New optimality conditions for the semivectorial bilevel optimization problem

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Abstract. The paper is concerned with the optimistic formulation of a bilevel optimization problem with multiobjective lower-level problem. Considering the scalarization approach for the multiobjective program, we transform our problem into a scalar-objective optimization problem with inequality constraints by means of the well-known optimal value reformulation. Completely detailed first-order necessary optimality conditions are then derived in the smooth and nonsmooth settings while using the generalized differentiation calculus of Mordukhovich. Our approach is different from the one previously used in the literature and the conditions obtained are new and furthermore, they reduce to those of a usual bilevel program if the lower-level objective function becomes single-valued.

Keywords: Semivectorial bilevel optimization, multiojective optimization, weak efficient solution, optimal value function, optimality conditions

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1 Introduction

In this paper, we are concerned with the following optimistic bilevel optimization problem

$$\begin{cases} \underset{x,z}{\text{Minimize}}\ F(x,z) \\ \text{subject to}:\ x \in X,\ z \in \Psi_{\text{wef}}(x), \end{cases} \tag{1.1}$$

where the nonempty closed set X denotes the upper-level feasible set and the function $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is the upper-level objective function. The set-valued mapping Ψ_{wef} represents the *weak efficient optimal solution* mapping of the multiobjective optimization problem

$$\begin{cases} \mathbb{R}^l_+ - \underset{z}{\text{Minimize}}\ f(x,z) \\ \text{subject to}:\ z \in K(x), \end{cases} \tag{1.2}$$

with $K : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^p$ being the lower-level feasible set-valued mapping, while $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^l$ is the lower-level multiobjective function. For simplicity in the exposition, it will be assumed throughout the paper that the upper- and lower-level feasible sets are respectively defined as

$$X := \{x \in \mathbb{R}^n |\, G(x) \leq 0\} \text{ and } K(x) := \{y \in \mathbb{R}^m |\, g(x,y) \leq 0\}. \tag{1.3}$$

with $G : \mathbb{R}^n \to \mathbb{R}^q$ and $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$. However, all the results here can easily be extended to the more general operator constraints in sense of Mordukhovich [12]. The term "$\mathbb{R}^l_+ - \underset{z}{\text{Minimize}}$" in (1.2) is used to symbolize that optimal vector-values in our lower-level problem are in the sense of weak Pareto minima w.r.t. an order induced by the positive orthant of $\mathbb{R}^l$. The definitions of efficient solutions and Pareto minima will be given in the next section.

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Problem (1.1) is in fact an optimization problem with multiobjective lower-level problem. It was labeled as "semivectorial bilevel optimization problem" by Bonnel and Morgan [2]. In this paper, a penalty approach was suggested to solve the problem in case of weakly efficient solutions in the lower-level problem (1.2). Another penalty method was developed in [20] in the case where the multiobjective lower-level problem is linear. In [10], problem (1.1) was also considered, now with the possibility for the upper-level objective function F to be also vector-valued. In this work, the feasible set of the investigated problem is shown to be the set of minimal points (w.r.t. a cone) of another (unperturbed) multiobjective optimization problem. Hence, the resulting problem is *simply* a multiobjective optimization problem over an efficient set, which is of course, also a very difficult class of problem. Using the Pascoletti and Serafini-type scalarisation, an adaptive parameter control approach based on sensitivity results is used to approximate the solution set of the problem.

In [1], Bonnel derived necessary optimality conditions for problem (1.1) in very general Banach spaces, while considering efficient and weakly efficient solutions for the lower-level problem (1.2). Broadly speaking, the method of Bonnel consists of inserting the weak or properly weak solution set-valued mapping of the lower-level problem in the upper-level objective function. The resulting problem is a set-valued optimization problem. Necessary optimality conditions are then derived by means of the notion of contingent derivative. The optimality conditions obtained are abstract in nature.

The aim of our paper is to also derive necessary optimality conditions for the semivectorial bilevel optimization problem (1.1). Our approach is completely different from that of Bonnel. Considering weakly efficient solutions for the lower-level problem (1.2), the classical scalarization technique is used to convert the problem into a usual bilevel optimization problem, that is, with a single-objective problem in the lower-level. Since the Pareto front would hardly reduce to a single point, the optimal value function reformulation appears to be one of the best approaches to transform the latter problem into a single-level optimization problem with inequality constraints. Karush-Kuhn-Tucker (KKT)-type optimality conditions are then derived for problem (1.1) in terms of initial problem data. A (non-classic) difficulty faced by our approach is that the full convexity assumption usually made on the lower-level objective function (see e.g. [5]) to obtain the Lipschitz continuity of the value function is not applicable here, considering the new (scalarization) parameter entering the new lower-level objective function. We consider this parameter to be a variable for the upper-level objective to be sure that all the weak Pareto points are taken into account while looking for the best choice for the leader.

The other two options we consider to ensure the Lipschitz continuity of the value function of the scalarized problem are the inner semicompactness and the inner semicontinuity (to be defined in the next section) of the solution set-valued mapping of the new lower-level problem. One strange thing that is observed is that under the inner semicontinuity assumption, the necessary optimality conditions of the aforementioned optimal value function reformulated problem are in fact independent of the restriction made on the scalarization parameter.

In the next section of the paper, we first present some basic notions of multiobjective optimization needed in the subsequent sections. Relevant notions and properties from variational analysis will be presented as well. Details on the transformation process of the semivectorial bilevel optimization problem into a single-level optimization problem are given in Section 3. In Section 4, KKT-type necessary optimality conditions are then derived for problem (1.1) while considering the instances where all functions involved are strictly differentiable and Lipschitz continuous, respectively. The special case where the lower-level multiobjective problem is linear in the lower-level variable is studied in the last section.

2 Preliminaries

2.1 Multiobjective optimization

Let $C \subset \mathbb{R}^p$ be a pointed ($C \cap (-C) = \{0\}$) closed convex cone with nonempty interior introducing a partial order $\preceq_C$ in $\mathbb{R}^p$ and let A be a nonempty subset of $\mathbb{R}^p$. $\overline{z} \in A$ is said to be a Pareto (resp. weak Pareto) minimal vector of A with respect to C if

$$A \subset \overline{z} + [(\mathbb{R}^p \setminus (-C)) \cup \{0\}] \, (\text{resp. } A \subset \overline{z} + (\mathbb{R}^p \setminus -\operatorname{int} C)), \tag{2.1}$$

where "int" denotes the topological interior of the set in question. Let us now consider the multiobjective optimization problem w.r.t. the partial order induced by the pointed, closed and convex cone C:

$$\begin{cases} C - \text{Minimize } f(x) \\ \text{subject to : } x \in E, \end{cases} \tag{2.2}$$

where f represents a vector-valued function and E the nonempty feasible set. Recall that for a nonempty set $A \subset E$, the image of A by f is defined by

$$f(A) := \{f(x) | \, x \in A\}.$$

A point $\overline{x} \in E$ is said to be an efficient (resp. weak efficient) optimal solution of problem (2.2) if $f(\overline{x})$ is a Pareto (resp. weak Pareto) minimal vector (2.1) of $f(E)$. The point $\overline{x} \in E$ is a local efficient (resp. weak local efficient) solution of problem (2.2) if there exists a neighborhood V of $\overline{x}$ such that $f(\overline{x})$ is a Pareto (resp. weak Pareto) minimal vector of $f(E \cap V)$.

To close this section, let us mention that a vector-valued function $f : \mathbb{R}^a \to \mathbb{R}^b$ will be said to be C-convex, that is, convex w.r.t. a partial order $\preceq_C$ induced by a pointed, closed convex cone C, if we have:

$$f(\lambda x_1 + (1-\lambda)x_2) \preceq_C \lambda f(x_1) + (1-\lambda) f(x_2), \, \forall x_1, x_2 \in \mathbb{R}^a, \, \forall \lambda \in (0,1).$$

It should be clear that by definition, $x \preceq_C y \Leftrightarrow y - x \in C$. Details on the above material and more generally on multiobjective optimization can be found in the book by Ehrgott [9] and references therein.

2.2 Tools from variational analysis

The material presented here is essentially taken from [12, 17]. We start with the Kuratowski-Painlevée outer/upper limit of a set-valued mapping $\Xi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, which is defined at a point $\overline{x}$ as

$$\operatorname*{Limsup}_{x \to \overline{x}} \Xi(x) := \{v \in \mathbb{R}^m | \exists x_k \to \overline{x}, \, v_k \to v \text{ with } v_k \in \Xi(x_k) \text{ as } k \to \infty\}. \tag{2.3}$$

For an extended real-valued function $\psi : \mathbb{R}^n \to \overline{\mathbb{R}}$, the Fréchet subdifferential of ψ at a point $\overline{x}$ of its domain is given by

$$\widehat{\partial}\psi(\overline{x}) := \left\{ v \in \mathbb{R}^n | \liminf_{x \to \overline{x}} \frac{\psi(x) - \psi(\overline{x}) - \langle v, x - \overline{x} \rangle}{\|x - \overline{x}\|} \geq 0 \right\}$$

whereas the basic/Mordukhovich subdifferential of ψ is the Kuratowski-Painlevée upper limit of the set-valued mapping $\widehat{\partial}\psi$ at $\overline{x}$:

$$\partial \psi(\overline{x}) := \operatorname*{Limsup}_{x \to \overline{x}} \widehat{\partial}\psi(x).$$

