The Triangle Closure is a Polyhedron

Amitabh Basu* Robert Hildebrand† Matthias Köppe‡

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Abstract

Recently, cutting planes derived from maximal lattice-free convex sets have been studied intensively by the integer programming community. An important question in this research area has been to decide whether the closures associated with certain families of lattice-free sets are polyhedra. For a long time, the only result known was the celebrated theorem of Cook, Kannan and Schrijver who showed that the split closure is a polyhedron. Although some fairly general results were obtained by Andersen, Louveaux and Weismantel [*An analysis of mixed integer linear sets based on lattice point free convex sets*, Math. Oper. Res. **35**, (2010) pp. 233–256], some basic questions have remained unresolved. For example, maximal lattice-free triangles are the natural family to study beyond the family of splits and it has been a standing open problem to decide whether the triangle closure is a polyhedron. In this paper, we resolve this by showing that the triangle closure is indeed a polyhedron, and its number of facets can be bounded by a polynomial in the size of the input data.

1 Introduction

We study the following system, introduced by Andersen et al. [2]:

$$\begin{aligned} &x = f + \sum_{j=1}^{k} r^j s_j \\ &x \in \mathbb{Z}^m \\ &s_j \geq 0 \quad \text{for all } j = 1, \dots, k. \end{aligned} \tag{1}$$

This model has been studied extensively with the purpose of providing a unifying theory for cutting planes and exploring new families of cutting planes [2, 3, 4, 5, 6, 9, 10, 11]. In this theory, an interesting connection is explored between valid inequalities for the convex hull of solutions to (1) (the integer hull) and maximal lattice-free convex sets in $\mathbb{R}^m$. A *lattice-free* convex set is a convex set which does not contain any integer point in its interior. A *maximal* lattice-free convex set is a lattice-free convex set which is maximal with respect to set

*Dept. of Mathematics, University of California, Davis, abasu@math.ucdavis.edu
†Dept. of Mathematics, University of California, Davis, rhildebrand@math.ucdavis.edu
‡Dept. of Mathematics, University of California, Davis, mkoeppe@math.ucdavis.edu
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inclusion. The integer hull can be obtained by intersecting all valid inequalities derived from the Minkowski functional of maximal lattice-free convex sets containing f in their interior.

It is also well-known that maximal lattice-free convex sets are polyhedra [4, 12]. The most primitive type of maximal lattice-free convex set in $\mathbb{R}^m$ is the *split*, which is of the form $\pi_0 \leq \pi \cdot x \leq \pi_0 + 1$ for some $\pi \in \mathbb{Z}^m$ and $\pi_0 \in \mathbb{Z}$. A famous theorem due to Cook, Kannan and Schrijver [8] shows that the intersection of all valid inequalities for (1) derived from splits is a polyhedron. It is a natural question to ask what happens to the closures for more complicated kinds of maximal lattice-free convex sets. Unfortunately, there are not too many satisfactory answers to this problem. The only other attack on the problem, apart from the split closure theorem, appears in the elegant results from [1]. Even so, some of the most basic questions have remained open. For instance, consider the case $m = 2$. For this case, the different types of maximal lattice-free convex sets have been classified quite satisfactorily. Lovász characterized the maximal lattice-free convex sets in $\mathbb{R}^2$ as follows.

Theorem 1.1 (Lovász [12]). *In the plane, a maximal lattice-free convex set with non-empty interior is one of the following:*

1. *A split* $c \leq ax_1 + bx_2 \leq c + 1$ *where* a *and* b *are co-prime integers and* c *is an integer;*
2. *A triangle with an integral point in the interior of each of its edges;*
3. *A quadrilateral containing exactly four integral points, with exactly one of them in the interior of each of its edges. Moreover, these four integral points are vertices of a parallelogram of area 1.*

Following Dey and Wolsey [10], the maximal lattice-free triangles can be further partitioned into three canonical types (see Figure 1):

- *Type 1 triangles*: triangles with integral vertices and exactly one integral point in the relative interior of each edge;
- *Type 2 triangles*: triangles with at least one fractional vertex v, exactly one integral point in the relative interior of the two edges incident to v and at least two integral points on the third edge;
- *Type 3 triangles*: triangles with exactly three integral points on the boundary, one in the relative interior of each edge.

Figure 1 shows these three types of triangles as well as a maximal lattice-free quadrilateral and a split satisfying the properties of Theorem 1.1.

Even for this simple case of $m = 2$, the only result regarding these families of lattice-free sets has been the original Cook–Kannan–Schrijver split closure result. It was not even known whether the triangle closure (the convex set formed by the intersection of all inequalities derived from maximal lattice-free triangles) is a polyhedron. In this paper, we finally settle this question in the affirmative under the assumption of rationality of all the data. The Cook–Kannan–Schrijver split closure result has been used repeatedly as a theoretical as well as practical tool in many diverse settings within the integer programming community. Our motivation for studying the corresponding question for the triangle closure is the conjecture

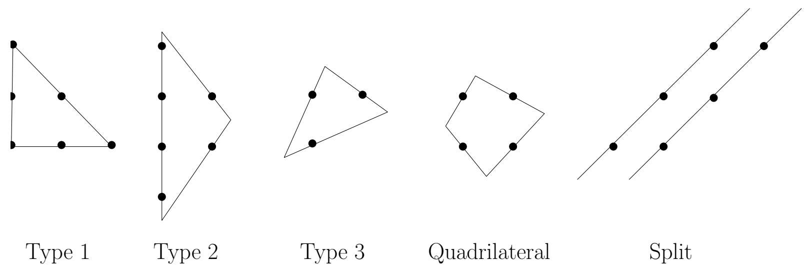

Figure 1: Types of maximal lattice-free convex sets in $\mathbb{R}^2$

that it will also prove to be a useful theorem in cutting plane theory, like its split closure counterpart.

For the remainder of the paper, we will consider (1) with $m = 2$. We will be concerned with maximal lattice-free convex sets in $\mathbb{R}^2$ with f in their interior; one can represent such sets in the following canonical manner.

Let $B \in \mathbb{R}^{n\times 2}$ be a matrix with n rows $b^1, \ldots, b^n \in \mathbb{R}^2$. We write $B = (b^1; \ldots; b^n)$. Our notation follows [5]. Let

$$M(B) = \{\, x \in \mathbb{R}^2 \,|\, B \cdot (x - f) \leq e \,\}, \tag{2}$$

where e is the vector of all ones. This is a polyhedron with f in its interior. We will denote its vertices by $\mathrm{vert}(B)$. In fact, any polyhedron with f in its interior can be given such a description. We will mostly deal with matrices B such that $M(B)$ is a maximal lattice-free convex set in $\mathbb{R}^2$. Define

$$\psi_B(r) = \max_{i\in\{1,\ldots,n\}} b^i \cdot r \quad \text{for } r \in \mathbb{R}^2.$$

If $B \in \mathbb{R}^{n\times 2}$ is a matrix such that $M(B)$ is a lattice-free convex set in $\mathbb{R}^2$, then the inequality $\sum_{j=1}^{k} \psi_B(r^j)s_j \geq 1$ is a valid inequality for (1) and in fact, it is well-known that the integer hull is given by the intersection of all inequalities derived in this manner from *maximal* lattice-free convex sets. We define the vector of coefficients as

$$\gamma(B) = (\psi_B(r^j))_{j=1}^{k}.$$

Given a real valued matrix $B \in \mathbb{R}^{3\times 2}$, if $M(B)$ is a lattice-free set, then it will be either a triangle or a split in $\mathbb{R}^2$ (not necessarily maximal); the latter case occurs when one row of B is a scaling of another row.

We define the *split closure* as

$$S = \{\, s \in \mathbb{R}^k_+ \,|\, \gamma(B) \cdot s \geq 1 \text{ for all } B \in \mathbb{R}^{3\times 2} \text{ such that } M(B) \text{ is a lattice-free split} \,\}.$$

Note that we are using a redundant description of convex sets that are splits, i.e., using 3 inequalities to describe it, instead of the standard 2 inequalities. It follows from the result of Cook, Kannan and Schrijver [8] that the split closure is a polyhedron. We are interested now in the closure using all inequalities derived from lattice-free triangles.

We define the *triangle closure*, first defined in [3], as

$$T = \{\, s \in \mathbb{R}^k_+ \mid \gamma(B)\cdot s \geq 1 \text{ for all } B \text{ such that } M(B) \text{ is a lattice-free triangle} \,\}.$$

It is proved in [3] that $T \subseteq S$, and therefore, $T = T \cap S$. This is because we can write a sequence of triangles whose limit is a split, and therefore all split inequalities are limits of triangle inequalities. Hence, using the fact that $T = T \cap S$, we can write the triangle closure as

$$T = \{\, s \in \mathbb{R}^k_+ \mid \gamma(B)\cdot s \geq 1 \text{ for all } B \in \mathbb{R}^{3\times 2} \text{ such that } M(B) \text{ is a lattice-free convex set} \,\}. \tag{3}$$

The reason we describe split sets using 3 inequalities is to write the pure triangle closure in a uniform manner using 3×2 matrices as in (3). We note here that in the definition of T, we do not insist that the lattice-free set $M(B)$ is maximal.

We will prove the following theorem.

Theorem 1.2. *Suppose that the data in* (1) *is rational, i.e.,* $f \in \mathbb{Q}^2$ *and* $r^j \in \mathbb{Q}^2$ *for all* $j = 1, \ldots, k$*. Then the triangle closure* T *is a polyhedron with only a polynomial number of facets with respect to the binary encoding sizes of* $f, r^1, \ldots, r^k$*.*

We will first use some convex analysis in Section 2 to illuminate the convex geometry of T by studying a well-defined dual convex set obtained from the defining inequalities of T. We will then demonstrate that it suffices to show that this dual convex set has finitely many extreme points. In Section 3, we prove that there are indeed only finitely many such extreme points, and in Section 4, we complete the proof of Theorem 1.2.

We make a remark about the proof structure here. The results in Section 3 are developed with the aim of proving Theorem 3.12, which is stated at the very end of Section 3. Theorem 3.12 can be viewed as the bridge between Section 2 and Section 4. The reader can follow the proof of Theorem 1.2 by reading only Sections 2 and 4, if Theorem 3.12 is taken on faith to be true. One can then return to Section 3 to see the proof of Theorem 3.12, which is rather technical.

2 Preliminaries: Convex Analysis and the Geometry of T

We will prove several preliminary convex analysis lemmas relating to the geometry of T. We show that we can write the triangle closure T using a smaller set of inequalities. We begin by defining the set of vectors which give the inequalities defining T,

$$\Delta = \{\, \gamma(B) \mid B \in \mathbb{R}^{3\times 2} \text{ such that } M(B) \text{ is a lattice-free convex set (not necessarily maximal)} \,\}.$$

It is easily verified that for any matrix $B \in \mathbb{R}^{3\times 2}$, if $M(B)$ is a lattice-free polytope, then $\psi_B(r) \geq 0$ for all $r \in \mathbb{R}^2$ and therefore $\Delta \subseteq \mathbb{R}^k_+$.

Let $\Delta' = \mathrm{cl}(\mathrm{conv}(\Delta)) + \mathbb{R}^k_+$ where $\mathrm{cl}(\mathrm{conv}(\Delta))$ denotes the closed convex hull of Δ, $\mathbb{R}^k_+$ denotes the nonnegative orthant and $+$ denotes the Minkowski sum. Δ' is convex as it is the Minkowski sum of two convex sets. In general the Minkowski sum of two closed sets is not closed. However, in this particular case, we show now that Δ' is closed. We will use the well-known fact that the Minkowski sum of two compact sets is indeed closed. We prove the following more general result.

Lemma 2.1. *Let $X, Y \subseteq \mathbb{R}^k_+$ be closed subsets of $\mathbb{R}^k_+$. Then $X + Y$ is closed.*

Proof. Let $Z = X + Y$ and let $(z^n) \in Z$ such that $z^n \to z \in \mathbb{R}^k$. We want to show that $z \in Z$. Let $A = \{ x \in \mathbb{R}^k_+ \mid \|x\|_\infty \le \|z\|_\infty + 1 \}$. Since $\|z^n - z\|_\infty \to 0$ as $n \to \infty$, for some $N \in \mathbb{N}$, we must have that $\|z^n\|_\infty \le \|z\|_\infty + 1$, that is, $z^n \in Z \cap A$, for all $n \ge N$. Since $X, Y \subseteq \mathbb{R}^k_+$, we see that $Z \cap A \subseteq (X \cap A) + (Y \cap A)$. Since $(X \cap A) + (Y \cap A)$ is a Minkowski sum of two closed and bounded subsets of $\mathbb{R}^k$, i.e., compact, $(X \cap A) + (Y \cap A)$ is closed. Therefore, the tail of (z^n) is contained in a closed set, so it must converge to a point in the set, that is, $z \in (X \cap A) + (Y \cap A)$. Since $(X \cap A) + (Y \cap A) \subseteq X + Y = Z$, we have that $z \in Z$. Therefore, Z is closed. □

Lemma 2.2. $T = \{ s \in \mathbb{R}^k_+ \mid \gamma \cdot s \ge 1$ *for all* $\gamma \in \Delta' \}$.

Proof. Since $\Delta \subseteq \Delta'$, we have that $\{ s \in \mathbb{R}^k_+ \mid \gamma \cdot s \ge 1 \quad \text{for all } \gamma \in \Delta' \} \subseteq T$. We now show the reverse inclusion. Consider any $s \in T$ and $\gamma \in \Delta'$. We show that $\gamma \cdot s \ge 1$.

