Subdifferential of the conjugate function in general Banach spaces*

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A Marco López amb mottu del seu seixanté aniversari
(To Marco López, on the occasion of his sixtieth birthday)

Abstract

We give explicit formulas for the subdifferential set of the conjugate of non necessarily convex functions defined on general Banach spaces. Even if such a subdifferential mapping takes its values in the bidual space, we show that up to a weak** closure operation it is still described by using only elements of the initial space relying on the behavior of the given function at the nominal point. This is achieved by means of formulas using the ε-subdifferential and an appropriate enlargement of the subdifferential of this function, revealing a useful relationship between the subdifferential of the conjugate function and its part lying in the initial space.

Key words. Conjugate function, subdifferential mapping, Banach spaces.
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1 Introduction

It is our aim in this paper to characterize the Fenchel subdifferential of the Legendre-Fenchel conjugate of a given function, not necessarily convex and defined on a general Banach space, by means only of primal information. This will be achieved in a number of explicit formulas by using the ε-subdifferential together with an appropriate enlargement of the Fenchel subdifferential of the initial function, which has been introduced and investigated in [2]. To compensate the lack of continuity assumptions, these formulas will also include the normal cone to the domain of the conjugate function, which describes the asymptotic behavior of the initial function. Hence, this analysis provides complete characterizations of the subdifferential of the conjugate function without requiring explicit expressions of the conjugate itself or its domain. The desired formulas will allow a comprehensive understanding of the behavior of the conjugate in a variety of interesting and practical situations which rely on the initial function and/or the underlying space.

The main feature of the present analysis is the ability to describe, up to a weak** closure process, the subdifferential of the conjugate function using only elements of the initial space

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relying on the behavior of this function at the corresponding point. Of course, all these questions make sense when the underlying space is not necessarily reflexive and its dual is endowed with its norm topology. However, the case when this Banach space and its dual form a dual topological pair has already been investigated in [2]. Nevertheless, a first attempt to deal with the general setting of Banach spaces has given rise in [4] to expressions of this subdifferential mapping by invoking an appropriate extension of the initial function to the bidual space. Next, the results of [2] were applied by considering the topological dual pair formed by the dual and the bidual spaces endowed with the norm and the weak** topologies, respectively. These results have been used to give integration criteria which provide the coincidence of the proper lsc convex hull of non necessarily convex functions. But, regarding the characterization of the subdifferential of the conjugate, the resulting formulas in [4] do not distinguish between the parts of this subdifferential set which do or do not lie in the initial space; see Section 3 for more comparisons. This is why we follow in the present work a direct approach invoking only the behavior of the initial function. It is also important in our analysis to characterize the subdifferential of the conjugate by only invoking the behavior of the initial function at the nominal point; that is, the point where the subdifferential of the conjugate is evaluated. Hence, these results may be also useful in the convex setting and can be compared with the classical result [9, Proposition 1], in which the subdifferential of the conjugate is described by means of subgradients of the initial function lying in the predual space at nearest points.

The summary of the paper is as follows: after we fix below the main notations which are used later on, we give in Section 2 the desired results which are stated in Theorems 3 by invoking an enlargement of the Fenchel subdifferential, and Theorem 5 which uses the ε-subdifferential. The proof of Theorem 3 is postponed to Section 4 at the end of the paper. For comparative purposes, in order to show the main advantages of the present formulas we make in Section 3 a short review of some of the recent results given in [2, 4].

Throughout the paper, X is a real Banach space endowed with a norm $\|\cdot\|$. The dual and bidual spaces are denoted by X^* and X^{**}, respectively. The null vector in all these spaces is denoted by θ. With abuse of language, in view of the canonical embedding of X in X^{**} we identify X to a subspace of X^{**}. We shall frequently endow X^* and X^{**} with the norm and the weak** topologies, respectively. The duality product in both pairs (X, X^*) and (X^*, X^{**}) is denoted by $\langle \cdot, \cdot \rangle$.

Let $f : X \to \overline{\mathbb{R}}$ or ($f : X^* \to \overline{\mathbb{R}}$) be an extended real-valued function. We say that f is *proper* if its *(effective) domain*

$$\operatorname{dom} f := \{x \in X \mid f(x) < +\infty\}$$

is nonempty and $f(x) > -\infty$ for all $x \in X$. The conjugate function of f is the weak* lsc convex function $f^* : X^* \to \overline{\mathbb{R}}$ given by

$$f^*(x^*) := \sup_{x \in X} \{\langle x, x^* \rangle - f(x)\}.$$

If $\varepsilon \geq 0$ is given, the *ε-subdifferential* mapping of f is the multifunction $\partial_\varepsilon f : X \rightrightarrows X^*$ which assigns to $x \in X$ the (possibly empty) set

$$\partial_\varepsilon f(x) := \{x^* \in X^* \mid f^*(x^*) + f(x) \leq \langle x^*, x \rangle + \varepsilon\}$$

(with the convention that $\partial_\varepsilon f(x) := \emptyset$ if $f(x) \notin \mathbb{R}$); hence, when $\varepsilon = 0$, we recover the usual *Fenchel subdifferential* mapping which we simply denote by $\partial f(x)$. In this way, the subdifferential of the conjugate function f^* is the mapping $\partial f^* : X^* \rightrightarrows X^{**}$ given by

$$\partial f^*(x^*) = \{x^{**} \in X^{**} \mid f^{**}(x^{**}) + f^*(x^*) \leq \langle x^*, x^{**}\rangle + \varepsilon\},$$

where $f^{**} : X^{**} \to \overline{\mathbb{R}}$ is the conjugate of f^*; that is, $f^{**}(x^{**}) = \sup_{x^* \in X^*} \{\langle x^*, x^{**}\rangle - f^*(x^*)\}$.

Finally, given subsets A, B in X (X^* or X^{**}), we use the Minkowsky sum of A and B defined as

$$A + B := \{a + b \mid a \in A, b \in B\}.$$

The *normal cone* to A at x is defined as

$$\mathrm{N}_A(x) := \{x^* \in X^* \mid \langle x^*, y - x\rangle \leq 0 \text{ for all } y \in A\} \text{ if } x \in A;\ \emptyset \text{ if } x \in X \setminus A.$$

By co$A$, cone$A$, aff$A$ and lin$A$, we denote the *convex, conic, affine* and *linear hulls* of A, respectively. By par$A$ we denote the *parallel subspace* to aff$A$; for instance, par$A$ = aff$A - a$ for $a \in A$. We use cl$A$, $\mathrm{cl}^w A$ and $\mathrm{cl}^{w^{**}} A$ (or, indistinctly, $\overline{A}$, $\overline{A}^w$ and $\overline{A}^{w^{**}}$) to respectively denote the norm, weak and weak** closures of A. Hence, we write $\overline{\mathrm{co}}A := \mathrm{cl}(\mathrm{co}A)$, $\overline{\mathrm{co}}^{w^{**}} A := \mathrm{cl}^{w^{**}}(\mathrm{co}A)$, etc.

2 Subdifferential of the conjugate function

We give in this section the desired formulas which express the subdifferential of the conjugate of any function defined on a given real Banach space X with norm $\|\cdot\|$.

For this aim, an important tool is the following enlargement of the usual Fenchel subdifferential, introduced and investigated in [2].

Definition 1 *Given $L \subset X^*$ and $f : X \to \overline{\mathbb{R}}$, a vector $x^* \in L$ is said to be a relative subgradient of f at $x \in X$ with respect to L, if $f^*(x^*) \in \mathbb{R}$ and there exists a net $(x_\alpha) \subset X$ such that*

$$\lim \langle x_\alpha - x, y^*\rangle = 0 \quad \forall y^* \in \overline{\mathrm{par}}\,(L \cap \mathrm{dom}\, f^*)\,, \text{ and}$$

$$\lim \langle x_\alpha, x^*\rangle - f(x_\alpha) = f^*(x^*).$$

The set of such relative subgradients, denoted by $\partial_L^r f(x)$, is called the relative subdifferential of f at x with respect to L. If $\mathrm{dom}\, f^* \subset L$, we omit the reference to L and simply write $\partial^r f(x)$.

Here, and throughout the paper, for $x^* \in X^*$ we use the notation

$$\mathcal{F}_\tau(x^*) := \{L \subset X^* \ \tau\text{-closed and convex} \mid x^* \in L,\ f^*_{|\tau\text{-}\mathrm{ri}(L \cap \mathrm{dom}\, f^*)} \text{ is finite and } \tau\text{-continuous}\}, \tag{1}$$

where τ-ri denotes the (topological) *relative interior* with respect to a given topology τ; that is, for $A \subset X^*$, τ-ri(A) is the interior relative to aff$A$ when aff$A$ is τ-closed, and the empty set otherwise (see, e.g., [10]). Hence, if the interior of A with respect to τ, denoted by τ-int$A$, is nonempty then τ-ri$(A) = \tau$-int$A$. The function $f^*_{|A}$ used above refers to the restriction of f^* to A with the convention that $f^*_{|\emptyset} \equiv +\infty$. In what follows, if $M : X \rightrightarrows X^*$ (or $M : X^* \rightrightarrows X$) is a multifunction, its inverse $M^{-1} : X^* \rightrightarrows X$ is given by $M^{-1}(x^*) := \{x \in X \mid x^* \in M(x)\}$.

Before characterizing the whole set $\partial f^*(x^*)$ in X^{**} we recall formulas providing the part of this subdifferential lying in X. The following result was given in [2] for the more general setting of locally convex spaces.