If ψ is convex, then $\partial\psi(\overline{x})$ reduces to the subdifferential in the sense of convex analysis, that is

$$\partial\psi(\overline{x}) := \{v \in \mathbb{R}^n |\, \psi(x) - \psi(\overline{x}) \geq \langle v, x - \overline{x}\rangle, \forall x \in \mathbb{R}^n\}. \tag{2.4}$$

For a local Lipschitz continuous function, $\partial\psi(\overline{x})$ is nonempty and compact. Moreover, its convex hull is the subdifferential of Clarke, that is, one can define the Clarke subdifferential $\overline{\partial}\psi(\overline{x})$ of ψ at $\overline{x}$ by

$$\overline{\partial}\psi(\overline{x}) := \operatorname{co} \partial\psi(\overline{x}) \tag{2.5}$$

where "co" stands for the convex hull of the set in question. Thanks to this link between the Mordukhovich and Clarke subdifferentials, we have the following convex hull property which plays and important role in this paper:

$$\operatorname{co} \partial(-\psi)(\overline{x}) = -\operatorname{co} \partial\psi(\overline{x}). \tag{2.6}$$

For this equality to hold, ψ should be Lipschitz continuous near $\overline{x}$. The *partial* basic (resp. Clarke) subdifferential of ψ w.r.t. x is defined by

$$\partial_x\psi(\overline{x}, \overline{y}) := \partial\psi(., \overline{y})(\overline{x}) \text{ (resp. } \overline{\partial}_x\psi(\overline{x}, \overline{y}) := \overline{\partial}\psi(., \overline{y})(\overline{x})).$$

The partial subdifferentials w.r.t. the variable y can be defined analogously.

We now introduce the basic/Mordukhovich normal cone to a set $\Omega \subset \mathbb{R}^n$ at one of its points $\overline{x}$

$$N_\Omega(\overline{x}) := \operatorname*{Limsup}_{x \to \overline{x}\, (x \in \Omega)} \widehat{N}_\Omega(\overline{x}) \tag{2.7}$$

where $\widehat{N}_\Omega(\overline{x})$ denotes the prenormal/Fréchet normal cone to Ω at $\overline{x}$ defined by

$$\widehat{N}_\Omega(\overline{x}) := \left\{ v \in \mathbb{R}^n |\, \limsup_{x \to \overline{x} (x \in \Omega)} \frac{\langle v, x - \overline{x}\rangle}{\|x - \overline{x}\|} \leq 0 \right\}$$

and "Limsup" stands for the Kuratowski-Painlevée upper limit defined in (2.3). The set Ω will be said to be regular at a point $\overline{x} \in \Omega$ if we have $N_\Omega(\overline{x}) = \widehat{N}_\Omega(\overline{x})$.

A set-valued mapping $\Xi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ will be said to be inner semicompact at a point $\overline{x}$, with $\Xi(\overline{x}) \neq \emptyset$, if for every sequence $x_k \to \overline{x}$ with $\Xi(x_k) \neq \emptyset$, there is a sequence of $y_k \in \Xi(x_k)$ that contains a convergent subsequence as $k \to \infty$. It follows that the inner semicompactness holds whenever Ξ is uniformly bounded around $\overline{x}$, i.e. there exists a neighborhood U of $\overline{x}$ and a bounded set $\Omega \subset \mathbb{R}^m$ such that

$$\Xi(x) \subset \Omega, \text{ for all } x \in U.$$

The mapping Ξ is inner semicontinuous at $(\overline{x}, \overline{y}) \in \operatorname{gph} \Xi$ if for every sequence $x_k \to \overline{x}$ there is a sequence of $y_k \in \Xi(x_k)$ that converges to $\overline{y}$ as $k \to \infty$. Obviously, if Ξ is inner semicompact at $\overline{x}$ with $\Xi(\overline{x}) = \{\overline{y}\}$, then Ξ is inner semicontinuous at $(\overline{x}, \overline{y})$. In general though, the inner semicontinuity is a property much stronger than the inner semicompactness and it is a necessary condition for the Lipschitz-like property to hold. For the definition of the latter notion and further details on these properties, see [12] and references therein.

A closed set Ω will be said to be semismooth at $\overline{x} \in \Omega$ if for any sequence $x_k \to \overline{x}$ with $x_k \in \Omega$ and $(x_k - \overline{x})\|x_k - \overline{x}\|^{-1} \to d$, it holds that $\langle \overline{x}_k^*, d\rangle \to 0$ for all selections $\overline{x}_k^* \in \overline{\partial} d_\Omega(x_k)$. Here, $d_\Omega(.)$ denotes the Euclidean distance from a given point to the set Ω. An example of semismooth set is the convex set [11]. For more on the semismoothness, the interested reader is referred to the latter paper.

3 Reformulation of the problem

Our aim in this section is to discuss the reformulation process of problem (1.1) into a single-level optimization problem with inequality constraints. We start by recalling that the notion of optimal solution for the upper-level problem is in the usual sense, that is, $(\overline{x}, \overline{z})$ is said to be a local optimal solution for problem (1.1) if and only if there exists a neighborhood U of this point such that

$$F(x, z) - F(\overline{x}, \overline{z}) \geq 0, \text{ for all } (x, y) \in U \cap \{(x, y) | \, x \in X, \, z \in \Psi_{\text{wef}}(x)\}.$$

The point $(\overline{x}, \overline{z})$ will be a global solution if U can be taken as large as possible. As for the lower-level problem, fix $x := \overline{x}$, according to (2.1), a point $\overline{z} \in \Psi_{\text{wef}}(\overline{x})$ if and only if

$$f(\overline{x}, z) - f(\overline{x}, \overline{z}) \notin -\text{int}\mathbb{R}^l_+, \text{ for all } z \in K(x).$$

This corresponds to the notion of global weak efficient solutions needed in this paper since it is a usual thing in bilevel programming to consider only global solutions at the lower-level. However, local Pareto optimal solutions can be defined analogously.

One way to transform the lower-level problem (1.2) into a usual one-level optimization problem is the so-called *scalarization technique*, which consists to solve the following further parameterized problem:

$$\begin{cases} \underset{z}{\text{Minimize}} \; f(x, y, z) := \langle y, f(x, z) \rangle \\ \text{subject to : } z \in K(x), \end{cases} \tag{3.1}$$

where the new parameter vector y is a nonnegative point of the unit sphere, that is, y belongs to

$$Y := \{y \in \mathbb{R}^l | \, y \geq 0, \, \|y\| = 1\}. \tag{3.2}$$

Since it is a difficult task to choose the best point $z(x)$ on the Parato front for a given upper-level strategy x, our approach in this paper consists to consider the set Y (3.2) as a new constraint set for the upper-level problem. To proceed in this way, denote by $\Psi(x, y)$ the solution set of problem (3.1) in the usual sense, for any given parameter couple $(x, y) \in X \times Y$. When weakly efficient solutions w.r.t. $\mathbb{R}^l_+$ are considered for the lower-level problem (1.2), the following relationship (see e.g. [9]) relates the solution set of this problem and that of (3.1):

Theorem 3.1 (link between solutions of problems (1.2) and (3.1))**.** *Assume that the set-valued mapping* $K : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ *is* $\mathbb{R}^p_+$*-convex and the function* $f(x, .)$ *is* $\mathbb{R}^l_+$*-convex for all* $x \in X$*. Then we have:*

$$\Psi_{\text{wef}}(x) = \Psi(x, Y) := \bigcup \{\Psi(x, y) | \, y \in Y\}. \tag{3.3}$$

Hence, the semivectorial bilevel optimization problem (1.1) can be replaced by the following bilevel optimization problem of the classical form:

$$\begin{cases} \underset{x,y,z}{\text{Minimize}} \; F(x, z) \\ \text{subject to : } \; (x, y) \in X \times Y, \, z \in \Psi(x, y) \end{cases} \tag{3.4}$$

where the restriction (3.2) on the new parameter of the lower-level problem acts like additional upper-level constraints. The link between problems (1.1) and (3.4) will be formalized in the next result. For this, note that a set-valued mapping $\Xi : \mathbb{R}^a \rightrightarrows \mathbb{R}^b$ is closed at a point $(u, v) \in \mathbb{R}^a \times \mathbb{R}^b$ if for any sequence $(u^k, v^k) \in \text{gph}\,\Xi$ with $(u^k, v^k) \to (u, v)$, one has $v \in \Xi(u)$. Ξ is said to be closed if it is closed at any point of $\mathbb{R}^a \times \mathbb{R}^b$.

Proposition 3.2 (link between problem (1.1) and problem (3.4))**.** *Consider problem* (1.1)-(1.2)*, where* $K : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ *is* $\mathbb{R}^p_+$*-convex and* $f(x, .)$ *is* $\mathbb{R}^l_+$*-convex for all* $x \in X$*. Then, the following assertions hold:*

(i) *Let* $(\overline{x}, \overline{z})$ *be a local* (*resp. global*) *optimal solution of problem* (1.1)*. Then, for all* $\overline{y} \in Y$ *with* $\overline{z} \in \Psi(\overline{x}, \overline{y})$*, the point* $(\overline{x}, \overline{y}, \overline{z})$ *is a local* (*resp. global*) *optimal solution of problem* (3.4)*.*

(ii) *Let* $(\overline{x}, \overline{y}, \overline{z})$ *be a local* (*resp. global*) *optimal solution of problem* (3.4) *and assume the set-valued mapping* Ψ *is closed. Then,* $(\overline{x}, \overline{z})$ *is a local* (*resp. global*) *optimal solution of problem* (1.1)*.*

Proof. We provide the proofs of (i) and (ii) in the local cases. The global ones can be obtained easily.