Since $\Delta' = \mathrm{cl}(\mathrm{conv}(\Delta)) + \mathbb{R}^k_+$, there exists $r \in \mathbb{R}^k_+$ and $a \in \mathrm{cl}(\mathrm{conv}(\Delta))$ such that $\gamma = a + r$. Moreover, there exists a sequence (a^n) such that (a^n) converges to a and (a^n) is in the convex hull of points in $p_j \in \Delta$, $j \in J$. Since $p_j \cdot s \ge 1$ for all $j \in J$, we have that $a_n \cdot s \ge 1$ for all $n \in \mathbb{N}$. Therefore $a \cdot s = \lim_{n\to\infty} a^n \cdot s \ge 1$. Since $r \in \mathbb{R}^k_+$, $r \cdot s \ge 0$ and so $\gamma \cdot s = (a + r) \cdot s \ge a \cdot s \ge 1$. □

We say that $a \in \Delta'$ is a *minimal* point if there does not exist $x \in \Delta'$ such that $a - x \in \mathbb{R}^k_+ \backslash \{0\}$. If such an x exists then we say that a is *dominated* by x. We introduce some standard terminology from convex analysis. Given a convex set $C \subseteq \mathbb{R}^k$, a *supporting hyperplane* for C is a hyperplane $H = \{ x \in \mathbb{R}^k \mid h \cdot x = d \}$ such that $h \cdot c \le d$ for all $c \in C$ and $H \cap C \ne \emptyset$. A point $x \in C$ is called *extreme* if there do not exist y^1 and y^2 in C different from x such that $x = \frac{1}{2}(y^1 + y^2)$. If such $y^1 \ne y^2$ exist, we say that x is a *strict convex combination* of y^1 and y^2. A point x is called *exposed* is there exists a supporting hyperplane H for C such that $H \cap C = \{x\}$. We will denote the closed ball of radius r around a point y as $\bar{B}(y, r)$. We denote the boundary of this ball by $\partial \bar{B}(y, r)$.

Lemma 2.3. *Δ' is a closed convex set with $\mathbb{R}^k_+$ as its the recession cone.*

Proof. Recall that $\Delta \subseteq \mathbb{R}^k_+$. Since $\mathbb{R}^k_+$ is closed and convex, $\mathrm{cl}(\mathrm{conv}(\Delta)) \subseteq \mathbb{R}^k_+$ and so $\Delta' = \mathrm{cl}(\mathrm{conv}(\Delta)) + \mathbb{R}^k_+$ is closed by Lemma 2.1. Since the Minkowski sum of two convex sets is convex, Δ' is convex. Moreover since $\Delta' \subseteq \mathbb{R}^k_+$, the recession cone of Δ' is $\mathbb{R}^k_+$. □

Lemma 2.4. *Let C be the set of extreme points of Δ'. Then*

$$T = \{ s \in \mathbb{R}^k_+ \mid a \cdot s \ge 1 \textit{ for all } a \in C \}.$$

Proof. Let $\hat{T} = \{ s \in \mathbb{R}^k_+ \mid a \cdot s \ge 1 \text{ for all } a \in C \}$. Since $C \subseteq \Delta'$, we have that $T \subseteq \hat{T}$. We show the reverse inclusion. Consider any $s \in \hat{T}$.

By Lemma 2.3, Δ' is a closed convex set with $\mathbb{R}^k_+$ as its the recession cone. Therefore, Δ' contains no lines. This implies that any point $a \in \Delta'$ can be represented as $a = z + \sum_j \lambda_j v_j$ where z is a recession direction of Δ', v_j's are extreme points of Δ', $\lambda_j \ge 0$ and $\sum_j \lambda_j = 1$ (see Theorem 18.5 in [13]). Moreover, since the v_j's are extreme points, $v_j \in C$ and therefore $v_j \cdot s \ge 1$ for all j because $s \in \hat{T}$. Since $z \in \mathbb{R}^k_+$, $s \in \mathbb{R}^k_+$, $\lambda_j \ge 0$ for all j and $\sum_j \lambda_j = 1$, $a \cdot s = z \cdot s + \sum_j \lambda_j (v_j \cdot s) \ge 1$. Therefore, for all $a \in \Delta'$, $a \cdot s \ge 1$. By Lemma 2.2, $s \in T$. □

Observation 2.5. *Since the recession cone of Δ' is $\mathbb{R}^k_+$ by Lemma 2.3, every extreme point of Δ' is minimal.*

Before we proceed, we need the following technical lemma.

Lemma 2.6. *Let A be any subset of $\mathbb{R}^k$ and let $A' = \mathrm{cl}(\mathrm{conv}(A))$. Then for any extreme point x of A', there exists a sequence of points $(a^n) \in A$ converging to x.*

Proof. We first show the following claim.

Claim α. For any exposed point a of A', there exists a sequence of points $(a^n) \in A$ converging to a.

Proof. Let $H = \{ x \in \mathbb{R}^k \,|\, h \cdot x = d \}$ be a supporting hyperplane for A' such that $H \cap A' = \{a\}$. Suppose to the contrary that there does not exist such a sequence in A. This implies that there exists $\epsilon > 0$ such that $\bar{B}(a, \epsilon) \cap A = \emptyset$. Let $D = \partial \bar{B}(a, \epsilon) \cap H$. Since $H \cap A' = \{a\}$, for any point $c \in D$, $\mathrm{dist}(c, A') > 0$. Since D is a compact set and the distance function is a Lipschitz continuous function, there exists $\delta > 0$ such that $\mathrm{dist}(c, A') > \delta$ for all $c \in D$. We choose δ' such that for any $y \in \partial \bar{B}(a, \epsilon)$ satisfying $d \geq h \cdot y > d - \delta'$, there exists $c \in D$ with $\mathrm{dist}(c, y) < \delta$.

Since $a \in \mathrm{cl}(\mathrm{conv}(A))$, there exists a sequence of points $b^n \in \mathrm{conv}(A)$ converging to a. This implies that $h \cdot b^n$ converges to $h \cdot a = d$. Therefore, we can choose b in this sequence such that $h \cdot b > d - \delta'$ and $b \in \bar{B}(a, \epsilon)$. Since $b \in \mathrm{conv}(A)$ there exist $v_j \in A, j = 0, \ldots, k$ such that $b = \mathrm{conv}\{v_0, \ldots, v_k\}$. Therefore, for some j, $h \cdot v_j > d - \delta'$. Moreover, since $v_j \in A$ and $\bar{B}(a, \epsilon) \cap A = \emptyset$, $v_j \notin \bar{B}(a, \epsilon)$. Since $b \in \bar{B}(a, \epsilon)$ and $v_j \notin \bar{B}(a, \epsilon)$, there exists a point $p \in \partial \bar{B}(a, \epsilon)$ such that p is a convex combination of b and v_j. Since $h \cdot b > d - \delta'$ and $h \cdot v_j > d - \delta'$, we have that $h \cdot p > d - \delta'$. Moreover $b \in \mathrm{conv}(A)$ implying $b \in A'$ and $v_j \in A'$, so we have $p \in A'$ and so $d \geq h \cdot p$ since H is a supporting hyperplane for A'. So by the choice of δ', we have that there exists $c \in D$ with $\mathrm{dist}(c, p) < \delta$. However, $\mathrm{dist}(c, A') > \delta$ for all $c \in D$ which is a contradiction because $p \in A'$. □

By Straszewicz's theorem (see for example Theorem 18.6 in [13]), for any extreme point x of A', there exists a sequence of exposed points converging to x. So for any $n \in \mathbb{N}$, there exists an exposed point e^n such that $\mathrm{dist}(e^n, x) < \frac{1}{2n}$ and using Claim α, there exists $a^n \in A$ such that $\mathrm{dist}(e^n, a^n) < \frac{1}{2n}$. Now the sequence a_n converges to x since $\mathrm{dist}(a^n, x) < \frac{1}{n}$. □

We now show that there are only a finite number of extreme points of Δ'. Lemma 2.4 would then imply that T is the intersection of a finite number of half-spaces and hence a polyhedron, proving Theorem 1.2.

3 Polynomially Many Extreme Points

In this section, we will use the tools and results from [5] to prove the following proposition.

Proposition 3.1. *There exists a finite set $\Xi \subseteq \Delta$, such that if $\gamma \in \Delta \backslash \Xi$, then γ is dominated by some $\gamma' \in \Delta$, or γ is the strict convex combination of γ^1 and $\gamma^2 \in \Delta$. Furthermore, the cardinality of Ξ is polynomially bounded in the binary encoding size of $f, r^1, \ldots, r^k$.*

The bulk of the proof comes from an analysis of the proof of the following result of [5].

Theorem 3.2 (Theorem 6.2, [5]). *For $m = 2$, the number of facets of the integer hull of* (1) *is polynomial in the size of the binary encoding of the problem.*

We now introduce the set Γ of all vectors $\gamma(B)$ that come from arbitrary (not necessarily maximal) lattice-free polyhedra in $\mathbb{R}^2$,

$$\Gamma = \bigcup_{n \in \mathbb{N}} \{ \gamma(B) \mid B \in \mathbb{R}^{n \times 2} \text{ such that } M(B) \text{ is a lattice-free convex set} \}.$$

Since we consider $B \in \mathbb{R}^{n \times 2}$ for all $n \in \mathbb{N}$, this includes all $\gamma(B)$ such that $M(B)$ is a lattice-free split, triangle, or quadrilateral and all other polyhedra that are lattice-free in $\mathbb{R}^2$. In fact, because of the correspondence between valid inequalities for the integer hull and lattice-free sets, one can show that Γ is actually the *blocking polyhedron* of the integer hull.

Basu, Hildebrand, and Köppe [5] prove Theorem 3.2 by first showing necessary conditions for each type of maximal lattice-free convex set $M(B)$ such that $\gamma(B)$ is an extreme point of Γ and then enumerating all possible maximal lattice-free convex sets with these necessary conditions. To show that a point $\gamma(B)$ is not extreme in Γ, they either show that it is the convex combination of two other points in Γ, or show that it is dominated by a point in Γ.

A very similar kind of analysis is needed to prove Proposition 3.1. The difference is that the set Γ is a convex set, so it makes sense to discuss its extreme points, whereas Δ is not necessarily convex. Instead of describing extreme points, we find a finite set $\Xi \subseteq \Delta$ with the property that if $\gamma(B) \in \Delta \setminus \Xi$, then $\gamma(B)$ is either dominated by another point in Δ or expressed as a convex combination of other points in Δ. The reason we cannot directly use Theorem 3.2 for this purpose is that, within its proof, some non-extreme points $\gamma(B)$ where $M(B)$ is a maximal lattice-free Type 2 triangle are expressed as a convex combination of $\gamma(T) \in \Delta$ and $\gamma(Q) \in \Gamma$ where $M(T)$ is a lattice-free triangle, but $M(Q)$ is a lattice-free quadrilateral, i.e., $Q \in \mathbb{R}^{4 \times 2}$. For these cases, we need to do a different analysis, which we present in this section.

The necessary conditions for a split, triangle, or quadrilateral $M(B)$ to yield an extreme point $\gamma(B)$ of Γ are stated and proved in Section 5 of [5]. The proof of Theorem 3.2 enumerates all possible extreme inequalities described by the necessary conditions. Then it shows that there are only polynomially many of them by using the following consequence of the Cook–Hartmann–Kannan–McDiarmid theorem on the polynomial-size description of the integer hulls of polyhedra in fixed dimension [7].

Lemma 3.3 (Remark 6.1 in [5]). *Given two rays r^1 and r^2 in $\mathbb{R}^2$, we define the cone*

$$C(r^1, r^2) = \{ x \in \mathbb{R}^2 \mid x = f + s_1 r^1 + s_2 r^2 \text{ for } s_1, s_2 \geq 0 \}.$$

The number of facets and vertices of the integer hull

$$(C(r^1, r^2))_{\mathrm{I}} = \operatorname{conv}(C(r^1, r^2) \cap \mathbb{Z}^2)$$

is bounded by a polynomial in the binary encoding sizes of f, r^1, r^2.

Here, we are interested in counting the triangles and splits $M(B)$ such that $\gamma(B)$ is not dominated by a point in Δ and is not a strict convex combination of points in Δ. We modify and adapt the necessary conditions and counting arguments of [5] to show Proposition 3.1.

Apart from the case when $M(B)$ is a Type 2 triangle, all the proofs of the necessary conditions proceed by showing that if $\gamma(B)$ is not extreme in Γ, then it is either dominated by a some point in $\Delta \subset \Gamma$, or that it is a convex combination of points from $\Delta \subset \Gamma$. Therefore, adaptations for these necessary conditions and the corresponding counting arguments follow directly for splits, Type 1 triangles and Type 3 triangles. We state these results below in a rather concise form which is most suited for the purposes of this paper and cite the appropriate results from [5] whose proofs imply these statements. To this end, we define the sets

$$\Delta_i = \{\, \gamma(B) : B \in \mathbb{R}^{3\times 2},\ M(B) \text{ is a Type } i \text{ triangle} \,\} \quad \text{for } i = 1, 2, 3$$

and

$$\Pi = \{\, \gamma(B) : B \in \mathbb{R}^{3\times 2},\ M(B) \text{ is a maximal lattice-free split} \,\}.$$

Note that these sets are not disjoint, as the same vector γ can be realized by maximal lattice-free convex sets of different kinds.

Proposition 3.4. *There exist finite subsets $\Xi_0 \subseteq \Pi$, $\Xi_1 \subseteq \Delta_1$, and $\Xi_3 \subseteq \Delta_3$ of cardinalities bounded polynomially in the binary encoding sizes of $f, r^1, \dots, r^k$ with the following properties:*

- (i) *For any $\gamma \in \Pi \setminus \Xi_0$, there exist $\gamma^1, \gamma^2 \in \Delta$ such that γ is a strict convex combination of γ^1 and γ^2.*
- (ii) *For any $\gamma \in \Delta_1 \setminus (\Xi_1 \cup \Pi)$, there exist $\gamma^1, \gamma^2 \in \Delta$ such that γ is a strict convex combination of γ^1 and γ^2 or there exists $\gamma' \in \Delta$ such that γ is dominated by γ'.*
- (iii) *For any $\gamma \in \Delta_3 \setminus \Xi_3$, there exist $\gamma^1, \gamma^2 \in \Delta$ such that γ is a strict convex combination of γ^1 and γ^2.*

Proof. This follows from the proofs of Lemma 5.10, Lemma 5.7, Lemma 5.3 and Theorem 6.2 in [5]. □

We now focus specifically on Type 2 triangles. In particular, we will prove the following proposition.

Proposition 3.5. *There exists a finite subset $\Xi_2 \subseteq \Delta_2$ of cardinality bounded polynomially in the binary encoding sizes of $f, r^1, \dots, r^k$ with the following property:*

For any $\gamma \in \Delta_2 \setminus (\Xi_2 \cup \Delta_3 \cup \Pi)$, there exist $\gamma^1, \gamma^2 \in \Delta$ such that γ is a strict convex combination of γ^1 and γ^2 or there exists $\gamma' \in \Delta$ such that γ is dominated by γ'.

