L^{\#}} \Omega[\lambda](\bar{x},\bar{u},\bar{v}) \geq 0, \quad \text{on } \mathcal{C}_2. \tag{36}$$

In order to prove Lemma 3.5 consider the matrix

$$\tfrac{1}{2}\begin{pmatrix} R_0[\lambda] & K[\lambda]^\top \\ K[\lambda] & 0 \end{pmatrix}, \tag{37}$$

and note that it is the coefficient of the quadratic term on $\begin{pmatrix}\bar{u}\\ \bar{v}\end{pmatrix}$ in $\Omega[\lambda]$. Hence $\Omega[\lambda]$ can be written as the sum of a weakly-continuous mapping on the space $\mathcal{H}_2$ and the quadratic operator given by

$$\int_0^T (\tfrac{1}{2}\bar{u}^\top R_0[\lambda]\bar{u} + \bar{v}^\top K[\lambda]\bar{u})\mathrm{d}t. \tag{38}$$

Recall next a characterization of weakly-l.s.c. forms.

Lemma 3.7. [19, Theorem 3.2] *Consider a real interval I and a quadratic form $\mathcal{Q}$ over the Hilbert space $L_2(I)$, given by*

$$\mathcal{Q}(y) := \int_I y_t^\top R_t y_t \mathrm{d}t.$$

Then $\mathcal{Q}$ is weakly l.s.c. over $L_2(I)$ iff

$$R_t \succeq 0, \quad \text{a.e. on } I. \tag{39}$$

Lemma 3.5 follows from last result and it yields, owing to Lemma 2.4:

$$\Lambda_P \subset \Lambda_L^{\#}. \tag{40}$$

Theorem 3.6 is a consequence of Remark 3.4, Lemma 3.5 and the following result on quadratic forms:

Lemma 3.8. [9, Theorem 5] *Given a Hilbert space H, and $a_1, a_2, \ldots, a_p$ in H, set*

$$K := \{x \in H : (a_i, x) \leq 0, \text{ for } i = 1, \ldots, p\}. \tag{41}$$

Let M be a convex and compact subset of $\mathbb{R}^s$, and let $\{Q^\psi : \psi \in M\}$ be a family of continuous quadratic forms over H, the mapping $\psi \to Q^\psi$ being affine. Set $M^{\#} := \{\psi \in M : Q^\psi$ is weakly-l.s.c.† on $H\}$ and assume that

$$\max_{\psi \in M} Q^\psi(x) \geq 0, \text{ for all } x \in K. \tag{42}$$

Then

$$\max_{\psi \in M^{\#}} Q^\psi(x) \geq 0, \text{ for all } x \in K. \tag{43}$$

4. Goh Transformation

In this section we introduce a linear transformation of variables $\bar{x}$, $\bar{u}$ and $\bar{v}$. Afterwards we define the critical cone in the new space of variables denoted by $\mathcal{P}_2$, and we show that performing the mentioned transformation in Ω yields a new quadratic operator called $\Omega_{\mathcal{P}_2}$ on the transformed space.

Consider hence the linear system in (18) and the change of variables

$$\begin{cases} \bar{y}_t := \int_0^t \bar{v}_s \mathrm{d}s, \\ \bar{\xi}_t := \bar{x}_t - B_t \bar{y}_t, \end{cases} \quad \text{for } t \in [0, T]. \tag{44}$$

This change of variables, first introduced by Goh [17], can be done in any linear system of differential equations, and it is often called *Goh's transformation.* Observe that $\bar{\xi}$ defined in that way satisfies the linear equation

$$\dot{\bar{\xi}} = A\bar{\xi} + D\bar{u} + B_1\bar{y}, \quad \bar{\xi}_0 = \bar{x}_0, \tag{45}$$

where A and D were given in (20), and

$$B_{1,t} := A_t B_t - \frac{\mathrm{d}}{\mathrm{d}t} B_t. \tag{46}$$

†weakly-lower semicontinuous

4.1. **Tranformed critical cones.** In this paragraph we present the critical cones obtained after Goh's transformation, that we use later on to formulate the optimality conditions. Recall the linearized constraints in (23)-(24) and the critical cone given by (25) in paragraph 2.3. Let $(\bar{x},\bar{u},\bar{v})\in\mathcal{C}_\infty$ be a critical direction. Define $(\bar{\xi},\bar{y})$ by transformation (44) and set $\bar{h}:=\bar{y}_T$. Note that (23)-(24) yields

$$D\eta_j(\hat{x}_0,\hat{x}_T)(\bar{\xi}_0,\bar{\xi}_T+B_T\bar{h})=0,\quad \text{for } j=1,\dots,d_\eta, \tag{47}$$

$$D\varphi_i(\hat{x}_0,\hat{x}_T)(\bar{\xi}_0,\bar{\xi}_T+B_T\bar{h})\le 0,\quad \text{for } i=0,\dots,d_\varphi. \tag{48}$$

Recall the definition of $\mathcal{W}_2$ in paragraph 2.3. Denote by $\mathcal{Y}$ the space $W^1_\infty(0,T;\mathbb{R}^m)$, and consider the cones

$$\mathcal{P}:=\{(\bar{\xi},\bar{u},\bar{y},\bar{h})\in\mathcal{W}\times\mathbb{R}^m:\ \bar{y}_0=0,\ \bar{y}_T=\bar{h},\ (45),\ (47)\text{-}(48)\text{ hold}\}, \tag{49}$$

$$\mathcal{P}_2:=\{(\bar{\xi},\bar{u},\bar{y},\bar{h})\in\mathcal{W}_2\times\mathbb{R}^m:\ (45),\ (47)\text{-}(48)\text{ hold}\}. \tag{50}$$

Remark 4.1. Notice that $\mathcal{P}$ consists of the directions obtained by transformating the elements of $\mathcal{C}_\infty$ via Goh's Transformation (44).

Lemma 4.2. *$\mathcal{P}$ is a dense subspace of $\mathcal{P}_2$ in the $\mathcal{W}_2\times\mathbb{R}^m-$topology.*

Proof. Notice that the inclusion is immediate. In order to prove the density, consider the linear spaces

$$X:=\{(\bar{\xi},\bar{u},\bar{y},\bar{h})\in\mathcal{W}_2\times\mathbb{R}^m:\ (45)\text{ holds}\}, \tag{51}$$

$$L:=\{(\bar{\xi},\bar{u},\bar{y},\bar{h})\in\mathcal{W}_\infty\times\mathbb{R}^m:\ \bar{y}(0)=0,\ \bar{y}(T)=\bar{h},\ \text{and (45) holds}\}, \tag{52}$$

and the cone

$$C:=\{(\bar{\xi},\bar{u},\bar{y},\bar{h})\in X:\ (47)-(48)\text{ holds}\}. \tag{53}$$

Since L is a dense linear subspace of X (by Lemma 6 in [13] or Lemma 7.1 in [1]), and C is a finite-faced cone of X, we get the desired density by Lemma 2.6. ☐

4.2. **Transformed second variation.** In Theorem 4.3 below we prove that performing the Goh transformation in Ω yields the new quadratic operator $\Omega_\mathcal{P}$ in variables $(\bar{\xi},\bar{u},\bar{y},\bar{v},\bar{h})$ defined below. Recall the definitions in (27) and set for $\lambda\in\Lambda_L^\#$,

$$\begin{aligned}\Omega_\mathcal{P}[\lambda](\bar{\xi},\bar{u},\bar{y},\bar{v},\bar{h}):=g[\lambda](\bar{\xi}_0,\bar{\xi}_T,\bar{h})+\int_0^T\Big(\tfrac12\bar{\xi}^\top Q[\lambda]\bar{\xi}+\bar{u}^\top E[\lambda]\bar{\xi}\\+\bar{y}^\top M[\lambda]\bar{\xi}+\tfrac12\bar{u}^\top R_0[\lambda]\bar{u}+\bar{y}^\top J[\lambda]\bar{u}+\tfrac12\bar{y}^\top R_1[\lambda]\bar{y}+\bar{v}^\top V[\lambda]\bar{y}\Big)\,\mathrm{d}t,\end{aligned} \tag{54}$$

where

$$M:=B^\top Q-\dot{C}-CA,\quad J:=B^\top E^\top-CD, \tag{55}$$

$$S:=\tfrac12(CB+(CB)^\top),\quad V:=\tfrac12(CB-(CB)^\top), \tag{56}$$

$$R_1:=B^\top QB-\tfrac12(CB_1+(CB_1)^\top)-\dot{S}, \tag{57}$$

$$g[\lambda](\zeta_0,\zeta_T,h):=\tfrac12\ell''(\zeta_0,\zeta_T+B_Th)^2+h^\top(C_T\zeta_T+\tfrac12S_Th). \tag{58}$$