(i) Let $(\overline{x}, \overline{z})$ be a local optimal solution of problem (1.1) and assume there exists $y^o \in Y$ with $\overline{z} \in \Psi(\overline{x}, y^o)$ such that $(\overline{x}, y^o, \overline{z})$ is not a local optimal solution of (3.4). Then, there exists a sequence (x^k, y^k, z^k) with $x^k \to \overline{x}$, $y^k \to y^o$, $z^k \to \overline{z}$, and $(x^k, y^k) \in X \times Y$, $z^k \in \Psi(x^k, y^k)$ such that $F(x^k, z^k) < F(\overline{x}, \overline{z})$. By equality (3.3), $[y^k \in Y,\ z^k \in \Psi(x^k, y^k)] \Rightarrow z^k \in \Psi_{\text{wef}}(x^k)$. Hence, there exists a sequence $(x^k, z^k) \to (\overline{x}, \overline{z})$ with $x^k \in X$, $z^k \in \Psi_{\text{wef}}(x^k)$ such that $F(x^k, z^k) < F(\overline{x}, \overline{z})$, which contradicts the initial statement that $(\overline{x}, \overline{z})$ is a local optimal solution of problem (1.1), given that $\overline{x} \in X$ (since X is closed) and $[\overline{y} \in Y,\ \overline{z} \in \Psi(\overline{x}, y^o)] \Rightarrow \overline{z} \in \Psi_{\text{wef}}(\overline{x})$ by equality (3.3).

(ii) Assume that $(\overline{x}, \overline{y}, \overline{z})$ is a local optimal solution of problem (3.4), but $(\overline{x}, \overline{z})$ is not a local optimal solution of problem (1.1). Then, there is a sequence $(x^k, z^k) \in \operatorname{gph} \Psi_{\text{wef}}$ with $x^k \in X$ and $(x^k, z^k) \to (\overline{x}, \overline{z})$ such that $F(x^k, z^k) < F(\overline{x}, \overline{z})$. Now consider the set-valued mapping:

$$\Phi(x, z) := \{y \in Y |\, z \in \Psi(x, y)\}$$

and observe that for any (x, z), we have $\Phi(x, z) \subset \mathbb{B}(0, 1)$ (unit ball of $\mathbb{R}^l$). Hence, Φ is uniformly bounded, thus inner semicompact at any point, cf. Section 2.2 (2.2). Since $(x^k, z^k) \in \operatorname{gph} \Psi_{\text{wef}}$, it follows from equality (3.3) that $\Phi(x^k, z^k) \neq \emptyset$ and with $(x^k, z^k) \to (\overline{x}, \overline{z})$, the inner semicompactness of Φ implies that, there exists a sequence $y^k \in \Phi(x^k, z^k)$, which has an accumulation point y^o with $y^o \in Y$ (given that Y is a closed set). Taking into account that Ψ is closed, we have $\overline{z} \in \Psi(\overline{x}, y^o)$. Combining all these facts, we have that there is a sequence $(x^k, y^k, z^k) \to (\overline{x}, y^o, \overline{z})$ with (x^k, y^k, z^k) feasible to problem (3.4) but with $F(x^k, z^k) < F(\overline{x}, \overline{z})$, where $(\overline{x}, y^o, \overline{z})$ is also a feasible point of problem (3.4) (since $\overline{x} \in X$, given that X is closed). This contradicts the fact that $(\overline{x}, \overline{y}, \overline{z})$ is a local optimal solution of problem (3.4). $\square$

Based on this result, we will attempt to derive necessary optimality conditions of the bilevel problem (1.1), by deriving those of the auxiliary problem (3.4). A now classical way to convert the latter problem into an optimization problem with more tractable constraints is the so-called optimal value reformulation

$$\begin{cases} \underset{x,y,z}{\text{Minimize}}\ F(x, z) \\ \quad \text{subject to :} \quad f(x, y, z) \leq \varphi(x, y) \\ \qquad\qquad\qquad\quad (x, y) \in X \times Y,\ z \in K(x), \end{cases} \tag{3.5}$$

introduced by Outrata [16]. This follows obviously from the fact that

$$\Psi(x, y) := \arg\min_{z} \{f(x, y, z) |\, z \in K(x)\} = \{z \in K(x) |\, f(x, y, z) \leq \varphi(x, y)\}, \tag{3.6}$$

where φ denotes the optimal value function of the scalarized lower-level problem (3.1):

$$\varphi(x, y) := \min_{z} \{\langle y, f(x, z)\rangle |\, z \in K(x)\}. \tag{3.7}$$

Other approaches like the generalized equation or Karush-Kuhn-Tucker reformulations can be considered for problem (3.4) as well, see for example [6, 15, 19] for details. But we focus our attention here on reformulation (3.5) in order to develop necessary optimality conditions for problem (1.1).

Remark 3.3 (The scalarization approach in previous works)**.** The scalarization approach used above for the multiobjective optimization problem (1.2) was also used by Bonnel and Morgan [1, 3]. In [1], the set-valued mapping Ψ (3.6) was then inserted in the upper-level objective function F and the notion of contingent derivative applied to derive optimality conditions for the resulting set-valued optimization problem. However, in the next section we investigate necessary optimality for problem (1.1) via problem (3.4) while using the link established in Proposition 3.2 (i). In [3], the same scalarization technique is used to reformulate a Semivectorial bilevel optimal control problem. An existence result is then provided for the resulting counterpart of problem (3.4).

4 Necessary optimality conditions

In this section, we derive necessary optimality conditions for the optimal value reformulation (3.5) of problem (1.1). To proceed, we first set

$$\mathcal{G}(x,y,z) := f(x,y,z) - \varphi(x,y) \text{ and } \Omega := \{(x,y,z) | (x,y) \in X \times Y,\ z \in K(x)\}.$$

Then, problem (3.5) takes the following much simpler form with an abstract constraint:

$$\begin{cases} \underset{x,y,z}{\text{Minimize}}\ F(x,z) \\ \text{subject to}:\ (x,y,z) \in \Omega,\ \mathcal{G}(x,y,z) \leq 0. \end{cases} \tag{4.1}$$

In order to apply the approach of [8], we consider the following weak form of the well-known basic constraint qualification (CQ)

$$\partial \mathcal{G}(\overline{x},\overline{y},\overline{z}) \cap -\text{bd}\, N_\Omega(\overline{x},\overline{y},\overline{z}) = \emptyset. \tag{4.2}$$

Here "bd" stands for the topological boundary of the set in question. If we drop this boundary from the normal cone N_Ω, one obtains the basic CQ introduced by Mordukhovich, see e.g. [12]. The latter condition can not be satisfied for problem (4.1) [8]. The weak basic CQ (4.2) emerged from [11] in the framework of the calmness property for set-valued mappings, and it was shown in [8] to work for the optimal value reformulation of bilevel optimization problem, in particular when the simple bilevel programming problem is considered.

It should be clear that CQ (4.2) is not enough to derive a completely detailed set of optimality conditions for problem (1.1) in terms of initial data. Before introducing further CQs that would be needed, we first state in the next Lemma the simple fact that the Mangasarian-Fromovitz constraint qualification (MFCQ) is automatically satisfied for any point of Y (3.2).

Lemma 4.1. *The MFCQ is satisfied for any point $y \in Y$.*

This lemma implies that the additional constraint set Y (3.2) does not induce any new constraint qualification (CQ) apart from (4.2) and the now often used lower- and upper-level regularity conditions labeled as such in [5], which are defined respectively as:

$$\left.\begin{array}{r} \sum_{i=1}^{p} \beta_i \nabla_z g_i(\overline{x},\overline{z}) = 0 \\ \beta_i \geq 0, \beta_i g_i(\overline{x},\overline{z}) = 0, i = 1,\dots,p \end{array}\right\} \Longrightarrow \beta_i = 0,\ i = 1,\dots,p. \tag{4.3}$$

$$\left.\begin{array}{r} \sum_{j=1}^{q} \alpha_j \nabla G_j(\overline{x}) = 0 \\ \alpha_j \geq 0, \alpha_j G_j(\overline{x}) = 0, j = 1,\dots,q \end{array}\right\} \Longrightarrow \alpha_j = 0,\ j = 1,\dots,q. \tag{4.4}$$

Clearly, these are the dual forms of the MFCQ for the lower-level constraints $g_i(\overline{x}, y) \leq 0$, $i = 1, \ldots, p$ (for a fixed parameter $x := \overline{x}$) and the upper-level constraint system $G_j(x) \leq 0$, $j = 1, \ldots, q$, respectively.

In the next lemma, we highlight another particularity of the new constraint set Y (3.2), that is, the fact that the related Lagrange multipliers can be completely eliminated from the optimality conditions. This point will be more clear in our main results.