Before the proof of Proposition 3.5, we introduce notation and results from [5]. We often refer to the set of *ray intersections*

$$P = \left\{\, p^j \in \mathbb{R}^2 \;\middle|\; p^j = f + \tfrac{1}{\psi_B(r^j)} r^j,\ \psi_B(r^j) > 0,\ j = 1, \dots, k \,\right\},$$

that is, the set of points p^j where the rays r^j meet the boundary of the set $M(B)$.

Whenever $\psi_B(r^j) > 0$, the set $I_B(r^j) = \arg\max_{i=1,\dots,3} b^i \cdot r$ is the index set of all inequalities of $M(B)$ that the ray intersection $p^j = f + \frac{1}{\psi_B(r^j)} r^j$ satisfies with equality. When $M(B)$

is a lattice-free triangle, $\#I_B(r^j) = 1$ when r^j points from f to the relative interior of a facet, and $\#I_B(r^j) = 2$ when r^j points from f to a vertex of $M(B)$. In this second case, we call r a *corner ray* of $M(B)$. Let $Y(B)$ be the set of integer points contained in $M(B)$. In our proofs, it is convenient to choose, for every $i = 1, 2, 3$, a certain subset $Y_i \subseteq Y(B) \cap F_i$ of the integer points on the facet F_i.

Definition 3.6. *Let $\mathcal{Y}$ denote the tuple (Y_1, Y_2, Y_3). The* tilting space *$\mathcal{T}(B, \mathcal{Y}) \subset \mathbb{R}^{3\times 2}$ is defined as the set of matrices $A = (a^1; a^2; a^3) \in \mathbb{R}^{3\times 2}$ that satisfy the following conditions:*

$$a^i \cdot (y - f) = 1 \qquad \text{for } y \in Y_i,\ i = 1, 2, 3, \tag{4a}$$

$$a^i \cdot r^j = a^{i'} \cdot r^j \qquad \text{for } i, i' \in I_B(r^j), \tag{4b}$$

$$a^i \cdot r^j > a^{i'} \cdot r^j \qquad \text{for } i \in I_B(r^j),\ i' \notin I_B(r^j). \tag{4c}$$

The tilting space $\mathcal{T}(B, \mathcal{Y})$ is defined for studying perturbations of the lattice-free set $M(B)$. This is done by changing or tilting the facets of $M(B)$ subject to certain constraints. Constraint (4a) requires that when we tilt facet F_i, the chosen subset Y_i of integer points continues to lie in the tilted facet; this obviously restricts how we can change the facet. Constraint (4b) implies that if a ray intersection $p^j = f + \frac{1}{\psi_B(r^j)} r^j$ lies on a facet F_i of $M(B)$, then the ray intersection $f + \frac{1}{\psi_A(r^j)} r^j$ for $M(A)$ needs to lie on the corresponding facet of $M(A)$. In particular, this means that if r^j is a corner ray of $M(B)$, then r^j must also be a corner ray for $M(A)$ if $A \in \mathcal{T}(B, \mathcal{Y})$. Constraint (4c) enforces that if a ray intersection for r^j does not lie in a facet F_i of $M(B)$, then it also does not have a ray intersection in the same facet of $M(A)$. Thus we have $I_A(r^j) = I_B(r^j)$ for all rays r^j if $A \in \mathcal{T}(B, \mathcal{Y})$.

Note that $\mathcal{T}(B, \mathcal{Y})$ is defined by linear equations and strict linear inequalities and, since $B \in \mathcal{T}(B, \mathcal{Y})$, it is non-empty. Thus it is a convex set whose dimension is the same as that of the affine space given by the equations (4a) and (4b) only. By $\mathcal{N}(B, \mathcal{Y}) \subset \mathbb{R}^{3\times 2}$ we denote the linear space parallel to this affine space, or in other words, the null space of these equations.

If $\dim \mathcal{N}(B, \mathcal{Y}) \geq 1$, we can find a matrix $\bar{A} \in \mathcal{N}(B, \mathcal{Y})$ such that $B \pm \epsilon \bar{A} \in \mathcal{T}(B, \mathcal{Y})$ for some ϵ small enough. This has the important consequence that $\gamma(B)$ can be expressed as the convex combination of $\gamma(B + \epsilon \bar{A})$ and $\gamma(B - \epsilon \bar{A})$. Thus, if both $M(B + \epsilon \bar{A})$ and $M(B - \epsilon \bar{A})$ are lattice-free polytopes and $\psi_{B+\epsilon\bar{A}}(r^j) \neq \psi_{B-\epsilon\bar{A}}(r^j)$ for some $j = 1, \dots, k$, i.e., $\gamma(B + \epsilon \bar{A}) \neq \gamma(B - \epsilon \bar{A})$, then $\gamma(B)$ is a strict convex combination of points in Δ.

The following lemma is proved in [5] using results from parametric linear programming.

Lemma 3.7 (Lemma 4.3, [5])**.** *Let $B \in \mathbb{R}^{3\times 2}$ be such that $M(B)$ is a bounded maximal lattice-free set. Then for every $\bar{A} \in \mathbb{R}^{3\times 2}$, there exists $\delta > 0$ such that for all $0 < \epsilon < \delta$, the set $Y(B + \epsilon \bar{A})$ of integer points contained in $M(B + \epsilon \bar{A})$ is a subset of $Y(B)$.*

This result, together with Lemma 4.2 and Observations 4.5 and 4.6 in [5], then implies the following lemma.

Lemma 3.8 (General Tilting Lemma)**.** *Let $B \in \mathbb{R}^{3\times 2}$ be such that $M(B)$ is a bounded maximal lattice-free set. Suppose $\mathcal{Y} = (Y_1, \dots, Y_n)$ is a covering of $Y(B)$. For any $\bar{A} \in \mathcal{N}(B, \mathcal{Y})$, there exists $\epsilon > 0$ such that:*

(i) $I_B(r^j) = I_{B+\epsilon\bar{A}}(r^j) = I_{B-\epsilon\bar{A}}(r^j)$ *for all $j = 1, \dots, k$.*

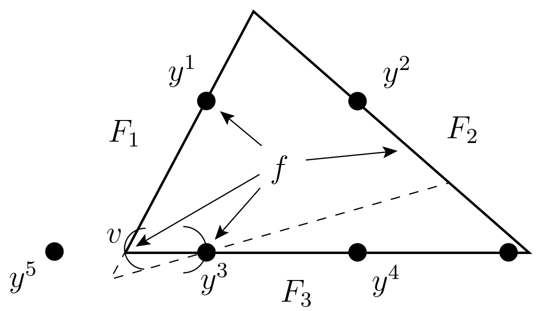

Figure 2: We depict the geometry of Lemma 3.10 and show how we can change a facet of $M(B)$ to find a new matrix $B' \in \mathbb{R}^{3\times 2}$ such that $M(B')$ is a Type 3 triangle and $\gamma(B')$ either dominates $\gamma(B)$, or $\gamma(B) = \gamma(B')$.

(ii) $\gamma(B) = \frac{1}{2}\gamma(B + \epsilon\bar{A}) + \frac{1}{2}\gamma(B - \epsilon\bar{A})$.

(iii) *Both* $M(B \pm \epsilon\bar{A})$ *are lattice-free.*

We will need the following lemma, which is a special case of Lemma 4.7 in [5].

Lemma 3.9 (Single Facet Tilt Lemma)**.** *Let* $M(B)$ *for some matrix* $B \in \mathbb{R}^{3\times 2}$ *be a maximal lattice-free triangle. Let* F *be a facet of* $M(B)$ *such that* $\operatorname{rel\,int}(F) \cap \mathbb{Z}^2 = \{y\}$ *and* $P \cap F \subset \operatorname{rel\,int}(F)$*, i.e., there are no ray intersections on the vertices of the facet* F*, i.e.,* $P \cap F \cap \operatorname{vert}(B) = \emptyset$*. If* $\operatorname{rel\,int}(F) \cap P \setminus \mathbb{Z}^2 \neq \emptyset$*, then* $\gamma(B)$ *is a strict convex combination of two points in* Δ*.*

Consider a matrix $B \in \mathbb{R}^{3\times 2}$ such that $M(B)$ is a Type 2 triangle. We label the rows of B with $i = 1, 2, 3$ and label the corresponding facets of $M(B)$ as F_1, F_2, F_3, such that F_3 is the facet containing multiple integer points. We label the unique integer points in the relative interiors of F_1 and F_2 as y^1 and y^2, respectively. The closed line segment between two points x^1 and x^2 will be denoted by $[x^1, x^2]$, and the open line segment will be denoted by (x^1, x^2). Within the case analysis of some of the proofs, we will refer to certain points lying within splits. For convenience, for $i = 1, 2, 3$, we define $S_i \in \mathbb{R}^{3\times 2}$ such that $M(S_i)$ is the maximal lattice-free split with the properties that one facet of $M(S_i)$ contains F_i and $M(S_i) \cap \operatorname{int}(M(B)) \neq \emptyset$.

Lemma 3.10 (Type 3 Dominating Type 2 Lemma)**.** *Consider any* $B \in \mathbb{R}^{3\times 2}$ *such that* $M(B)$ *is a Type 2 triangle. Denote the vertex* $F_1 \cap F_3$ *by* v *and let* $y^3 \in F_3$ *be the integer point in* $\operatorname{rel\,int}(F_3)$ *closest to* v*. Suppose* $P \cap F_3$ *is a subset of the line segment connecting* v *and* y^3*. Then there exists a matrix* $B' \in \mathbb{R}^{3\times 2}$ *such that* $M(B')$ *is a Type 3 triangle and either* $\gamma(B)$ *is dominated by* $\gamma(B')$*, or* $\gamma(B) = \gamma(B')$*.*

Proof. Choose $\bar{a}^3$ such that $\bar{a}^3 \cdot (y^3 - f) = 0$ and $\bar{a}^3 \cdot (y^3 - v) > 0$. Consider tilting F_3 by adding $\epsilon\bar{A} = \epsilon(0; 0; \bar{a}^3)$ to B for some small enough $\epsilon > 0$, so that the following two conditions are met. Firstly, ϵ is chosen small enough such that the set of integer points contained in $M(B + \epsilon\bar{A})$ is a subset of $Y(B)$; this can be done by Lemma 3.7. Secondly, since $P \cap F_3 \subset [y^3, v]$, we can choose ϵ small enough such that for all rays r such that $2 \in I_B(r)$, $I_{B+\epsilon\bar{A}}(r) = I_B(r)$. This means that if there is a ray such that its corresponding ray intersection is on F_2 in $M(B)$, then it continues to have a ray intersection on the corresponding facet in $M(B+\epsilon\bar{A})$.

Since $P\cap F_3\subset[y^3,v]$, if r is a ray pointing from f to F_3, then $r=\alpha_1(y^3-f)-\alpha_2(y^3-v)$ for some $\alpha_1,\alpha_2\geq 0$, and hence

$$\psi_{B+\epsilon\bar{A}}(r)=\psi_B(r)+\epsilon\bar{a}^3\cdot\big(\alpha_1(y^3-f)-\alpha_2(y^3-v)\big)\leq\psi_B(r). \tag{5}$$

Moreover, $\psi_{B+\epsilon\bar{A}}(r)=\psi_B(r)$ for all $r\in P\cap((F_1\cup F_2)\setminus F_3)$ since by construction $I_B(r)=I_{B+\epsilon\bar{A}}(r)$ for all such rays. Also, note that for any $y\in F_3\cap\mathbb{Z}^2$, $y=y^3+\beta(y^3-v)$ for some $\beta\geq 0$. Therefore,

$$(b^3+\epsilon\bar{a}^3)\cdot(y-f)\geq b^3\cdot(y-f)=1,$$

meaning that none of these integer points are contained in the interior of $M(B+\epsilon\bar{A})$. Since the set of integer points contained in $M(B+\epsilon\bar{A})$ is a subset of $Y(B)$ and facets F_1 and F_2 were not changed, $M(B+\epsilon\bar{A})$ is lattice-free; in fact, it is a Type 3 triangle. See Figure 2. Thus, we can choose $B'=B+\epsilon\bar{A}$. $\gamma(B)$ is dominated by $\gamma(B')$ when the inequality (5) is strict for some r; otherwise, $\gamma(B)=\gamma(B')$. □

Lemma 3.11 (Type 2 Triangles, cf. Lemma 5.11 in [5])**.** *Let $\gamma\in\Delta_2\setminus(\Delta_3\cup\Pi)$ (i.e., there does not exist a maximal lattice-free split or Type 3 triangle $M(B)$ such that $\gamma(B)=\gamma$) such that γ is not dominated by any $\gamma'\in\Delta$, and γ is not a strict convex combination of any $\gamma^1,\gamma^2\in\Delta$. Then there exists $\bar{B}\in\mathbb{R}^{3\times 2}$ with $\gamma=\gamma(\bar{B})$ such that $M(\bar{B})$ is a Type 2 triangle satisfying one of the following:*

Case a. *$P\subset\mathbb{Z}^2$.*

Case b. *$P\not\subset\mathbb{Z}^2$ and there exist $p^1\in P\cap F_1\cap F_3$ (i.e., there is a corner ray pointing from f to $F_1\cap F_3$) and $p^2\in P\cap F_3$ with $\#([p^1,p^2]\cap\mathbb{Z}^2)\geq 2$. Moreover, if $P\cap\operatorname{rel\,int}(F_2)\setminus\mathbb{Z}^2\neq\emptyset$, then there is a corner ray in $M(\bar{B})$ pointing to a vertex different from $F_1\cap F_3$. Also, one of the following holds:*

Case b1. *$f\notin M(S_3)$.*

Case b2. *$f\in M(S_3)$ and $P\not\subset F_3$.*

Case c. *$P\not\subset\mathbb{Z}^2$ and there exist $p^1\in P\cap F_1\cap F_3\cap\mathbb{Z}^2$ (i.e., there is a corner ray pointing from f to $F_1\cap F_3\subset\mathbb{Z}^2$) and $p^2\in P\cap F_1$ with $\#([p^1,p^2]\cap\mathbb{Z}^2)\geq 2$. Moreover, if $P\cap\operatorname{rel\,int}(F_2)\setminus\mathbb{Z}^2\neq\emptyset$, then p^2 can be chosen such that $p^2\in F_1\cap F_2$ (i.e., there is a corner ray pointing from f to $F_1\cap F_2$). Also, one of the following holds:*

Case c1. *$f\notin M(S_1)$.*

Case c2. *$f\in M(S_1)$ and $P\not\subset M(S_1)$.*

Case d. *$P\not\subset\mathbb{Z}^2$, for all $i\in\{1,2,3\}$ and all $p^1,p^2\in P\cap F_i$ we have $\#([p^1,p^2]\cap\mathbb{Z}^2)\leq 1$, there exists a corner ray pointing from f to $F_1\cap F_3$, and $F_1\cap F_3\not\subset\mathbb{Z}^2$. Let $y^3,y^4\in F_3$ such that y^3 is the closest integer point in $\operatorname{rel\,int}(F_3)$ to $F_1\cap F_3$, and y^4 is the next closest integer point. Let $H_{2,4}$ be the half-space adjacent to $[y^2,y^4]$ and containing y^1.*

Then, we further have $P\cap(y^3,y^4)\neq\emptyset$. Moreover, one of the following holds:

Case d1. *$f\notin H_{2,4}$, there exists a corner ray from f to $F_1\cap F_2$.*

Case d2. *$f\notin H_{2,4}$, there exists a ray pointing from f through (y^1,y^2) to F_1 and there are no rays pointing from f to $\operatorname{rel\,int}(F_2)\setminus\mathbb{Z}^2$.*

Case d3. *$f\in H_{2,4}$, $P\not\subset H_{2,4}$, and there exists a corner ray pointing from f to $F_1\cap F_2$.*

Furthermore, the number of vectors $\gamma(B)$ such that $M(B)$ is a Type 2 triangle satisfying the conditions in Cases a, b, c and d, is polynomial in the binary encoding sizes of $f,r^1,\ldots,r^k$.

