Deterministic Kalman Filtering on Semi-infinite Interval

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Abstract

We relate a deterministic Kalman filter on semi-infinite interval to linear-quadratic tracking control model with unfixed initial condition.

1 Introduction

In [4], E. Sontag considered the deterministic analogue of Kalman filtering problem on finite interval. The deterministic model allows a natural extension to semi-infinite interval. It is of a special interest because for the standard linear-quadratic stochastic control problem extension to semi-infinite interval leads to complications with the standard quadratic objective function (see e.g. [1]). According to [4], the model which we are going to consider has the following form:

$$J(x, u, x_0) = \int_0^{+\infty} [u^T Ru + (Cx - \bar{y})^T Q(Cx - \bar{y})] \, dt, \tag{1}$$

$$\dot{x} = Ax + Bu, \tag{2}$$

$$x(0) = x_0. \tag{3}$$

Here we assume that the pair $(x, u) \in a(x_0) + Z$, where Z is a vector subspace of the Hilbert space $L_2^n[0, +\infty) \times L_2^m[0, +\infty)$ (with $L_2^n[0, +\infty)$ a Hilbert space of R^n-value square integrable functions) defined as follows:

$$\begin{aligned} Z \quad = \quad & \{(x, u) \in L_2^n[0, +\infty) \times L_2^m[0, +\infty) : x \text{ is absolutely continuous}, \\ & \dot{x} \in L_2^n[0, +\infty), \dot{x} = Ax + Bu, x(0) = 0\} \end{aligned}$$

Here A is an n by n matrix; B is an n by m matrix; $R = R^T$ is an n by n and positive definite; $Q = Q^T$ is an r by r and positive definite; C is an r

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by n matrix; $\bar{y} \in L_2^r[0, +\infty)$. Notice that in (1) - (3) x_0 is not fixed and we minimize over all triple $(x, u, x_0) \in L_2^n[0, +\infty) \times L_2^m[0, +\infty) \times R^n$ satisfying our assumption.

Notice also that we interpret (1) - (3) as an estimation problem of the form

$$\dot{x} = Ax + Bu,$$

$$\bar{y} = Cx + v,$$

where we try to estimate x with the help of observation $\bar{y}$ by minimizing perturbations u, v and choosing an appropriate initial condition x_0.

2 Solution of the deterministic problem

Consider the algebraic Riccati equation

$$KA + A^T K + KLK - C^T QC = 0, \tag{4}$$

where $L = BR^{-1}B^T$. Assuming that the pair (A, B) is stabilizable and the pair (C, A) is detectable, there exists a negative definite symmetric solution K_{st} to (4) such that the matrix $A + LK_{st}$ is stable (see e.g. Theorem 12.3 in [5]). Using the result of [2], we can describe the optimal solution to (1) - (3) with fixed x_0 as follows.

There exists a unique solution $\rho_0 \in L_2^n[0, \infty)$ satisfying the differential equation

$$\dot{\rho} = -(A + LK_{st})^T \rho - C^T Q\bar{y}. \tag{5}$$

Moreover, ρ_0 can be explicitly described as follows:

$$\rho_0(t) = \int_0^{+\infty} \exp[(A + LK_{st})^T \tau] C^T Q\bar{y}(t + \tau)\, d\tau. \tag{6}$$

The optimal solution (x, u) to (1) - (3) has the form

$$\dot{x} = (A + LK_{st})x + L\rho_0, \;\; x(0) = x_0, \tag{7}$$

$$u = R^{-1}B^T(K_{st}x + \rho_0). \tag{8}$$

For details see [2].

Notice that ρ_0 does not depend on x_0. To solve the original problem (1) - (3) we need to express the minimal value of the functional (1) in term of x_0.

Theorem 1. *Let (x, u) be an optimal solution of (1) - (3) with fixed x_0 given by (5) - (8). Then*

$$J(x, u, x_0) = -x_0^T K_{st} x_0 - 2\rho_0(0)^T x_0 + \int_0^{+\infty} [\bar{y}^T Q\bar{y} - \rho_0^T L\rho_0]\, dt. \tag{9}$$

Remark. Notice that $J(x, u, x_0)$ is a strictly convex function of x_0 and hence minimum of J as a function of x_0 is attained at

$$x_0^{opt} = -K_{st}^{-1}\rho_0(0). \tag{10}$$

Hence (5) - (8) gives a complete solution of the original problem (1) - (3).

Proof. Let $(y, w) \in a(x_0) + Z$ be feasible solution to (1) - (3), where x_0 is fixed. Consider

$$\Delta(y, w) = [w - R^{-1}B^T(K_{st}y + \rho_0)]^T R[w - R^{-1}B^T(K_{st}y + \rho_0)].$$

where we suppressed an explicit dependence on time. Notice that

$$\Delta(x, u) \equiv 0.$$

Furthermore, $\Delta(y, w) = \Delta_1 + \Delta_2 + \Delta_3$, where
$\Delta_1 = w^T Rw$,
$\Delta_2 = -2(K_{st}y + \rho_0)^T Bw$,
$\Delta_3 = (K_{st}y + \rho_0)^T L(K_{st}y + \rho_0)$.
Now $Bw = \dot{y} - Ay$ and consequently