Lemma 4.2. *The set of vectors* $(\overline{x}, \overline{y}, \overline{z}) \in \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m$, $\gamma, z_s \in \mathbb{R}^l$ *and* $\mu, r, v_s \in \mathbb{R}$ *with* $s = 1, \ldots, n+l+1$ *satisfies the system*

$$\begin{cases} rf(\overline{x}, \overline{z}) - r\sum_{s=1}^{n+l+1} v_s f(\overline{x}, z_s) - \gamma + \mu.\overline{y} = 0 \\ \gamma \geq 0, \ \gamma^\top \overline{y} = 0, \ \|\overline{y}\| = 1 \end{cases} \tag{4.5}$$

if and only if the following inequality is satisfied:

$$r\left\{\Big[\sum_{k=1}^{l} y_k\Big(f_k(\overline{x}, \overline{z}) - \sum_{s=1}^{n+l+1} v_s f_k(\overline{x}, z_s)\Big)\Big].\overline{y} - \big[f(\overline{x}, \overline{z}) - \sum_{s=1}^{n+l+1} v_s f(\overline{x}, z_s)\big]\right\} \leq 0. \tag{4.6}$$

Proof. We have from the first equality of (4.5) that

$$\gamma = rf(\overline{x}, \overline{z}) - r\sum_{s=1}^{n+l+1} v_s f(\overline{x}, z_s) + \mu.\overline{y}. \tag{4.7}$$

Inserting this value of γ in the equation $\gamma^\top \overline{y} = 0$, we have that

$$\mu = -r\overline{y}^\top \Big[f(\overline{x}, \overline{z}) - \sum_{s=1}^{n+l+1} v_s f(\overline{x}, z_s))\Big] = -r\sum_{i=1}^{l} y_i\Big(f_i(\overline{x}, \overline{z}) - \sum_{s=1}^{n+l+1} v_s f_i(\overline{x}, z_s)\Big)$$

taking into account the fact that $\|y\| = 1$. Inserting the latter value of μ in γ (4.7), while noting that $\gamma \geq 0$, one has the result. □

We are now ready to state one of the main results of this paper, which provides necessary optimality conditions for the auxiliary problem (4.1). We first concentrate on the case where all the functions are strictly differentiable. The proof technique is exactly that of [8, Theorem 3.5].

Theorem 4.3 (optimality conditions for problem (4.1) when the functions are strictly differentiable)**.** *Let* $(\overline{x}, \overline{y}, \overline{z})$ *be a local optimal solution of problem* (4.1)*, where the functions* f *and* g *are strictly differentiable at* $(\overline{x}, z)$*,* $z \in \Psi(\overline{x}, \overline{y})$*, whereas* F *and* G *are strictly differentiable at* $(\overline{x}, \overline{z})$ *and* $\overline{x}$*, respectively. Assume that the solution set-valued mapping* Ψ (3.6) *is inner semicompact at* $(\overline{x}, \overline{y})$ *while for all* $z \in \Psi(\overline{x}, \overline{y})$*, the point* $(\overline{x}, z)$ *is lower-level regular* (4.3)*. Furthermore, let the set* Ω *be regular and semismooth at* $(\overline{x}, \overline{y}, \overline{z})$*, while the point* $\overline{x}$ *is upper-level regular* (4.4) *and the weak basic CQ* (4.2) *is satisfied at* $(\overline{x}, \overline{y}, \overline{z})$*. Then, there exist* $r \geq 0$*,* α, β*,* β^s, v_s *and* $z_s \in \Psi(\overline{x}, \overline{y})$*, with* $s = 1, \ldots, n+l+1$ *such that relationship* (4.6) *together with the following conditions are* *satisfied:*

$$\nabla_x F(\overline{x},\overline{z}) + r\sum_{k=1}^{l}\overline{y}_k\nabla_x f_k(\overline{x},\overline{z}) + \sum_{j=1}^{q}\alpha_j\nabla G_j(\overline{x}) + \sum_{i=1}^{p}\beta_i\nabla_x g_i(\overline{x},\overline{z})$$

$$-r\sum_{s=1}^{n+l+1} v_s\Big(\sum_{k=1}^{l}\overline{y}_k\nabla_x f_k(\overline{x},z_s) + \sum_{i=1}^{p}\beta_i^s\nabla_x g_i(\overline{x},z_s)\Big) = 0, \tag{4.8}$$

$$\nabla_z F(\overline{x},\overline{z}) + r\sum_{k=1}^{l}\overline{y}_k\nabla_z f_k(\overline{x},\overline{z}) + \sum_{i=1}^{p}\beta_i\nabla_z g_i(\overline{x},\overline{z}) = 0, \tag{4.9}$$

$$\forall s = 1,\dots,n+l+1,\ \sum_{k=1}^{l}\overline{y}_k\nabla_z f_k(\overline{x},z_s) + \sum_{i=1}^{p}\beta_i^s\nabla_z g_i(\overline{x},z_s) = 0, \tag{4.10}$$

$$\forall s = 1,\dots,n+l+1,\ i = 1,\dots,p,\ \beta_i^s \geq 0, \beta_i^s g_i(\overline{x},z_s) = 0, \tag{4.11}$$

$$\forall j = 1,\dots,q,\ \alpha_j \geq 0, \alpha_j G_j(\overline{x}) = 0, \tag{4.12}$$

$$\forall i = 1,\dots,p,\ \beta_i \geq 0, \beta_i g_i(\overline{x},\overline{z}) = 0, \tag{4.13}$$

$$\forall s = 1,\dots,n+l+1, v_s \geq 0,\ \sum_{s=1}^{n+l+1} v_s = 1. \tag{4.14}$$

Proof. Under the assumptions of the theorem, it follows from [8, Theorem 3.1] that there exists $r \geq 0$ such that

$$0 \in \nabla_{x,y,z}F(\overline{x},\overline{z}) + r\partial\mathcal{G}(\overline{x},\overline{y},\overline{z}) + N_\Omega(\overline{x},\overline{y},\overline{z}). \tag{4.15}$$

What simply remains to be done is evaluating the basic subdifferential of $\mathcal{G}$ and the basic normal cone to Ω. Starting with Ω, let us note that it can be reformulated as:

$$\Omega = \{(x,y,z)|\, a(x,y,z) \leq 0,\ b(x,y,z) = 0\},$$

with $a(x,y,z) := [G(x), g(x,z), -y]^\top$ and $b(x,y,z) := \|y\| - 1$. Applying Theorem 6.14 in [17], while performing some calculations, one obtains

$$N_\Omega(\overline{x},\overline{y},\overline{z}) \subset \left\{ \begin{bmatrix} \sum_{j=1}^{q}\alpha_j\nabla G_j(\overline{x}) + \sum_{i=1}^{p}\beta_i\nabla_x g_i(\overline{x},\overline{z}) \\ -\gamma + \mu\overline{y} \\ \sum_{i=1}^{p}\beta_i\nabla_z g_i(\overline{x},\overline{z}) \end{bmatrix} \left| \begin{array}{l} \alpha_j \geq 0, \alpha_j G_j(\overline{x}) = 0,\ j = 1,\dots,q \\ \beta_i \geq 0, \beta_i g_i(\overline{x},\overline{z}) = 0,\ i = 1,\dots,p \\ \gamma_k \geq 0,\ \gamma_k\overline{y}_k = 0,\ k = 1,\dots,l \end{array} \right. \right\}. \tag{4.16}$$

provided the following CQ holds

$$\big[\nabla a(\overline{x},\overline{y},\overline{z})^\top u + \nabla b(\overline{x},\overline{y},\overline{z})^\top v = 0,\ u \geq 0, u^\top a(\overline{x},\overline{y},\overline{z}) = 0\big] \Longrightarrow u = 0,\ v = 0.$$

Considering Lemma 4.1, it is a simple exercise to check that the fulfilment of both the lower-level (4.3) and upper-level (4.4) regularity conditions implies the satisfaction of the latter implication.

As far as the function $\mathcal{G}$ is concerned, one can easily check that

$$\partial\mathcal{G}(\overline{x},\overline{y},\overline{z}) \subset \begin{bmatrix} \sum_{k=1}^{l}\overline{y}_k\nabla_x f_k(\overline{x},\overline{z}) \\ f(\overline{x},\overline{z}) \\ \sum_{k=1}^{l}\overline{y}_k\nabla_z f_k(\overline{x},\overline{z}) \end{bmatrix} + \partial(-\varphi)(\overline{x},\overline{y}) \times \{0\}. \tag{4.17}$$

For the estimation of the basic subdifferential of $-\varphi$, first note that since the solution set-valued mapping Ψ is inner semicompact at $(\overline{x}, \overline{y})$ and for all $z \in \Psi(\overline{x}, \overline{y})$, the point $(\overline{x}, z)$ is lower-level regular (4.3), then it follows from [14, Theorem 7] that we have

$$\partial\varphi(\overline{x}, \overline{y}) \subset \bigcup_{z\in\Psi(\overline{x},\overline{y})} \bigcup_{\beta\in\Lambda(\overline{x},\overline{y},z)} \left\{ \begin{bmatrix} \sum_{k=1}^{l} \overline{y}_k \nabla_x f_k(\overline{x}, z) + \sum_{i=1}^{p} \beta_i \nabla_x g_i(\overline{x}, z) \\ f(\overline{x}, z) \end{bmatrix} \right\}. \tag{4.18}$$

Here, the lower-level Lagrange multipliers set $\Lambda(\overline{x}, \overline{y}, \overline{z})$ is given by:

$$\begin{aligned} \Lambda(\overline{x}, \overline{y}, \overline{z}) := \{\beta \in \mathbb{R}^p | \quad & \beta_i \geq 0,\ \beta_i g_i(\overline{x}, \overline{z}) = 0,\ i = 1, \ldots, p \\ & \textstyle\sum_{k=1}^{l} \overline{y}_k \nabla_z f_k(\overline{x}, \overline{z}) + \sum_{i=1}^{p} \beta_i \nabla_z g_i(\overline{x}, \overline{z}) = 0\}. \end{aligned} \tag{4.19}$$

The result then follows from a combination of (4.15)-(4.19) while using Carathédory's Theorem to compute an element of $-\text{co}\, \partial\varphi(\overline{x}, \overline{y}) \supseteq \partial(-\varphi)(\overline{x}, \overline{y})$. □