Proof. Consider any $\gamma \in \Delta_2$. By definition of Δ_2, there exists a matrix $B \in \mathbb{R}^{3\times 2}$ such that $\gamma(B) = \gamma$ and $M(B)$ is a Type 2 triangle. Recall the labeling of the facets of $M(B)$ as F_1, F_2, F_3 with corresponding labels for the rows of B.

Let P denote set of the ray intersections in $M(B)$. If $P \subset \mathbb{Z}^2$, then we set $\bar{B} = B$ and we are in Case a. Therefore, in the remainder of the proof, we always assume $P \not\subset \mathbb{Z}^2$.

Proof steps 1 and 2: Dominated, convex combination, or Case d.

Suppose $P \not\subset \mathbb{Z}^2$ and for all $i \in \{1, 2, 3\}$ and all $p^1, p^2 \in P \cap F_i$, we have $\#([p^1, p^2] \cap \mathbb{Z}^2) \leq 1$. We will show that at least one of the following occurs:

(i) $\gamma(B)$ is dominated by some $\gamma' \in \Delta$, or is a strict convex combination of some $\gamma^1, \gamma^2 \in \Delta$, or there exists a maximal lattice-free split or Type 3 triangle $M(B')$ such that $\gamma(B') = \gamma(B)$.

(ii) Either Case d1, Case d2, or Case d3 occurs.

First note that there cannot exist corner rays pointing to different vertices of F_3 because there are multiple integer points on F_3. Otherwise, if r^1, r^2 are corner rays that point to different vertices of F_3 with ray intersections p^1, p^2, then $\#([p^1, p^2] \cap \mathbb{Z}^2) \geq 2$, violating the assumptions.

Consider the sub-lattice of $\mathbb{Z}^2$ contained in the linear space parallel to F_3. We use the notation $v(F_3)$ to denote the primitive lattice vector which generates this one-dimensional lattice and lies in the same direction as the vector pointing from $F_1 \cap F_3$ to $F_2 \cap F_3$. Since $\#([p^1, p^2] \cap \mathbb{Z}^2) \leq 1$ for all $p^1, p^2 \in P \cap F_3$, there exists $y^3 \in F_3 \cap \mathbb{Z}^2$ such that $P \cap F_3 \subset (y^3 - v(F_3), y^3 + v(F_3))$. Let $y^4 = y^3 + v(F_3)$ and let $y^5 = y^3 - v(F_3)$ and so $P \cap F_3$ is a subset of the *open* segment (y^5, y^4). Note that y^4, y^5 are not necessarily contained in F_3. In Step 1 we will analyze the case with no corner rays on F_3 and see that we always arrive in conclusion (i), whereas in Step 2 we will analyze the case with a corner ray on F_3 and see that we will also arrive in conclusion (i), except for the last step, Step 2d, where we arrive in conclusion (ii).

Step 1. Suppose that F_3 has no corner rays, i.e., $\text{vert}(B) \cap P \cap F_3 = \emptyset$.

Step 1a. Suppose $P \cap (F_1 \cup F_2) \setminus \mathbb{Z}^2 \neq \emptyset$. We will use the tilting space to show that $\gamma(B)$ is a strict convex combination of points in Δ. Let $Y_1 = \{y^1\}, Y_2 = \{y^2\}, Y_3 = F_3 \cap \mathbb{Z}^2$, $\mathcal{Y} = \{Y_1, Y_2, Y_3\}$ and consider $\mathcal{T}(B, \mathcal{Y})$.

We first count the equations that define $\mathcal{T}(B, \mathcal{Y})$. The equation $a^3 = b^3$ is implicit in $\mathcal{T}(B, \mathcal{Y})$ since there are multiple integer points on F_3. There are two other equations for integer points on F_1 and F_2. Now there are two cases depending on whether there is a corner ray pointing from f to $F_1 \cap F_2$.

Suppose first that such a corner ray does exist; call it r^1. The defining equations for the set $\mathcal{T}(B, \mathcal{Y})$ are

$$a^1 \cdot (y^1 - f) = 1, \qquad a^2 \cdot (y^2 - f) = 1, \qquad a^1 \cdot r^1 = a^2 \cdot r^1, \qquad a^3 = b^3.$$

Since $\mathcal{N}(B, \mathcal{Y}) \subset \mathbb{R}^6$ and there are 5 equations (note that $a_3 = b_3$ is actually two equations), we see $\dim(\mathcal{N}(B, \mathcal{Y})) \geq 1$. Let $\bar{A} = (\bar{a}^1; \bar{a}^2; \bar{a}^3) \in \mathcal{N}(B, \mathcal{Y}) \setminus \{0\}$. Since $\mathcal{Y}$ is a covering

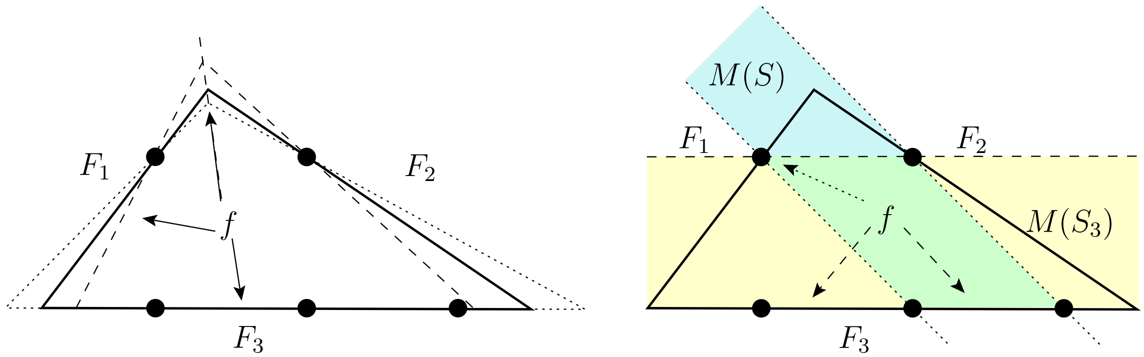

Figure 3: Steps 1a and 1b1. The left figure depicts Step 1a where $P \cap (F_1 \cup F_2) \neq \emptyset$ and there are no corner rays on F_3, and shows that $\gamma(B)$ is a strict convex combination of other points in Δ by finding two lattice-free triangles through tilting the facets F_1 and F_2. The right figure depicts Step 1b1 where we find a split $M(S)$ such that $\gamma(S)$ dominates $\gamma(B)$.

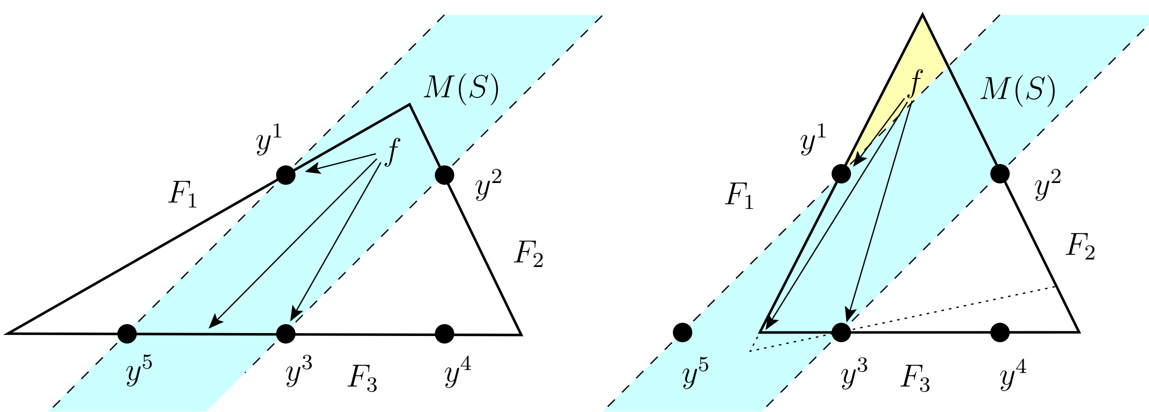

Figure 4: Step 1b2. In this step we consider $f \notin M(S_3)$. On the left we see that $P \cap F_3 \subset [y^3, y^4]$ and both $y^3, y^4 \in F_3$, which allows $\gamma(B)$ to be dominated by $\gamma(S)$. This split satisfies $\gamma(S) \in \Delta$ because $f \notin M(S_3)$, meaning that f is located somewhere on the top of the triangle, which is completely contained by $M(S)$. On the right, $y^4 \notin F_3$, which means that the split S cuts off the top corner of the triangle, potentially leaving f outside the split. This is problematic, so instead, we use Lemma 3.10 to create a new Type 3 triangle $M(B')$ such that $\gamma(B')$ dominates $\gamma(B)$.

of the lattice points in $M(B)$, by Lemma 3.8, there exists an $\epsilon > 0$ such that $\gamma(B) = \frac{1}{2}\gamma(B+\epsilon\bar{A}) + \frac{1}{2}\gamma(B-\epsilon\bar{A})$ and $M(B \pm \epsilon\bar{A})$ are both lattice-free. See Figure 3 for these possible triangles.

We next show that $\gamma(B-\epsilon\bar{A}) \neq \gamma(B+\epsilon\bar{A})$. Observe that $\bar{a}^3 = 0$ since we are restricted by the equation $a^3 = b^3$. If $\bar{a}^1 = 0$, then $\bar{a}^2$ must satisfy $\bar{a}^2 \cdot r^1 = 0$ and $\bar{a}^2 \cdot (y^2 - f) = 0$, which implies that $\bar{a}^2 = 0$ since r^1 and $y^2 - f$ are linearly independent (since $y^2 \in \operatorname{rel\,int}(F_2)$ and r^1 points to a corner of F_2). Similarly, if $\bar{a}^2 = 0$, then $\bar{a}^1 = 0$. Since $\bar{A} \neq 0$, we must have both $\bar{a}^1, \bar{a}^2 \neq 0$. Moreover, this argument shows that $\bar{a}^1 \cdot r^1 = \bar{a}^2 \cdot r^1 \neq 0$. Since $I_B(r) = I_{B+\epsilon\bar{A}}(r)$ by Lemma 3.8, we get that $\psi_{B-\epsilon\bar{A}}(r^1) = (b^1 - \epsilon\bar{a}^1)\cdot r^1 \neq (b^1 + \epsilon\bar{a}^1)\cdot r^1 = \psi_{B+\epsilon\bar{A}}(r^1)$. Thus, we have the explicit strict convex combination $\gamma(B) = \frac{1}{2}\gamma(B+\epsilon\bar{A}) + \frac{1}{2}\gamma(B-\epsilon\bar{A})$ and $M(B \pm \epsilon\bar{A})$ are lattice-free triangles.

Now suppose such a corner ray does not exist. Then we have 4 defining equations for $\mathcal{T}(B,\mathcal{Y})$:

$$a^1 \cdot (y^1 - f) = 1, \qquad a^2 \cdot (y^2 - f) = 1, \qquad a^3 = b^3.$$

This implies $\dim(\mathcal{N}(B,\mathcal{Y})) = 2$ since we only have 4 independent equations defining $\mathcal{T}(B,\mathcal{Y})$. Since $P \cap (F_1 \cup F_2) \setminus \mathbb{Z}^2 \neq \emptyset$, there exists a ray r such that the corresponding ray intersection $p \in \operatorname{rel\,int}(F_i) \setminus \mathbb{Z}^2$ for $i = 1$ or $i = 2$. Without loss of generality, assume $i = 1$. Consider the linear space $\bar{\mathcal{N}} = \mathcal{N}(B,\mathcal{Y}) \cap \{ A \in \mathbb{R}^{3\times 2} \mid a^2 = 0 \}$. Now $\dim(\bar{\mathcal{N}}) = 1$ and we pick $\bar{A} \in \bar{\mathcal{N}} \setminus \{0\}$. Then we must have $\bar{a}^1 \neq 0$. Since $\mathcal{Y}$ is a covering of the lattice points in $M(B)$, by Lemma 3.8, there exists an $\epsilon > 0$ such that $\gamma(B) = \frac{1}{2}\gamma(B+\epsilon\bar{A}) + \frac{1}{2}\gamma(B-\epsilon\bar{A})$ and $M(B \pm \epsilon\bar{A})$ are both lattice-free. Moreover, $\bar{a}^1 \cdot r \neq 0$ since r and $y^1 - f$ are linearly independent and $\bar{a}^1 \cdot (y^1 - f) = 0$. Hence, $\psi_{B-\epsilon\bar{A}}(r) \neq \psi_{B+\epsilon\bar{A}}(r)$. Therefore, we again conclude that $\gamma(B)$ is the strict convex combination of $\gamma(B \pm \epsilon\bar{A})$.