Theorem 4.3. *Let* $(\bar{x},\bar{u},\bar{v}) \in \mathcal{H}_2$ *(given in* (22)*) and* $(\bar{\xi},\bar{y})$ *defined by the transformation* (44)*. Then*

$$\Omega[\lambda](\bar{x},\bar{u},\bar{v}) = \Omega_{\mathcal{P}}[\lambda](\bar{\xi},\bar{u},\bar{y},\bar{v},\bar{y}_T). \tag{59}$$

Proof. We omit the dependence on λ for the sake of simplicity. Replacing $\bar{x}$ in the definition (26) of Ω by its expression in (44) yields

$$\begin{aligned}\Omega(\bar{x},\bar{u},\bar{v}) = \; & \tfrac{1}{2}\ell''(\hat{x}_0,\hat{x}_T)(\bar{\xi}_0,\bar{\xi}_T + B_T\bar{y}_T)^2 + \int_0^T \left[\tfrac{1}{2}(\bar{\xi}+B\bar{y})^\top Q(\bar{\xi}+B\bar{y})\right.\\ & \left. + \bar{u}^\top E(\bar{\xi}+B\bar{y}) + \bar{v}^\top C(\bar{\xi}+B\bar{y}) + \tfrac{1}{2}\bar{u}^\top R_0\bar{u}\right]\mathrm{d}t.\end{aligned} \tag{60}$$

Integrating by parts the first term containing $\bar{v}$ yields, owing to (45),

$$\int_0^T \bar{v}^\top C\bar{\xi}\mathrm{d}t = [\bar{y}^\top C\bar{\xi}]_0^T - \int_0^T \bar{y}^\top\{\dot{C}\bar{\xi} + C(A\bar{\xi}+D\bar{u}+B_1\bar{y})\}\mathrm{d}t. \tag{61}$$

Using the descomposition of CB introduced in (56) we get

$$\begin{aligned}\int_0^T \bar{v}^\top CB\bar{y}\mathrm{d}t &= \int_0^T \bar{v}^\top (S+V)\bar{y}\mathrm{d}t\\ &= \tfrac{1}{2}[\bar{y}^\top S\bar{y}]_0^T + \int_0^T(-\tfrac{1}{2}\bar{y}^\top\dot{S}\bar{y} + \bar{v}^\top V\bar{y})\mathrm{d}t.\end{aligned} \tag{62}$$

Combining (60), (61) and (62), the identity (59) follows. $\square$

Recall Theorem 3.6. Observe that by performing Goh's transformation (44) in (36) and in view of Remark 4.1, we obtain the following form for the second order necessary condition:

Corollary 4.4. *If* $\hat{w}$ *is a weak minimum of problem (P), then*

$$\max_{\lambda\in\Lambda_L^{\#}} \Omega_{\mathcal{P}}[\lambda](\bar{\xi},\bar{u},\bar{y},\dot{\bar{y}},\bar{h}) \geq 0, \quad \text{on } \mathcal{P}. \tag{63}$$

5. New second order necessary condition

We aim to remove the dependence of $\bar{v}$ from the formulation of the second order necessary condition. Note that in (63), given that the considered multipliers are in $\Lambda_L^{\#}$, the matrix $K[\lambda]$ vanishes and that is why we do not include it in $\Omega_{\mathcal{P}}$. However, there still remains the term $\bar{v}^\top V[\lambda]\bar{y}$. Next we prove that we can restrict the maximum in (63) to the subset of $\Lambda_L^{\#}$ consisting of the multipliers for which $V[\lambda]$ vanishes. We use, in an essential way, some techniques introduced by Dmitruk [8, 10] for the proof of similar results.

Definition 5.1. Given $M \subset \mathbb{R}^s$, define

$$G(M) := \{\lambda \in M : V_{ij}[\lambda] \equiv 0 \text{ on } [0,T]\}.$$

Theorem 5.2. *Let $M \subset \mathbb{R}^s$ be convex and compact, and assume that*

$$\max_{\lambda \in M} \Omega_{\mathcal{P}}[\lambda](\bar{\xi}, \bar{u}, \bar{y}, \dot{\bar{y}}, \bar{y}_T) \geq 0, \quad \text{on } \mathcal{P}. \tag{64}$$

Then

$$\max_{\lambda \in G(M)} \Omega_{\mathcal{P}}[\lambda](\bar{\xi}, \bar{u}, \bar{y}, \dot{\bar{y}}, \bar{y}_T) \geq 0, \quad \text{on } \mathcal{P}. \tag{65}$$

The proof of Theorem 5.2 is based on some techniques introduced in Dmitruk [8, 10], and was given in detail in Aronna et al. [1] for a system that is affine in all the control variables. For the case treated here, the same proof holds with minor modifications and hence there is no point in writing it again.

When $\hat{w}$ has a unique associated Lagrange multiplier, as a consequence of Theorem 5.2 we get the corollary below. This corollary is one of the necessary conditions stated by Goh in [16].

Corollary 5.3. *When $\hat{w}$ is a weak minimum having a unique associated multiplier, $V \equiv 0$ or, equivalently, CB is symmetric.*

From theorems 3.6 and 5.2 we get

Theorem 5.4 (New necessary condition)**.** *If $\hat{w}$ is a weak minimum of problem (P), then*

$$\max_{\lambda \in G(\operatorname{co} \Lambda_L^{\#})} \Omega_{\mathcal{P}}[\lambda](\bar{\xi}, \bar{u}, \bar{y}, \dot{\bar{y}}, \bar{y}_T) \geq 0, \quad \text{on } \mathcal{P}. \tag{66}$$

Observe that for $\lambda \in G(\operatorname{co} \Lambda_L^{\#})$, the quadratic form $\Omega[\lambda]$ does not depend on $\bar{v}$ since its coefficients vanish. We can then consider its continuous extension to $\mathcal{P}_2$, given by

$$\begin{aligned} \Omega_{\mathcal{P}_2}[\lambda](\bar{\xi}, \bar{u}, \bar{y}, \bar{h}) := g[\lambda](\bar{\xi}_0, \bar{\xi}_T, \bar{h}) + \int_0^T \Big(\tfrac{1}{2} \bar{\xi}^\top Q[\lambda] \bar{\xi} + \bar{u}^\top E[\lambda] \bar{\xi} \\ + \bar{y}^\top M[\lambda] \bar{\xi} + \tfrac{1}{2} \bar{u}^\top R_0[\lambda] \bar{u} + \bar{y}^\top J[\lambda] \bar{u} + \tfrac{1}{2} \bar{y}^\top R_1[\lambda] \bar{y} \Big) \mathrm{d}t. \end{aligned} \tag{67}$$

Applying Theorem 5.4 we obtain

Theorem 5.5. *If $\hat{w}$ is a weak minimum of problem (P), then*

$$\max_{\lambda \in G(\operatorname{co} \Lambda_L^{\#})} \Omega_{\mathcal{P}_2}[\lambda](\bar{\xi}, \bar{u}, \bar{y}, \bar{h}) \geq 0, \quad \text{on } \mathcal{P}_2. \tag{68}$$

Remark 5.6. The latter optimality condition does not involve variable $\bar{v}$. It is stated in the space of variables $(\bar{\xi}, \bar{u}, \hat{y}, \bar{h})$.

6. Second order sufficient condition for the unconstrained weak minimum

This section provides a second order sufficient condition for strict weak-weak optimality with quadratic growth. The proof is an adaptation of the proof in [1], with important simplifications due to the absence of control constraints.

The quadratic growth above mentioned will be established with respect to the *order*

$$\gamma(\bar{x}_0, \bar{u}, \bar{y}, \bar{h}) := |\bar{x}_0|^2 + |\bar{h}|^2 + \int_0^T (|\bar{u}_t| + |\bar{y}_t|)^2 \mathrm{d}t. \tag{69}$$

for $(\bar{x}_0, \bar{u}, \bar{y}, \bar{h}) \in \mathcal{U}_2 \times \mathcal{V}_2 \times \mathbb{R}^{n+m}$. It can also be considered as a function of $(\bar{x}_0, \bar{u}, \bar{v}) \in \mathbb{R}^n \times \mathcal{U}_2 \times \mathcal{V}_2$ by the identity

$$\tilde{\gamma}(\bar{x}_0, \bar{u}, \bar{v}) := \gamma(\bar{x}_0, \bar{u}, \bar{y}, \bar{y}_T), \tag{70}$$

with $\bar{y}$ being the primitive of $\bar{v}$ defined in (44).
Notation: We write γ to refer to either γ or $\tilde{\gamma}$.

Definition 6.1. [Quadratic Growth] We say that $\hat{w}$ satisfies $\gamma-$*quadratic growth condition in the weak sense* if there exists $\rho > 0$ such that

$$J(w) \geq J(\hat{w}) + \rho\gamma(x_0 - \hat{x}_0, u - \hat{u}, v - \hat{v}), \tag{71}$$

for every feasible trajectory w satisfying $\|w - \hat{w}\|_\infty < \varepsilon$.

