On t-branch split cuts for mixed-integer programs

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Abstract

In this paper we study the t-branch split cuts introduced by Li and Richard (2008). They presented a family of mixed-integer programs with n integer variables and a single continuous variable and conjectured that the convex hull of integer solutions for any n has unbounded rank with respect to $(n-1)$-branch split cuts. It was shown earlier by Cook, Kannan and Schrijver (1990) that this conjecture is true when $n = 2$, and Li and Richard proved the conjecture when $n = 3$. In this paper we show that this conjecture is also true for all $n > 3$.

1 Introduction and definitions

In their seminal paper, Cook, Kannan and Schrijver [1] introduced split cuts and showed that the split closure of a polyhedral mixed-integer set is again a polyhedron. They also presented a very simple mixed-integer set (with two integer variables and one continuous variable) whose convex hull cannot be obtained with split cuts alone. As Gomory mixed-integer (GMI) cuts are split cuts, this example also proves that Gomory's cutting plane algorithm for mixed-integer programs [3] will fail to terminate on an associated three variable MIP. Li and Richard [4] introduced a generalization of split cuts which they call t-branch split cuts. They studied a specific family of mixed-integer sets (one for each $n \geq 2$, with n integer variables and one continuous variable). When $n = 2$, the mixed-integer set is the same as the Cook, Kannan and Schrijver example mentioned above. Li and Richard showed that each set in their family with $n \geq 3$ has a valid inequality with unbounded rank with respect to 2-branch split cuts. They also conjectured that $(n-1)$-branch split cuts are also not enough to obtain finite rank. In this paper, we prove that their conjecture is correct.

1.1 Split cuts and the Cook-Kannan-Schrijver example

Consider a polyhedral mixed-integer set of the form

$$P = \Big\{(x, y) \in \mathbb{Z}^n \times \mathbb{R}^l \ : \ Ax + Gy \leq b\Big\},$$

where A, G and b have m rows and rational components. We call P a "polyhedral" mixed-integer set as its continuous relaxation, denoted by P^{LP}, is defined by linear inequalities.

A *split disjunction* is a set of the form $D(\pi, \gamma) = D_1 \cup D_2$, where

$$D_1 = \big\{(x, y) \in \mathbb{R}^{n+l} \ : \ \pi^T x \leq \gamma\big\} \text{ and } D_2 = \big\{(x, y) \in \mathbb{R}^{n+l} \ : \ \pi^T x \geq \gamma + 1\big\}$$

for some $\pi \in \mathbb{Z}^n, \gamma \in \mathbb{Z}$. Notice that if $x \in \mathbb{Z}^n$ then $\pi^T x \in \mathbb{Z}$ and consequently $\mathbb{Z}^n \times \mathbb{R}^l \subseteq D(\pi, \gamma)$. Therefore $P \subseteq D(\pi, \gamma)$ and $D(\pi, \gamma)$ is called a valid disjunction for P for any $\pi \in \mathbb{Z}^n, \gamma \in \mathbb{Z}$. We define the *split set* associated with the disjunction $D(\pi, \gamma)$ as follows,

$$S(\pi, \gamma) = \{(x, y) \in \mathbb{R}^{n+l} \ : \ \gamma < \pi^T x < \gamma + 1\}$$

and note that $D(\pi, \gamma) = \mathbb{R}^{n+l} \setminus S(\pi, \gamma)$. Therefore, $S(\pi, \gamma) \cap (\mathbb{Z}^n \times \mathbb{R}^l) = \emptyset$. We denote the closure of the set $S(\pi, \gamma)$ by $\bar{S}(\pi, \gamma)$ and refer to it as a *closed split set*. If the coefficients of the vector π are not co-prime, the split set is strictly contained in (or *dominated* by) another split set.

An inequality $c^T x + d^T y \leq f$ is called a *split cut* for P if it is valid for both $P^{LP} \cap D_1$ and $P^{LP} \cap D_2$. Multiple split cuts can be generated from the same split disjunction. The points in P^{LP} satisfying all split cuts that can be generated from disjunctions $D(\pi, \gamma)$ for all $\pi \in \mathbb{Z}^n, \gamma \in \mathbb{Z}$ form the split closure of P which is denoted by $P^{[1]}$. It is clearly possible to repeat this procedure; we use $P^{[k+1]}$ to denote the split closure of $P^{[k]}$ for $k \geq 1$.