Step 1b. Suppose $P \cap (F_1 \cup F_2) \setminus \mathbb{Z}^2 = \emptyset$, i.e., there only exist rays pointing from f to F_3, y^1, y^2. Therefore, $P \subset M(S_3)$.

Step 1b1. If $f \in M(S_3)$, then $\gamma(B)$ is either dominated by or equal to $\gamma(S_3)$. If $P \cap F_3 = \emptyset$, then $P \subset \{y^1, y^2\} \subset \mathbb{Z}^2$, which is a contradiction with the assumption of Step 1 that $P \not\subset \mathbb{Z}^2$.

Step 1b2. Suppose that $f \notin M(S_3)$ and $P \cap F_3 \neq \emptyset$. Suppose further that either $P \cap F_3 \subset [y^3, y^4]$ or $P \cap F_3 \subset [y^5, y^3]$, and without loss of generality, assume $P \subset [y^5, y^3]$.

If both $y^5, y^3 \in F_3$, then $\gamma(B)$ is dominated by $\gamma(S)$ where S is the maximal lattice-free split with its two facets along $[y^3, y^2]$ and $[y^5, y^1]$. Note that we have domination because $P \not\subset \mathbb{Z}^2$ and so there exists a ray intersection lying in the open segment (y^5, y^3).

Otherwise, suppose $y^5 \notin F_3$. Note that $y^3 \notin \operatorname{vert}(B)$, because otherwise since $P \cap F_3 \neq \emptyset$, we find that $P \cap F_3 \subset [y^5, y^3] \cap F_3 = \{y^3\}$, and therefore, $y^3 \in P$, contradicting the fact that there are no corner rays on F_3. Thus y^3 is the integer point in $\operatorname{rel\,int}(F_3)$ closest to $F_1 \cap F_3$. This implies that $M(B)$ satisfies the hypotheses of Lemma 3.10. Hence there exists B' such that $M(B')$ is a Type 3 triangle and either $\gamma(B)$ is dominated by $\gamma(B')$ or $\gamma(B) = \gamma(B')$.

Step 1b3. Suppose that $f \notin M(S_3)$, $P \cap F_3 \neq \emptyset$, and $P \not\subset [y^3, y^4]$, $P \not\subset [y^5, y^3]$, i.e., $\operatorname{conv}(P \cap F_3)$ contains the integer point y^3 in its relative interior.

Let $F_3' = F_3 \cap [y^5, y^4]$. Let F_1' and F_2' be given by lines from the endpoints of F_3' through y^1 and y^2, respectively, and let $B' \in \mathbb{R}^{3\times 2}$ such that $M(B')$ has facets F_1', F_2', F_3'. See Figure 5.

Claim α. $M(B')$ is lattice-free.

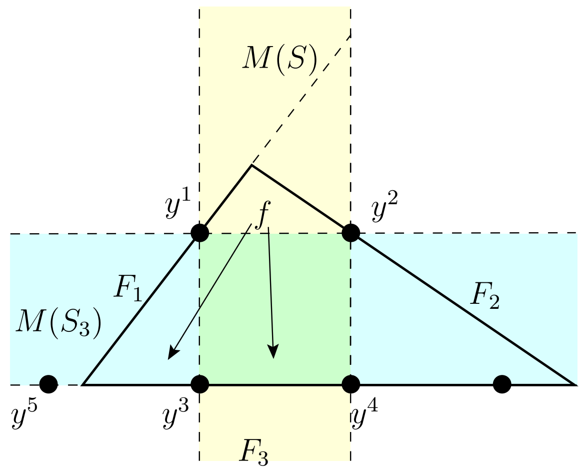

Figure 5: Step 1b3. This figure demonstrates a new triangle $M(B')$ that yields the same inequality as $M(B)$.

Proof. First note that $\#(F_3 \cap [y^5, y^4] \cap \mathbb{Z}^2) \geq 2$ since y^3 is in the relative interior of F_3 and F_3 contains multiple integer points. Without loss of generality, suppose $y^4 \in F_3$. Let $S \in \mathbb{R}^{3\times 2}$ such that $M(S)$ is the maximal lattice-free split with facets on $[y^1, y^3]$ and $[y^2, y^4]$. Then $M(B')$ is lattice-free since $M(B') \subset M(S) \cup M(B)$, which are both lattice-free sets. □

Claim β. $f \in M(B')$ and $\gamma(B') = \gamma(B)$.

Proof. Since $f \in M(B) \cap M(S)$ and $M(B) \cap M(S) \subset M(B') \cap M(S)$, it follows that $f \in M(B')$. Recall that we are under the assumption that all rays point to y^1, y^2 or F_3. Moreover, all ray intersections on F_3 are contained in (y^5, y^4) and hence the ray intersections are contained in F_3'. Therefore, the set of ray intersections P' with respect to $M(B')$ is the same as P, and therefore $\gamma(B) = \gamma(B')$. □

Since $\text{conv}(P \cap F_3)$ contains y^3 in its relative interior and $P \cap F_3$ is contained in the open segment (y^5, y^4), we must have $P \cap \text{rel int}(F_3) \setminus \mathbb{Z}^2 \neq \emptyset$. Furthermore, if P' is the set of ray intersections for $M(B')$, then $P' \cap F_3' = P \cap F_3$ by definition of F_3'. Therefore, $P' \cap \text{rel int}(F_3') \setminus \mathbb{Z}^2 \neq \emptyset$. Furthermore, $P' \cap \text{vert}(B') \cap F_3' = \emptyset$ since $M(B)$ has no corner rays pointing to F_3 and there cannot exist rays pointing to y^4 or y^5 since $P \cap F_3$ is contained in the open segment (y^5, y^4). Moreover, $\text{rel int}(F_3') \cap \mathbb{Z}^2 = \{y^3\}$. Therefore, Lemma 3.9 can be applied to $M(B')$ with $F = F_3'$, which shows that $\gamma(B') = \gamma(B)$ is a strict convex combination of other points in Δ.

Step 2. Suppose there is a corner ray in F_3 and, if necessary, relabel the facets of $M(B)$ (and the rows of B) such that this corner ray points from f to the intersection $F_1 \cap F_3$. Recall that we label the integer points $y^1 \in F_1$, $y^2 \in F_2$. Since $F_1 \cap F_3 \subseteq P$, observe that $y^3 \in F_3$ (as defined in the paragraph before Step 1) is the closest integer point in F_3 to $F_1 \cap F_3$, and since $M(B)$ is a Type 2 triangle, we have $y^4 \in F_3$. Let $H_{2,4}$ be the half-space with boundary containing the segment $[y^2, y^4]$ and with interior containing y^1. See Figure 7.

Step 2a. Suppose $y^3 \in F_1 \cap F_3$ and recall that there is a corner ray pointing from f to $F_1 \cap F_3$. Note that this implies that $P \cap F_2 \cap \text{vert}(B) = \emptyset$, because $\#([p^1, p^2] \cap \mathbb{Z}^2) \leq 1$ for all

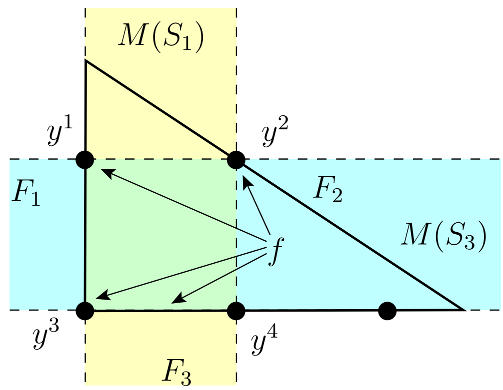

Figure 6: Step 2a. Depending on where f is located in the triangle, at least one of $\gamma(S_3)$ or $\gamma(S_1)$ either dominates or realizes $\gamma(B)$. In this picture, $\gamma(S_3)$ realizes $\gamma(B)$, while $\gamma(S_1) \notin \Delta$ since $f \notin \operatorname{int} M(S_1)$.

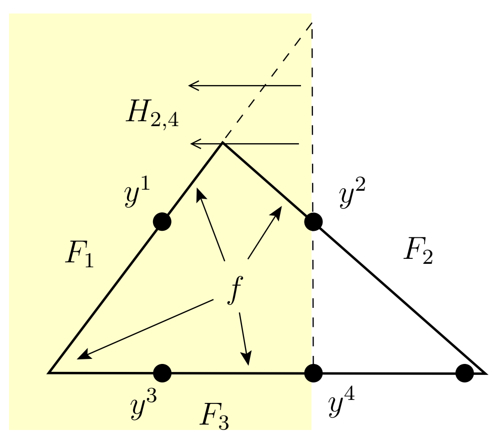

Figure 7: Step 2b. We show that if $P \cup \{f\} \subset H_{2,4}$, then we can create a different Type 2 triangle $M(B')$ such that $\gamma(B')$ dominates $\gamma(B)$. If $\gamma(B) = \gamma(B')$, i.e., the ray pointing from f to the facet F_2 does not exist in the above picture, then the new triangle is a Type 2 triangle that was considered in Step 1a.

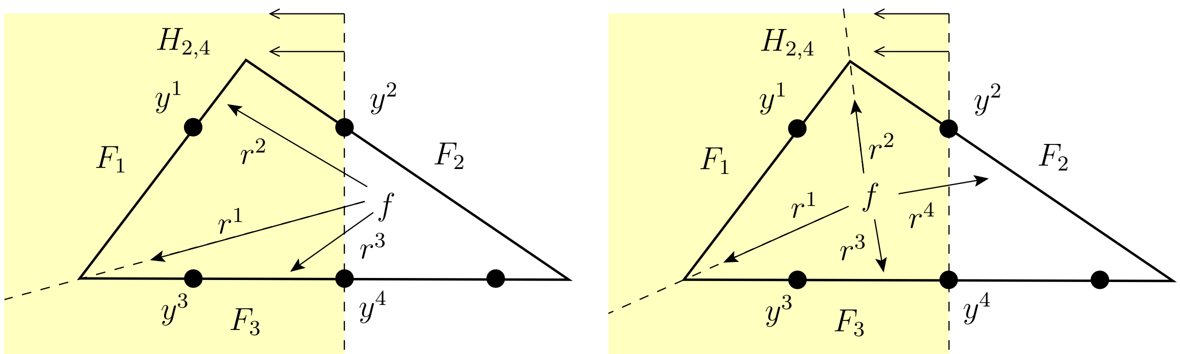

Figure 8: Steps 2d1 and 2d2. This figure depicts the defining features of Cases d1, d2, and d3. Other rays may also exist. Cases d1 and d2, because of their commonalities, are represented in one picture on the left, and Case d3 is on the right.

$p^1, p^2 \in P \cap F_i$ for any $i \in \{1, 2, 3\}$ and including any corner ray pointing from f to $F_1 \cap F_2$ or $F_2 \cap F_3$ would contradict this. Therefore, $P \cap F_2 \subset \operatorname{rel\,int}(F_2)$.

If $P \cap F_2 \setminus \mathbb{Z}^2 = P \cap \operatorname{rel\,int}(F_2) \setminus \mathbb{Z}^2 \neq \emptyset$, then $M(B)$ satisfies the hypotheses of Lemma 3.9 with $F = F_2$ and $\gamma(B)$ is a strict convex combination of points in Δ.

If instead $P \cap F_2 \setminus \mathbb{Z}^2 = \emptyset$, then $P \cap F_2 \subset \{y^2\}$, and since $F_1 \cap F_3 \subseteq P$ and no two ray intersections within a facet can contain two integer points between them, we must have $P \cap F_1 \subseteq [y^3, y^1]$ and $P \cap F_3 \subseteq [y^3, y^4]$. Therefore, $P \subset \operatorname{conv}(\{y^1, y^2, y^3, y^4\})$. Since $M(S_1) \cup M(S_3) \supset M(B)$, we must have $P \cup \{f\} \subset M(S_i)$ for $i = 1$ or 3, and hence $\gamma(B)$ is either dominated by or equal to $\gamma(S_i)$. See Figure 6.

Step 2b. Suppose $y^3 \notin F_1 \cap F_3$ and $P \cup \{f\} \subset H_{2,4}$. Let $B' \in \mathbb{R}^{3\times 2}$ such that $M(B')$ is the lattice-free Type 2 triangle with base F_3' along $[y^2, y^4]$, the facet F_1' given by the line defining F_1 for $M(B)$ and the facet F_2' given by the line defining F_3 for $M(B)$. Let P' be the set of ray intersections for $M(B')$. See Figure 7.

If $P \cap \operatorname{rel\,int}(F_2) \setminus \{y^2\} \neq \emptyset$, then $\gamma(B')$ dominates $\gamma(B)$ because $P \cup \{f\} \subset H_{2,4}$.

Otherwise, $\gamma(B) = \gamma(B')$ and $P \cap \operatorname{rel\,int}(F_2) \setminus \{y^2\} = \emptyset$. This implies that no ray points from f to the corner $F_1' \cap F_3'$ of $M(B')$. Recall that $P \cap F_3$ is a subset of the open segment (y_5, y_4), therefore, $y^4 \notin P$. Hence, $M(B')$ has no corner rays on F_3'. Also, since there exists a corner ray pointing from f to $F_1 \cap F_3 = F_1' \cap F_2'$, we see that $P' \cap (F_1' \cup F_2') \setminus \mathbb{Z}^2 \neq \emptyset$. Hence, $M(B')$ is a Type 2 triangle satisfying the conditions considered in Step 1a, and using the same reasoning from that step, $\gamma(B') = \gamma(B)$ can be shown to be a strict convex combination of points in Δ.

Step 2c. Suppose $P \cap F_3 \subset [F_1 \cap F_3, y^3]$, $y^3 \notin F_1 \cap F_3$, and that $y^5 \notin F_3$. This implies again that y^3 is the closest integer point in F_3 to the corner $F_1 \cap F_3$. Then $M(B)$ satisfies the hypotheses of Lemma 3.10 and we can find a Type 3 triangle $M(B')$ such that $\gamma(B)$ is dominated by $\gamma(B')$ or $\gamma(B) = \gamma(B')$.