Theorem 6.2 (Sufficient condition)**.** *Assume that there exists $\rho > 0$ such that*

$$\max_{\lambda \in G(\mathrm{co}\,\Lambda_L^\#)} \Omega_{\mathcal{P}_2}[\lambda](\bar{\xi}, \bar{u}, \bar{y}, \bar{h}) \geq \rho\gamma(\bar{\xi}_0, \bar{u}, \bar{y}, \bar{h}), \quad \textit{on } \mathcal{P}_2. \tag{72}$$

Then $\hat{w}$ is a weak minimum satisfying $\gamma-$quadratic growth in the weak sense.

The remainder of this section is devoted to the proof of Theorem 6.2. We shall start by establishing some technical results that will be needed for the main result.

For the lemma below recall the definition of the space $\mathcal{H}_2$ in (22).

Lemma 6.3. *There exists $\rho > 0$ such that*

$$|\bar{x}_0|^2 + \|\bar{x}\|_2^2 + |\bar{x}_T|^2 \leq \rho\gamma(\bar{x}_0, \bar{u}, \bar{v}), \tag{73}$$

for every linearized trajectory $(\bar{x}, \bar{u}, \bar{v}) \in \mathcal{H}_2$. The constant ρ depends on $\|A\|_\infty$, $\|B\|_\infty$, $\|D\|_\infty$ and $\|B_2\|_\infty$.

Proof. Every time we put ρ_i we refer to a positive constant depending on $\|A\|_\infty$, $\|B\|_\infty$, $\|D\|_\infty$, and/or $\|B_2\|_\infty$. Let $(\bar{x}, \bar{u}, \bar{v}) \in \mathcal{H}_2$ and $(\bar{\xi}, \bar{y})$ be defined by Goh's Transformation (44). Thus $(\bar{\xi}, \bar{u}, \bar{y})$ is solution of (45) having $\bar{\xi}_0 = \bar{x}_0$. Gronwall's Lemma and Cauchy-Schwartz inequality yield

$$\|\bar{\xi}\|_\infty \leq \rho_1(|\bar{\xi}_0|^2 + \|\bar{u}\|_2^2 + \|\bar{y}\|_2^2)^{1/2} \leq \rho_1\gamma(\bar{x}_0, \bar{u}, \bar{y}, \bar{y}_T)^{1/2}, \tag{74}$$

with $\rho_1 = \rho_1(\|A\|_1, \|D\|_\infty, \|B_1\|_\infty)$. This last inequality together with the relation between $\bar{\xi}$ and xb provided by (44) imply

$$\|\bar{x}\|_2 \leq \|\bar{\xi}\|_2 + \|B\|_\infty\|\bar{y}\|_2 \leq \rho_2\gamma(\bar{x}_0, \bar{u}, \bar{y}, \bar{y}_T)^{1/2}, \tag{75}$$

for $\rho_2 = \rho_2(\rho_1, \|B\|_\infty)$. On the other hand, (44) and estimation (74) lead to

$$|\bar{x}_T| \leq |\bar{\xi}_T| + \|B\|_\infty|\bar{y}_T| \leq \rho_1\gamma(\bar{x}_0, \bar{u}, \bar{y}, \bar{y}_T)^{1/2} + \|B\|_\infty|\bar{y}_T|. \tag{76}$$

Then, in view of inequality '$ab \leq \frac{a^2+b^2}{2}$,'

$$|\bar{x}_T|^2 \leq \rho_3 \gamma(\bar{x}_0, \bar{u}, \bar{y}, \bar{y}_T), \tag{77}$$

for some $\rho_3 = \rho_3(\rho_1, \|B\|_\infty)$, The desired estimation follows from (75) and (77). □

Notice that Lemma 6.3 above gives an estimation of the linearized state in terms of γ. The following result shows that the analogous property holds for the variation of the state variable as well. Recall the state dynamics (2).

Lemma 6.4. *For every $C > 0$ there exists $\rho > 0$ such that*

$$|\delta x_0|^2 + \|\delta x\|_2^2 + |\delta x_T|^2 \leq \rho \gamma(\delta x_0, \delta u, \delta v). \tag{78}$$

for every (x, u, v) solution of (2) having $\|v\|_2 \leq C$, and where we denote $\delta w := w - \hat{w}$. The constant ρ depends on C, $\|B\|_\infty$, $\|\dot{B}\|_\infty$ and the Lipschitz constants of f_i.

Proof. Consider (x, u, v) solution of (2) with $\|v\|_2 \leq C$. Let $\delta w := w - \hat{w}$, and $\xi := \delta x - B\delta y$, with B given in (21) and $y_t := \int_0^t v_s \mathrm{d}s$. Note that

$$\begin{aligned} \dot{\xi} &= \sum_{i=0}^m [v_i f_i(x, u) - \hat{v}_i f_i(\hat{x}, \hat{u})] - \dot{B}\delta y - \sum_{i=1}^m \delta v_i f_i(\hat{x}, \hat{u}) \\ &= \sum_{i=0}^m v_i [f_i(\hat{x} + \xi + B\delta y, \hat{u} + \delta u) - f_i(\hat{x}, \hat{u})] - \dot{B}\delta y, \end{aligned} \tag{79}$$

where $v_0 \equiv 1$. In view of the Lipschitz-continuity of f_i,

$$|f_i(\hat{x} + \xi + B\delta y, \hat{u} + \delta u) - f_i(\hat{x}, \hat{u})| \leq L(|\xi| + \|B\|_\infty |\delta y| + |\delta u|), \tag{80}$$

for some $L > 0$. Thus, from (79) it follows

$$\begin{aligned} |\dot{\xi}| &\leq L(|\xi| + \|B\|_\infty |\delta y| + |\delta u|)(1 + |v|) + \|\dot{B}\|_\infty |\delta y| \\ &= L(|\xi|(1 + |v|) + \|B\|_\infty |\delta y| + |\delta u| + \|B\|_\infty |\delta y||v| + |\delta u||v|) + \|\dot{B}\|_\infty |\delta y|. \end{aligned}$$

Applying Gronwall's Lemma and Cauchy-Schwartz inequality to previous estimation yields

$$\|\xi\|_\infty \leq \rho_1(|\xi_0| + \|\delta u\|_1 + \|\delta y\|_1 + \|\delta y\|_2 \|v\|_2 + \|\delta u\|_2 \|v\|_2), \tag{81}$$

for $\rho_1 = \rho_1(L, C, \|B\|_\infty, \|\dot{B}\|_\infty)$. Hence, since $\|\delta x\|_2 \leq \|\xi\|_2 + \|B\|_\infty \|\delta y\|_2$, by Cauchy-Schwartz inequality and previous estimation, the desired follows. □

Finally, the following lemma gives an estimation for the difference between the variation of the state variable and the linearized state.

Lemma 6.5. *Consider $C > 0$ and $w = (x, u, v) \in \mathcal{W}$ a feasible trajectory having $\|w - \hat{w}\|_\infty < C$. Set $(\delta x, \bar{u}, \bar{v}) := w - \hat{w}$ and $\bar{x}$ its corresponding linearized state. Consider*

$$\eta := \delta x - \bar{x}. \tag{82}$$

Then,

$$\dot\eta = \sum_{i=0}^m \hat v_i D_x f_i(\hat x, \hat u)\eta + \sum_{i=1}^m \bar v_i D f_i(\hat x,\hat u)(\delta x, \bar u) + \zeta, \tag{83}$$

with

$$\|\zeta\|_\infty < \mathcal{O}(C), \quad \|\zeta\|_2 < \mathcal{O}(\gamma). \tag{84}$$

If in addition, $\|\bar u\|_2 + \|\bar v\|_2 \to 0$, *the following estimations for hold:*

$$\|\eta\|_\infty < o(\sqrt{\gamma}), \quad \|\dot\eta\|_2 < o(\sqrt{\gamma}). \tag{85}$$

Proof. Let us begin by observing that the variation of the state variable satisfies the differential equation:

$$\dot{\delta x} = \sum_{i=1}^m \bar v_i f_i(\hat x, \hat u) + \sum_{i=0}^m v_i[f_i(x,u) - f_i(\hat x, \hat u)]. \tag{86}$$

Consider the following Taylor expansions for f_i :

$$f_i(x,u) = f_i(\hat x, \hat u) + Df_i(\hat x,\hat u)(\delta x, \bar u) + \tfrac{1}{2} D^2 f_i(\hat x, \hat u)(\delta x, \bar u)^2 + o(|(\delta x, \bar u)|^2). \tag{87}$$

Combining (86) and (87) yields

$$\dot{\delta x} = \sum_{i=1}^m \bar v_i f_i(\hat x, \hat u) + \sum_{i=0}^m v_i D f_i(\hat x, \hat u)(\delta x, \bar u) + \zeta, \tag{88}$$

with the remainder being

$$\zeta := \tfrac{1}{2} \sum_{i=0}^m v_i [D^2 f_i(\hat x, \hat u)(\delta x, \bar u)^2 + o(|(\delta x, \bar u)|^2)]. \tag{89}$$

The linearized equation together with (88) lead to (83), and, in view of (89), it can be seen that the estimations in (84) hold. Applying Gronwall's Lemma in (83), and using Cauchy-Schwartz inequality afterwards lead to:

$$\|\eta\|_\infty \leq \rho_1 \left\| \sum_{i=1}^m \bar v_i D f_i(\hat x, \hat u)(\delta x, \bar u) + \zeta \right\|_1 \leq \rho_2 \left\{ \|\bar v\|_2 (\|\delta x\|_2 + \|\bar u\|_2) + \|\zeta\|_2 \right\}, \tag{90}$$

for some positive ρ_1, ρ_2. Finally, using estimation of Lemma 6.4 and (84) just obtained, the inequalities in (85) follow. □

In view of Lemmas 3.1, 6.3, 6.4 and 6.5 we can justify the following technical result that is an essential point in the proof of Theorem 6.2.