Remark 4.4 (optimality conditions for (4.1) in the smooth case under the inner semicontinuity of Ψ)**.** In case one replaces the inner semicompactness of Ψ (3.6) in the above theorem by the stronger inner semicontinuity, the optimality conditions obtained correspond to those of Theorem 4.3 when $\Psi(\overline{x}, \overline{y}) = \{\overline{z}\}$ and $\Lambda(\overline{x}, \overline{y}, \overline{z}) = \{\gamma\}$. More formally, if we replace the inner semicompactness of Ψ at $(\overline{x}, \overline{y})$ by its inner semicontinuity at $(\overline{x}, \overline{y}, \overline{z})$ while assuming the satisfaction of the lower level regularity only at $(\overline{x}, \overline{z})$, then there exist $r \geq 0$, α, β, γ such that (4.9), (4.12), (4.13) and the following conditions are satisfied:

$$\nabla_x F(\overline{x}, \overline{z}) + \sum_{j=1}^{q} \alpha_j \nabla_x G_j(\overline{x}) + \sum_{i=1}^{p} (\beta_i - r\gamma_i) \nabla_x g_i(\overline{x}, \overline{z}) = 0, \tag{4.20}$$

$$\sum_{k=1}^{l} \overline{y}_k \nabla_z f_k(\overline{x}, \overline{z}) + \sum_{i=1}^{p} \gamma_i \nabla_z g_i(\overline{x}, \overline{z}) = 0, \tag{4.21}$$

$$i = 1, \ldots, p,\ \gamma_i \geq 0, \gamma_i g_i(\overline{x}, \overline{z}) = 0. \tag{4.22}$$

This is due to the fact that if the solution set-valued mapping Ψ is inner semicontinuous at $(\overline{x}, \overline{y}, \overline{z})$ and the point $(\overline{x}, \overline{z})$ is lower-level regular (4.3), then the value function φ is also Lipschitz continuous near $(\overline{x}, \overline{y})$ and the Clarke subdifferential of φ can be estimated as [13]:

$$\overline{\partial}\varphi(\overline{x}, \overline{y}) \subset \bigcup_{\beta\in\Lambda(\overline{x},\overline{y},\overline{z})} \left\{ \begin{bmatrix} \sum_{k=1}^{l} \overline{y}_k \nabla_x f_k(\overline{x}, \overline{z}) + \sum_{i=1}^{p} \beta_j \nabla_x g_i(\overline{x}, \overline{z}) \\ f(\overline{x}, \overline{z}) \end{bmatrix} \right\}.$$

One can easily check that the above necessary optimality conditions of problem (4.1) (under the inner semicontinuity of the lower-level solution set-valued mapping Ψ (3.6)) are in fact those of the problem:

$$\begin{cases} \underset{x,y,z}{\text{Minimize}}\ F(x, z) \\ \text{subject to}:\ x \in X,\ z \in \Psi(x, y). \end{cases} \tag{4.23}$$

This means that under the above framework, the constraints described by Y (3.2) can be dropped while deriving the necessary optimality conditions of problem (1.1), which is a strange phenomenon.

In the next result, we extend the result in Theorem 4.3 to the case where the functions involved in (1.1) are locally Lipschitz continuous. To proceed, we need the following nonsmooth counterparts of the lower- and upper-level regularity conditions defined respectively as:

$$\left. \begin{array}{r} \sum_{i=1}^{p} \beta_i x_i^* = 0 \\ \beta_i \geq 0,\ \beta_i g_i(\overline{x}, \overline{z}) = 0,\ i = 1, \ldots, p \\ (x_i^*, z_i^*) \in \overline{\partial} g_i(\overline{x}, \overline{z}),\ i = 1, \ldots, p \end{array} \right\} \Longrightarrow \beta_i = 0,\ i = 1, \ldots, p, \tag{4.24}$$

$$\left.\begin{array}{r}0\in\sum_{j=1}^{q}\alpha_j\partial G_j(\overline{x})\\ \alpha_j\geq 0,\ \alpha_jG_j(\overline{x})=0,\ j=1,\ldots,q\end{array}\right\}\Longrightarrow\alpha_j=0,\ j=1,\ldots,q. \tag{4.25}$$

Theorem 4.5 (optimality conditions for (4.1) in the case where the functions are Lipschitzian)**.** *Let $(\overline{x},\overline{y},\overline{z})$ be a local optimal solution of problem* (4.1)*, where the functions F and $G_j, j=1,\ldots,q$ are Lipschitz continuous around $(\overline{x},\overline{z})$ and $\overline{x}$, respectively. Assume that Ω is regular and semismooth at $(\overline{x},\overline{y},\overline{z})$, where the weak basic CQ* (4.2) *is also satisfied, and let $\overline{x}$ be upper-level regular in the sense of* (4.25)*. Then, the following assertions hold:*

(i) *Let the solution set-valued mapping Ψ* (3.6) *be inner semicompact at $(\overline{x},\overline{y})$ while for all $z\in\Psi(\overline{x},\overline{y})$, the functions f_k, $k=1,\ldots,l$ and g_i, $i=1,\ldots,p$ are Lipschitz continuous near the point $(\overline{x},z)$, where the lower-level regularity in the sense of* (4.24) *is also satisfied. Then, there exist $(x_F^*,z_F^*)\in\partial F(\overline{x},\overline{z})$, $r\geq 0$, α,β,β^s,v_s and $z_s\in\Psi(\overline{x},\overline{y})$, with $s=1,\ldots,n+l+1$ such that relationships* (4.6) *and* (4.11)-(4.14)*, together with the following conditions are satisfied:*

$$x_F^*+r\sum_{k=1}^{l}\overline{y}_kx_k^*-r\sum_{s=1}^{n+l+1}v_s\Big(\sum_{k=1}^{l}\overline{y}_kx_{ks}^*+\sum_{i=1}^{p}\beta_{is}u_{is}^*\Big)+\sum_{i=1}^{p}\beta_iu_i^*+\sum_{j=1}^{q}\alpha_jx_{Gj}^*=0, \tag{4.26}$$

$$z_F^*+r\sum_{k=1}^{l}\overline{y}_kz_k^*+\sum_{i=1}^{p}\beta_iv_i^*=0, \tag{4.27}$$

$$\text{for } s=1,\ldots,n+l+1,\ \sum_{k=1}^{l}\overline{y}_kz_{ks}^*+\sum_{i=1}^{p}\beta_iv_{is}^*=0, \tag{4.28}$$

where we have the following inclusions:

$$\text{for } j=1,\ldots,q,\ x_{Gj}^*\in\partial G_j(\overline{x}), \tag{4.29}$$

$$\text{for } k=1,\ldots,l,\ (x_k^*,z_k^*)\in\partial f_k(\overline{x},\overline{z}), \tag{4.30}$$

$$\text{for } i=1,\ldots,p,\ (u_i^*,v_i^*)\in\partial g_i(\overline{x},\overline{z}), \tag{4.31}$$

$$\text{for } k=1,\ldots,l;\ s=1,\ldots,n+l+1,\ (x_{ks}^*,z_{ks}^*)\in\partial f_k(\overline{x},z_s), \tag{4.32}$$

$$\text{for } i=1,\ldots,p;\ s=1,\ldots,n+l+1,\ (u_{is}^*,v_{is}^*)\in\partial g_i(\overline{x},z_s). \tag{4.33}$$

(ii) *Let the solution set-valued mapping Ψ* (3.6) *be inner semicontinous at $(\overline{x},\overline{y},\overline{z})$ while the functions f_k, $k=1,\ldots,l$ and g_i, $i=1,\ldots,p$ are Lipschitz continuous near $(\overline{x},\overline{z})$, where the lower-level regularity in the sense of* (4.24) *is also satisfied. Then, there exist $(x_F^*,z_F^*)\in\partial F(\overline{x},\overline{z})$, $r\geq 0$, α,β,γ such that relationships* (4.12), (4.13), (4.22), *and* (4.27)*, together with the following conditions are satisfied:*

$$0\in x_F^*+r\sum_{k=1}^{l}\overline{y}_kx_k^*-r\Big(\sum_{k=1}^{l}\overline{y}_k\widetilde{x}_k^*+\sum_{i=1}^{p}\gamma_i\widetilde{u}_i^*\Big)+\sum_{i=1}^{p}\beta_iu_i^*+\sum_{j=1}^{q}\alpha_jx_{Gj}^*=0, \tag{4.34}$$

$$\sum_{k=1}^{l}\overline{y}_k\widetilde{z}_k^*+\sum_{i=1}^{p}\gamma_i\widetilde{v}_i^*=0, \tag{4.35}$$

where inclusions (4.29) $-$ (4.31)*, together with the following ones hold:*

$$\text{for } k=1,\ldots,l,\ (\widetilde{x}_k^*,\widetilde{z}_k^*)\in\overline{\partial}f_k(\overline{x},\overline{z}), \tag{4.36}$$

$$\text{for } i=1,\ldots,p,\ (\widetilde{u}_i^*,\widetilde{v}_i^*)\in\overline{\partial}g_i(\overline{x},\overline{z}). \tag{4.37}$$