Step 2d. We can now assume that $P \not\subset \mathbb{Z}^2$ (the assumption for Steps 1 and 2), there is a corner ray pointing from f to $F_1 \cap F_3$ (assumption in beginning of Step 2), $y^3 \notin F_1 \cap F_3$ (negation of the assumption in Step 2a), $P \cup \{f\} \not\subset H_{2,4}$ (negation of the second assumption in Step 2b), and $(y^3, y^4) \cap P \neq \emptyset$ (negation of the assumption in Step 2c), which implies

$y^3 \in \text{int}(\text{conv}(P \cap F_3))$. Furthermore, we may be in one of the following subcases.

Step 2d1. $f \notin H_{2,4}$. This implies $f \in M(S_3)$ since $M(B) \setminus H_{2,4} \subset M(S_3)$. If P is also contained in $M(S_3)$, then either $\gamma(B)$ is dominated by $\gamma(S_3)$, or $\gamma(B) = \gamma(S_3)$. Therefore, we assume $P \not\subset M(S_3)$, and so there must exist a ray r pointing from f through (y^1, y^2).

Suppose there is a ray that points from f to $\text{rel int}(F_2)$. If there is no corner ray pointing from f to $F_1 \cap F_2$, then $M(B)$ would satisfy the hypotheses of Lemma 3.9 with $F = F_2$ since no ray points to $F_2 \cap F_3$. Therefore, $\gamma(B)$ can be expressed as the strict convex combination of points from Δ. On the other hand, if there is a corner ray pointing to $F_1 \cap F_2$, then we satisfy the statement of Case d1.

Suppose now that no ray points from f to $\text{rel int}(F_2)$. This implies that the ray r points from f to F_1 through (y^1, y^2) and $P \cap \text{rel int}(F_2) \setminus \mathbb{Z}^2 = \emptyset$. This is Case d2.

Step 2d2. $f \in H_{2,4}$ and $P \not\subset H_{2,4}$. Because also $P \cap F_3 \subseteq H_{2,4}$, this implies that $P \cap \text{rel int}(F_2) \setminus \mathbb{Z}^2 \neq \emptyset$.

If there is no corner ray pointing from f to $F_1 \cap F_2$, then $M(B)$ satisfies the hypotheses of Lemma 3.9 with $F = F_2$ because there is no ray intersection in $F_2 \cap F_3$. Then $\gamma(B)$ can be expressed as the strict convex combination of points from Δ. On the other hand, if there is a corner ray pointing from f to $F_1 \cap F_2$, then we satisfy the statement of Case d3.

From the analysis of Steps 1 and 2, we can set $\bar{B} = B$ and conclude that if $P \not\subset \mathbb{Z}^2$, γ is not dominated by any $\gamma' \in \Delta$, is not a strict convex combination of any $\gamma^1, \gamma^2 \in \Delta$, and there does not exist a maximal lattice-free split or Type 3 triangle $M(B')$ such that $\gamma(B') = \gamma$, then one of the following holds:

(i) There exist ray intersections $p^1, p^2 \in P \cap F_i$ with $\#([p^1, p^2] \cap \mathbb{Z}^2) \geq 2$ for some $i \in \{1, 2, 3\}$.

(ii) We are in Case d.

Proof steps 3 and 4: Remaining cases.

We now assume that $\gamma = \gamma(B)$ is not dominated by any $\gamma' \in \Delta$, is not a strict convex combination of any $\gamma^1, \gamma^2 \in \Delta$, and there does not exist a maximal lattice-free split or Type 3 triangle $M(B')$ such that $\gamma(B') = \gamma(B)$, and we are not in Case d, and we are not in Case a (so $P \not\subset \mathbb{Z}^2$). Therefore, from our previous analysis, there exist ray intersections $p^1, p^2 \in P \cap F_i$ with $\#([p^1, p^2] \cap \mathbb{Z}^2) \geq 2$ for some $i \in \{1, 2, 3\}$. We will show that either Case b1, b2, c1, or c2 occurs. In Step 3 below, we analyze the case when $i = 3$ and in Step 4, we analyze the case when $i = 1$ or $i = 2$.

Step 3. Suppose $P \not\subset \mathbb{Z}^2$ and there exist $p^1, p^2 \in P \cap F_3$ with $\#([p^1, p^2] \cap \mathbb{Z}^2) \geq 2$. Without loss of generality, we label p^1, p^2 such that $P \cap F_3 \subseteq [p^1, p^2]$.

Step 3a. We first show that there exists a matrix B' such that $M(B')$ is a lattice-free Type 2 triangle that has a corner ray in F_3, and $\gamma(B) = \gamma(B')$.

If either p^1 or p^2 is a vertex of $M(B)$, then we let $B' = B$ and move to Step 3b.

We now deal with the case that $p^1, p^2 \notin \text{vert}(B)$, i.e., there are no corner rays pointing from f to F_3.

Suppose first that there exists $\hat{r} \in \{r^1, \ldots, r^k\}$ such that its ray intersection $\hat{p} \in F_1 \cap F_2$, i.e., $\hat{r}$ is a corner ray on F_1 and F_2. We now use the tilting space to argue that $\gamma(B)$ is a strict

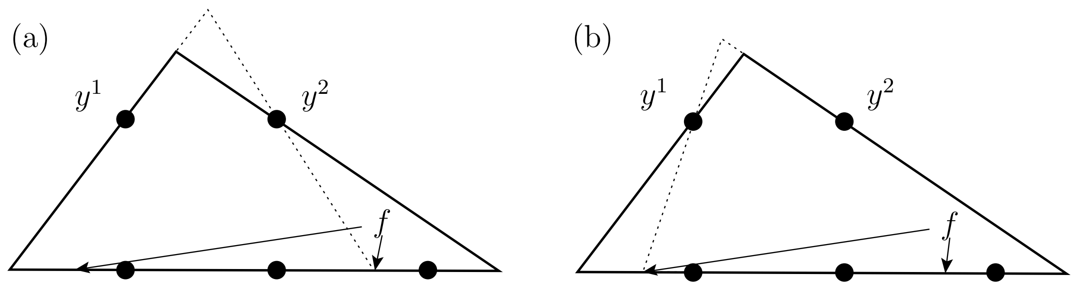

Figure 9: Step 3a, either F_1 or F_2 is tilted to give a new triangle $M(B')$ (dotted). (a) Here F_2 cannot be used because tilting would remove f from the interior. (b) Instead, F_1 needs to be used.

convex combination of other points in Δ. We define $\mathcal{Y} = (Y_1, Y_2, Y_3)$ as $Y_1 = \{y^1\}$, $Y_2 = \{y^2\}$ and $Y_3 = F_3 \cap \mathbb{Z}^2$. Hence, $\mathcal{Y}$ is a covering of $Y(B)$. Since there is no corner ray pointing to a vertex other than $F_1 \cap F_2$, there is only one independent equation coming from a corner ray condition in the system defining $\mathcal{T}(B, \mathcal{Y})$. Y_1 and Y_2 each contribute one equation. Since Y_3 contains two integer points, it contributes a system of equalities involving a^3 with rank 2. Therefore, $\dim \mathcal{N}(B, \mathcal{Y}) = 6 - 5 = 1$.

We pick any $\bar{A} \in \mathcal{N}(B, \mathcal{Y}) \setminus \{0\}$. By Lemma 3.8, there exists $\epsilon > 0$ such that $\gamma(B \pm \epsilon \bar{A}) \in \Delta$ and $\gamma(B)$ is a convex combination of these two vectors. We now show that $\psi_{B+\epsilon\bar{A}}(\hat{r}) \neq \psi_{B-\epsilon\bar{A}}(\hat{r})$. Note that the equations from Y_3 impose that $\bar{a}^3 = 0$. Therefore, either $\bar{a}^1 \neq 0$ or $\bar{a}^2 \neq 0$. Without loss of generality, assume $\bar{a}^1 \neq 0$. Observe now that $y^1 - f$ and $\hat{r}$ are linearly independent since y^1 is in the relative interior of F_1 and $\hat{p}$ is a vertex of F_1. Since Y_1 imposes $\bar{a}^1 \cdot (y^1 - f) = 0$, this implies that $\bar{a}^1 \cdot \hat{r} \neq 0$. Therefore, $\psi_{B+\epsilon\bar{A}}(\hat{r}) = (b^1 + \epsilon\bar{a}^1) \cdot \hat{r} \neq (b^1 - \epsilon\bar{a}^1) \cdot \hat{r} = \psi_{B-\epsilon\bar{A}}(\hat{r})$; the equalities follow from the fact that $I_{B\pm\epsilon\bar{A}}(\hat{r}) = I_B(\hat{r})$ by Lemma 3.8.

So we can assume that $p^1, p^2 \notin \text{vert}(B)$ and $F_1 \cap F_2 \not\subset P$, i.e., $M(B)$ has no corner rays. Since F_1 and F_2 do not have corner rays, we must have $\operatorname{rel\,int}(F_i) \cap P \setminus \mathbb{Z}^2 = \emptyset$ for $i = 1, 2$ because otherwise Lemma 3.9, applied to F_1 or F_2, shows that $\gamma(B)$ is a strict convex combination of points in Δ.

For $i = 1, 2$, since $\operatorname{rel\,int}(F_i) \cap (P \setminus \mathbb{Z}^2) = \emptyset$, tilting F_i to now lie on the line through p^i and y^i does not change $\psi_B(r^j)$ for $j = 1, \ldots, k$, unless f is no longer in the interior of the set. At most one of these facet tilts puts f outside the perturbed set, thus at least one of them is possible. This is illustrated in Figure 9. We can assume that the tilt of facet F_1 is possible (with a relabeling of the facets of $M(B)$ and the rows of B, if necessary). Let the set after tilting be $M(B')$ and B' be the corresponding matrix. We label the facets of $M(B')$ as F_1', F_2' and F_3', where F_1' corresponds to the new tilted F_1 and F_2', F_3' correspond to F_2, F_3 respectively.

We claim that $M(B')$ is lattice-free. To see this, let $y^3, y^4 \in [p^1, p^2] \cap \mathbb{Z}^2$ be distinct integer points adjacent to each other. Then consider the maximal lattice-free split $M(S)$, where $S \in \mathbb{R}^{3\times 2}$, with facets through $[y^3, y^1]$ and $[y^4, y^2]$. Since $[y^3, y^4] \subset [p^1, F_2 \cap F_3]$ is a strict subset, the new intersection at $F_1' \cap F_2'$ is a subset of the split, and hence $M(B') \subset M(B) \cup M(S)$. Therefore $M(B')$ is lattice-free.

Step 3b. If $P \cap \operatorname{rel\,int}(F_2') \setminus \mathbb{Z}^2 \neq \emptyset$, then there is a corner ray on F_2' (and thus pointing to a vertex different from $F_1' \cap F_3'$); otherwise, $M(B')$ satisfies the hypotheses of Lemma 3.9 and $\gamma(B) = \gamma(B')$ could be expressed as a strict convex combination of points in Δ, which is a contradiction.

Therefore, if we set $\bar{B} = B'$, the conditions of Case b are met. Furthermore, if $P \cup \{f\} \subset M(S_3)$, then $\gamma(B) = \gamma(B')$ is dominated by or equal to $\gamma(S_3)$, hence either Case b1 or Case b2 occurs.

Thus, from the analysis of Step 3, when there exist $p^1, p^2 \in P \cap F_3$ with $\#([p^1, p^2] \cap \mathbb{Z}^2) \geq 2$, we can find a matrix $\bar{B}$ such that $M(\bar{B})$ is a Type 2 triangle satisfying the statement of Case b.

Step 4. Suppose $P \not\subset \mathbb{Z}^2$ and there exist $p^1, p^2 \in P \cap F_i$ with $\#([p^1, p^2] \cap \mathbb{Z}^2) \geq 2$, for $i = 1$ or $i = 2$. After a relabeling of the facets of $M(B)$ and the rows of B, we can assume $i = 1$. In order for $\#([p^1, p^2] \cap \mathbb{Z}^2) \geq 2$, it has to equal exactly two, and one of the points, say p^1, must lie in $F_1 \cap F_3 \cap \mathbb{Z}^2$. Thus, p^1 corresponds to a corner ray.

If $P \cap \operatorname{rel\,int}(F_2) \setminus \mathbb{Z}^2 \neq \emptyset$, then again, there must be a corner ray on F_2; otherwise, Lemma 3.9 shows that $\gamma(B)$ is a strict convex combination of points in Δ. We can assume that this corner ray points from f to $F_1 \cap F_2$, otherwise we are back to the assumptions in Step 3 and $M(B)$ will satisfy the conditions of Case b. Thus p^2 can be chosen such that $p^2 \in F_1 \cap F_2$.

As in Case b, if $P \cup \{f\} \subset M(S_1)$, then $\gamma(B)$ is dominated by or equal to $\gamma(S_1)$. Hence, if we set $\bar{B} = B$, we are either in Case c1 or Case c2.

Proof step 5: Polynomially many cases.

Recall that we have a set of k rays $\{r^1, \ldots, r^k\}$ and P is the set of ray intersections. Given this set of rays, we count how many distinct vectors $\gamma(B)$ can arise when $M(B)$ satisfies the conditions in Cases a, b, c and d. We will apply Lemma 3.3 to show that there are only polynomially many possibilities for $\gamma(B)$ in each case.

Case a. We need to count the vectors $\gamma(B)$ with corresponding $M(B)$ such that $P \subset \mathbb{Z}^2$. Consider the set Q of closest integer points that the rays $\{r^1, \ldots, r^k\}$ point to from f. If $\operatorname{conv}(Q)$ is a lattice-free set and it is contained in one or more Type 2 triangles, then we choose any such Type 2 triangle and we will have $P = Q$. Moreover, all of these triangles yield the same vector $\gamma(B)$. If $\operatorname{conv}(Q)$ is not lattice-free or is not contained in a Type 2 triangle, then there does not exist a Type 2 triangle whose set of ray intersections is P. Therefore there is at most one possibility for $\gamma(B)$ which arises from Case a.

Before we move onto Cases b, c and d, we make an important observation, which will be used repeatedly below. Given vectors $\bar{r}^1, \bar{r}^2 \in \mathbb{R}^2$, recall the notations $C(\bar{r}^1, \bar{r}^2)$ and $(C(\bar{r}^1, \bar{r}^2))_I$ from Lemma 3.3.