Lemma 6.6. *Let* $w \in \mathcal{W}$ *be a feasible variation. Set* $(\delta x, \bar u, \bar v) := w - \hat w$, *and* $\bar x$ *its corresponding linearized state, i.e., the solution of* (18)-(19) *associated to* $\bar u$, $\bar v$ *and* δx_0. *Assume that* $\|w - \hat w\|_\infty \to 0$. *Then*

$$\Phi[\lambda](w) = \Phi[\lambda](\hat w) + \Omega[\lambda](\bar x, \bar u, \bar v) + o(\gamma). \tag{91}$$

Proof. Omit the dependence on λ for the sake of simplicity. Recall the expansion of the Lagrangian function given in Lemma 3.1. Notice that by Lemma 6.4, $\Phi(w)=\Phi(\hat{w})+\Omega(\delta x,\bar{u},\bar{v})+o(\gamma)$. And hence,

$$\Phi(w)=\Phi(\hat{w})+\Omega(\bar{x},\bar{u},\bar{v})+\Delta\Omega+o(\gamma), \tag{92}$$

with $\Delta\Omega:=\Omega(\delta x,\bar{u},\bar{v})-\Omega(\bar{x},\bar{u},\bar{v})$. The next step is using Lemmas 6.3, 6.4 and 6.5 to prove that

$$\Delta\Omega=o(\gamma). \tag{93}$$

Note that $\mathcal{Q}(a,a)-\mathcal{Q}(b,b)=\mathcal{Q}(a+b,a-b)$, for any bilinear mapping $\mathcal{Q}$, and any pair a,b. Put $\eta:=\delta x-\bar{x}$ as it is done in Lemma 6.5. Hence,

$$\Delta\Omega=\tfrac{1}{2}\ell''((\delta x_0+\bar{x}_0,\delta x_T+\bar{x}_T),(0,\eta_T))+\int_0^T[\tfrac{1}{2}(\delta x+\bar{x})^\top Q\eta+\bar{u}^\top E\eta+\bar{v}^\top C\eta]\mathrm{d}t. \tag{94}$$

The estimations in Lemmas 6.3 and 6.4 yield $\Delta\Omega=\int_0^T\bar{v}^\top C\eta\mathrm{d}t+o(\gamma)$. Integrating by parts in the latter expression and using (85) lead to

$$\int_0^T\bar{v}^\top C\eta\mathrm{d}t=[\bar{y}^\top C\eta]_0^T-\int_0^T\bar{y}^\top(\dot{C}\eta+C\dot{\eta})\mathrm{d}t=o(\gamma), \tag{95}$$

and hence the desired result follows. □

Proof. [of Theorem 6.2]

We shall prove that if (72) holds for some $\rho>0$, then $\hat{w}$ satisfies $\gamma-$quadratic growth in the weak sense. By the contrary assume that the quadratic growth condition (71) is not satisfied. Consequently, there exists a sequence of feasible trajectories $\{w_k\}$ converging to $\hat{w}$ in the weak sense (i.e., in the $L_\infty-$norm), such that

$$J(w_k)\leq J(\hat{w})+o(\gamma_k), \tag{96}$$

with $\delta w_k:=w_k-\hat{w}$ and $\gamma_k:=\gamma(\delta x_{k,0},\bar{u}_k,\bar{v}_k)$. Let $(\bar{\xi}_k,\bar{u}_k,\bar{y}_k)$ be the transformed directions defined by (44). We divide the remainder of the proof in two steps:

(I) First we prove that the sequence given by

$$(\tilde{\xi}_k,\tilde{u}_k,\tilde{y}_k,\tilde{h}_k):=(\bar{\xi}_k,\bar{u}_k,\bar{y}_k,\bar{h}_k)/\sqrt{\gamma_k} \tag{97}$$

contains a subsequence converging to an element $(\tilde{\xi},\tilde{u},\tilde{y},\tilde{h})$ of $\mathcal{P}_2$ in the weak topology, i.e., $(\tilde{u}_k,\tilde{y}_k)\rightharpoonup(\tilde{u},\tilde{y})$ in the weak topology of $\mathcal{U}_2\times\mathcal{V}_2$ and $(\tilde{\xi}_k,\tilde{h}_k)\to(\tilde{\xi},\tilde{h})$ in the strong sense in $\mathcal{X}_2\times\mathbb{R}^m$.

(II) Afterwards, employing the latter sequence and its weak limit, we show that (72) together with (96) lead to a contradiction.

We shall begin by **Part (I).** Take any Lagrange multiplier λ in $\Lambda_L^\#$. Multiply inequality (96) by α_0, then add the nonpositive term

$$\sum_{i=0}^{d_\varphi}\alpha_i\varphi_i(x_{k,0},x_{k,T})+\sum_{j=1}^{d_\eta}\beta_j\eta_j(x_{k,0},x_{k,T}), \tag{98}$$

to its left-hand side and obtain the inequality

$$\Phi[\lambda](w_k) \leq \Phi[\lambda](\hat{w}) + o(\gamma_k). \tag{99}$$

Recall now expansion (91) given in Lemma 6.6. Note that the elements of the sequence $(\tilde{\xi}_{k,0}, \tilde{u}_k, \tilde{y}_k, \tilde{h}_k)$ have unit $\mathbb{R}^n \times \mathcal{U}_2 \times \mathcal{V}_2 \times \mathbb{R}^m-$norm. The Banach-Alaoglu Theorem (see e.g. [7, Theorem III.15]), implies that, extracting if necessary a subsequence, there exists $(\tilde{\xi}_0, \tilde{u}, \tilde{y}, \tilde{h}) \in \mathbb{R}^n \times \mathcal{U}_2 \times \mathcal{V}_2 \times \mathbb{R}^m$ such that

$$\tilde{\xi}_{k,0} \to \tilde{\xi}_0, \quad \tilde{u}_k \rightharpoonup \tilde{u}, \quad \tilde{y}_k \rightharpoonup \tilde{y}, \quad \tilde{h}_k \to \tilde{h}, \tag{100}$$

where the two limits indicated with $\rightharpoonup$ are taken in the weak topology of $\mathcal{U}_2$ and $\mathcal{V}_2$, respectively. Let $\tilde{\xi}$ be the solution of equation (45) associated with $(\tilde{\xi}_0, \tilde{u}, \tilde{y})$. Note that $\tilde{\xi}$ is the limit of $\tilde{\xi}_k$ in $\mathcal{X}_2$. For the aim of proving that $(\tilde{\xi}, \tilde{u}, \tilde{v}, \tilde{h})$ belongs to $\mathcal{P}_2$, we shall check that the initial-final conditions (47)-(48) are verified. For each index $0 \leq i \leq d_\varphi$,

$$D\varphi_i(\hat{x}_0, \hat{x}_T)(\tilde{\xi}_0, \tilde{\xi}_T + B_T\tilde{h}) = \lim_{k\to\infty} D\varphi_i(\hat{x}_0, \hat{x}_T)\left(\frac{\bar{x}_{k,0}, \bar{x}_{k,T}}{\sqrt{\gamma_k}}\right). \tag{101}$$

In order to prove that the right hand-side of (101) is non-positive, consider the following first order Taylor expansion of function φ_i around $(\hat{x}_0, \hat{x}_T)$:

$$\varphi_i(x_{k,0}, x_{k,T}) = \varphi_i(\hat{x}_0, \hat{x}_T) + D\varphi_i(\hat{x}_0, \hat{x}_T)(\delta x_{k,0}, \delta x_{k,T}) + o(|(\delta x_{k,0}, \delta x_{k,T})|).$$

Previous equation and Lemmas 6.3 and 6.5 imply

$$\varphi_i(x_{k,0}, x_{k,T}) = \varphi_i(\hat{x}_0, \hat{x}_T) + D\varphi_i(\hat{x}_0, \hat{x}_T)(\bar{x}_{k,0}, \bar{x}_{k,T}) + o(\sqrt{\gamma_k}).$$