Proposition 1 [2, Theorem 4] *Given a function $f : X \to \overline{\mathbb{R}}$, for any $x^* \in X^*$ we have that*

$$\partial f^*(x^*) \cap X = \bigcap_{L \in \mathcal{F}_\tau(x^*)} \overline{\text{co}} \left\{ (\partial_L^r f)^{-1}(x^*) + X \cap \text{N}_{L \cap \text{dom}\, f^*}(x^*) \right\},$$

where τ is a topology on X^* compatible with the duality pair (X, X^*). In particular, the following formula holds provided that $X^* \in \mathcal{F}_\tau(x^*)$,

$$\partial f^*(x^*) \cap X = \overline{\text{co}} \left\{ (\partial^r f)^{-1}(x^*) + X \cap \text{N}_{\text{dom}\, f^*}(x^*) \right\}.$$

We also give the following extension of the above proposition established in [5]. We recall that $\text{I}_C : X^* \to \overline{\mathbb{R}}_+$ denotes the indicator function of a subset $C \subset X^*$; that is,

$$\text{I}_C(x^*) := 0 \;\text{ if }\; x^* \in C \;\text{ and }\; +\infty \quad \text{otherwise.}$$

Proposition 2 *Let be given function $f : X \to \overline{\mathbb{R}}$ and τ-closed convex set $C \subset X^*$, where τ is a locally convex topology on X^* compatible with the duality pair (X, X^*). Then, for every $x^* \in X^*$ we have that*

$$\partial(f^* + \text{I}_C)(x^*) \cap X = \bigcap_{L \in \mathcal{F}_\tau(C, x^*)} \overline{\text{co}} \left\{ (\partial_{L \cap C}^r f)^{-1}(x^*) + X \cap \text{N}_{L \cap C \cap \text{dom}\, f^*}(x^*) \right\},$$

where $\mathcal{F}_\tau(C, x^*) := \{L \subset X^*$ τ-closed and convex $\mid x^* \in L$, $f^*_{|\tau\text{-}\text{ri}(L \cap C \cap \text{dom}\, f^*)}$ is finite and τ-continuous$\}$. In particular, provided that $C \in \mathcal{F}_\tau(C, x^*)$ we get

$$\partial(f^* + \text{I}_C)(x^*) \cap X = \overline{\text{co}} \left\{ (\partial_C^r f)^{-1}(x^*) + X \cap \text{N}_{C \cap \text{dom}\, f^*}(x^*) \right\}.$$

Now, we give the main result of the paper in which we characterize the whole set $\partial f^*(x^*)$ in X^{**}. Its proof is postponed to Section 4 at the end of the paper.

Theorem 3 *Let $f : X \to \overline{\mathbb{R}}$ be any function. Then, for every $x^* \in X^*$ we have that*

$$\partial f^*(x^*) = \bigcap_{L \in \mathcal{F}_\tau(x^*)} \overline{\text{co}}^{w^{**}} \left\{ (\partial_L^r f)^{-1}(x^*) + X \cap \text{N}_{L \cap \text{dom}\, f^*}(x^*) \right\},$$

where τ is any topology on X^* compatible with the duality pair (X, X^*).

Remark 1 It is worth observing that the term $X \cap \text{N}_{L \cap \text{dom}\, f^*}(x^*)$ in the formula above does not require an explicit knowledge of the domain of f^* nor the values of the function f^* itself. Indeed, on the one hand, such a term describes the asymptotic behavior of the initial function f as it can be easily seen from the straightforward relationship (assuming that the involved functions are proper)

$$X \cap \text{N}_{L \cap \text{dom}\, f^*}(x^*) = (\partial(\overline{\text{co}} f)^\infty)^{-1}(x^*),$$

where $(\overline{\text{co}} f)^\infty$ denotes the usual recession function in the sense of convex analysis (see, e.g., [8]) of the lsc convex hull of the function f, $\overline{\text{co}} f$. In this respect, a general relation between the normal cone to $\text{dom}\, f^*$ and an appropriate concept of asymptotic function of f can be found in [3]. On the other hand, it can be also checked that the formula above can be equivalently written as

$$\partial f^*(x^*) = \bigcap_{L \in \mathcal{F}_{x^*}} \overline{\text{co}}^{w^{**}} \left\{ (\partial_L^r f)^{-1}(x^*) + X \cap \text{N}_{L \cap \text{dom}\, f^*}(x^*) \right\},$$

where $\mathcal{F}_{x^*} := \{L \subset X^* \mid L$ is a finite-dimensional subspace containing $x^*\}$, confirming that the current formulas do not depend on any explicit knowledge of the conjugate function f^*.

The following corollary illustrates Theorem 3.

Corollary 4 *Let $f : X \to \overline{\mathbb{R}}$ be weakly lsc such that f^* is finite and continuous at some point with respect to a topology on X^* compatible with the pair (X, X^*). Then, for every $x^* \in X^*$,*

$$\partial f^*(x^*) = \overline{\text{co}}^{w^{**}} \left\{ (\partial f)^{-1}(x^*) + X \cap \text{N}_{\text{dom}\, f^*}(x^*) \right\}.$$

Proof. Let τ be a topology on X^* as stated in the theorem and fix $x^* \in X^*$. Then, as $X^* \in \mathcal{F}_\tau(x^*)$ and f is weakly lsc we can easily check that $\partial^r f = \partial f$ and, so, the inclusion "$\subset$" follows by applying Theorem 3. This finishes the proof since the opposite inclusion is straightforward. ■

The following result gives an alternative for Theorem 3 where one uses the ε-subdifferential mapping instead of $\partial_L^r f$.

Theorem 5 *Let $f : X \to \overline{\mathbb{R}}$ and the topology τ be as in Theorem 3. Then, for every $x^* \in X^*$ we have the formula*

$$\partial f^*(x^*) = \bigcap_{\substack{\varepsilon>0 \\ L\in\mathcal{F}_\tau(x^*)}} \overline{\text{co}}^{w^{**}} \left\{ (\partial_\varepsilon f)^{-1}(x^*) + X \cap \text{N}_{L\cap\text{dom}\, f^*}(x^*) \right\}.$$

Moreover, provided that f^* is finite and τ-continuous at some point, the formula above reduces to

$$\partial f^*(x^*) = \bigcap_{\varepsilon>o} \overline{\text{co}}^{w^{**}} \left\{ (\partial_\varepsilon f)^{-1}(x^*) + X \cap \text{N}_{\text{dom}\, f^*}(x^*) \right\}.$$

Proof. It suffices to prove the inclusion "$\subset$" of the main statement when $x^* \in X^*$ is such that $\partial f^*(x^*) \neq \emptyset$. For this aim, according to Theorem 3 we only need to show that for every given $\varepsilon > 0$ and $L \in \mathcal{F}_\tau(x^*)$ it holds $(\partial_L^r f)^{-1}(x^*) \subset \overline{\text{co}}^{w^{**}} \left\{ (\partial_\varepsilon f)^{-1}(x^*) + X \cap \text{N}_{L\cap\text{dom}\, f^*}(x^*) \right\}$. Equivalently, it suffices to show that

$$\sigma_{(\partial_L^r f)^{-1}(x^*)}(w^*) \leq \sigma_{(\partial_\varepsilon f)^{-1}(x^*)+X\cap\text{N}_{L\cap\text{dom}\, f^*}(x^*)}(w^*) \quad \text{for all } w^* \in X^*, \tag{2}$$

where σ refers to the support function with the convention that $\sigma_\emptyset = -\infty$. Indeed, if $w^* \in L\cap\text{dom}\, f^* - x^*$ we pick $x \in (\partial_L^r f)^{-1}(x^*)$ and let $(x_\alpha) \subset X$ be a net such that $\lim_\alpha \langle x_\alpha - x, y^*\rangle = 0$, for all $y^* \in \overline{\text{par}}\,(L \cap \text{dom}\, f^*)$, and $\lim_\alpha f(x_\alpha) + f^*(x^*) - \langle x_\alpha, x^*\rangle = 0$. Hence, we may suppose that $x_\alpha \in (\partial_\varepsilon f)^{-1}(x^*)$ so that

$$\langle x, w^*\rangle = \lim_\alpha \langle x_\alpha, w^*\rangle \leq \sigma_{(\partial_\varepsilon f)^{-1}(x^*)}(w^*) \leq \sigma_{(\partial_\varepsilon f)^{-1}(x^*)+X\cap\text{N}_{L\cap\text{dom}\, f^*}(x^*)}(w^*).$$

Therefore, (2) follows by the arbitrariness of x in $(\partial_L^r f)^{-1}(x^*)$. Moreover, by the positive homogeneity of the support function it follows that (2) also holds for every $w^* \in \text{cone}(L\cap\text{dom}\, f^* - x^*)$. Now, if $w^* \in \overline{\text{cone}}(L \cap \text{dom}\, f^* - x^*)$, we pick $w_0^* \in \tau\text{-ri}\,(\text{cone}(L \cap \text{dom}\, f^* - x^*))$ (this set being nonempty by assumption) so that by the accessibility lemma for each $\lambda \in (0, 1)$ it holds $w_\lambda^* := \lambda w^* + (1 - \lambda)w_0^* \in \text{cone}(L \cap \text{dom}\, f^* - x^*)$. Then, invoking the convexity of the support function, from the paragraph above we obtain that

$$\sigma_{(\partial_L^r f)^{-1}(x^*)}(w_\lambda^*) \leq \lambda\sigma_{(\partial_\varepsilon f)^{-1}(x^*)+X\cap\text{N}_{L\cap\text{dom}\, f^*}(x^*)}(w^*) + (1 - \lambda)\sigma_{(\partial_\varepsilon f)^{-1}(x^*)+X\cap\text{N}_{L\cap\text{dom}\, f^*}(x^*)}(w_0^*). \tag{3}$$

But, writing $w_0^* = \gamma(u^* - x^*)$ for some $\gamma \geq 0$ and $u^* \in L \cap \operatorname{dom} f^*$, and observing that $(\partial_\varepsilon f)^{-1}(x^*) \subset \partial_\varepsilon f^*(x^*)$, we get

$$\begin{aligned}\sigma_{(\partial_\varepsilon f)^{-1}(x^*)+X\cap \mathrm{N}_{L\cap \operatorname{dom} f^*}(x^*)}(w_0^*) &\leq \gamma \sigma_{(\partial_\varepsilon f)^{-1}(x^*)+\mathrm{N}_{L\cap \operatorname{dom} f^*}(x^*)}(u^* - x^*) \\ &\leq \gamma \sigma_{\partial_\varepsilon f^*(x^*)}(u^* - x^*) \leq \gamma(f^*(u^*) - f^*(x^*)) < +\infty.\end{aligned}$$

So, by taking the limit as $\lambda \searrow 0$ in (3) in view of the lsc of the support function we get $\sigma_{(\partial_L^r f)^{-1}(x^*)}(w^*) \leq \liminf_{\lambda \searrow 0} \sigma_{(\partial_L^r f)^{-1}(x^*)}(w_\lambda^*) \leq \sigma_{(\partial_\varepsilon f)^{-1}(x^*)+X\cap \mathrm{N}_{L\cap \operatorname{dom} f^*}(x^*)}(w^*)$, showing that (2) holds.