Claim γ. Consider any Type 2 triangle $M(B)$. Suppose there exist two rays $\bar{r}^1, \bar{r}^2$ such that the corresponding ray intersections $\bar{p}^1, \bar{p}^2$ are on a facet F of $M(B)$.

(i) If $\operatorname{conv}(\bar{p}^1, \bar{p}^2) \cap \mathbb{Z}^2 = \{y\}$ and y is in the relative interior of $\operatorname{conv}(\bar{p}^1, \bar{p}^2)$, then y is a vertex of the integer hull $(C(\bar{r}^1, \bar{r}^2))_I$. Moreover, the line $\operatorname{aff}(F)$ is a supporting hyperplane for $(C(\bar{r}^1, \bar{r}^2))_I$, i.e., $(C(\bar{r}^1, \bar{r}^2))_I$ lies on one side of this line.

(ii) If $\operatorname{conv}(\bar{p}^1, \bar{p}^2) \cap \mathbb{Z}^2$ contains at least two points, then the line $\operatorname{aff}(F)$ defines a facet of the integer hull $(C(\bar{r}^1, \bar{r}^2))_I$.

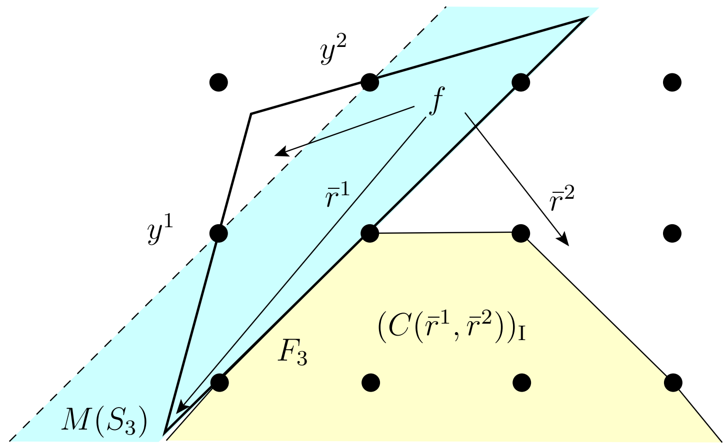

Figure 10: Counting a polynomial number of Type 2 triangles in Case b

Proof. Suppose H is the halfspace corresponding to F that contains f. Then $H \cap C(\bar{r}^1, \bar{r}^2) \subset M(B)$ and since $M(B)$ does not contain any integer points in its interior, neither does $H \cap C(\bar{r}^1, \bar{r}^2)$. Since we assume $\mathrm{conv}(\bar{p}^1, \bar{p}^2) \cap \mathbb{Z}^2$ is non-empty and $\bar{p}^1, \bar{p}^2$ lie on the line defining H (and also F), this line is a supporting hyperplane for $(C(\bar{r}^1, \bar{r}^2))_I$.

If $\mathrm{conv}(\bar{p}^1, \bar{p}^2) \cap \mathbb{Z}^2$ contains the single point y and $y \in \mathrm{rel\,int}(\mathrm{conv}(\bar{p}^1, \bar{p}^2))$, then clearly y is an extreme point of $(C(\bar{r}^1, \bar{r}^2))_I$.

If $\mathrm{conv}(\bar{p}^1, \bar{p}^2) \cap \mathbb{Z}^2$ contains two (or more) points, then the line defining H (and also F) defines a facet of $(C(\bar{r}^1, \bar{r}^2))_I$. □

With this in mind, we proceed to analyze Cases b, c and d.

Case b. We now count the number of $\gamma(B)$ such that $M(B)$ satisfies the conditions of Case b with respect to our set of rays $\{r^1, \dots, r^k\}$. Consider any such $M(B)$. From the conditions stated in Case b, we can assume that $M(B)$ has a corner ray on F_3. We label as $\bar{r}^1, \bar{r}^2$ the two rays whose corresponding ray intersections are on F_3, so that $\bar{r}^1$ points to $F_1 \cap F_3$ and the ray intersection of $\bar{r}^2$ is closest to $F_2 \cap F_3$; and so $\bar{r}^1$ is a corner ray by the statement of Case b. There are $2 \times \binom{k}{2}$ ways to choose $\bar{r}^1, \bar{r}^2$ from the set $\{r^1, \dots, r^k\}$ with one of them as the corner ray. See Figure 10. By Claim γ, this means that $\mathrm{aff}(F_3)$ defines a facet of $(C(\bar{r}^1, \bar{r}^2))_{\mathrm{I}}$. By Lemma 3.3, we have polynomially many choices for $\mathrm{aff}(F_3)$. Once we make a choice for $\mathrm{aff}(F_3)$, we look at the possible choices for y^1, y^2, which are the integer points on F_1, F_2, respectively.

In Case b1, where $f \notin M(S_3)$, y^1, y^2 are given uniquely by where f is. To see this, we observe a few things. Let y^3 and y^4 be the integer points on F_3 that are closest to $F_1 \cap F_3$. The split with one side going through y^1, y^3 and the other side going through y^2, y^4 contains f. Now consider the family of maximal lattice-free splits with one side going through y^3 and the other side going through y^4. Observe that since $f \notin M(S_3)$, only one member of this family of splits contains f. This then uniquely determines y^1 and y^2.

In Case b2, $P \not\subset M(S_3)$, which implies that there exists a ray $\bar{r}^3$ such that $\bar{r}^3$ points between y^1 and y^2. Moreover, since y^1, y^2 have to lie on the lattice plane adjacent to F_3, we have a unique choice for y^1, y^2 once we choose $\bar{r}^3$ from our set of k rays. Now $\bar{r}^3$ can be chosen in $O(k)$ ways and so there are $O(k)$ ways to pick y^1, y^2.

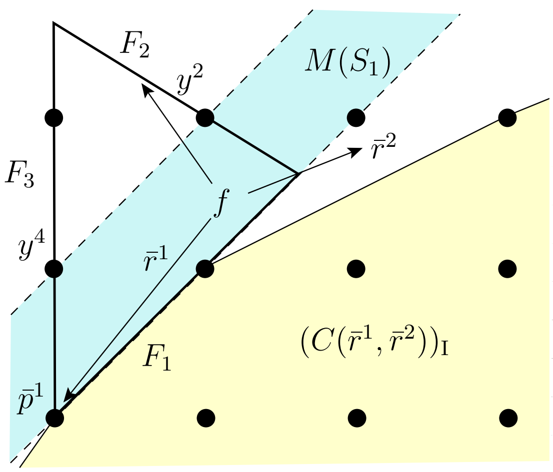

Figure 11: Counting a polynomial number of Type 2 triangles in Case c

We already know there is a corner ray pointing to $F_1 \cap F_3$. By the statement of Case b, either $P \cap \operatorname{rel\,int}(F_2) \setminus \mathbb{Z}^2 \neq \emptyset$, in which case we have a corner ray in $M(B)$ pointing to a different vertex, or $P \cap \operatorname{rel\,int}(F_2) \setminus \mathbb{Z}^2 = \emptyset$. If $M(B)$ has a corner ray pointing to a vertex different from $F_1 \cap F_3$, then we can choose it in $O(k)$ ways, and the triangle is uniquely determined by these two corner rays, $\operatorname{aff}(F_3)$, y^1, and y^2.

On the other hand, if $M(B)$ has corner rays pointing only to $F_1 \cap F_3$ (one of which is $\bar{r}^1$), then the facet F_2 has no non-integer ray intersections in its relative interior. Therefore, any possible choice of this facet such that no ray points to $\operatorname{rel\,int}(F_2) \setminus \mathbb{Z}^2$ will give a triangle that yields the same vector $\gamma(B)$.

Hence, there are only polynomially many possibilities for Case b.

Case c. We now count the vectors $\gamma(B)$ such that $M(B)$ satisfies the conditions of Case c with respect to our set of rays $\{r^1, \dots, r^k\}$. Consider any such $M(B)$. Then there exist $\bar{r}^1, \bar{r}^2$ such that $\bar{p}^1, \bar{p}^2 \in F_1$ and $\#([\bar{p}^1, \bar{p}^2] \cap \mathbb{Z}^2) \geq 2$, where $\bar{p}^1, \bar{p}^2$ are the ray intersections for $\bar{r}^1, \bar{r}^2$, respectively. Moreover, $\bar{p}^1$ is an integer point on the facet F_3. There are $2 \times \binom{k}{2}$ ways to choose $\bar{r}^1, \bar{r}^2$ from the set $\{r^1, \dots, r^k\}$ with $\bar{r}^1$ pointing from f to $F_1 \cap F_3$. See Figure 11.

We next choose $\operatorname{aff}(F_1)$ as the affine hull of a facet of $(C(\bar{r}^1, \bar{r}^2))_{\mathrm{I}}$, using Claim γ. There is a unique choice for $\operatorname{aff}(F_1)$ because $\bar{p}^1$ is an integer point and so $\bar{p}^1$ is the vertex of the unbounded facet of $(C(\bar{r}^1, \bar{r}^2))_{\mathrm{I}}$ that lies on the ray $f + \mathbb{R}_+ \bar{r}^1$. Hence $\operatorname{aff}(F_1)$ is equal to the affine hull of the other, bounded, facet of $(C(\bar{r}^1, \bar{r}^2))_{\mathrm{I}}$ that is incident with the vertex $\bar{p}^1$.

Now we pick the integer points y^2, y^4 where y^2 is the integer point on the facet F_2 of $M(B)$ and y^4 is the integer point in the relative interior of F_3 that is closest to $\bar{p}^1$. This analysis is the same as with Cases b1 and b2. In Case c1, these points are uniquely determined by f. In Case c2, these are uniquely determined by one of the rays pointing between them. There are $O(k)$ ways of choosing this ray.

The statement of Case c implies that either there is also a corner ray pointing to $F_1 \cap F_2$, or $P \cap \operatorname{rel\,int}(F_2) \setminus \mathbb{Z}^2 = \emptyset$.

If there is a corner ray pointing to $F_1 \cap F_2$, then the triangle is uniquely determined by the two corner rays, $\operatorname{aff}(F_1)$, y^2, and y^4.

On the other hand, if $P \cap \operatorname{rel\,int}(F_2) \setminus \mathbb{Z}^2 = \emptyset$, then F_2 can be chosen in any possible way such that no ray points to $\operatorname{rel\,int}(F_2) \setminus \mathbb{Z}^2$. Then the triangle is uniquely determined by $r^1, \operatorname{aff}(F_1), \operatorname{aff}(F_2), y^2$, and y^4.

Therefore, there are only polynomially many possibilities for Case c.

Case d. We consider Type 2 triangles with a corner ray $\bar{r}^1$ pointing from f to $F_1 \cap F_3$. We label the closest integer point in $\operatorname{rel\,int}(F_3)$ to $F_1 \cap F_3$ as y^3, and the next closest integer point in $\operatorname{rel\,int}(F_3)$ as y^4. Also, since $P \cap (y^3, y^4) \neq \emptyset$, there exists a ray $\bar{r}^3$ that points from f through (y^3, y^4) (we use the notation $\bar{r}^3$ to remind ourselves that it points to F_3). Moreover, the condition that no two ray intersections on F_3 can contain two (or more) integer points between them implies that the ray intersections on F_3 are contained in the segment $[F_1 \cap F_3, y^4]$. As before, y^1 and y^2 will denote the integer points on the facets F_1 and F_2.

Case d1 and Case d2. For these two cases, there exists a ray $\bar{r}^2$ that points from f through (y^1, y^2) to F_1 (for example, in Case d1 this will be the corner ray pointing from f to $F_1 \cap F_2$). Observe that $\operatorname{conv}(\bar{p}^1, \bar{p}^2) \cap \mathbb{Z}^2 = \{y^1\}$ and $\operatorname{conv}(\bar{p}^1, \bar{p}^3) \cap \mathbb{Z}^2 = \{y^3\}$, where y^1 and y^3 lie in the relative interiors of $\operatorname{conv}(\bar{p}^1, \bar{p}^2)$ and $\operatorname{conv}(\bar{p}^1, \bar{p}^3)$, respectively. We now count the choices of these triangles.

First pick rays $\bar{r}^1, \bar{r}^2, \bar{r}^3$, for which there are $\binom{k}{3}$ ways to do this. Pick y^1 as a vertex of $(C(\bar{r}^1, \bar{r}^2))_{\mathrm{I}}$ and pick y^3 as a vertex of $(C(\bar{r}^1, \bar{r}^3))_{\mathrm{I}}$, utilizing Claim γ (i). By Lemma 3.3, there are only polynomially many ways to do this.

Claim δ. The vector $\gamma(B)$ is uniquely determined by the choices of $\bar{r}^1, \bar{r}^2, \bar{r}^3, y^1$, and y^3 in Case d1 and Case d2.

Proof. First note that $[y^2, y^4]$ is necessarily parallel to $[y^1, y^3]$. Therefore, regardless of the choice of y^2, y^4, the half-space $H_{2,4}$ is already determined by $[y^1, y^3]$. Recall that, by assumption, $f \notin H_{2,4}$. Define the family of splits

$$\mathcal{H} = \{\, S \mid y^1, y^3 \in S, \text{ and } S \cap \operatorname{int} H_{2,4} \neq \emptyset, \ S \text{ is a maximal lattice-free split} \,\}.$$

For any distinct $S_1, S_2 \in \mathcal{H}$, since they both contain $[y^1, y^3]$, we find that $S_1 \cap S_2 \setminus H_{2,4} = \emptyset$. Since $f \notin H_{2,4}$, there exists a unique $S \in \mathcal{H}$ such that $f \in S$. Therefore, the unique choices for y^2, y^4 are the two points given by $\partial S \cap \partial H_{2,4}$ where ∂ denotes the boundary of a set.

Now we show how to choose $M(B)$. The affine hull of facet F_3 is determined by the segment $[y^3, y^4] \subset F_3$, and $\operatorname{aff}(F_1)$ is determined by $[\bar{p}^1, y^1] \subset F_1$, where $\bar{p}^1$ is the corner ray intersection of $\bar{r}^1$ on F_3. Lastly, $\operatorname{aff}(F_2)$ must be chosen. For Case d1, $\bar{r}^2$ is chosen as a corner ray pointing from f to $F_1 \cap F_2$, and therefore, $\operatorname{aff}(F_2)$ is determined by the ray intersection $\bar{p}^2$ of $\bar{r}^2$ on F_1, and by y^2, i.e., by the segment $[\bar{p}^2, y^2]$. For Case d2, any choice of $\operatorname{aff}(F_2)$ such that there are no rays pointing from f to $\operatorname{rel\,int}(F_2) \setminus \mathbb{Z}^2$ and such that $f \in M(B)$ will yield the same vector $\gamma(B)$, thus, we only need to consider one such triangle. □

Since the vector $\gamma(B)$ is uniquely determined by these choices, there are only polynomially many possibilities for this case.