Split disjunctions and split cuts were studied by Cook, Kannan, and Schrijver in [1] where they show that $P^{[1]}$, and consequently, $P^{[k]}$ for any finite k, is a polyhedral set. This is not a straightforward result as there are infinitely many split disjunctions and consequently, infinitely many split cuts might potentially be needed to define $P^{[1]}$. Furthermore, they also show that a mixed-integer set P can have facet-defining inequalities that are not valid for $P^{[k]}$ for any finite k. More precisely, they argue that for any fixed $\epsilon > 0$, the convex hull of the set

$$P = \Big\{(x, y) \in \mathbb{Z}^2 \times \mathbb{R} \ : \ (x, y) \in \text{conv}\big\{(0,0,0), (2,0,0), (0,2,0), (1/2, 1/2, \epsilon)\big\}\Big\},$$

cannot be obtained with split cuts. Notice that $\text{conv}\,(P) = \text{conv}\{(0,0,0), (2,0,0), (0,2,0)\}$, and consequently $y \leq 0$ is a valid inequality for P. Cook, Kannan, and Schrijver argue that this inequality is not valid for $P^{[k]}$ for any finite k by showing that for any $k \geq 1$, there exists a number $\epsilon_k > 0$, such that $(1/2, 1/2, \epsilon_k) \in P^{[k]}$ (here $\epsilon > \epsilon_1$ and $\epsilon_k > \epsilon_{k+1}$ for any k).

1.2 *t-branch split cuts and the Li-Richard example*

Li and Richard [4] defined a generalization of split disjunctions called *t-branch split disjunctions* which are obtained by intersecting t split disjunctions. Given $\pi_i \in \mathbb{Z}^n$ and $\gamma_i \in \mathbb{Z}$ for $i \in T = \{1, \ldots, t\}$, the associated t-branch split disjunction can be defined as

$$D(\pi_1, \ldots, \pi_t, \gamma_1, \ldots, \gamma_t) = \mathbb{R}^{n+l} \ \setminus \ \bigcup_{i \in T} S(\pi_i, \gamma_i).$$

As $S(\pi, \gamma) \cap (\mathbb{Z}^n \times \mathbb{R}^l) = \emptyset$ for each $i \in T$, clearly $\mathbb{Z}^n \times \mathbb{R}^l \subseteq D(\pi_1, \ldots, \pi_t, \gamma_1, \ldots, \gamma_t)$ and therefore t-branch split disjunctions are valid disjunctions. Also note that in this terminology a split disjunction is called a 1-branch split disjunction. A t-branch split disjunction is defined in [4] as

$$D(\pi_1, \ldots, \pi_t, \gamma_1, \ldots, \gamma_t) = \bigcup_{U \subseteq T} \big\{(x, y) \in \mathbb{R}^{n+l} \ : \pi_i^T x \leq \gamma_i \text{ if } i \in U,\ \pi_i^T x \geq \gamma_i + 1 \text{ if } i \notin U\big\}, \qquad (1)$$

and is equivalent to the previous definition.

Given a fixed t-branch split disjunction D, we define *the disjunctive hull of P with respect to D* as

$$P_D = \text{conv}\left(P^{LP} \cap D\right) = \text{conv}\Big(P^{LP} \setminus \bigcup_{i \in T} S(\pi_i, \gamma_i)\Big) \supseteq P.$$

A *t-branch split cut* is an inequality valid for P_D for some t-branch split disjunction D. Clearly, any t-branch split cut is valid for P. The *t-branch split closure* of P, denoted by $P_t^{[1]}$, is the set of points satisfying all t-branch split cuts, i.e., $P_t^{[1]} = \cap_{D \in \mathcal{D}} P_D$ where $\mathcal{D}$ is the set of all t-branch split disjunctions. As in the case of split disjunctions, it is possible to repeat this procedure and define $P_t^{[k+1]}$ to denote the t-branch split closure of $P_t^{[k]}$ for any integer $k \geq 1$. Clearly $P \subseteq P_t^{[k]} \subseteq P_t^{[k-1]} \subseteq \ldots \subseteq P^{LP}$. It is not known if $P_t^{[1]}$ is a polyhedon for $t > 1$.

In [4], Li and Richard give a set $P \subseteq \mathbb{Z}^n \times \mathbb{R}$ and show that a certain valid inequality for this set is not valid for $P_2^{[k]}$ for any finite k. We use the same set in this paper and present it in the next section.