Finally, it remains to check that (2) holds when $w^* \notin \overline{\operatorname{cone}}(L \cap \operatorname{dom} f^* - x^*)$. Indeed, in this case, by the classical bipolar theorem there exists $w \in X \cap \mathrm{N}_{L\cap \operatorname{dom} f^*}(x^*)$ such that $\langle w^*, w \rangle > 0$ and, so, $\sigma_{(\partial_\varepsilon f)^{-1}(x^*)+X\cap \mathrm{N}_{L\cap \operatorname{dom} f^*}(x^*)}(w^*) = +\infty$. Thus, (2) trivially holds. ■

Remark 2 The following formula, significantly different to the one given in Theorem 5, has been established in [4, Proposition 4]:

$$\partial f^*(x^*) = \bigcap_{\substack{\varepsilon>0 \\ L\in \mathcal{F}_\tau(x^*)}} \overline{\operatorname{co}}^{w^{**}} \left\{(\partial_\varepsilon f)^{-1}(x^*) + \mathrm{N}_{L\cap \operatorname{dom} f^*}(x^*)\right\},$$

for every $x^* \in X^*$, where τ is any topology on X^* compatible with the duality pair (X, X^*). Indeed, while Theorem 5 uses only the part of $\mathrm{N}_{L\cap \operatorname{dom} f^*}(x^*)$ lying in X, the term $\mathrm{N}_{L\cap \operatorname{dom} f^*}(x^*)$ in the last formula is a subset of X^{**} which possibly contains points that are not in the predual space X. In this respect, the formulas in Theorems 3 and 5 agree with the classical result in convex analysis [9, Proposition 1], corresponding to f being convex, where the subgradients of f^* at x^* are written as a weak** limit of subgradients of f^*, at nearby points of x^*, that belong to X. However, note that in Theorems 3 and 5 only the nominal point x^* is concerned in the closure process. We refer to Section 3 for more comparisons.

Remark 3 If $\operatorname{Argmin} f^{**}$ denotes the minimizer set of the biconjugate function f^{**}, and f^* is proper, then in view of the relationship $\operatorname{Argmin} f^{**} = \partial f^*(\theta)$ Theorem 5 easily leads us to the following characterization of $\operatorname{Argmin} f^{**}$, by means of the ε-minimizer sets ε-$\operatorname{Argmin} f$ of f,

$$\operatorname{Argmin} f^{**} = \bigcap_{\substack{\varepsilon>0 \\ L\in \mathcal{F}_\tau(\theta)}} \overline{\operatorname{co}}^{w^{**}} \left\{\varepsilon\text{-}\operatorname{Argmin} f + X \cap \mathrm{N}_{L\cap \operatorname{dom} f^*}(\theta)\right\},$$

where τ is any topology on X^* compatible with the duality pair (X, X^*). Similarly, invoking Corollary 4, if f is weakly lsc and f^* is finite and τ-continuous at some point, then we have the following relationship which gives the characterization of $\operatorname{Argmin} f^{**}$ by means of the minimizer set $\operatorname{Argmin} f$ of f,

$$\operatorname{Argmin} f^{**} = \overline{\operatorname{co}}^{w^{**}} \left\{\operatorname{Argmin} f + X \cap \mathrm{N}_{\operatorname{dom} f^*}(\theta)\right\}.$$

We close this section by giving the finite-dimensional counterpart of Theorem 3. Namely, the following corollary has already been stated in [2, Corollary 7] where a small gap appeared in the proof.

Corollary 6 ([2, Corollary 7]) *Let* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *be such that* $\operatorname{int}(\operatorname{dom} f^*) \neq \emptyset$. *Then, for every* $x^* \in X^*$ *we have the formula*

$$\partial f^*(x^*) = \operatorname{co}\left\{(\partial^r f)^{-1}(x^*)\right\} + \mathrm{N}_{\operatorname{dom} f^*}(x^*).$$

In addition, if f *is lsc then*

$$\partial f^*(x^*) = \operatorname{co}\left\{(\partial f)^{-1}(x^*)\right\} + \mathrm{N}_{\operatorname{dom} f^*}(x^*).$$

Proof. We shall denote $B_\gamma(z)$ (B_γ if $z = 0$) the ball of radius $\gamma > 0$ centered at z, and $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^n$. It is enough to prove the inclusion "$\subset$" when $x^* = \theta$, $\partial f^*(\theta) \neq \emptyset$ and $f^*(\theta) = 0$; thus, f^* is proper and $\inf_{\mathbb{R}^n} f = 0$. By assumption, we fix $x_0 \in \operatorname{int}(\operatorname{dom} f^*)$ and $\rho > 0$ such that

$$f^*(x_0 + v) \leq f^*(x_0) + 1 \quad \text{for all } v \in B_\rho. \tag{4}$$

Then, according to [2, Corolrary 6], it suffices to show that

$$\overline{\operatorname{co}}\left\{(\partial^r f)^{-1}(\theta)\right\} \subset \operatorname{co}\left\{(\partial^r f)^{-1}(x^*)\right\} + \mathrm{N}_{\operatorname{dom} f^*}(x^*). \tag{5}$$

For we pick a sequence $(x_k)_k$ in $\operatorname{co}\left\{(\partial^r f)^{-1}(\theta)\right\}$ which converges to a given x. By taking into account Carathéodory's Theorem, for each $k \geq 1$ there are $x_{k,1}, \cdots, x_{k,n+1} \in (\partial^r f)^{-1}(\theta)$ and

$$(\lambda_{k,1}, \cdots, \lambda_{k,n+1}) \in \Delta_{n+1} := \{(\delta_1, \cdots, \delta_{n+1}) \in \mathbb{R}^{n+1} \mid \delta_1, \cdots, \delta_{n+1} \geq 0, \delta_1 + \cdots + \delta_{n+1} = 1\} \tag{6}$$

such that $x_k = \lambda_{k,1}x_{k,1} + \cdots + \lambda_{k,n+1}x_{k,n+1}$, $\langle x_0, x_k\rangle \geq \langle x_0, x\rangle - 1$ and $\lambda_{k,i} > 0$ (without loss of generality). We also may assume that the sequence $(\lambda_{k,1}, \cdots, \lambda_{k,n+1})_k$ converges to some $(\lambda_1, \cdots, \lambda_{n+1}) \in \Delta_{n+1}$. By the definition of $(\partial^r f)^{-1}(\theta)$, for each $i \in \{1, \cdots, n+1\}$ there exists $y_{k,i} \in B_{\frac{1}{k}}(x_{k,i})$ such that $f(y_{k,i}) \leq \frac{1}{k}$ and, so, by Fenchel inequality,

$$\langle x_0, x_{k,i}\rangle \leq \langle x_0, y_{k,i}\rangle + k^{-1}\|x_0\| \leq f^*(x_0) + \|x_0\| + 1 \ \text{ for all } k. \tag{7}$$

Let us denote $I := \{i \mid \lambda_i > 0\}$, $J := \{i \mid \lambda_i = 0\}$ so that $I \neq \emptyset$. If $i \in I$, taking into account (7), by multiplying the equation $x_k = \lambda_{k,1}x_{k,1} + \cdots + \lambda_{k,n+1}x_{k,n+1}$ by x_0 for each k we get $\lambda_{k,i}\langle x_0, x_{k,i}\rangle \geq \langle x_0, x\rangle - 1 - \max\{f^*(x_0) + \|x_0\| + 1, 1\}$. So, given that $\lambda_{k,i} \to \lambda_i > 0$ there exists $\alpha > 0$ independent of k such that $\langle x_0, x_{k,i}\rangle \geq \alpha$. Thus, by invoking once again Fenchel inequality together with (4) and (7), for every $v \in B_\rho$ we get $\langle v, x_{k,i}\rangle \leq f^*(x_0+v) + f(y_{k,i}) - \alpha + k^{-1}\|x_0\| + \rho \leq f^*(x_0) - \alpha + \|x_0\| + \rho + 2 < +\infty$; that is, we may suppose that the sequence $(x_{k,i})_k$ converges to some x_i and, consequently, the corresponding sequence $(y_{k,i})_k$ also converges to x_i. Therefore, since $\lim_{k\to+\infty} f(y_{k,i}) = 0$ we infer that $x_i \in (\partial^s f)^{-1}(\theta)$.

Now, we suppose that $i \in J$. If $(\lambda_{k,i}x_{k,i})_k$ is bounded, then the sequence $(\lambda_{k,i}x_{k,i})_k$ has an accumulation point $\tilde{x}_i$ which is also an accumulation point of the sequence $(\lambda_{k,i}y_{k,i})_k$, also by the fact that $y_{k,i} \in B_{\frac{1}{k}}(x_{k,i})$. Hence, since $(\overline{\operatorname{co}}f)(y_{k,i}) \leq f(y_{k,i}) \leq \frac{1}{k}$ and $\lambda_{k,i} \to 0$ we infer that (recall that $\overline{\operatorname{co}}f \in \Gamma_0(\mathbb{R}^n)$ is the lsc convex hull of f)

$$\begin{aligned}\sigma_{\operatorname{dom} f^*}(\tilde{x}_i) = \sigma_{\operatorname{dom}(\overline{\operatorname{co}}f)^*}(\tilde{x}_i) = (\overline{\operatorname{co}}f)^\infty(\tilde{x}_i) &\leq \liminf_k \lambda_{k,i}(\overline{\operatorname{co}}f)(\lambda_{k,i}^{-1}\lambda_{k,i}y_{k,i}) \\ &= \liminf_k \lambda_{k,i}(\overline{\operatorname{co}}f)(y_{k,i}) \leq \lim_k k^{-1}\lambda_{k,i} = 0,\end{aligned}$$

showing that $\tilde{x}_i \in \mathrm{N}_{\operatorname{dom} f^*}(\theta)$. Next, we denote $J_0 := \{i \in J \mid (\lambda_{k,i}x_{k,i})_k \text{ is not bounded}\}$; hence, as $y_{k,i} \in B_{\frac{1}{k}}(x_{k,i})$ we get $J_0 = \{i \in J \mid (\lambda_{k,i}y_{k,i})_k \text{ is not bounded}\}$. We also choose $j \in J_0$ such that $\lambda_{k,j}\|y_{k,j}\| = \max\{\lambda_{k,i}\|y_{k,i}\| \mid i \in J_0\}$. Thus, for each $i \in J_0$ there exists an accumulation point z_i of the sequence $(z_{k,i})_k$ defined by $z_{k,i} := \frac{\lambda_{k,i} y_{k,i}}{\lambda_{k,j} \|y_{k,j}\|}$ such that $\|z_i\| \leq 1$ (with equality when $i = j$). Moreover, writing