Case d3. For this case, there exists a corner ray $\bar{r}^2$ that points from f to $F_1 \cap F_2$. Since $P \not\subset H_{2,4}$, there also exists a ray $\bar{r}^4$ such that it points from f through (y^2, y^4). Since $\bar{r}^1$ is a corner ray pointing from f to $F_1 \cap F_3$ and the ray intersections are contained in $[F_1 \cap F_3, y^4]$, $\bar{r}^4$ must be chosen to point from f to F_2.

We now count triangles of this description. First pick rays $\bar{r}^1, \bar{r}^2, \bar{r}^3, \bar{r}^4$ from the set $\{r^1, \dots, r^k\}$. There are at most $\binom{k}{4}$ ways to do this. Pick y^1 as a vertex of $(C(\bar{r}^1, \bar{r}^2))_{\mathrm{I}}$, y^2 as a vertex of $(C(\bar{r}^2, \bar{r}^4))_{\mathrm{I}}$, and y^3 as a vertex of $(C(\bar{r}^1, \bar{r}^3))_{\mathrm{I}}$. By Lemma 3.3, there are only polynomially many ways to do this. Then y^4 is uniquely determined since y^1, y^2, y^3, y^4 form an area 1 parallelogram. The affine hull of F_3 is uniquely determined, since it runs along $[y^3, y^4]$. Since $\bar{r}^1$ is a corner ray pointing to $F_1 \cap F_3$, the choice of y^1 fixes $\mathrm{aff}(F_1)$. Finally, since $\bar{r}^2$ is a corner ray pointing to $F_1 \cap F_2$, the choice of y^2 fixes $\mathrm{aff}(F_2)$. Therefore, there are only polynomially Type 2 triangles satisfying the conditions of this case.

This concludes the proof of the fact that there are only a polynomial (in the binary encoding sizes of $f, r^1, \dots, r^k$) number of vectors $\gamma(B)$ such that $M(B)$ is a Type 2 triangle satisfying Cases a, b, c and d. $\square$

Proof of Proposition 3.5. Let Ξ_2 be the set of vectors $\gamma(B)$ for $B \in \mathbb{R}^{3\times 2}$ such that $M(B)$ satisfies cases a, b, c, or d of Lemma 3.11. Then Ξ_2 has the desired properties. $\square$

Proof of Proposition 3.1. Let $\Xi = \Xi_0 \cup \Xi_1 \cup \Xi_2 \cup \Xi_3$ using the sets Ξ_i from Propositions 3.4 and 3.5. We show that for any $\gamma \in \Delta \setminus \Xi$, γ is dominated by some $\gamma' \in \Delta$, or γ is a strict convex combination of some $\gamma_1, \gamma_2 \in \Delta$.

If $\gamma \notin \Pi \cup \Delta_1 \cup \Delta_2 \cup \Delta_3$, then $\gamma = \gamma(B)$ for some $B \in \mathbb{R}^{3\times 2}$ such that $M(B)$ is *not* a maximal lattice-free convex set, and further, γ cannot be realized by a maximal lattice-free split or triangle. This implies that there exists $B' \in \mathbb{R}^{3\times 2}$ such that $M(B')$ is a maximal lattice-free convex set containing $M(B)$ and γ is dominated by $\gamma(B')$.

So we consider $\gamma \in \Pi \cup \Delta_1 \cup \Delta_2 \cup \Delta_3$. Observe that $\Pi \cup \Delta_1 \cup \Delta_2 \cup \Delta_3 = \Pi \cup (\Delta_1 \setminus \Pi) \cup \Delta_3 \cup (\Delta_2 \setminus (\Delta_3 \cup \Pi))$ and so γ is in one of the sets Π, $\Delta_1 \setminus \Pi$, Δ_3 or $\Delta_2 \setminus (\Delta_3 \cup \Pi)$. Since $\gamma \notin \Xi_0 \cup \Xi_1 \cup \Xi_2 \cup \Xi_3$, we have that γ is in one of the four sets $(\Pi \setminus \Xi_0)$, $(\Delta_1 \setminus (\Xi_1 \cup \Pi))$, $(\Delta_3 \setminus \Xi_3)$ or $(\Delta_2 \setminus (\Xi_2 \cup \Delta_3 \cup \Pi))$. Now it follows from Propositions 3.4 and 3.5 that γ is dominated by some $\gamma' \in \Delta$, or γ is a strict convex combination of some $\gamma^1, \gamma^2 \in \Delta$. Furthermore, the cardinality $\#\Xi$, being bounded above by the sum of the cardinalities of Ξ_i, $i = 0, \dots, 3$, is polynomial in the binary encoding sizes of $f, r^1, \dots, r^k$. $\square$

Theorem 3.12. *There exists a finite set $\Xi \subset \Delta$ such that if γ is an extreme point of Δ', then $\gamma \notin \Delta \setminus \Xi$. Furthermore, the cardinality $\#\Xi$ is polynomial in the encoding sizes of $f, r^1, \dots, r^k$.*

Proof. Let Ξ be the set from Proposition 3.1. Since $\Delta \subseteq \Delta'$, together with Observation 2.5 and the definition of extreme point, this implies that $\Delta \setminus \Xi$ does not contain any extreme points of Δ'. $\square$

4 Proof of Theorem 1.2

We first state the following proposition.

Proposition 4.1. *Let $\mathcal{B}$ be a family of matrices in $\mathbb{R}^{3\times 2}$. If there exists $\epsilon > 0$ such that $\bar{B}(f, \epsilon) \subseteq M(B)$ for all $B \in \mathcal{B}$, then there exists a real number M depending only on ϵ such that $\|B\| < M$ for all $B \in \mathcal{B}$.*

Proof. Since $\bar{B}(f,\epsilon) \subseteq M(B)$, the point $f + \epsilon b_i \in M(B)$, where b^i is the i-th row of B. Therefore, $b^i \cdot (f + \epsilon b^i - f) \le 1$. Therefore, $\|b^i\| \le \frac{1}{\sqrt{\epsilon}}$. Since this holds for every row b^i, there exists M depending only on ϵ such that $\|B\| < M$. □

We will use the following set to derive a bound on a sequence of matrices to show there exists a convergent subsequence. For any vector $\gamma \in \mathbb{R}^k_+$, define

$$M_\gamma = \operatorname{conv}(\{f\} \cup \{\, f + \tfrac{1}{\gamma_j} r^j \mid \gamma_j \neq 0 \,\}) + \operatorname{cone}(\{\, r^j \mid \gamma_j = 0 \,\}).$$

Observation 4.2. *For all $B \in \mathbb{R}^{3\times 2}$ we have the inclusion $M_{\gamma(B)} \subseteq M(B)$.*

Proof. Clearly $f \in M(B)$. Next observe that $f + \frac{1}{\psi_B(r^j)} r^j \in M(B)$ if $\psi_B(r^j) > 0$. Finally, $\psi_B(r^j) = 0$ implies that r^j is in the recession cone of $M(B)$. The claim follows. □

Theorem 4.3. *Assume that $f \in \mathbb{Q}^2$ and $r^j \in \mathbb{Q}^2$ for all $j \in \{1,\dots,k\}$. If $\operatorname{cone}(\{r^1,\dots,r^k\}) = \mathbb{R}^2$, then Δ' has a polynomial (in the binary encoding sizes of $f, r^1, \dots, r^k$) number of extreme points.*

Proof. Consider any extreme point x of Δ'. By Observation 2.5, $x \in \operatorname{cl}(\operatorname{conv}(\Delta))$. By Lemma 2.6, there exists a sequence of points a^n from Δ such that a^n converges to x.

Claim α. There exists a bounded sequence of matrices $B_n \in \mathbb{R}^{3\times 2}$ such that $\gamma(B_n) = a^n$ and $M(B_n)$ is a lattice-free.

Proof. Since a^n converges to x, there exists $N \in \mathbb{N}$ such that $a^n_i \le x_i + 1$ for every $n \ge N$ and $i \in \{1,\dots,k\}$ where the notation is that y_i denotes the i-th component of a vector $y \in \mathbb{R}^k$. Since $a^n \in \Delta$, there exists a sequence of matrices B_n such that $a^n = \gamma(B_n)$ and $M(B_n)$ is lattice-free. Consider the sequence of polyhedra $M_{\gamma(B_n)}$. Let $\epsilon = \frac{1}{1+\max_i x_i}$.

By the definition of N, for every $n \ge N$, we have that $\frac{1}{a^n_i} \ge \epsilon$. Since the conical hull of the rays $r^1,\dots,r^k$ is $\mathbb{R}^2$, this implies that there exists $\bar{\epsilon}$ such that $B(f,\bar{\epsilon}) \subseteq M_{\gamma(B_n)}$ for all $n \ge N$. By Observation 4.2, $M_{\gamma(B_n)} \subseteq M(B_n)$. Therefore, for every $n \ge N$, $B(f,\bar{\epsilon}) \subseteq M(B_n)$. Proposition 4.1 implies that there exists a real number M depending only on $\bar{\epsilon}$ such that $\|B_n\| \le M$ for all $n \ge N$. This implies that B_n is a bounded sequence. □

By the Bolzano–Weierstrass theorem, we can extract a convergent subsequence $\bar{B}_n$ converging to a point $\bar{B}$. The map $B \mapsto \gamma(B)$ is continuous because $\psi_B(r)$ is continuous in B for every fixed r. Therefore, $\gamma(\bar{B}_n)$ converges to $\gamma(\bar{B})$. By assumption $a^n = \gamma(B_n)$ converges to x and therefore $\gamma(\bar{B}) = x$. Moreover, since $M(\bar{B}_n)$ is lattice-free for all $n \in \mathbb{N}$, $M(\bar{B})$ is also lattice-free and hence it is a lattice-free triangle or a lattice-free split. Thus, for every extreme point x of Δ', we have shown that $x \in \Delta$.

Let Ξ be the set from Theorem 3.12. Since every extreme point x of Δ' is in Δ, Theorem 3.12 implies that the set of extreme points of Δ' is a subset of Ξ. Since $\#\Xi$ is polynomial in the encoding sizes of $f, r^1, \dots, r^k$, we have shown this property for the number of extreme points Δ' as well. □

This implies the following corollary.

Corollary 4.4. *Assume that $f \in \mathbb{Q}^2$ and $r^j \in \mathbb{Q}^2$ for all $j \in \{1, \ldots, k\}$. If* $\operatorname{cone}(\{r^1, \ldots, r^k\}) = \mathbb{R}^2$*, then the triangle closure T is a polyhedron with a polynomial (in the binary encoding sizes of $f, r^1, \ldots, r^k$) number of facets.*

Proof. Lemma 2.4 and Theorem 4.3 together imply the corollary. □

We can now finally prove Theorem 1.2.

Proof of Theorem 1.2. If $\operatorname{cone}(\{r^1, \ldots, r^k\}) = \mathbb{R}^2$ then Corollary 4.4 gives the result. Otherwise we add "ghost" rays $r^{k+1}, \ldots, r^{k'}$ such that $\operatorname{cone}(\{r^1, \ldots, r^k, r^{k+1}, \ldots, r^{k'}\}) = \mathbb{R}^2$. Now consider the system (1) with the rays $r^1, \ldots, r^{k'}$. We can similarly define the triangle closure T' for this extended system. We use the notation $\gamma_k(B) = \gamma(B)$ to emphasize the dimension and set $\gamma_{k'}(B) = (\psi_B(r^i))_{i=1}^{k'}$. So T' is defined as

$$T' = \left\{ s \in \mathbb{R}^{k'}_+ \;\middle|\; \gamma_{k'}(B) \cdot s \geq 1 \text{ for all } B \text{ such that } M(B) \text{ is a lattice-free triangle} \right\}.$$

Claim α. $T' \cap \{s_{k+1} = 0, \ldots, s_{k'} = 0\} = T \times \{0^{k'-k}\}$.

Proof. Consider any point $s \in T' \cap \{s_{k+1} = 0, \ldots, s_{k'} = 0\}$ and let $s^k = (s_1, \ldots, s_k)$ be the truncation of s to the first k coordinates. Consider any $a \in \mathbb{R}^k$ such that $a = \gamma_k(B)$ for some matrix B where $M(B)$ is a lattice-free triangle. Consider $a' = \gamma_{k'}(B)$. Clearly, $a'_i = a_i$ for $i \in \{1, \ldots, k\}$. Since $a' \cdot s \geq 1$ and $a' \cdot s = a \cdot s^k$, we have that $a \cdot s^k \geq 1$. So, $s \in T \times \{0^{k'-k}\}$.

For the reverse inclusion, consider a point $s \in T \times \{0^{k'-k}\}$ and let $s^k = (s_1, \ldots, s_k)$ be the truncation of s to the first k coordinates. Consider any $a' \in \mathbb{R}^{k'}$ such that $a' = \gamma_{k'}(B)$ for some matrix B where $M(B)$ is a lattice-free triangle. Let $a = \gamma_k(B)$. As before, $a'_i = a_i$ for $i \in \{1, \ldots, k\}$. Since $a \cdot s^k \geq 1$ and $a' \cdot s = a \cdot s^k$, we have that $a' \cdot s \geq 1$. So, $s \in T' \cap \{s_{k+1} = 0, \ldots, s_{k'} = 0\}$. □

Since $\operatorname{cone}(\{r^1, \ldots, r^k, r^{k+1}, \ldots, r^{k'}\}) = \mathbb{R}^2$, Corollary 4.4 says that T' is a polyhedron with a polynomial (in the binary encoding sizes of $f, r^1, \ldots, r^k$) number of facets. Since $T' \cap \{s_{k+1} = 0, \ldots, s_{k'} = 0\} = T \times \{0^{k'-k}\}$, this shows that T is a polyhedron with a polynomial number of facets. □

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