Proof. (i) Considering the Lipschitz continuity assumptions on the functions involved in problem (4.1), it follows as in the proof of Theorem 4.3 that under CQ (4.2), while considering [12, Proposition 5.3], we have

$$0\in\partial_{x,y,z}F(\overline{x},\overline{z})+r\partial f(\overline{x},\overline{y},\overline{z})+r\partial(-\varphi)(\overline{x},\overline{y})\times\{0\}+N_\Omega(\overline{x},\overline{y},\overline{z}), \tag{4.38}$$

taking into account the fulfilment of the lower-level regularity of $(\overline{x}, z)$, $z \in \Psi(\overline{x}, \overline{y})$ and the inner semicompactness of Ψ at $(\overline{x}, \overline{y})$, which both ensure the Lipschitz continuity of the value function φ near $(\overline{x}, \overline{y})$. As in the previous theorem, the only thing that remains to be done is estimating the basic subdifferential of f, $-\varphi$ and the basic normal cone to Ω. For the first term, note that since for all $k = 1, \ldots, l$, f_k is Lipschitz continuous near $(\overline{x}, z)$, $z \in \Psi(\overline{x}, \overline{y})$, then applying the basic subdifferential product rule of [12, Corollary 1.111] to $f(x, y, z) := \langle y, f(x, z)\rangle := \sum_k^l y_k f_k(x, z)$, we have the following inclusion after some calculations:

$$\partial f(\overline{x}, \overline{y}, \overline{z}) \subset f^o(\overline{x}, \overline{z}) + \Big\{ \sum_{k=1}^{l} \overline{y}_k (x_k^*, 0, z_k^*) | \, (x_k^*, z_k^*) \in \partial f_k(\overline{x}, \overline{z}),\, k = 1, \ldots, l \Big\}, \tag{4.39}$$

$$\text{where } f^o(\overline{x}, \overline{z}) := \big(\underbrace{0, \ldots, 0}_{n-\text{times}}, f(\overline{x}, \overline{z}), \underbrace{0, \ldots, 0}_{m-\text{times}}\big),$$

while taking into account that $\overline{y}_k \geq 0$ for $k = 1, \ldots, l$.

On the other hand, it follows from [14, Theorem 7(ii)] that under the assumptions in Theorem 4.5(i), we have

$$\partial\varphi(\overline{x}, \overline{y}) \subset \bigcup_{\overline{z} \in \Psi(\overline{x}, \overline{y})} \Big\{ (x^*, y^*) \big| \quad (x^*, y^*, 0) \in \partial f(\overline{x}, \overline{y}, \overline{z}) + \textstyle\sum_{i=1}^p \beta_i \partial_{x,y,z} g_i(\overline{x}, \overline{z}), \\ \beta_i \geq 0, \beta_i g_i(\overline{x}, \overline{z}) = 0, i = 1, \ldots, p \Big\}. \tag{4.40}$$

Combining inclusions (4.39) and (4.40) we have

$$\partial\varphi(\overline{x}, \overline{y}) \subset \bigcup_{\overline{z} \in \Psi(\overline{x}, \overline{y})} \left\{ \begin{bmatrix} \sum_{k=1}^l \overline{y}_k x_k^* + \sum_{i=1}^p \beta_i u_i^* \\ f(\overline{x}, \overline{z}) \end{bmatrix} \middle| \begin{array}{r} (x_k^*, z_k^*) \in \partial f_k(\overline{x}, \overline{z}),\, k = 1, \ldots, l, \\ (u_i^*, v_i^*) \in \partial g_i(\overline{x}, \overline{z}),\, i = 1, \ldots, p, \\ \sum_{k=1}^l \overline{y}_k z_k^* + \sum_{i=1}^p \beta_i v_i^* = 0, \\ \beta_i \geq 0, \beta_i g_i(\overline{x}, \overline{z}) = 0, i = 1, \ldots, p \end{array} \right\}. \tag{4.41}$$

Thus applying Carathédory's Theorem to $\operatorname{co} \varphi(\overline{x}, \overline{y})$, an upper estimate of the basic subdifferential of $-\varphi$ is obtained from the latter inclusion:

$$\partial(-\varphi)(\overline{x}, \overline{y}) \subset \left\{ \begin{bmatrix} -\sum_{s=1}^{n+l+1} v_s \Big(\sum_{k=1}^l \overline{y}_k x_{ks}^* + \sum_{i=1}^p \beta_i u_{is}^* \Big) \\ -\sum_{s=1}^{n+l+1} v_s f(\overline{x}, z_s) \end{bmatrix} \middle| \begin{array}{r} \sum_{s=1}^{n+l+1} v_s = 1 \text{ and } z_s \in \Psi(\overline{x}, \overline{y}),\, v_s \geq 0,\, s = 1, \ldots, n+l+1, \\ (x_{ks}^*, z_{ks}^*) \in \partial f_k(\overline{x}, z_s),\, k = 1, \ldots, l;\, s = 1, \ldots, n+l+1, \\ (u_{is}^*, v_{is}^*) \in \partial g_i(\overline{x}, z_s),\, i = 1, \ldots, p;\, s = 1, \ldots, n+l+1, \\ \sum_{k=1}^l \overline{y}_k z_{ks}^* + \sum_{i=1}^p \beta_i v_{is}^* = 0,\, s = 1, \ldots, n+l+1, \\ \beta_{is} \geq 0, \beta_{is} g_i(\overline{x}, z_s) = 0,\, i = 1, \ldots, p;\, s = 1, \ldots, n+l+1 \end{array} \right\}. \tag{4.42}$$

Now, considering Lemma 4.1 and [17, Theorem 6.14], the upper- and lower-level regularity conditions in the sense of (4.25) and (4.24), respectively, are sufficient to derive the following upper bound of the basic normal cone to Ω:

$$N_\Omega(\overline{x}, \overline{y}, \overline{z}) \subset \bigcup \left\{ \begin{bmatrix} \sum_{i=1}^p \beta_i u_i^* + \sum_{j=1}^q \alpha_j \partial G_j(\overline{x}) \\ -\gamma + \mu\overline{y} \\ \sum_{i=1}^p \beta_i v_i^* \end{bmatrix} \middle| \begin{array}{r} (u_i^*, v_i^*) \in \partial g_i(\overline{x}, \overline{z}),\, i = 1, \ldots, p, \\ \alpha_j \geq 0, \alpha_j G_j(\overline{x}) = 0,\, j = 1, \ldots, q \\ \beta_i \geq 0, \beta_i g_i(\overline{x}, \overline{z}) = 0,\, i = 1, \ldots, p \\ \gamma_k \geq 0,\, \gamma_k \overline{y}_k = 0,\, k = 1, \ldots, l \end{array} \right\}. \tag{4.43}$$

The result of Theorem 4.5(i) follows from the combination of (4.38)–(4.39) and (4.42)–(4.43), while noting that summing up only the middle terms (that is, the y-components) gives the system in (4.5), which is equivalent to (4.6) by Lemma 4.2.

(ii) The proof of this case is similar to the previous one with the difference that the convexification process of $\partial(-\varphi)$ in (4.42) can be avoided by applying [13, Theorem 5.9], which gives the following upper bound for the Clarke subdifferential of φ

$$\overline{\partial}\varphi(\overline{x},\overline{y}) \subset \bigcup \Big\{(x^*,y^*)\big| \begin{array}{l} (x^*,y^*,0) \in \overline{\partial} f(\overline{x},\overline{y},\overline{z}) + \sum_{i=1}^{p} \beta_i \overline{\partial}_{x,y,z} g_i(\overline{x},\overline{z}),\\ \beta_i \geq 0, \beta_i g_i(\overline{x},\overline{z}) = 0, i = 1,\ldots,p\Big\}, \end{array} \tag{4.44}$$

since Ψ is inner semicontinuous at $(\overline{x},\overline{y},\overline{z})$ and the lower-level regularity in the sense of (4.24) holds at $(\overline{x},\overline{z})$. Finally, we get the result by considering the Clarke counterpart of the subdifferential product rule in (4.39):

$$\overline{\partial} f(\overline{x},\overline{y},\overline{z}) \subset f^o(\overline{x},\overline{z}) + \Big\{ \sum_{k=1}^{l} y_k(x_k^*,0,z_k^*) |\, (x_k^*,z_k^*) \in \overline{\partial} f_k(\overline{x},\overline{z}),\, k = 1,\ldots,l\Big\}, \tag{4.45}$$

which is obtained from [4, Proposition 2.3.13]. □

Clearly, the optimality conditions in this theorem coincide with those of Theorem 4.3 and Remark 4.4, respectively, provided the functions are assumed to be strictly differentiable.

As mentioned above, the weak basic CQ (4.2) has been shown to work in particular for the simple bilevel programming problem. For further discussions on this CQ, see [8] and references therein. Next, we consider another CQ, namely, the partial calmness condition introduced in [18] and which has recently been highly investigated and used to derive necessary optimality conditions for a classical optimistic bilevel program via its optimal value reformulation.