$$\sigma_{\operatorname{dom} f^*}(z_i) = \sigma_{\operatorname{dom}(\overline{\operatorname{co}} f)^*}(z_i) = (\overline{\operatorname{co}} f)^\infty(z_i) \leq \liminf_k \frac{\lambda_{k,i}}{\lambda_{k,j} \|y_{k,j}\|} (\overline{\operatorname{co}} f) \left(\frac{\lambda_{k,j} \|y_{k,j}\|}{\lambda_{k,i}} z_{k,i} \right)$$
$$= \liminf_k \frac{\lambda_{k,i}}{\lambda_{k,j} \|y_{k,j}\|} (\overline{\operatorname{co}} f)(y_{k,i}) \leq \liminf_k \frac{\lambda_{k,i}}{k \lambda_{k,j} \|y_{k,j}\|} = 0,$$

we deduce that $z_i \in \mathrm{N}_{\operatorname{dom} f^*}(\theta)$. On the other hand, observing that $x_k = \lambda_{k,1} y_{k,1} + \cdots + \lambda_{k,n+1} y_{k,n+1}$, by dividing both sides of this last equality by $\lambda_{k,j} \|y_{k,j}\|$ and next passing to the limit on k we get $\sum_{i \in J_0} z_i = \theta$. But, invoking (4), since $z_i \in \mathrm{N}_{\operatorname{dom} f^*}(\theta)$ for each $i \in J_0$, we can write $\langle z_i, x_0 \rangle \leq -\frac{\varrho}{2} \|z_i\|$, and so by summing up on $i \in J_0$ we get the contradiction $0 \leq -\frac{\varrho}{2} \sum_{i \in J_0} \|z_i\| \leq -\frac{\varrho}{2} \|z_j\| = -\frac{\varrho}{2} < 0$. Consequently, $J_0 = \emptyset$ and so by passing to the limit on k in the equation $x_k = \lambda_{k,1} x_{k,1} + \cdots + \lambda_{k,n+1} x_{k,n+1}$ we get

$$x = \lim_k \sum_{i \in J \setminus J_0} \lambda_{k,i} x_{k,i} + \sum_{i \in I} \lambda_{k,i} x_{k,i} = \sum_{i \in J \setminus J_0} \tilde{x}_i + \sum_{i \in I} \lambda_i x_i \in \mathrm{N}_{\operatorname{dom} f^*}(\theta) + \operatorname{co} \left\{ (\partial^r f)^{-1}(\theta \right\},$$

establishing the desired relation (5). ■

3 Further remarks and comparisons with previous results

We give in this short section some remarks and compare the preceding formulas of Section 2 with those previously established in [2, 4].

Hereafter, $f : X \to \overline{\mathbb{R}}$ is a given function defined on the Banach space X, and τ is a topology on X^* which is compatible with the duality pair (X, X^*).

Remark 4 (i) Proposition 1 can be immediately obtained from Theorem 3 in the following way: if $x^* \in X^*$ is fixed we write

$$X \cap \partial f^*(x^*) = X \bigcap \left[\bigcap_{L \in \mathcal{F}_\tau(x^*)} \overline{\operatorname{co}}^{w^{**}} \left\{ (\partial_L^r f)^{-1}(x^*) + X \cap \mathrm{N}_{L \cap \operatorname{dom} f^*}(x^*) \right\} \right]$$
$$= \bigcap_{L \in \mathcal{F}_\tau(x^*)} \left[X \bigcap \overline{\operatorname{co}}^{w^{**}} \left\{ (\partial_L^r f)^{-1}(x^*) + X \cap \mathrm{N}_{L \cap \operatorname{dom} f^*}(x^*) \right\} \right].$$

Hence, since the weak** topology coincides with the weak topology on X, by invoking Mazur's Theorem we obtain that

$$X \cap \partial f^*(x^*) = \bigcap_{L \in \mathcal{F}_\tau(x^*)} \overline{\operatorname{co}} \left\{ (\partial_L^r f)^{-1}(x^*) + X \cap \mathrm{N}_{L \cap \operatorname{dom} f^*}(x^*) \right\},$$

which is the statement of Proposition 1.

(ii) Similarly, as in (i) above the main formula in Theorem 5 yields the following characterization for every $x^* \in X^*$,

$$X \cap \partial f^*(x^*) = \bigcap_{\substack{\varepsilon>0 \\ L\in\mathcal{F}_\tau(x^*)}} \overline{\operatorname{co}} \left\{ (\partial_\varepsilon f)^{-1}(x^*) + X \cap \mathrm{N}_{L\cap \operatorname{dom} f^*}(x^*) \right\};$$

this last formula is also a simple consequence of the characterization given in [6] for the subdifferential of the supremum of convex functions.

In the following remark we exhibit the relationship between the subdifferential set of the conjugate function and its part lying in the initial space.

Remark 5 Remark 4(i) above together with Theorems 3-5 and Proposition 1 provide a quite natural relationship between the subdifferential set $\partial f^*(x^*)$ and its part lying in the initial space, $X \cap \partial f^*(x^*)$. Indeed, it is well known that the set $\partial f^*(x^*)$ may in general be strictly larger than $\overline{X \cap \partial f^*(x^*)}^{w^{**}}$; for example [7, Example 1.4 (b)], the conjugate of $\|\cdot\|_\infty$ in $X = c_0(\mathbb{N})$ is $\|\cdot\|_1$, while $X^* = l_1$ and $X^{**} = l_\infty$. Moreover, $f^* \equiv \|\cdot\|_1$ is Gâteaux-differentiable at any $x^* = (x_n)$, with $x_n \neq 0$ for all n, and $\partial f^*(x^*) = \{(\operatorname{sgn} x_n)\} \not\subset c_0(\mathbb{N})$. Nevertheless, our analysis shows that the set $\partial f^*(x^*)$ can be still recovered by a weak** closure procedure on subsets entering the expression of $X \cap \partial f^*(x^*)$. To put this in one picture, for instance Theorems 3 and Proposition 1 respectively give us

$$\partial f^*(x^*) = \bigcap_{L\in\mathcal{F}_\tau(x^*)} \overline{\operatorname{co}}^{w^{**}} \left\{ (\partial_L^r f)^{-1}(x^*) + X \cap \mathrm{N}_{L\cap \operatorname{dom} f^*}(x^*) \right\},$$

$$X \cap \partial f^*(x^*) = \bigcap_{L\in\mathcal{F}_\tau(x^*)} \overline{\operatorname{co}} \left\{ (\partial_L^r f)^{-1}(x^*) + X \cap \mathrm{N}_{L\cap \operatorname{dom} f^*}(x^*) \right\},$$

showing that $\partial f^*(x^*)$ and $X \cap \partial f^*(x^*)$ are built upon the same elements of the initial space X but with closures invoking different topologies. So, the choice of the topology when taking the closure of the sets $\operatorname{co}\left\{ (\partial_L^r f)^{-1}(x^*) + X \cap \mathrm{N}_{L\cap \operatorname{dom} f^*}(x^*) \right\}$ is decisive in the structure of $\partial f^*(x^*)$: the norm topology gives us the set $X \cap \partial f^*(x^*)$, while the weak** topology provides us with the whole subdifferential set $\partial f^*(x^*)$. In other words, in order that the equality $\partial f^*(x^*) = \overline{X \cap \partial f^*(x^*)}^{w^{**}}$ hold one needs to manage the intersection over the sets $L \in \mathcal{F}_\tau(x^*)$. For example, according to Corollary 4, in the simple case when f^* is finite and continuous at some point, with respect to a topology compatible with the duality pair (X, X^*), we have that

$$\partial f^*(x^*) = \overline{X \cap \partial f^*(x^*)}^{w^{**}} \quad \text{for every } x^* \in X^*.$$

Let us recall that when $x^* \in \operatorname{int}(\operatorname{dom} f^*)$, a characterization of (proper lsc convex) functions f whose the conjugate satisfies the last relationship is given in [1, Proposition 5.2] by means of the behavior at 0 of the multifunction $\varepsilon \rightrightarrows X \cap \partial_\varepsilon f^*(x^*)$.

Remark 6 In the current Banach space setting, the approach of [2] applies when X^* is endowed with a topology which is compatible with the duality pair (X, X^*); in particular, the norm topology on X^* when X is a reflexive Banach space. The resulting formulas of this method provide different characterizations of the part of the subdifferential set $\partial f^*(x^*)$ lying in X; see Proposition 1. There is another way to overcome the difficulty raised in the case when the norm topology is considered on X^*. Indeed, as we explained in the introduction, an alternative approach was undertaken in [4] by using an appropriate extension to X^{**} of the function f,

$\hat{f} : X^{**} \to \overline{\mathbb{R}}$, given by

$$\hat{f}(x^{**}) := f(x^{**}), \text{ if } x^{**} \in X; \ \ +\infty, \text{ otherwise.}$$

So, if we respectively endow X^* and X^{**} with the norm and the weak**-topologies so that (X^{**}, X^*) forms a dual pair of locally convex spaces, according to [4, Lemma 2] both the functions f and $\hat{f}$ have the same conjugate and the same ε-subdifferential mapping. Consequently, we can characterize the subdifferential of f^* by using either [2] (evoking the subdifferential enlargement as in Proposition 1) or [6] (by means of the ε-subdifferential). In this way, it was recently shown in [4, Proposition 3] that, for every $x^* \in X^*$,

$$\partial f^*(x^*) = \bigcap_{L \in \mathcal{F}_\tau(x^*)} \overline{\text{co}}^{w^{**}} \left\{ (\partial_L^r \hat{f})^{-1}(x^*) + \mathrm{N}_{L \cap \operatorname{dom} f^*}(x^*) \right\},$$

or, equivalently, according to [4, Proposition 4],

$$\partial f^*(x^*) = \bigcap_{\substack{\varepsilon > 0 \\ L \in \mathcal{F}_\tau(x^*)}} \overline{\text{co}}^{w^{**}} \left\{ (\partial_\varepsilon f)^{-1}(x^*) + \mathrm{N}_{L \cap \operatorname{dom} f^*}(x^*) \right\}.$$

It is clear that the main notable difference between these two formulas and the assertions of Theorems 3 and 5 is that the last formulas evoke the terms $\mathrm{N}_{L \cap \operatorname{dom} f^*}(x^*)$ and $(\partial_L^r \hat{f})^{-1}(x^*)$ which may contain points in $X^{**} \setminus X$. So, in some sense Theorems 3 and 5 could provide an accurate estimation of the subdifferential of the conjugate function since they only require the access to the $(\partial_L^r f)^{-1}(x^*)$, $(\partial_\varepsilon f)^{-1}(x^*)$ and $X \cap \mathrm{N}_{L \cap \operatorname{dom} f^*}(x^*)$ which lie in X.