$$\begin{aligned}
\Delta_2 &= -2(y^T K_{st} + \rho_0^T)(\dot{y} - Ay) \\
&= y^T(K_{st}A + A^T K_{st})y - 2y^T K_{st}\dot{y} - 2\rho_0^T\dot{y} + 2\rho^T Ay, \\
\Delta_3 &= y^T K_{st}LK_{st}y + \rho_0^T L\rho_0 + 2\rho_0^T LK_{st}y.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\Delta(y, w) &= w^T Rw + \rho_0^T L\rho_0 + y^T(K_{st}LK_{st} + K_{st}A + A^T K_{st})y \\
&\quad -\frac{d}{dt}(y^T K_{st}y) - 2\frac{d}{dt}(\rho_0^T y) + 2\dot{\rho}_0^T y + 2\rho_0^T LK_{st}y + 2\rho_0^T Ay.
\end{aligned}$$

Using (4) and (5), we obtain

$$\begin{aligned}
\Delta(y, w) &= w^T Rw + \rho_0^T L\rho_0 + y^T C^T QCy - 2\frac{d}{dt}(\rho_0^T y) \\
&\quad -\frac{d}{dt}(y^T K_{st}y) - 2(C^T Q\bar{y})^T y \\
&= w^T Rw + \rho_0^T L\rho_0 - 2\frac{d}{dt}(\rho_0^T y) - \frac{d}{dt}(y^T K_{st}y) \\
&\quad +(\bar{y} - Cy)^T Q(\bar{y} - Cy) - \bar{y}^T Q\bar{y}.
\end{aligned}$$

Hence, taking into account that $\rho_0(t) \to 0,\ y(t) \to 0,\ t \to +\infty$ (see for details [2]), we obtain

$$
\begin{aligned}
\int_0^{+\infty} \Delta(y,w)\ dt &= \int_0^{+\infty} [w^T R w + +(\bar{y} - Cy)^T Q(\bar{y} - Cy)]\ dt \\
&\quad + \int_0^{+\infty} [\rho_0^T L \rho_0 - \bar{y}^T Q \bar{y}]\ dt + 2\rho_0(0)^T x_0 + x_0 K_{st} x_0 \\
&= J(y,w,x_0) + 2\rho_0(0)^T x_0 + x_0 K_{st} x_0 + c
\end{aligned}
$$

where $c = \int_0^{+\infty} [\rho_0^T L \rho_0 - \bar{y}^T Q \bar{y}]\ dt$.
Notice, that $\Delta(y,w) \geq 0$ and $\Delta(x,u) \equiv 0$. This shows that, indeed, (x,u) is an optimal solution to (1) - (3) (with fixed x_0) and proves (9). □

3 Steady state deterministic Kalman filtering

In light of (10), it is natural to consider the process

$$z(t) = -K_{st}^{-1}\rho_0(t),\ t \in [0,+\infty) \tag{11}$$

as a natural estimate for the optimal solution to problem (1) - (3). Let us find the differential equation for z.

Proposition 1.

$$\dot{z} = Az + K_{st}^{-1} C^T Q(\bar{y} - Cz). \tag{12}$$

Remark: Notice that K_{st}^{-1} is a solution to the algebraic equation

$$L - PC^T QCP + AP + PA^T = 0 \tag{13}$$

In other words, the differential equation (12) is a precise deterministic analogue for the stochastic differential equation describing the optimal (steady-state) estimation in Kalman filtering problem. See e.g. [1].

Proof. Using (5) and (11), we obtain:

$$
\begin{aligned}
\dot{z} &= K_{st}^{-1}(A + LK_{st})^T \rho_0 + K_{st}^{-1} C^T Q \bar{y} \\
&= -(K_{st}^{-1} A^T + L)(K_{st} z) + K_{st}^{-1} C^T Q \bar{y}.
\end{aligned}
$$

Since K_{st} is a solution to (4), we have

$$-K_{st}^{-1} A^T K_{st} - LK_{st} = A - K_{st}^{-1} C^T Q C.$$

Hence,

$$\dot{z} = Az - K_{st}^{-1} C^T Q C z + K_{st}^{-1} C^T Q \bar{y}.$$

Hence, we obtain (12). □

Remark: Notice that due to (11) $\Delta(z,0) \equiv 0$ and consequently $(z,0)$ would be an optimal solution to (1) - (3) if it were feasible for this problem.

4 Concluding remarks

The differential equation (12) allows one to recursively estimate of z based on observation $\bar{y}$ provided we know $z(0) = x_0^{opt}$. Notice that according to (6)

$$z(0) = -K_{st}^{-1} \int_0^{\infty} \exp[(A + LK_{st})^T \tau] C^T Q \bar{y}(\tau)\ d\tau,$$

requires the knowledge of the whole $\bar{y}$. However, due to the integral nature of this relationship the infinite integral can be very well approximated by the integral over the finite interval $[0, \bar{T}]$ (since the matrix $A + LK_{st})^T$ is stable) and hence the recursive nature of (12) is recovered for $t \geq \bar{T}$ for some finite $\bar{T}$. Notice, further, that similar problem with discrete time can be solved using the formalism developed in [3].

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