Definition 4.6 (partial calmness condition)**.** *According to [18], problem* (3.5) *will be partially calm at one of its local optimal solutions* $(\overline{x},\overline{y},\overline{z})$ *if and only if there exists* $r > 0$ *such that* $(\overline{x},\overline{y},\overline{z})$ *is a local optimal solution of the problem*

$$\begin{cases} \underset{x,y,z}{Minimize}\ F(x,z) + r(f(x,y,z) - \varphi(x,y)) \\ \ \ subject\ to:\ \ (x,y) \in X \times Y,\, z \in K(x). \end{cases} \tag{4.46}$$

Theorem 4.7 (optimality conditions for (3.5) in the Lipschitzian case under the partial calmness)**.** *Let* $(\overline{x},\overline{y},\overline{z})$ *be a local optimal solution of problem* (3.5)*, where the functions* F *and* $G_j, j = 1,\ldots,q$ *are Lipschitz continuous around* $(\overline{x},\overline{z})$ *and* $\overline{x}$*, respectively. Assume that problem* (3.5) *is partially calm at* $(\overline{x},\overline{y},\overline{z})$*, while* $\overline{x}$ *is upper-level regular in the sense of* (4.25)*. Then, the following assertions hold:*

(i) *Let the solution set-valued mapping* Ψ (3.6) *be inner semicompact at* $(\overline{x},\overline{y})$ *while for all* $\overline{z} \in \Psi(\overline{x},\overline{y})$*, the functions* f_k*,* $k = 1,\ldots,l$ *and* g_i*,* $i = 1,\ldots,p$ *are Lipschitz continuous near the point* $(\overline{x},\overline{z})$*, where the lower-level regularity in the sense of* (4.24) *is also satisfied. Then, there exist* $(x_F^*,z_F^*) \in \partial F(\overline{x},\overline{z})$*,* $r > 0$*,* α,β,β^s,v_s *and* $z_s \in \Psi(\overline{x},\overline{y})$*, with* $s = 1,\ldots,n+l+1$ *such that relationships* (4.11)–(4.14) *and* (4.26)–(4.33) *together with the following condition are satisfied:*

$$\Big[\sum_{k=1}^{l} y_k\Big(f_k(\overline{x},\overline{z}) - \sum_{s=1}^{n+l+1} v_s f_k(\overline{x},z_s)\Big)\Big].\overline{y} - \big[f(\overline{x},\overline{z}) - \sum_{s=1}^{n+l+1} v_s f(\overline{x},z_s)\big] \leq 0. \tag{4.47}$$

(ii) *Let the solution set-valued mapping* Ψ (3.6) *be inner semicontinous at* $(\overline{x},\overline{y},\overline{z})$ *while the functions* f_k*,* $k = 1,\ldots,l$ *and* g_i*,* $i = 1,\ldots,p$ *are Lipschitz continuous near* $(\overline{x},\overline{z})$*, where the lower-level regularity in the sense of* (4.24) *is also satisfied. Then, there exist* $(x_F^*,z_F^*) \in \partial F(\overline{x},\overline{z})$*,* $r > 0$*,* α,β,γ *such that relationships* (4.12)–(4.13), (4.22), (4.27), (4.29)–(4.31) *and* (4.34)–(4.37) *hold all together.*

Proof. Under the partial calmness condition, $(\overline{x}, \overline{y}, \overline{z})$ is a local optimal solution of problem (4.46). Considering the estimates of the basic subdifferentials of f and $-\varphi$, and the basic normal cone to Ω in the proof of the previous theorem, one has the result by applying [12, Proposition 5.3] to problem (4.46). $\square$

Some few observations can be made on the link between Theorem 4.5 and Theorem 4.7. Firstly, the additional assumptions imposed on the set Ω (that is, the regularity and semismoothness) in Theorem 4.5 are not necessary in Theorem 4.7. Secondly, recall that the weak basic CQ (4.2) strictly implies the partial calmness condition [8]. It is also clear that the multiplier r is simply nonnegative under CQ (4.2) whereas it is strictly positive under the partial calmness condition. Hence, the reason why condition (4.6) takes the form (4.47). More detail on the link between (4.2) and the partial calmness condition can be found in [8]. A class of bilevel optimization problem with multiobjective lower-level problem which is automatically partially calm in the sense of Definition 4.6 will be discussed in the next section.

Combining the results of Proposition 3.2 (i), Theorem 4.5 and Theorem 4.7, we can now deduce necessary optimality conditions of our initial semivectorial bilevel optimization problem (1.1) in the case where the functions involved are locally Lipschitz continuous. When the functions are strictly differentiable the corresponding optimality conditions can be derived similarly by means of Theorem 4.3 and Remark 4.4.

Corollary 4.8 (necessary optimality conditions for the semivectorial bilevel program (1.1))**.** *Let $(\overline{x}, \overline{z})$ be a local optimal solution of problem* (1.1)*, where the functions F and $G_j, j = 1, \ldots, q$ are Lipschitz continuous around $(\overline{x}, \overline{z})$ and $\overline{x}$, respectively; for all $x \in X$, the functions $f(x, .)$ and $g(x, .)$ are $\mathbb{R}^l_+$- and $\mathbb{R}^p_+$-convex, respectively. Let $\overline{x}$ be upper-level regular in the sense of* (4.25) *and assume that for all $\overline{y} \in Y$ with $\overline{z} \in \Psi(\overline{x}, \overline{y})$, Ω is regular and semismooth at $(\overline{x}, \overline{y}, \overline{z})$, where the weak basic CQ* (4.2) *also holds (**resp.** problem* (4.1) *is partially calm at $(\overline{x}, \overline{y}, \overline{z})$). Then, the following assertions are satisfied:*

(i) *Assume that for all $\overline{y} \in Y$ with $\overline{z} \in \Psi(\overline{x}, \overline{y})$, the solution set-valued mapping Ψ* (3.6) *is inner semicompact at $(\overline{x}, \overline{y})$ while for all $\overline{z} \in \Psi(\overline{x}, \overline{y})$, the functions f_k, $k = 1, \ldots, l$ and g_i, $i = 1, \ldots, p$ are Lipschitz continuous near the point $(\overline{x}, \overline{z})$, where the lower-level regularity in the sense of* (4.24) *is also satisfied. Then, for all $\overline{y} \in Y$ with $\overline{z} \in \Psi(\overline{x}, \overline{y})$, there exist $(x^*_F, z^*_F) \in \partial F(\overline{x}, \overline{z})$, $r \geq 0$ (**resp.** $r > 0$), $\alpha, \beta, \beta^s, v_s$ and $z_s \in \Psi(\overline{x}, \overline{y})$, with $s = 1, \ldots, n + l + 1$ such that relationships* (4.6) *(**resp.** (4.47)),* (4.11)*–*(4.14) *and* (4.26)*–*(4.33) *are satisfied.*

(ii) *Assume that for all $\overline{y} \in Y$ with $\overline{z} \in \Psi(\overline{x}, \overline{y})$, the solution set-valued mapping Ψ* (3.6) *is inner semicontinous at $(\overline{x}, \overline{y}, \overline{z})$ while the functions f_k, $k = 1, \ldots, l$ and g_i, $i = 1, \ldots, p$ are Lipschitz continuous near $(\overline{x}, \overline{z})$, where the lower-level regularity in the sense of* (4.24) *is also satisfied. Then, for all $\overline{y} \in Y$ with $\overline{z} \in \Psi(\overline{x}, \overline{y})$, there exist $(x^*_F, z^*_F) \in \partial F(\overline{x}, \overline{z})$, $r \geq 0$ (**resp.** $r > 0$), α, β, γ such that relationships* (4.12)*–*(4.13)*,* (4.22)*,* (4.27)*,* (4.29)*–*(4.31) *and* (4.34)*–*(4.37) *hold all together.*

Remark 4.9 (single-valued lower-level objective function)**.** If we assume that the lower-level objective function is single-valued, that is, if $l := 1$ in (1.2), then problem (1.1) reduces to a classical optimistic bilevel optimization problem. In this case, the optimality conditions in Remark 4.4 and Theorem 4.5 (ii) are exactly those known for the problem

$$\begin{cases} \underset{x,z}{\text{Minimize}} \; F(x, z) \\ \text{subject to} : \; x \in G(x) \leq 0, \; z \in \Psi(x), \end{cases} \tag{4.48}$$

$$\text{where } \Psi(x) := \arg\min_{y} \{f(x, z) | \, g(x, z) \leq 0\}, \text{ with } f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R},$$

obtained in the corresponding framework. Considering the optimality conditions of the latter problem in a framework analog to that of Theorem 4.3 and Theorem 4.5 (i), the slight difference that occurs is the number of terms in the convex combination which should be $n+1$ for problem (4.48) instead of $n+1+1$ if the latter results are used. The additional 1 here represents the dimension of the space $\mathbb{R}$, which can be intuitively be adjusted to 0 considering the fact that the variable y is no more part of the problem. For results on necessary optimality conditions of problem (4.48) via the optimal value reformulation, see e.g. [5, 8, 13, 18]. For the Karush-Kuhn-Tucker approach, see e.g. [6, 12, 15, 19].

5 Case where the lower-level is a linear multiobjective problem

In this section, we consider a semivectorial bilevel programming problem where the lower-level is a multiobjective optimization problem which is linear in the lower-level variable:

$$\begin{cases} \underset{x,z}{\text{Minimize}}\ F(x,z) \\ \text{subject to}:\ x \in X,\ z \in \Psi_{\text{wef}}(x), \end{cases} \tag{5.1}$$

with set X and the function F defined as in Section 1 while the set-valued mapping Ψ_{wef} represents the weak Pareto optimal solution mapping of the problem:

$$\begin{cases} \mathbb{R}^l_+ - \underset{z}{\text{Minimize}}\ A(x)z + b(x) \\ \text{subject to}:\ C(x)z - d(x) = 0,\ z \geq 0, \end{cases} \tag{5.2}$$

with $b : \mathbb{R}^n \to \mathbb{R}^l$ and $d : \mathbb{R}^n \to \mathbb{R}^p$ being strictly differentiable functions. On the other hand, $A : \mathbb{R}^n \to \mathbb{R}^{l\times m}$ and $C : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ are defined by

$$A(x) := (a_{kt}(x))_{1\leq k\leq l,\, 1\leq t\leq m} \text{ and } C(x) := (c_{it}(x))_{1\leq i\leq p,\, 1\leq t\leq m},$$

where the real-valued functions $a_{kt}\,(1 \leq k \leq l,\ 1 \leq t \leq m)$ and $c_{it}\,(1 \leq i \leq p,\ 1 \leq t \leq m)$ are strictly differentiable.