4 Proof of Theorem 3

This section is devoted to complete the proof of Theorem 3. We recall that $f : X \to \overline{\mathbb{R}}$ is a given function and τ is a locally convex topology on X^* which is compatible with the duality pair (X, X^*). The family $\mathcal{F}_\tau(x^*)$, $x^* \in X^*$, is defined in (1). If $g : X \to \overline{\mathbb{R}}$ is another function, we denote $f \Box g : X \to \overline{\mathbb{R}}$ the inf-convolution of f and g; that is, $f \Box g := \inf_{x \in X}\{f(x) + g(\cdot - x)\}$.

Lemma 7 *For every $x^* \in X^*$ we have that*

$$\partial f^*(x^*) = \bigcap_{L \in \mathcal{F}_\tau(x^*)} \partial (f \Box \sigma_L)^*(x^*).$$

Moreover, if $L \in \mathcal{F}_\tau(x^*)$ then we get

$$\partial (f \Box \sigma_L)^*(x^*) \cap X = \overline{\text{co}} \left\{ (\partial^r (f \Box \sigma_L))^{-1}(x^*) + X \cap \mathrm{N}_{L \cap \operatorname{dom} f^*}(x^*) \right\}.$$

Proof. We fix $x^* \in X^*$ and denote by S the subset on the right-hand side in the first formula. We pick $L \in \mathcal{F}_\tau(x^*)$. Then, for $z \in S$ we have that $\emptyset \neq \partial (f \Box \sigma_L)^*(x^*) = \partial (f^* + \mathrm{I}_L)(x^*)$ and, so, $x^* \in L \cap \operatorname{dom} f^*$. Given $y^* \in \operatorname{dom} f^*$, we denote $M := \operatorname{aff}\{x^*, y^*\}$ so that $M \in \mathcal{F}_\tau(x^*)$. Then, since $z \in S \subset \partial (f \Box \sigma_M)^*(x^*)$ we obtain that $\langle z, y^* - x^* \rangle \leq (f \Box \sigma_M)^*(y^*) - (f \Box \sigma_M)^*(x^*) = f^*(y^*) - f^*(x^*)$, showing that $z \in \partial f^*(x^*)$. Conversely, we pick $z \in \partial f^*(x^*)$ and $L \in \mathcal{F}_\tau(x^*)$ so that $x^* \in L \cap \operatorname{dom} f^*$ and $(f \Box \sigma_L)^*(x^*) = f^*(x^*) + \mathrm{I}_L(x^*) = f^*(x^*)$. Thus, since $f^* \leq f^* + \mathrm{I}_L = (f \Box \sigma_L)^*$ we deduce that $z \in \partial (f \Box \sigma_L)^*(x^*)$ and, so, by the arbitrariness of $L \in \mathcal{F}_\tau(x^*)$ it follows that $z \in S$.

The second formula remains to be checked. For this aim we fix $x^* \in X^*$ and $L \in \mathcal{F}_\tau(x^*)$ so that $x^* \in L \cap \operatorname{dom} f^*$. Then, in view of the relationship $(f \square \sigma_L)^* = f^* + \mathrm{I}_L$, the desired conclusion follows by applying Proposition 1 to the function $f \square \sigma_L$. ■

In the remainder of the proof we fix $(x^*, x^{**}) \in \partial f^*$, $L \in \mathcal{F}_\tau(x^*)$ and define the sets A_L and A as

$$A_L := (\partial_L^r f)^{-1}(x^*) + X \cap \mathrm{N}_{L \cap \operatorname{dom} f^*}(x^*), \; A := \bigcap_{L \in \mathcal{F}_\tau(x^*)} \overline{\operatorname{co}}^{w^{**}}(A_L). \tag{8}$$

Lemma 8 *We have that $A \subset \partial f^*(x^*)$.*

Proof. According to the first statement of Lemma 7, we write

$$A_L \subset \overline{\operatorname{co}} \left\{ (\partial^r (f \square \sigma_L)^{-1}(x^*) + X \cap \mathrm{N}_{\operatorname{dom}(f \square \sigma_L)^*}(x^*) \right\} = \partial (f \square \sigma_L)^*(x^*) \cap X \subset \partial (f \square \sigma_L)^*(x^*), \tag{9}$$

which gives us $A \subset \bigcap_{L \in \mathcal{F}_\tau(x^*)} \partial (f \square \sigma_L)^*(x^*)$. So, the desired inclusion follows by invoking the second statement of Lemma 7. ■

We continue with the proof of Theorem to prove the opposite of the inclusion given in the lemma above. Equivalently, we shall establish the inequality

$$\langle x^{**}, w^* \rangle \leq \sigma_{A_L}(w^*) \quad \text{for all } w^* \in X^*. \tag{10}$$

In the following corollary we show that it is enough to prove this last inequality on the set $(L \cap \operatorname{dom} f^* - x^*) \cap \operatorname{dom} \sigma_{A_L}$.

Lemma 9 *Inequality* (10) *holds if and only if it holds for w^* lying in $(L \cap \operatorname{dom} f^* - x^*) \cap \operatorname{dom} \sigma_{A_L}$.*

Proof. Let us first observe that for every $z_1 \in X \cap \mathrm{N}_{L \cap \operatorname{dom} f^*}(x^*)$ and $z_2 \in (\partial_L^r f)^{-1}(x^*)$ ($\subset (\partial^r (f \square \sigma_L))^{-1}(x^*) \subset \partial (f \square \sigma_L)^*(x^*)$, by (9)), and every $y^* \in L \cap \operatorname{dom} f^*$, it holds

$$\langle z_1 + z_2, y^* - x^* \rangle \leq \langle z_2, y^* - x^* \rangle \leq (f \square \sigma_L)^*(y^*) - (f \square \sigma_L)^*(x^*) = f^*(y^*) - f^*(x^*) < +\infty,$$

showing that $\operatorname{cone}(L \cap \operatorname{dom} f^* - x^*) \subset \operatorname{dom} \sigma_{A_L}$. Now, we fix $w^* \in \operatorname{dom} \sigma_{A_L} \cap \overline{\operatorname{cone}}^\tau (L \cap \operatorname{dom} f^* - x^*)$ and pick $\bar{w}^* \in \tau\text{-ri}(\operatorname{cone}(L \cap \operatorname{dom} f^* - x^*))$ ($\subset \operatorname{dom} \sigma_{A_L}$) so that $w_\lambda^* := (1-\lambda) w^* + \lambda \bar{w}^* \in \operatorname{cone}(L \cap \operatorname{dom} f^* - x^*)$ ($\subset \operatorname{dom} \sigma_{A_L}$, as shown in the last inequality). Hence, since (10) holds on $\operatorname{cone}(L \cap \operatorname{dom} f^* - x^*)$ for every $\lambda \in (0,1)$ we can write

$$\langle x^{**}, w_\lambda^* \rangle \leq \sigma_{A_L}(w_\lambda^*) \leq (1-\lambda) \sigma_{A_L}(w^*) + \lambda \sigma_{A_L}(\bar{w}^*),$$

which leads us, as $\lambda \to 0^+$, to $\langle x^{**}, w^* \rangle \leq \lim_{\lambda \to 0^+} (1-\lambda) \sigma_{A_L}(w^*) + \lambda \sigma_{A_L}(\bar{w}^*) = \sigma_{A_L}(w^*)$. Therefore, (10) holds on $\overline{\operatorname{cone}}^\tau (L \cap \operatorname{dom} f^* - x^*)$. Finally, if $w^* \in X^* \setminus \overline{\operatorname{cone}}^\tau (L \cap \operatorname{dom} f^* - x^*)$, then by the separation theorem applied in the (locally convex) space (X^*, τ) we find $\tilde{x} \in X \setminus \{\theta\}$ and $\alpha \in \mathbb{R}$ such that

$$\langle \tilde{x}, w^* \rangle > \alpha \geq \langle \tilde{x}, v^* \rangle \;\; \text{for all } v^* \in \overline{\operatorname{cone}}^\tau (L \cap \operatorname{dom} f^* - x^*).$$

Hence, $\tilde{x} \in (X \cap \mathrm{N}_{L \cap \operatorname{dom} f^*}(x^*)) \setminus \{\theta\}$ and $\langle \tilde{x}, w^* \rangle > 0$ so that $\sigma_{A_L}(w^*) \geq \sigma_{(\partial_L^r f)^{-1}(x^*)}(w^*) + \sup_{n \geq 1} n \langle \tilde{x}, w^* \rangle = +\infty$; that is, (10) trivially holds. ■

From now on, we shall use the notation $Y := \operatorname{aff}(L \cap \operatorname{dom} f^* - x^*)$ so that, by the choice of L ($\in \mathcal{F}_\tau(x^*)$), Y is a τ-closed (a fortiori, norm-closed) subspace of X^* and there exist $\bar{z}^* \in$

$L \cap \operatorname{dom} f^*$ and a convex symmetric τ-neighborhood $V \subset X^*$ of θ such that

$$(\bar{z}^* + V) \cap \operatorname{aff}(L \cap \operatorname{dom} f^*) \subset L \cap \operatorname{dom} f^*, \;\; f^*(\bar{z}^* + z^*) \leq f^*(\bar{z}^*) + \frac{1}{2} \;\text{ for all } z^* \in V \cap Y. \quad (11)$$