Thus, the following approximation for the right hand-side in (101) holds:

$$D\varphi_i(\hat{x}_0, \hat{x}_T)\left(\frac{\bar{x}_{k,0}, \bar{x}_{k,T}}{\sqrt{\gamma_k}}\right) = \frac{\varphi_i(x_{k,0}, x_{k,T}) - \varphi_i(\hat{x}_0, \hat{x}_T)}{\sqrt{\gamma_k}} + o(1). \tag{102}$$

Since w_k is a feasible trajectory, it satisfies (4), and then equations (101) and (102) yield, for $1 \leq i \leq d_\varphi$, $D\varphi_i(\hat{x}_0, \hat{x}_T)(\tilde{\xi}_0, \tilde{\xi}_T + B_T\tilde{h}) \leq 0$. For $i = 0$ use inequality (96) to get the corresponding inequality. Analogously,

$$D\eta_j(\hat{x}_0, \hat{x}_T)(\tilde{\xi}_0, \tilde{\xi}_T + B_T\tilde{h}) = 0, \quad \text{for } j = 1, \dots, d_\eta. \tag{103}$$

Thus $(\tilde{\xi}, \tilde{u}, \tilde{y}, \tilde{h})$ satisfies (47)-(48), and hence it belongs to $\mathcal{P}_2$.

Let us deal with **Part (II).** Notice that from (91) and (99) we get

$$\Omega_{\mathcal{P}_2}[\lambda](\tilde{\xi}_k, \tilde{u}_k, \tilde{y}_k, \tilde{h}_k) \leq o(1), \tag{104}$$

and thus

$$\liminf_{k\to\infty} \Omega_{\mathcal{P}_2}[\lambda](\tilde{\xi}_k, \tilde{u}_k, \tilde{y}_k, \tilde{h}_k) \leq 0. \tag{105}$$

Consider the subset of $G(\operatorname{co}\Lambda_L^\#)$ given by

$$\Lambda_L^{\#,\rho} := \{\lambda \in G(\operatorname{co}\Lambda_L^\#) : \Omega_{\mathcal{P}_2}[\lambda] - \rho\gamma \text{ is weakly l.s.c. on } \mathcal{H}_2 \times \mathbb{R}^m\}. \tag{106}$$

Applying Lemma 3.8 to the inequality (72) yields

$$\max_{\lambda \in \Lambda_L^{\#,\rho}} \Omega_{\mathcal{P}_2}[\lambda](\bar{\xi},\bar{u},\bar{y},\bar{h}) \geq \rho\gamma(\bar{\xi}_0,\bar{u},\bar{y},\bar{h}), \quad \text{on } \mathcal{P}_2. \tag{107}$$

Take $\tilde{\lambda} \in \Lambda_L^{\#,\rho}$ that attains the maximum in (107) for the direction $(\tilde{\xi},\tilde{u},\tilde{y},\tilde{h})$. Hence

$$\begin{aligned} 0 &\leq \Omega_{\mathcal{P}_2}[\tilde{\lambda}](\tilde{\xi},\tilde{u},\tilde{y},\tilde{h}) - \rho\gamma(\tilde{\xi}_0,\tilde{u},\tilde{y},\tilde{h}) \\ &\leq \liminf_{k\to\infty} \Omega_{\mathcal{P}_2}[\tilde{\lambda}](\tilde{\xi}_k,\tilde{u}_k,\tilde{y}_k,\tilde{h}_k) - \rho\gamma(\tilde{\xi}_{k,0},\tilde{u}_k,\tilde{y}_k,\tilde{h}_k) \leq -\rho, \end{aligned} \tag{108}$$

since $\Omega_{\mathcal{P}_2}[\tilde{\lambda}] - \rho\gamma$ is weakly-l.s.c., $\gamma(\tilde{\xi}_{k,0},\tilde{u}_k,\tilde{y}_k,\tilde{h}_k) = 1$ for every k and inequality (105) holds. This leads us to a contradiction, since $\rho > 0$, and so the desired result follows.

□

7. Shooting algorithm

The purpose of this section is to present an appropriated numerical scheme to solve the problem given by equations (1)-(3), that we denote with (SP). Notice that no inequality constraints are considered. Suppose that the following assumption holds:

Assumption 7.1. The extremal $\hat{w}$ has a unique associated multiplier $\hat{\lambda} = (\hat{p},\hat{\alpha}_0,\hat{\beta})$.

This implies that $\hat{w}$ verifies the classical *qualification hypothesis:* the function that associates to each $(\bar{x}_0,\bar{u},\bar{v}) \in \mathbb{R}^n \times \mathcal{U} \times \mathcal{V}$ the value $D\eta(\hat{x}_0,\hat{x}_T)(\bar{x}_0,\bar{x}_T) \in \mathbb{R}^{d_\eta}$ is surjective. In other words, $D\eta(\hat{x}_0,\hat{x}_T)$ is onto when considered as a function of $(\bar{x}_0,\bar{u},\bar{v})$. And hence, $\hat{w}$ is a *normal extremal*, i.e., $\hat{\alpha}_0 > 0$. In particular, we may assume

$$\hat{\alpha}_0 = 1. \tag{109}$$

7.1. **Optimality system.**

We aim to provide an optimality system for (SP) in the form of a boundary value problem. First, call back condition (11) given by the Pontryagin maximum principle (PMP) in Section 2.

We shall recall that for the case where all the control variables appear nonlinearly ($m = 0$), the classical technique is using the stationarity equation

$$H_u[\hat{\lambda}](\hat{w}) = 0, \tag{110}$$

to solve $\hat{u}$ as a function of $(\hat{x},\hat{\lambda})$. One is able to do this by assuming, for instance, the *strengthened Legendre-Clebsch condition*

$$H_{uu}[\hat{\lambda}](\hat{w}) \succ 0. \tag{111}$$

In this case, in view of the Implicit Function Theorem, we can write $\hat{u} = U[\hat{\lambda}](\hat{x})$ with U being differentiable. Hence, replacing the occurrences of $\hat{u}$ by $U[\hat{\lambda}](\hat{x})$ in the conditions provided by the PMP yields a two point boundary value problem.

When the system is affine in all the control variables ($l = 0$), we cannot eliminate the control from the equation $H_v = 0$ and then a different technique is implemented (see [3]). Let then $1 \leq i \leq m$, and $d^{M_i} H_v / dt^{M_i}$ be the lowest order derivative of H_v in which $\hat{v}_i$ appears with a coefficient that is not identically zero. Kelley et al. in [21] proved that M_i is even when the investigated extremal is normal (i.e., (109) holds). This implies that $\dot{H}_v$ depends only on $\hat{x}$ and $\hat{\lambda}$. Hence, it is differentiable and the expression

$$\ddot{H}_v[\hat{\lambda}](\hat{w}) = 0 \tag{112}$$

is well-defined. The control $\hat{v}$ can be retrieved from (112) provided that, for instance, the *strengthened generalized Legendre-Clebsch condition* (see Goh [16])

$$-\frac{\partial \ddot{H}_v}{\partial v}[\hat{\lambda}](\hat{w}) \succ 0 \tag{113}$$

holds. In this case, we can write $\hat{v} = V[\hat{\lambda}](\hat{x})$ with V being differentiable. An optimality system in the form of a boundary value problem can be then obtained by replacing $\hat{v}$ by $V[\hat{\lambda}](\hat{x})$ in the PMP.

In the problem studied here, where $l > 0$ and $m > 0$, we aim to use both equations (110) and (112) to retrieve the control $(\hat{u}, \hat{v})$ as a function of the state and costate variables. We next describe a procedure to achieve this elimination that was given in Goh [18]. Let us show that H_v can be differentiated two times in the time variable as it was done in the totally affine case. We shall start by proving that we can use (110) to write $\dot{\hat{u}}$ as a function of $(\hat{\lambda}, \hat{w})$. In fact, since $H_{uv} = 0$ (see Lemma 2.4), the coefficient of $\dot{\hat{v}}$ in $\dot{H}_u$ is zero and hence,

$$\dot{H}_u = \dot{H}_u[\hat{\lambda}](\hat{x}, \hat{u}, \hat{v}, \dot{\hat{u}}) = 0. \tag{114}$$

Thus, if the strong Legendre-Clebsch condition (111) holds, $\dot{\hat{u}}$ can be eliminated from (114) yielding

$$\dot{\hat{u}} = \Gamma[\hat{\lambda}](\hat{x}, \hat{u}, \hat{v}). \tag{115}$$

Observe that $H_v = H_v[\hat{\lambda}](\hat{x}, \hat{u})$, i.e., it does not depend on v. In view of (115) we can write $\dot{H}_v = \dot{H}_v[\hat{\lambda}](\hat{x}, \hat{u}, \hat{v})$. Thus, a priori, $\dot{H}_v$ is a function of $(\hat{\lambda}, \hat{x}, \hat{u}, \hat{v})$. However, since Goh in [16] stated that

$$\frac{\partial \dot{H}_v}{\partial v} = 0 \tag{116}$$

on the reference trajectory, $\dot{H}_v$ does not depend on $\hat{v}$, and so

$$\dot{H}_v = \dot{H}_v[\hat{\lambda}](\hat{x}, \hat{u}) = 0. \tag{117}$$

We can then differentiate one more time (117) obtaining (112) as it was desired.