This problem (with $b(x) = 0$) was treated in [1] as a special case. Considering the approach developed in the previous section, we derive not only new necessary optimality conditions for the problem but also from a perspective completely different from that of [1]. We start by recalling the optimal value reformulation of problem (5.1) according to Section 3:

$$\begin{cases} \underset{x,y,z}{\text{Minimize}}\ F(x,z) \\ \quad \text{subject to}: \quad f(x,y,z) \leq \varphi(x,y) \\ \qquad\qquad\qquad\quad C(x)z - d(x) = 0 \\ \qquad\qquad\qquad\quad (x,y,z) \in X \times Y \times \mathbb{R}^m_+, \end{cases} \tag{5.3}$$

where Y is given in (3.2) and f and φ are defined respectively as:

$$\begin{aligned} &f(x,y,z) := \langle y, A(x)z\rangle + \langle y, b(x)\rangle \text{ and} \\ &\varphi(x,y) := \min_z\{f(x,y,z)|\, C(x)z - d(x) = 0,\ z \geq 0\}. \end{aligned} \tag{5.4}$$

As in the previous section, the following solution set-valued mapping of problem (5.2) also plays an important role in analyzing the problem:

$$\Psi(x,y) := \arg\min_z\{f(x,y,z)|\, C(x)z - d(x) = 0,\ z \geq 0\}. \tag{5.5}$$

We show in the next result that the partial calmness condition in the sense of Definition 4.6 is automatically satisfied for problem (5.3).

Proposition 5.1 (validity of the partial calmness property)**.** *Let* $(\overline{x}, \overline{y}, \overline{z})$ *be an optimal solution of problem* (5.3)*, the function* F *be Lipschitz continuous and* $\operatorname{dom} \Psi = X \times Y$*. Then, problem* (5.3) *is partially calm at* $(\overline{x}, \overline{y}, \overline{z})$*.*

Proof. Observe that the function f in (5.4) can be written as

$$f(x, y, z) := \langle z, A(x)y \rangle + \langle y, b(x) \rangle.$$

Then proceeding as in the proof of [7, Theorem 4.2], we have the result. □

This result can be seen as an extension of [7, Theorem 4.2], where considering a usual bilevel optimization problem, it was required that $X := \mathbb{R}^n$. Clearly, one simply needs to replace the condition $\operatorname{dom} \Psi = \mathbb{R}^n$ in [7, Theorem 4.2] by $\operatorname{dom} \Psi = X$. As it will be obvious in the next result, the main implication of this result is that for any given bilevel optimization problem where the lower-level problem is linear in the lower-level variable, only the upper- and lower-level regularity conditions are needed to derive necessary optimality. Hence, we define the lower-level regularity condition corresponding to our problem (5.1):

$$\begin{array}{c} \exists \widehat{z} : \; C(\overline{x})\widehat{z} = d(\overline{x}), \; \widehat{z}_k > 0, \; k = 1, \ldots, l \\ \text{and } C(\overline{x}) \text{ has full row rank.} \end{array} \tag{5.6}$$

The following result is a consequence of Theorem 4.3.

Corollary 5.2 (necessary optimality conditions for problem (5.1))**.** *Let* $(\overline{x}, \overline{z})$ *be a local optimal solution of problem* (5.1)*, where* F *and* G *are strictly differentiable at* $(\overline{x}, \overline{z})$ *and* $\overline{x}$*, respectively. Assume that* $\overline{x}$ *is upper-level regular* (4.4) *and satisfies* (5.6)*. Furthermore, let* Ψ (5.5) *be inner semicompact at* $(\overline{x}, \overline{y})$*, for* $\overline{y} \in Y$ *with* $\overline{z} \in \Psi(\overline{x}, \overline{y})$*. Then, for all* $\overline{y}$ *with* $\overline{z} \in \Psi(\overline{x}, \overline{y})$*, there exist* $r \geq 0$*,* $\alpha, \beta, \beta^s, v_s$ *and* $z_s \in \Psi(\overline{x}, \overline{y})$*, with* $s = 1, \ldots, n + l + 1$ *such that relationships* (4.6) (*with* $f(x, z) := A(x)z + b(x)$) *and* (4.12)*,* (4.14)*, together with the following conditions are satisfied:*

$$\begin{aligned} \nabla_x F(\overline{x}, \overline{z}) + r \sum_{k=1}^{l} \overline{y}_k \sum_{t=1}^{m} \overline{z}_t \nabla a_{kt}(\overline{x}) + \sum_{j=1}^{q} \alpha_j \nabla G_j(\overline{x}) + \sum_{i=1}^{p} \beta_i \Big(\sum_{t=1}^{m} z_t \nabla c_{it}(\overline{x}) - \nabla d_i(\overline{x}) \Big) & \\ - r \sum_{s=1}^{n+l+1} v_s \Big[\sum_{k=1}^{l} \overline{y}_k \sum_{t=1}^{m} z_{ts} \nabla a_{kt}(\overline{x}) + \sum_{i=1}^{p} \beta_i^s \Big(\sum_{t=1}^{m} z_{ts} \nabla c_{it}(\overline{x}) - \nabla d_i(\overline{x}) \Big) \Big] &= 0, \\ \nabla_z F(\overline{x}, \overline{z}) + r \sum_{k=1}^{l} \overline{y}_k a_k(\overline{x}) + \sum_{i}^{p} \beta_i c_i(\overline{x}) &\leq 0, \\ \overline{z}^\top \Big[\nabla_z F(\overline{x}, \overline{z}) + r \sum_{k=1}^{l} \overline{y}_k a_k(\overline{x}) + \sum_{i}^{p} \beta_i c_i(\overline{x}) \leq 0 \Big] &= 0, \\ \forall s = 1, \ldots, n + l + 1, \; \sum_{k=1}^{l} \overline{y}_k a_k(\overline{x}) + \sum_{i}^{p} \beta_{si} c_i(\overline{x}) &\leq 0, \\ \forall s = 1, \ldots, n + l + 1, \; \overline{z}^\top \Big[\sum_{k=1}^{l} \overline{y}_k a_k(\overline{x}) + \sum_{i}^{p} \beta_{si} c_i(\overline{x}) \leq 0 \Big] &= 0. \end{aligned}$$

References

[1] H. Bonnel, *Optimality conditions for the semivectorial bilevel optimization problem*, Pac. J. Optim. **2** (2006), no. 3, 447–467.

[2] H. Bonnel and J. Morgan, *Semivectorial bilevel optimization problem: penalty approach*, J. Optim. Theory Appl. **131** (2006), no. 3, 365–382.

[3] H. Bonnel and J. Morgan, *Semivectorial bilevel convex optimal control problems: An existence result*, working paper.

[4] F.H. Clarke, *Optimization and nonsmooth analysis*, SIAM Classics in Applied Mathematics 5, Wiley, New York, 1984, Reprint. Philadelphia, 1994.

[5] S. Dempe, J. Dutta and B.S. Mordukhovich, *New necessary optimality conditions in optimistic bilevel programming*, Optimization **56** (2007), no. 5-6, 577–604.

[6] S. Dempe and A.B. Zemkoho, O*n the Karush-Kuhn-Tucker reformulation of the bilevel optimization problem*, Nonlinear Analysis (2011), in press, doi:10.1016/j.na.2011.05.097.

[7] S. Dempe and A.B. Zemkoho, *The bilevel programming problem: reformulations, constraint qualifications and optimality conditions*, Sumitted.

[8] S. Dempe and A.B. Zemkoho, *The generalized Mangasarian-Fromowitz constraint qualification and optimality conditions for bilevel programs*, J. Optim. Theory Appl. **148** (2011), 433–441.

[9] M. Ehrgott, *Multicriteria optimization. 2nd ed.*, Berlin: Springer, 2005.

[10] G. Eichfelder, *Multiobjective bilevel optimization*, Math. Program. **123** (2010), 419–449.

[11] R. Henrion and J.V. Outrata, *A subdifferential condition for calmness of multifunctions*, J. Math. Anal. Appl. **258** (2001), no. 1, 110–130.

[12] B.S. Mordukhovich, *Variational analysis and generalized differentiation. I: Basic theory. II: Applications*, Berlin: Springer, 2006.

[13] B.S. Morukhovich, M.N. Nam and H.M. Phan, *Variational analysis of marginal function with applications to bilevel programming problems*, Sumitted.

[14] B.S. Morukhovich, M.N. Nam and N.D. Yen, *Subgradients of marginal functions in parametric mathematical programming*, Math. Program. **116** (2009), no. 1-2, 369–396.

[15] B.S. Mordukhovich and J.V. Outrata, *Coderivative analysis of quasi-variational inequalities with applications to stability and optimization*, SIAM J. Optim. **18** (2007), no. 2, 389–412.

[16] J.V. Outrata, *A note on the usage of nondifferentiable exact penalties in some special optimization problems*, Kybernetika **24** (1988), no. 4, 251–258.

[17] R.T. Rockafellar and R.J.-B. Wets, *Variational analysis*, Berlin: Springer, 1998.

[18] J.J. Ye and D.L. Zhu, *Optimality conditions for bilevel programming problems*, Optimization **33** (1995), no. 1, 9–27 (with Erratum in Optimization **39** (1997), no. 4, 361-366)

[19] J.J. Ye and X.Y. Ye, *Necessary optimality conditions for optimization problems with variational inequality constraints*, Math. Oper. Res. **22** (1997), no. 4, 977-997.

[20] Y. Zheng and Z. Wan, *A solution method for semivectorial bilevel programming problem via penalty method*, J. Appl. Math. Comput. **37** (2011), 207-219.