Then, according to Lemma 9, we only need to show in the following lemmas that for every fixed $w^* \in L \cap \operatorname{dom} f^*$ it holds

$$\langle x^{**}, w^* - x^* \rangle \leq \sigma_{A_L}(w^* - x^*). \quad (12)$$

Lemma 10 *Let $(x^*, x^{**}) \in \partial f^*$ and $L \in \mathcal{F}_\tau(x^*)$ be fixed as in (8) above. Then, there exist nets $(x_\alpha, x_\alpha^*) \subset X \times L$, $\alpha \in (S_1, \leq)$, such that $x_\alpha \in \overline{\text{co}} \left\{ (\partial_L^r f)^{-1}(x_\alpha^*) + X \cap \mathrm{N}_{L \cap \operatorname{dom} f^*}(x_\alpha^*) \right\}$,*

$$\|x_\alpha^* - x^*\|_* \to 0, \; (\|x_\alpha\|) \text{ bounded}, \; x_\alpha \rightharpoonup^{w^{**}} x^{**}; \quad (13)$$

$$\liminf_\alpha f^*(x_\alpha^*) \geq f^*(x^*); \quad (14)$$

$$f^*(x_\alpha^*) \geq f^*(x^*) - \frac{1}{2} \quad \text{for all } \alpha. \quad (15)$$

Proof. We recall that cl and $\overline{\text{co}}$ also refer to the closed and the closed convex hulls of functions. By the current assumptions $\partial f^*(x^*) \neq \emptyset$ and $x^* \in L$ $(\in \mathcal{F}_\tau(x^*))$, from the relationships $\partial f^*(x^*) \subset \partial(f^* + \mathrm{I}_L)(x^*) = \partial((\overline{\text{co}} f) \square \sigma_L)^*(x^*) = \partial(\text{cl}((\overline{\text{co}} f) \square \sigma_L))^*(x^*)$ we infer that the functions $\overline{\text{co}} f$ and $\text{cl}((\overline{\text{co}} f) \square \sigma_L)$ are necessarily proper. Then, writing $x^{**} \in \partial f^*(x^*) \subset \partial(\text{cl}((\overline{\text{co}} f) \square \sigma_L))^*(x^*)$, by [9, Proposition 1] there exists a net $(x_\alpha, x_\alpha^*)_\alpha \subset X \times X^*$, $\alpha \in S_1$, satisfying (13) together with

$$x_\alpha \in X \cap \partial(\text{cl}((\overline{\text{co}} f) \square \sigma_L))^*(x_\alpha^*) = X \cap \partial((\overline{\text{co}} f) \square \sigma_L)^*(x_\alpha^*) = X \cap \partial(f^* + \mathrm{I}_L)(x_\alpha^*);$$

hence, $x_\alpha^* \in L$ and (14), (15) hold in view of the weak* lsc of f^*. Finally, by invoking Proposition 2 we deduce that $x_\alpha \in X \cap \partial(f^* + \mathrm{I}_L)(x_\alpha^*) = \overline{\text{co}} \left\{ (\partial_L^r f)^{-1}(x_\alpha^*) + X \cap \mathrm{N}_{L \cap \operatorname{dom} f^*}(x_\alpha^*) \right\}$. ■

Lemma 11 *We fix $\rho > 0$ and let $L \in \mathcal{F}_\tau(x^*)$ and $(x_\alpha, x_\alpha^*) \subset X \times X^*$ be as in Lemma 10. Then, there exist nets $u_\alpha \in X \cap \mathrm{N}_{L \cap \operatorname{dom} f^*}(x_\alpha^*)$, $v_\alpha \in \text{co} \left\{ (\partial_L^r f)^{-1}(x_\alpha^*) \right\}$ ($\alpha \in S_1$) and $u, v \in Y^*$ such that*

$$u_\alpha + v_\alpha - x_\alpha \in \rho B_X; \quad (16)$$

$$v_\alpha \in \partial(f^* + \mathrm{I}_L)(x_\alpha^*); \quad (17)$$

$$\langle v_\alpha - v, z^* \rangle \to 0 \quad \text{for all } z^* \in Y; \quad (18)$$

$$\langle u_\alpha - u, z^* \rangle \to 0 \quad \text{for all } z^* \in Y. \quad (19)$$

Proof. Since $x_\alpha \in \overline{\text{co}} \left\{ (\partial_L^r f)^{-1}(x_\alpha^*) + X \cap \mathrm{N}_{L \cap \operatorname{dom} f^*}(x_\alpha^*) \right\}$ as shown in Lemma 10, we find $u_\alpha \in X \cap \mathrm{N}_{L \cap \operatorname{dom} f^*}(x_\alpha^*)$ and $v_\alpha \in \text{co} \left\{ (\partial_L^r f)^{-1}(x_\alpha^*) \right\} \subset X \cap \partial(f^* + \mathrm{I}_L)(x_\alpha^*)$ (taking into account Proposition 2) such that (16) and (17) hold. Consequently, on the one hand, without loss of generality on α, for each $z^* \in V \cap Y$ (recall that $\bar{z}^*$ and V, Y were defined in (11)) we write

$$\begin{aligned}
\langle v_\alpha, z^* \rangle &\leq \langle v_\alpha, x_\alpha^* - \bar{z}^* \rangle + f^*(\bar{z}^* + z^*) + \mathrm{I}_L(\bar{z}^* + z^*) - f^*(x_\alpha^*) \\
&\leq \langle v_\alpha, x_\alpha^* - \bar{z}^* \rangle + f^*(\bar{z}^*) - f^*(x^*) + 1 \qquad \text{(by (11) and (15))} \\
&\leq \langle v_\alpha + u_\alpha, x_\alpha^* - \bar{z}^* \rangle + f^*(\bar{z}^*) - f^*(x^*) + 1 \qquad \text{(as } u_\alpha \in \mathrm{N}_{L \cap \operatorname{dom} f^*}(x_\alpha^*)\text{)}.
\end{aligned}$$

In other words, in view of (13) and (16) we have that $\sup_{z^* \in V \cap Y} \sup_\alpha |\langle v_\alpha, z^* \rangle| < +\infty$. Therefore, invoking Banach-Alaoglu Theorem, by passing to a subnet if necessary we get the existence of $v \in Y^*$ such that $\langle v_\alpha - v, z^* \rangle \to 0$, for all $z^* \in Y$; that is (18) holds. Moreover, since $u_\alpha \in \mathrm{N}_{L\cap \mathrm{dom}\, f^*}(x^*_\alpha)$, for each $z^* \in V \cap Y$ we have that

$$\begin{aligned}\langle u_\alpha, z^* \rangle &\le \langle u_\alpha, x^*_\alpha - \bar{z}^* \rangle && \text{(by (11))}\\ &\le \langle v_\alpha + u_\alpha, x^*_\alpha - \bar{z}^* \rangle + f^*(x^*) + \mathrm{I}_L(x^*) - f^*(x^*_\alpha) - \mathrm{I}_L(x^*_\alpha) + \langle v_\alpha, \bar{z}^* - x^* \rangle && \text{(by (17))}\\ &\le \langle v_\alpha + u_\alpha, x^*_\alpha - \bar{z}^* \rangle + \langle v_\alpha, \bar{z}^* - x^* \rangle + \frac{1}{2} && \text{(by (15)).}\end{aligned}$$

This, together with $\sup_\alpha \{ |\langle v_\alpha, \bar{z}^* - x^* \rangle|, |\langle v_\alpha + u_\alpha, x^*_\alpha - \bar{z}^* \rangle| \} < +\infty$ (recall (13), (16) and (18)), gives us $\sup_{z^* \in V \cap Y} \sup_\alpha |\langle u_\alpha, z^* \rangle| < +\infty$. Hence, by arguing as above we show the existence of $u \in Y^*$ which satisfies (19). ■

Lemma 12 *The vector u which appears in Lemma* 11 (19) *can be extended to X and this extension, denoted in the same way, satisfies $u \in \mathrm{N}_{L\cap \mathrm{dom}\, f^*}(x^*)$.*

Proof. First, the extension of u to X can be easily done by using the Hahn-Banach extension theorem in the space (X^*, τ). So, we only need to show that such an extension which is denoted in the same way belongs to $\mathrm{N}_{L\cap \mathrm{dom}\, f^*}(x^*)$. Indeed, since $u_\alpha \in \mathrm{N}_{L\cap \mathrm{dom}\, f^*}(x^*_\alpha)$ and $v_\alpha \in \partial(f^* + \mathrm{I}_L)(x^*_\alpha)$ (recall Lemma 11), for every $z^* \in L \cap \mathrm{dom}\, f^*$ we have that

$$\langle u_\alpha, z^* - x^* \rangle \le \langle u_\alpha, x^*_\alpha - x^* \rangle \le \langle u_\alpha + v_\alpha, x^*_\alpha - x^* \rangle + f^*(x^*) - f^*(x^*_\alpha).$$

Thus, combining (13), (14), (16) and (19), by taking limits on α we get $\langle u, z^* - x^* \rangle \le 0$, showing that $u \in \mathrm{N}_{L\cap \mathrm{dom}\, f^*}(x^*)$. ■

Lemma 13 *Let (x^*_α) and (v_α) be the nets defined in Lemmas* 10 *and* 11*, respectively. Then, for each $\alpha \in S_1$ there exist $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \Delta_4$ (see* (6)*), $(\lambda_{1,\alpha}, \lambda_{2,\alpha}, \lambda_{3,\alpha}, \lambda_{4,\alpha})_\alpha \subset \Delta_4$ and $(v_{1,\alpha}), \cdots, (v_{4,\alpha}) \subset (\partial^r_L f)^{-1}(x^*_\alpha)$ such that*

$$v_{i,\alpha} \in X \cap \partial(f^* + \mathrm{I}_L)(x^*_\alpha) \quad \text{for } i = 1, \ \cdots, 4; \tag{20}$$

$$\lim_\alpha \lambda_{i,\alpha} = \lambda_i \ge 0 \quad \text{for } i = 1, \ \cdots, 4; \tag{21}$$

$$\langle v_\alpha, x^*_\alpha - x^* \rangle = \sum_{i=1,\ \cdots,4} \lambda_{i,\alpha} \langle v_{i,\alpha}, x^*_\alpha - x^* \rangle; \tag{22}$$

$$\langle v_\alpha, w^* - x^*_\alpha \rangle = \sum_{i=1,\ \cdots,4} \lambda_{i,\alpha} \langle v_{i,\alpha}, w^* - x^*_\alpha \rangle; \tag{23}$$

$$\langle v_\alpha, \bar{z}^* - x^* \rangle = \sum_{i=1,\ \cdots,4} \lambda_{i,\alpha} \langle v_{i,\alpha}, \bar{z}^* - x^* \rangle; \tag{24}$$

$$\sup_\alpha \{ |\langle v_\alpha, w^* - x^*_\alpha \rangle|, \ |\langle v_{i,\alpha}, w^* - x^*_\alpha \rangle|, \ i = 1, \ \cdots, 4\} < +\infty. \tag{25}$$

Furthermore, one may suppose without loss of generality that $\lambda_{1,\alpha}, \lambda_{2,\alpha}, \lambda_{3,\alpha}, \lambda_{4,\alpha} > 0$ for all α.