Remark 7.2. Observe that if the system has the special structure

$$\dot{x} = f_0(x,u) + \sum_{i=1}^m v_i f_i(x), \tag{118}$$

then in the expression (115), $\dot{\hat{u}}$ is affine on $\hat{v}$. In fact, by differentiating H_u we get

(119)

$$\begin{aligned} \dot{H}_u &= H_{ux}H_p - H_x H_{up} + H_{uu}\dot{\hat{u}} \\ &= \hat{p}\left(\textstyle\sum_{i=0}^m \hat{v}_i f_{i,ux}\right)\left(\textstyle\sum_{j=0}^m \hat{v}_j f_j\right) - \hat{p}\left(\textstyle\sum_{j=0}^m \hat{v}_j f_{j,x}\right)\left(\textstyle\sum_{i=0}^m \hat{v}_i f_{i,u}\right) + H_{uu}\dot{\hat{u}} \\ &= \hat{p}\textstyle\sum_{i,j=0}^m \hat{v}_i\hat{v}_j (f_{i,ux}f_j - f_{j,x}f_{i,u}) + H_{uu}\dot{\hat{u}}. \end{aligned}$$

In our case, we have $f_{i,u} = 0$ for $i = 1,\dots,m$, and hence it follows from (119),

$$\dot{H}_u = \hat{p}\sum_{j=0}^m \hat{v}_j (f_{0,ux}f_j - f_{j,x}f_{0,u}) + H_{uu}\dot{\hat{u}}. \tag{120}$$

Thus, given that H_{uu} does not depend on $\hat{v}$, we can deduce from (120) that $\dot{\hat{u}}$ is affine on $\hat{v}$.

Observe now that the derivative of the function

$$\begin{pmatrix} H_u \\ -\ddot{H}_v \end{pmatrix} \tag{121}$$

with respect to (u,v) is given by

$$\mathcal{J} := \begin{pmatrix} H_{uu} & 0 \\ -\dfrac{\partial \ddot{H}_v}{\partial u} & -\dfrac{\partial \ddot{H}_v}{\partial v} \end{pmatrix}, \tag{122}$$

where we used (16). Therefore, the equations (110) and (112) can be used to solve $(\hat{u},\hat{v})$ in terms of $(\hat{\lambda},\hat{x})$ provided that $\mathcal{J}$ is invertible. Notice that if (111) and (113) are verified, $\mathcal{J}$ is definite positive and consequently, non-singular.

Finally, see that (112) together with the boundary conditions

$$H_v[\hat{\lambda}](\hat{w}_T) = 0, \tag{123}$$

$$\dot{H}_v[\hat{\lambda}](\hat{w}_0) = 0, \tag{124}$$

guarantee the second identity in (11).

Notation: Denote by (OS) the set of equations consisting of (2)-(3), (7), (9)-(10), (110), (112) and the boundary conditions (123)-(124).

Remark 7.3. Instead of (123)-(124), we could choose another pair of endpoint conditions among the four possible ones: $H_v(0) = 0$, $H_v(T) = 0$, $\dot{H}_v(0) = 0$ and $\dot{H}_v(T) = 0$, always including at least one of order zero. The choice we made will simplify the presentation of the well-posedness result afterwards.

The rest of this article is very close to what was done in Aronna el at- [3]. The presentation here is then more concise, and the reader is referred to the mentioned sections for further details.

7.2. **The algorithm.** The aim of this section is to present an appropriated numerical scheme to solve system (OS). Let us define the *shooting function*

$$\begin{aligned}\mathcal{S} : \mathrm{D}(\mathcal{S}) := \mathbb{R}^n \times \mathbb{R}^{n+d_\eta,*} &\to \mathbb{R}^{d_\eta} \times \mathbb{R}^{2n+2m,*}, \\ \big(x_0, p_0, \beta\big) =: \nu &\mapsto \mathcal{S}(\nu) := \begin{pmatrix} \eta(x_0, x_T) \\ p_0 + D_{x_0}\ell[\lambda](x_0, x_T) \\ p_T - D_{x_T}\ell[\lambda](x_0, x_T) \\ H_v[\lambda](w_T) \\ \dot{H}_v(w_0) \end{pmatrix},\end{aligned} \tag{125}$$

where (x, u, v, p) is a solution of (2),(9),(110),(112) with initial conditions x_0 and p_0, and $\lambda := (p, \beta)$. Note that solving (OS) consists of finding $\hat{\nu} \in \mathrm{D}(\mathcal{S})$ such that

$$\mathcal{S}(\hat{\nu}) = 0. \tag{126}$$

This procedure is called the *shooting method.* Since the number of equations in (126) is greater than the number of unknowns, the Gauss-Newton method is a suitable approach to solve it (see [3] for further details of Gauss-Newton method). As we already know, Gauss-Newton is applicable when the derivative of the shooting function $\mathcal{S}$ is one-to-one in a neighborhood of $\hat{\nu}$, and in this case it is said that the algorithm is *well-posed.*

The main result of this last part of the article is presenting a condition that guarantees the well-posedness of the shooting method around the optimal local solution $(\hat{w}, \hat{\lambda})$. This condition involves the sufficient optimality condition of Theorem 6.2 in Section 6.

8. Well-posedness of the shooting algorithm

The purpose of this section is proving the following result:

Theorem 8.1 (Well-posedness)**.** *If $\hat{w}$ is a weak minimum of (SP) satisfying* (72) *then the shooting algorithm is well-posed in some neighborhood of $\hat{w}$. Furthermore, the approximating sequence defined by the Gauss-Newton method converges quadratically to $\hat{\nu}$.*

8.1. **Linearized optimality system.** In this paragraph we shall compute the linearization of (OS). The reader is referred to the Appendix for a definition of linearized system of differential algebraic equations and for a commutation property between the linearization and time differentiation. We denote by $\mathrm{Lin}\,\mathcal{F}$ the linearization of function $\mathcal{F}$. Each time the argument of a function is missing, assume that it is evaluated on $(\hat{w}, \hat{\lambda})$. Recall the definitions of B in (21) and of C in (27). Notice that, since $H_v = pB$,

$$\mathrm{Lin}\, H_v = \bar{p}B + \bar{x}^\top C^\top. \tag{127}$$

The linearization of system (OS) at point $(\hat{x}, \hat{u}, \hat{v}, \hat{\lambda})$ consists of the linearized state equation (18) with initial-final condition (23), the linearized costate equation

$$-\dot{\bar{p}}_t = \bar{p}_t A_t + \bar{x}_t^\top Q_t + \bar{u}_t^\top E_t + \bar{v}_t^\top C_t, \quad \text{a.e. on } (0,T), \tag{128}$$

with initial-final conditions

$$\bar{p}_0 = -\left[\bar{x}_0^\top D^2_{x_0^2}\ell + \bar{x}_T^\top D^2_{x_0 x_T}\ell + \sum_{j=1}^{d_\eta} \bar{\beta}_j D_{x_0}\eta_j\right]_{(\hat{x}_0,\hat{x}_T)}, \tag{129}$$

$$\bar{p}_T = \left[\bar{x}_T^\top D^2_{x_T^2}\ell + \bar{x}_0^\top D^2_{x_0 x_T}\ell + \sum_{j=1}^{d_\eta} \bar{\beta}_j D_{x_T}\eta_j\right]_{(\hat{x}_0,\hat{x}_T)}, \tag{130}$$

and the algebraic equations

$$0 = \text{Lin } H_u = \bar{p}D + \bar{x}^\top E^\top + \bar{u}^\top R_0, \tag{131}$$

$$0 = \text{Lin } \ddot{H}_v = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\bar{p}B + \bar{x}^\top C^\top), \quad \text{a.e. on } [0,T], \tag{132}$$

$$0 = (\text{Lin } H_v)_T = \bar{p}_T B_T + \bar{x}_T^\top C_T^\top, \tag{133}$$

$$0 = (\text{Lin } \dot{H}_v)_0 = -\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(\bar{p}B + \bar{x}^\top C^\top), \tag{134}$$

where we used equation (127) and commutation property of Lemma 9.2. There is no need to detail the derivatives in (133) and (134) since we will not make use of them later. Observe that (132) -(134) and Lemma 9.2 yield

$$0 = \text{Lin } H_v = \bar{p}B + \bar{x}^\top C^\top, \quad \text{a.e. on } [0,T]. \tag{135}$$

Note that equation (135) holds everywhere on $[0,T]$ since all the involved functions are continuous in time.