Proof. We fix $\alpha \in S_1$. By Lemma 11 we have that $v_\alpha \in \mathrm{co}\left\{(\partial^r_L f)^{-1}(x^*_\alpha)\right\}$ and, so, there exist $l_\alpha \in \mathbb{N}^*$, $(\tilde{\lambda}_{1,\alpha} \cdots, \tilde{\lambda}_{l_\alpha,\alpha}) \in \Delta_{l_\alpha}$ and $v_{1,\alpha}, \cdots, v_{l_\alpha,\alpha} \in (\partial^r_L f)^{-1}(x^*_\alpha)$ ($\subset X \cap \partial(f^* + \mathrm{I}_L)(x^*_\alpha)$, by (17)) such that $v_\alpha = \sum_{1\le i\le l_\alpha} \tilde{\lambda}_{i,\alpha} v_{i,\alpha}$. Moreover, invoking Carathéodory's Theorem applied in $\mathbb{R}^3$, by reordering if necessary we find $(\lambda_{1,\alpha}, \lambda_{2,\alpha}, \lambda_{3,\alpha}, \lambda_{4,\alpha}) \in \Delta_4$ such that (22)–(24) hold. In particular, we may assume that each $(\lambda_{i,\alpha})_\alpha$ ($\subset [0, 1]$) converges to $\lambda_i \ge 0$ so that $\sum_{1\le i\le 1} \lambda_i = 1$; that is,

(21) follows. We also may suppose that $\lambda_{1,\alpha}, \lambda_{2,\alpha}, \lambda_{3,\alpha}, \lambda_{4,\alpha} >$; for otherwise, we consider only the nonnul elements. Now, from (20) together with (15) we infer that (recall that $w^* \in L \cap \mathrm{dom}\, f^*$ and $x^*_\alpha \in L$)

$$\langle v_{i,\alpha}, w^* - x^*_\alpha \rangle \leq f^*(w^*) - f^*(x^*_\alpha) \leq f^*(w^*) - f^*(x^*) + 1. \tag{26}$$

But, as $u_\alpha \in \mathrm{N}_{L\cap \mathrm{dom}\, f^*}(x^*_\alpha)$ and $v_\alpha \in \partial(f^* + \mathrm{I}_L)(x^*_\alpha)$ (recall Lemma 11), by (15) we have that $\langle v_\alpha + u_\alpha, w^* - x^*_\alpha \rangle \leq \langle v_\alpha, w^* - x^*_\alpha \rangle \leq f^*(w^*) - f^*(x^*) + 1$, and so $\sup_\alpha |\langle v_\alpha, w^* - x^*_\alpha \rangle| < +\infty$, accordingly to (13) and (16). Therefore, by combining (26) together with (23) we deduce that $\sup_\alpha |\langle v_{i,\alpha}, w^* - x^*_\alpha \rangle| < +\infty, i = 1, \cdots, 4$; that is, (25) follows. ■

Lemma 14 *For each $\alpha \in S_1$ and $i \in \{1, \cdots, 4\}$ there exists a net $(v^\beta_{i,\alpha})_\beta \subset X$, $\beta \in (S_2, \leq)$, such that*

$$\lim_\beta \langle v^\beta_{i,\alpha} - v_{i,\alpha}, z^* \rangle = 0 \quad \text{for all } z^* \in Y, \tag{27}$$

$$\lim_\beta f(v^\beta_{i,\alpha}) + f^*(x^*_\alpha) - \langle v^k_{i,\alpha}, x^*_\alpha \rangle = 0. \tag{28}$$

Consequently, by passing to subnets if necessary,

$$\lim_\alpha \langle v_{i,\alpha}, x^*_\alpha - x^* \rangle = \lim_\alpha \langle v_\alpha, x^*_\alpha - x^* \rangle = 0; \tag{29}$$

$$\lim_\alpha \lim_\beta (f(v^\beta_{i,\alpha}) - \langle v^\beta_{i,\alpha}, x^* \rangle) = -f^*(x^*). \tag{30}$$

Proof. We fix α and $i = \{1, \cdots, 4\}$. Since $v_{i,\alpha} \in (\partial^r_L f)^{-1}(x^*_\alpha)$, according to Lemma 13, we find a net $(v^\beta_{i,\alpha})_\beta \subset X$, $\beta \in (S_2, \leq)$, such that (27)-(28) hold. Thus, as x^*_α, $x^* \in L \cap \mathrm{dom}\, f^*$, we can write $-f^*(x^*_\alpha) = \lim_\beta (f(v^\beta_{i,\alpha}) - \langle v^\beta_{i,\alpha}, x^*_\alpha \rangle) = -\langle v_{i,\alpha}, x^*_\alpha - x^* \rangle + \lim_\beta (f(v^\beta_{i,\alpha}) - \langle v^\beta_{i,\alpha}, x^* \rangle)$ so that, invoking Fenchel inequality,

$$-f^*(x^*) \leq \lim_\beta (f(v^\beta_{i,\alpha}) - \langle v^\beta_{i,\alpha}, x^* \rangle) = -f^*(x^*_\alpha) + \langle v_{i,\alpha}, x^*_\alpha - x^* \rangle. \tag{31}$$

Hence, in view of (14) we deduce that $\liminf_\alpha \langle v_{i,\alpha}, x^*_\alpha - x^* \rangle \geq 0$ and, so, by appealing to (22) together with (13) and the fact that $u_\alpha \in \mathrm{N}_{L\cap \mathrm{dom}\, f^*}(x^*_\alpha)$ we get

$$\begin{aligned} 0 \leq \lambda_i \liminf_\alpha \langle v_{i,\alpha}, x^*_\alpha - x^* \rangle &\leq \liminf_\alpha \sum_{i=1,\cdots,4} \tilde{\lambda}_{i,\alpha} \langle v_{i,\alpha}, x^*_\alpha - x^* \rangle \\ &= \liminf_\alpha \langle v_\alpha, x^*_\alpha - x^* \rangle \leq \limsup_\alpha \langle v_\alpha + u_\alpha, x^*_\alpha - x^* \rangle = 0. \end{aligned}$$

But $\lambda_i > 0$ (recall Lemma 13), and so we deduce that $\liminf_\alpha \langle v_{i,\alpha}, x^*_\alpha - x^* \rangle = \lim_\alpha \langle v_\alpha, x^*_\alpha - x^* \rangle =$; that is, (29) follows up to a subnet. Finally, (30) follows by combining (29) and (31). ■

Lemma 15 *For each $i \in \{1, \cdots, 4\}$ there exists $v_i \in Y^*$ such that*

$$\lim_\alpha \langle v_{i,\alpha} - v_i, z^* \rangle = 0 \quad \text{for all } z^* \in Y, \tag{32}$$

where the net $(v_{i,\alpha})_\alpha$ appears in Lemma 13.

Proof. For each $\alpha \in S_1$, by (17) we have that $v_{i,\alpha} \in \partial(f^* + \mathrm{I}_L)(x^*_\alpha)$ and so, in view of (11) together with (15), for every given $z^* \in V \cap Y$ we have that

$$\langle v_{i,\alpha}, \bar{z}^* + z^* - x^*_\alpha \rangle \leq f^*(\bar{z}^* + z^*) + \mathrm{I}_L(\bar{z}^* + z^*) - f^*(x^*_\alpha) - \mathrm{I}_L(x^*_\alpha) \leq f^*(\bar{z}^*) - f^*(x^*) + 1. \tag{33}$$

Moreover, by invoking Fenchel inequality together with (27) and (30) we get

$$\langle v_{i,\alpha}, \bar{z}^* - x^* \rangle = \lim_\beta \langle v_{i,\alpha}^\beta, \bar{z}^* - x^* \rangle \le f^*(\bar{z}^*) + \limsup_\beta \left(f(v_{i,\alpha}^\beta) - \langle v_{i,\alpha}^\beta, x^* \rangle \right) = f^*(\bar{z}^*) - f^*(x^*) + 1. \tag{34}$$

Now, as $u \in \mathrm{N}_{L \cap \operatorname{dom} f^*}(x^*)$ (recall Lemma 12) we observe that

$$\begin{aligned}\langle v_\alpha, \bar{z}^* - x^* \rangle &= \langle v_\alpha + u_\alpha, \bar{z}^* - x^* \rangle + \langle u_\alpha - u, x^* - \bar{z}^* \rangle + \langle u, x^* - \bar{z}^* \rangle \\ &\ge \langle v_\alpha + u_\alpha, \bar{z}^* - x^* \rangle + \langle u_\alpha - u, x^* - \bar{z}^* \rangle.\end{aligned}$$

But these last terms are bounded independently of α (according to (13), (16) and (18)) and, so, from (24) and (34) together with (29) we infer that $\sup_\alpha \{ |\langle v_{i,\alpha}, \bar{z}^* - x^* \rangle|, |\langle v_{i,\alpha}, \bar{z}^* - x_\alpha^* \rangle| \} < +\infty$. Consequently, in view of (33) we get $\sup_{z^* \in V \cap Y} \langle v_{i,\alpha}, z^* \rangle < +\infty$ so that, taking subnets if necessary, by arguing as in the proof of Lemma 11 we find $v_i \in Y^*$ such that $\langle v_{i,\alpha} - v_i, z^* \rangle \to 0$ for all $z^* \in Y$. Thus, (32) holds. ■

Lemma 16 *For each $i \in \{1, \cdots, 4\}$, the vector $v_i \in Y^*$ which appears in Lemma 15 can be extended to X and this extension, denoted in the same way, satisfies $v_i \in (\partial_L^r f)^{-1}(x^*)$.*

Proof. Let $(x_\alpha^*)_\alpha$, $(v_\alpha)_\alpha$, $(v_{i,\alpha})_\alpha$ and $(v_{i,\alpha}^\beta)_{(\alpha,\beta)}$, $i \in \{1, \cdots, 4\}$, $(\alpha, \beta) \in S_1 \times S_2$, be the nets defined in the previous lemmas. Then, by successively invoking (15), (29) and (32) we may assume that for all $i \in \{1, \cdots, 4\}$ and all $\alpha \in S_1$ it holds

$$f^*(x_\alpha^*) \ge f^*(x^*) - \frac{1}{2}, \;\; |\langle v_{i,\alpha}, x_\alpha^* - x^* \rangle| \le \frac{1}{3}, \;\; |\langle v_{i,\alpha}, \bar{z}^* - x^* \rangle| \le |\langle v_i, \bar{z}^* - x^* \rangle| + \frac{1}{3}.$$

Moreover, accordingly to (27), (28) and (29) all together, for all $z^* \in Y$ we have that

$$\lim_\beta \langle v_{i,\alpha}^\beta - v_{i,\alpha}, z^* \rangle = \lim_\beta (f(v_{i,\alpha}^\beta) + f^*(x_\alpha^*) - \langle v_{i,\alpha}^\beta, x_\alpha^* \rangle) = \lim_\alpha \langle v_{i,\alpha}, x_\alpha^* - x^* \rangle = \lim_\alpha \langle v_\alpha, x_\alpha^* - x^* \rangle = 0 \tag{35}$$

and, so, for every given $\alpha \in S_1$ there exists $\beta_\alpha \in S_2$ such that the following statement holds for all $\beta \ge \beta_\alpha$,

$$|\langle v_{i,\alpha}, x_\alpha^* - x^* \rangle| \le \frac{1}{2}, \;\; f(v_{i,\alpha}^\beta) - \langle v_{i,\alpha}^\beta, x_\alpha^* \rangle \le -f^*(x_\alpha^*) + 1; \tag{36}$$

hence, $(f^*)^*(v_{i,\alpha}^\beta) - \langle v_{i,\alpha}^\beta, x_\alpha^* \rangle \le f(v_{i,\alpha}^\beta) - \langle v_{i,\alpha}^\beta, x_\alpha^* \rangle \le -f^*(x_\alpha^*) + 1$ and, so, $v_{i,\alpha}^\beta \in \partial_1 f^*(x_\alpha^*)$. Moreover, by taking into account Lemma 15 we may assume that $|\langle v_{i,\alpha}, \bar{z}^* - x^* \rangle| + |\langle v_{i,\alpha}, \bar{z}^* - x_\alpha^* \rangle| \le m$ for some $m \in \mathbb{R}$ so that, for all $\beta \ge \beta_\alpha$,

$$\left| \langle v_{i,\alpha}^\beta, x_\alpha^* - x^* \rangle \right| \le 1, \;\; \left| \langle v_{i,\alpha}^\beta, \bar{z}^* - x^* \rangle \right| \le m + 1, \;\; \left| \langle v_{i,\alpha}^\beta, \bar{z}^* - x_\alpha^* \rangle \right| \le m + 1, \tag{37}$$

at the same time as

$$f(v_{i,\alpha}^\beta) - \langle v_{i,\alpha}^\beta, x_\alpha^* \rangle \le -f^*(x_\alpha^*) + 1. \tag{38}$$

At this moment, for fixed $i \in \{1, \cdots, 4\}$ we define the net $(w_{i,\alpha}^\beta)_{(\alpha,\beta) \in S}$ as $w_{i,\alpha}^\beta := v_{i,\alpha}^\beta$ where the corresponding index set

$$S := \{(\alpha, \beta) \in S_1 \times S_2 \mid \beta \ge \beta_\alpha \;\text{ for some } \beta_\alpha \;\text{ satisfying (36)–(38)}\}$$

is directed via the order relation $(\alpha_1, \beta_1) \le (\alpha_2, \beta_2) \iff \alpha_1 \le \alpha_2$ and $\beta_1 \le \beta_2$. Then, by recalling (11) and (15) together with the fact that $w_{i,\alpha}^\beta \in \partial_1 f^*(x_\alpha^*)$ (see (36)), for each $(\alpha, \beta) \in S$ and all

$z^* \in V \cap Y$ we have that

$$\langle w^{\beta}_{i,\alpha}, \bar{z}^* + z^* - x^*_{\alpha} \rangle \leq f^*(\bar{z}^* + z^*) - f^*(x^*_{\alpha}) + 1 \leq f^*(\bar{z}^*) - f^*(x^*) + 2.$$

Hence, by (37) we get

$$\langle w^{\beta}_{i,\alpha}, z^* \rangle \leq \langle w^{\beta}_{i,\alpha}, x^*_{\alpha} - \bar{z}^* \rangle + f^*(\bar{z}^*) - f^*(x^*) + 2 \leq m + f^*(\bar{z}^*) - f^*(x^*) + 3,$$

and so, by taking a subnet if necessary we find $w_i \in Y^*$ such that the net $(w^{\beta}_{i,\alpha})_{(\alpha,\beta)\in S}$ weak* converges to w_i in Y^*. Let us show that $w_i = v_i$. Indeed, if this was not the case, there would exist open neighborhoods in Y^*, W_1 of v_i and W_2 of w_i such that $W_1 \cap W_2 = \emptyset$. Then, from one hand we find $(\alpha_0, \beta_0) \in S$ such that $w^{\beta}_{i,\alpha} \in W_1$ for every $(\alpha, \beta) \in S$ satisfying $(\alpha, \beta) \geq (\alpha_0, \beta_0)$. On the other hand, by Lemma 15 there exists α_1 such that $v_{i,\alpha_1} \in W_2$; we may assume without loss of generality that $\alpha_1 \geq \alpha_0$. We also find $\beta_1 \geq \beta_{\alpha_1}$ with $\beta_1 \geq \beta_0$ such that $v^{\beta_1}_{i,\alpha_1} \in W_2$. Therefore, $(\alpha_1, \beta_1) \in S$, $(\alpha_1, \beta_1) \geq (\alpha_0, \beta_0)$ and we have that $w^{\beta_1}_{i,\alpha_1} = v^{\beta_1}_{i,\alpha_1} \in W_2 \cap W_1$, leading us to a contradiction. Hence, we must have $w_i = v_i$ on Y.

Now, we are going to show that

$$\lim_{(\alpha,\beta)\in S} f(w^{\beta}_{i,\alpha}) - \langle w^{\beta}_{i,\alpha}, x^* \rangle + f^*(x^*) = 0. \tag{39}$$

Proceeding by contradiction, if this last inequality doesn't hold, by taking into account Fenchel inequality there would exist $\eta > 0$ such that

$$\lim_{(\alpha,\beta)\in S} f(w^{\beta}_{i,\alpha}) - \langle w^{\beta}_{i,\alpha}, x^* \rangle + f^*(x^*) \geq \eta.$$

Hence, we find $(\alpha_2, \beta_2) \in S$ such that $f(w^{\beta}_{i,\alpha}) - \langle w^{\beta}_{i,\alpha}, x^* \rangle + f^*(x^*) \geq \eta$ for every $(\alpha, \beta) \in S$ verifying $(\alpha, \beta) \geq (\alpha_1, \beta_2)$. Moreover, since for all $\beta \geq \beta_2$ we have that $(\alpha_2, \beta) \in S$ and $(\alpha_2, \beta) \geq (\alpha_2, \beta_2)$, by taking the limit on β in the inequality $f(v^{\beta}_{i,\alpha_2}) + f^*(x^*_{\alpha_2}) - \langle v^{\beta}_{i,\alpha_2}, x^*_{\alpha_2} \rangle \geq \eta$ from (35) we obtain the contradiction

$$0 < \eta \leq \lim_{\beta} f(w^{\beta}_{i,\alpha_2}) - \langle w^{\beta}_{i,\alpha_2}, x^* \rangle + f^*(x^*) = 0,$$

showing that (39) holds. Finally, to conclude that $v_i \in (\partial^r_L f)^{-1}(x^*)$ we only need to observe that v_i can be extended using the Hahn-Banach Theorem in the space (X^*, τ). ■

Now, we are able to conclude the proof of Theorem 3:

Lemma 17 *The inequality* (12) *holds.*

Proof. By invoking (29) together with Lemma 15 we take the limit over α in (23) to obtain that

$$\begin{aligned}\langle v, w^* - x^* \rangle &= \lim_{\alpha} \langle v_{\alpha}, w^* - x^* \rangle + \langle v_{\alpha}, x^* - x^*_{\alpha} \rangle = \lim_{\alpha} \langle v_{\alpha}, w^* - x^* \rangle \\ &= \lim_{\alpha} \sum_{i=1,\cdots,4} \lambda_{i,\alpha} \langle v_{i,\alpha}, w^* - x^*_{\alpha} \rangle = \sum_{i=1,\cdots,4} \lambda_i \langle v_i, w^* - x^* \rangle.\end{aligned}$$

But, by Lemma 16 we have that $v_i \in (\partial^r_L f)^{-1}(x^*)$, and so

$$\langle v, w^* - x^* \rangle \leq \sigma_{\mathrm{co}\left\{(\partial^r_L f)^{-1}(x^*)\right\}}(w^* - x^*).$$

Therefore, taking into account Lemma 12, by (16) together with (18) and (19) we get

$$\begin{aligned}\langle x^{**}, w^* - x^* \rangle &\leq \lim_{\alpha} \langle u_\alpha + v_\alpha, w^* - x^* \rangle + \rho \left\| w^* - x^* \right\| \\ &= \langle u + v, w^* - x^* \rangle + \rho \left\| w^* - x^* \right\| \\ &\leq \sigma_{\mathrm{co}\left\{ (\partial_L^r f)^{-1}(x^*) + X \cap \mathrm{N}_{L \cap \mathrm{dom}\, f^*}(x^*) \right\}}(w^* - x^*) + \rho \left\| w^* - x^* \right\|.\end{aligned}$$

Thus, the desired inequality holds when $\rho \to 0$. ■

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