Notation: denote by (LS) the set of equations consisting of (18), (23), (128)-(134).

Once we have computed the linearized system (LS), we can write the derivative of $\mathcal{S}$ in the direction $\bar{\nu} := \left(\bar{x}_0, \bar{p}_0, \bar{\beta}\right)$ as follows:

$$\mathcal{S}'(\hat{\nu})\bar{\nu} = \begin{pmatrix} D\eta(\hat{x}_0,\hat{x}_T)(\bar{x}_0,\bar{x}_T) \\ \bar{p}_0 + \left[\bar{x}_0^\top D^2_{x_0^2}\ell + \bar{x}_T^\top D^2_{x_0 x_T}\ell + \sum_{j=1}^{d_\eta} \bar{\beta}_j D_{x_0}\eta_j\right]_{(\hat{x}_0,\hat{x}_T)} \\ \bar{p}_T - \left[\bar{x}_T^\top D^2_{x_T^2}\ell + \bar{x}_0^\top D^2_{x_0 x_T}\ell + \sum_{j=1}^{d_\eta} \bar{\beta}_j D_{x_T}\eta_j\right]_{(\hat{x}_0,\hat{x}_T)} \\ \bar{p}_T B_T + \bar{x}_T^\top C_T \\ \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(\bar{p}B + \bar{x}^\top C^\top) \end{pmatrix}, \tag{136}$$

where $(\bar{x}, \bar{u}, \bar{v}, \bar{p})$ is the solution of (18),(128),(131),(132) associated to the initial condition $(\bar{x}_0, \bar{p}_0)$ and the multiplier $\bar{\beta}$. Thus, $\mathcal{S}'(\hat{\nu})$ is one-to-one if the only solution of (18),(128),(131).(132) with the initial conditions $\bar{x}_0 = 0$, $\bar{p}_0 = 0$ and with $\bar{\beta} = 0$ is $(\bar{x}, \bar{u}, \bar{v}, \bar{p}) = 0$.

8.2. **Additional LQ problem.** In this paragraph we present a linear-quadratic control problem (LQ) in the variables $(\bar{\xi}, \bar{u}, \bar{y}, \bar{h})$ having $\Omega_{\mathcal{P}_2}$ (defined in (67)) as cost functional. Afterwards we note that condition (72) yields the strong convexity of the pre-Hamiltonian of (LQ) and hence the uniqueness of the optimal solution. Furthermore, the unique optimal solution will be characterized by its first order optimality system, i.e., by the Pontryagin maximum principle. Finally we present a one-to-one linear mapping that transforms each solution of (LS) (introduced in paragraph 8.1 above) into a solution of this new optimality system. Theorem 8.1 will follow.

Let us consider the linear-quadratic problem (LQ) given by:

$$\Omega_{\mathcal{P}_2}(\bar{\xi}, \bar{u}, \bar{y}, \bar{h}) \to \min, \tag{137}$$

$$(45)\text{-}(47), \tag{138}$$

$$\dot{h} = 0. \tag{139}$$

Here $\bar{u}$ and $\bar{y}$ are the control variables, $\bar{\xi}$ and $\bar{h}$ are the state variables. Denote by $\bar{\chi}$ and $\bar{\chi}_h$ the costate variables corresponding to $\bar{\xi}$ and $\bar{h}$, respectively; and by ρ_0^{LQ} and β^{LQ} the multipliers associated to the cost function (137) and the initial-final state constraint (47), respectively. Put λ^{LQ} for the element $(\bar{\chi}, \bar{\chi}_h, \rho_0^{LQ}, \beta^{LQ})$. The pre-Hamiltonian for (LQ) is:

$$\begin{aligned} &\mathcal{H}[\lambda^{LQ}](\bar{\xi}, \bar{u}, \bar{y}) := \bar{\chi}(A\bar{\xi} + D\bar{u} + B_1\bar{y}) \\ &+(\tfrac{1}{2}\bar{\xi}^\top Q\bar{\xi} + \bar{u}^\top E\bar{\xi} + \bar{y}^\top M\bar{\xi} + \tfrac{1}{2}\bar{u}^\top R_0\bar{u} + \bar{y}^\top J\bar{u} + \tfrac{1}{2}\bar{y}^\top R_1\bar{y}). \end{aligned} \tag{140}$$

Observe that $\mathcal{H}$ does not depend on $\bar{h}$ since it has zero dynamics and does not appear in the running cost. The initial-final Lagrangian is

$$\ell^{LQ}[\lambda^{LQ}](\bar{\xi}_0, \bar{\xi}_T, \bar{h}_T) := \rho_0^{LQ} g(\bar{\xi}_0, \bar{\xi}_T, \bar{h}_T) + \sum_{j=1}^{d_\eta} \beta_j^{LQ} D\eta_j(\bar{\xi}_0, \bar{\xi}_T + B_T\bar{h}_T). \tag{141}$$

The costate equation for $\bar{\chi}$ is:

$$-\dot{\bar{\chi}}_t = D_{\bar{\xi}}\mathcal{H}[\lambda^{LQ}] = \bar{\chi}A + \bar{\xi}^\top Q + \bar{u}^\top E + \bar{y}^\top M, \tag{142}$$

with the boundary conditions:

$$\begin{aligned} \bar{\chi}_0 = &\ -D_{\bar{\xi}_0}\ell^{LQ}[\lambda^{LQ}] \\ = &\ -\rho_0^{LQ}\left[\bar{\xi}_0^\top D^2_{x_0^2}\ell + (\bar{\xi}_T + B_T\bar{h})^\top D^2_{x_0x_T}\ell\right] - \textstyle\sum_{j=1}^{d_\eta}\beta_j^{LQ} D_{x_0}\eta_j, \end{aligned} \tag{143}$$

$$\begin{aligned} \bar{\chi}_T = &\ D_{\xi_T}\ell^{LQ}[\lambda^{LQ}] \\ = &\ \rho_0^{LQ}\left[\bar{\xi}_0^\top D^2_{x_0x_T}\ell + (\bar{\xi}_T + B_T\bar{h})^\top D^2_{x_T^2}\ell\right] + \bar{h}^\top C_T + \textstyle\sum_{j=1}^{d_\eta}\beta_j^{LQ} D_{x_T}\eta_j. \end{aligned} \tag{144}$$

For costate variable $\bar\chi_h$ we get the equation

$$-\dot{\bar\chi}_h = 0, \tag{145}$$

$$\bar\chi_{h,0} = 0, \tag{146}$$

$$\bar\chi_{h,T} = D_{\bar h}\ell^{LQ}[\lambda^{LQ}]. \tag{147}$$

Hence, $\bar\chi_h \equiv 0$ and thus (147) yields

(148)

$$0 = \rho_0^{LQ}\left[\bar\xi_0^\top D^2_{x_0 x_T}\ell\, B_T + (\bar\xi_T + B_T\bar h)^\top (D^2_{x_T^2}\ell\, B_T + C_T^\top)\right] + \sum_{j=1}^{d_\eta}\beta_j^{LQ} D_{x_T}\eta_j B_T.$$

The **stationarity with respect to the new control** $\bar y$ implies

$$0 = \mathcal{H}_{\bar u} = \bar\chi D + \xi^\top E^\top + \bar u^\top R_0 + \bar y^\top J, \tag{149}$$

$$0 = \mathcal{H}_{\bar y} = \bar\chi B_1 + \bar\xi^\top M^\top + \bar u^\top J^\top + \bar y^\top R_1. \tag{150}$$

Notation: Denote by (LQS) the set of equations consisting of (138)-(139), (142)-(144),(148) and (150), and observe that (LQS) is an **optimality system** of problem (137)-(139).

Note that condition (72) is equivalent to the strong convexity of $\mathcal{H}$ on the set of feasible trajectories of problem (137)-(139). If $\mathcal{H}$ is strongly convex, its unique critical point is a strict global minimum and it is characterized by the first optimality system (LQS).

8.3. **The transformation.** In this paragraph we show how to transform a solution of (LS) into a solution of (LQS) via a one-to-one linear mapping. Given $(\bar x, \bar u, \bar v, \bar p, \bar\beta) \in \mathcal{W} \times W_{1,\infty} \times \mathbb{R}^{d_\eta,*}$, define

$$\bar y_t := \textstyle\int_0^t \bar v_s ds,\ \bar\xi := \bar x - B\bar y,\ \bar\chi := \bar p + \bar y^\top C,\ \bar\chi_h := 0,\ \bar h := \bar y_T, \\ \rho_0^{LQ} := 1, \quad \beta_j^{LQ} := \bar\beta_j. \tag{151}$$

The next lemma shows that the point $(\bar\xi, \bar u, \bar y, \bar h, \bar\chi, \bar\chi_h, \rho_0^{LQ}, \beta^{LQ})$ is solution of (LQS) provided that $(\bar x, \bar u, \bar v, \bar p, \bar\beta)$ is solution of (LS).

Lemma 8.2. *The one-to-one linear mapping defined by* (151) *converts each solution of (LS) into a solution of (LQS).*

Remark 8.3. Recall Corollary 5.3 for the proof below.

Proof. Let $(\bar x, \bar u, \bar v, \bar p, \bar\beta)$ be a solution of (LS), and set $(\bar\xi, \bar u, \bar y, \bar h, \bar\chi, \rho_0^{LQ}, \beta^{LQ})$ by (151).

Part I. We shall prove that $(\bar\xi, \bar u, \bar y, \bar h, \bar\chi, \rho_0^{LQ}, \beta^{LQ})$ satisfies conditions (138). Equation (45) follows differentiating expression of $\bar\xi$ in (151), and equation (47) follows from (23).

Part II. We shall prove that $(\bar{\xi}, \bar{u}, \bar{y}, \bar{h}, \bar{\chi}, \rho_0^{LQ}, \beta^{LQ})$ verifies (142)-(144) and (148). Differentiate $\bar{\chi}$ in (151), use equations (128) and (151), recall definition of M in (55) and obtain:

$$\begin{aligned} -\dot{\bar{\chi}} = & \ -\dot{\bar{p}} - \bar{v}^\top C - \bar{y}^\top \dot{C} \\ = & \ \bar{p}A + \bar{x}^\top Q + \bar{u}^\top E - \bar{y}^\top \dot{C} \\ = & \ \bar{\chi}A + \bar{\xi}^\top Q + \bar{u}^\top E + \bar{y}^\top(-CA + B^\top Q - \dot{C}) \\ = & \ \bar{\chi}A + \xi^\top Q + \bar{u}^\top E + y^\top M. \end{aligned} \tag{152}$$

Hence (142) holds. Equations (143)-(144) follow from (129)-(130). Combine (130) and (133) to get

$$\begin{aligned} 0 = & \ \bar{p}_T B_T + \bar{x}_T^\top C_T^\top \\ = & \ \left[\bar{p}_T^\top D^2_{x_T^2}\ell + \bar{x}_0^\top D^2_{x_0 x_T}\ell + \textstyle\sum_{j=1}^{d_\eta} \bar{\beta}_j D_{x_T}\eta_j\right]_{(\hat{x}_0,\hat{x}_T)} B_T + \bar{x}_T^\top C_T^\top. \end{aligned} \tag{153}$$

Performing transformation (151) in the previous equation yields (148).
Part III. Let us show that the stationarity with respect to $\bar{y}$ in (149) is verified. The transformation in (151) together with equation (131) imply

$$\begin{aligned} 0 = & \ (\bar{\chi} - \bar{y}^\top C)D + (\bar{\xi} + B\bar{y})^\top E^\top + \bar{u}^\top R_0 \\ = & \ \bar{\chi}D + \bar{\xi}^\top E^\top + \bar{u}^\top R_0 + \bar{y}^\top(B^\top E^\top - CD). \end{aligned} \tag{154}$$

Calling back definition of J in (55), stationarity condition (149) follows.
Part IV. Finally, we shall prove that (150) holds. Perform the transformation (151) in equation (135) to obtain

$$0 = (\bar{\chi} - \bar{y}^\top C)B + (\bar{\xi} + B\bar{y})^\top C^\top = \bar{\chi}B + \bar{\xi}^\top C^\top, \tag{155}$$

since Corollary 5.3 holds when the multiplier in unique. Differentiating previous expression we obtain

$$\begin{aligned} 0 = & \ -(\bar{\chi}A + \bar{\xi}^\top Q + \bar{u}^\top E + \bar{y}^\top M)\, B + +\bar{\chi}\dot{B} \\ & +(A\bar{\xi} + D\bar{u} + B_1\bar{y})^\top C^\top + \bar{\xi}^\top \dot{C}^\top. \end{aligned} \tag{156}$$

Recall the definitions of B_1 in (46) and of J in (55), of R_1 in (57), and use then in (156) to get (150).

Parts I to IV show that $(\bar{\xi}, \bar{u}, \bar{y}, \bar{h}, \bar{\chi}, \rho_0^{LQ}, \beta^{LQ})$ is a solution of (LQS), and hence the result follows.

□

Remark 8.4. Observe that the unique assumption we needed in previous proof was the symmetry condition in Corollary 5.3.

Proof. [of Theorem 8.1] We shall prove the first statement. Let $(\bar{x}, \bar{u}, \bar{v}, \bar{p}, \bar{\beta})$ be a solution of (LS), and let $(\bar{\xi}, \bar{u}, \bar{y}, \bar{h}, \bar{\chi}, \bar{\chi}_h, \rho_0^{LQ}, \beta^{LQ})$ be defined by the transformation in (151). Hence we know by Lemma 8.2 that $(\bar{\xi}, \bar{u}, \bar{y}, \bar{h}, \bar{\chi}, \bar{\chi}_h, \rho_0^{LQ}, \beta^{LQ})$ is solution of (LQS). Since (72) holds, the cost function of problem (137)-(139) is strongly convex and thus the unique solution of its optimality system (LQS) is 0. Hence $(\bar{\xi}, \bar{u}, \bar{y}, \bar{h}, \bar{\chi}, \bar{\chi}_h, \rho_0^{LQ}, \beta^{LQ}) = 0$ and thus $(\bar{x}, \bar{u}, \bar{v}, \bar{p}, \bar{\beta}) = 0$.

Conclude that the unique solution of (LQ) is 0. This implies the injectivity of $\mathcal{S}'$ around a neighborhood of $\hat{\nu}$, and hence the result follows.

The quadratic convergence follows from $\mathcal{S}(\hat{\nu}) = 0$, i.e., the residual is zero. See Fletcher [14] or Bonnans [5] for a proof of this quadratic convergence. □

9. Conclusion

We provided a set of necessary and sufficient conditions for a problem having a part of the control variable entering linearly in the pre-Hamiltonian. These conditions apply to a weak minimum and do not assume the uniqueness of multipliers.

We proposed a shooting algorithm based on the procedure described by Goh [18] to compute the control variables in terms of the state and costate variables. We proved that the sufficient condition above-mentioned guarantees the well-posedness of the shooting algorithm.

Appendix

Linearization of a Differential Algebraic System. For the aim of finding an expression of $\mathcal{S}'(\hat{\nu})$, we make use of the linearization of (OS) and thus we recall the following concept:

Definition 9.1 (Linearization of a Differential Algebraic System). Consider a system of differential algebraic equations:

$$\dot{\zeta}_t = \mathcal{F}(\zeta_t, \alpha_t), \tag{157}$$

$$0 = \mathcal{G}(\zeta_t, \alpha_t), \tag{158}$$

$$0 = \mathcal{I}(\zeta_0, \zeta_T), \tag{159}$$

with $\mathcal{F} : \mathbb{R}^{m+n} \to \mathbb{R}^n$, $\mathcal{G} : \mathbb{R}^{m+n} \to \mathbb{R}^{n_\mathcal{G}}$, and $\mathcal{I} : \mathbb{R}^{2n} \to \mathbb{R}^{n_\mathcal{I}}$. Let (ζ^0, α^0) be a solution. We call *linearized system* at point (ζ^0, α^0) the following DAE in the variables $\bar{\zeta}$ and $\bar{\alpha}$:

$$\dot{\bar{\zeta}}_t = \operatorname{Lin} \mathcal{F} \mid_{(\zeta_t^0, \alpha_t^0)} (\bar{\zeta}_t, \bar{\alpha}_t), \tag{160}$$

$$0 = \operatorname{Lin} \mathcal{G} \mid_{(\zeta_t^0, \alpha_t^0)} (\bar{\zeta}_t, \bar{\alpha}_t), \tag{161}$$

$$0 = \operatorname{Lin} \mathcal{I} \mid_{(\zeta_0^0, \zeta_T^0)} (\bar{\zeta}_0, \bar{\zeta}_T), \tag{162}$$

where

$$\operatorname{Lin} \mathcal{F} \mid_{(\zeta_t^0, \alpha_t^0)} (\bar{\zeta}_t, \bar{\alpha}_t) := \mathcal{F}'(\zeta_t^0, \alpha_t^0)(\bar{\zeta}_t, \bar{\alpha}_t), \tag{163}$$

and the analogous definitions for $\operatorname{Lin} \mathcal{G}$ and $\operatorname{Lin} \mathcal{H}$.

The technical result below will simplify the computation of the linearization of (OS). Its proof is immediate.

Lemma 9.2 (Commutation of linearization and differentiation)**.** *Given $\mathcal{G}$ and $\mathcal{F}$ as in the previous definition, it holds:*

$$\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Lin} \mathcal{G} = \operatorname{Lin} \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{G}, \quad \frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Lin} \mathcal{F} = \operatorname{Lin} \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{F}. \tag{164}$$

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