1.3 Valid inequalities as t-branch split cuts

Let $c^T x + d^T y \leq f$ be a given valid inequality for P and let $V \subseteq \mathbb{R}^n$ be the points in P^{LP} that violate this inequality. In other words,

$$V = \{(x, y) \in P^{LP} \ : \ c^T x + d^T y > f\}. \tag{2}$$

Let D be a t-branch split disjunction given by the split sets $S(\pi_i, \gamma_i)$ (for $i \in T$) of appropriate dimension. In this case, it is easy to see that

$$V \subseteq \bigcup_{i \in T} S(\pi_i, \gamma_i) \Longleftrightarrow c^T x + d^T y \leq f \text{ is valid for conv} \left(P^{LP} \cap D\right).$$

In other words, it is relatively easy to verify if a given inequality is a t-branch split cut implied by a given t-branch split disjunction D or not. Verifying if a given inequality is a t-branch split cut or not is more complicated as it requires checking all possible t-branch split disjunctions. Verifying if a given inequality is valid for the t-branch split closure $P_t^{[1]}$ is yet more difficult as multiple t-branch split cuts might imply the inequality at hand. In the next section, we analyze the Li-Richard mixed-integer set contained in $\mathbb{Z}^n \times \mathbb{R}$, and show that the valid inequality considered by them is not valid for the kth t-branch split closure for any finite k if $t \leq n - 1$.

2 Main result

In $\mathbb{R}^n$, let $e_1, \ldots, e_n$ stand for the n unit vectors (e_i has a one in the ith component and zeros elsewhere). Let X_k^n be the simplex in $\mathbb{R}^n$ defined as the convex hull of $0, ke_1, ke_2, \ldots, ke_n$, where k is a positive real number. Note that for any integer $n \geq 1$, the simplex X_n^n does not contain any integer points in its interior.

Given a polytope P in $\mathbb{R}^n$, a point $\bar{x}$ in its interior, and a number $\epsilon > 0$, we define a polyhedron in $\mathbb{R}^{n+1}$ as follows:

$$L(P, \bar{x}, \epsilon) = \text{conv} \left((P \times \{0\}) \cup \{(\bar{x}, \epsilon\} \right).$$

Let $\mathbf{1}_n$ stand for the all-ones vector in $\mathbb{R}^n$. For any positive integer n, let $w_n = \frac{n-1}{n} \mathbf{1}_n$ and note that w_n is contained in the interior of X_n^n. Using w_n and a fixed $\epsilon > 0$, we next define a family of mixed-integer polyhedral sets follows:

$$P_n = \{(x, y) \in \mathbb{Z}^n \times \mathbb{R} : (x, y) \in L(X_n^n, w_n, \epsilon)\}.$$

All mixed-integer solutions of P_n satisfy $y \leq 0$ (in fact, the convex hull of solutions equals $X_n^n \times \{0\}$). This is the set considered by Li and Richard [4]. More precisely, what is called $P(n+1, (n-1)/n, \epsilon)$ in their paper is the same as P_n. Also notice that the set considered by Li and Richard is a generalization of the set in $\mathbb{R}^3$ constructed earlier by Cook, Kannan and Schrijver [1] which in our notation is P_2.

Our main result in this section is to establish that for any $n \geq 2$, the inequality $y \leq 0$ is not valid for the kth t-branch split closure of P_n for any finite k if $t \leq n-1$. In fact, though we prove the result for $w_n = \frac{n-1}{n}\mathbf{1}_n$, our result holds when w_n is chosen as an arbitrary point in the interior of X_n^n. To start off, we show that $y \leq 0$ cannot be obtained as a $(n-1)$-branch split cut. The discussion in Section 1.3 implies that this result is equivalent to showing that the points in P_n with $y > 0$ cannot be contained in the union of $n-1$ split sets defined on the first n integral variables of P_n. This latter result in turn is equivalent to showing that X_n^n is not contained in the union of any $n-1$ split sets, which follows from the next lemma.

For a given set K, we say that a collection of split (or closed split) sets $S_1, \dots, S_k$ *weakly covers* K, if $\mathrm{vol}(K \setminus \cup_{i=1}^k S_i) = 0$ (here vol stands for volume), whereas a set is *covered* by the split sets if it is contained in their union. We denote the Euclidean norm of a by $||a||$.

Lemma 2.1. *Let n, k be positive integers. The simplex X_k^n in $\mathbb{R}^n$ cannot be weakly covered by $k-1$ closed split sets in $\mathbb{R}^n$.*

Proof. The proof is by induction on n. We will show that $\mathrm{int}(X_k^n)$ is not contained in the union of any $k-1$ closed split sets. This is equivalent to the desired result as $\mathrm{vol}(X_k^n) = \mathrm{vol}(\mathrm{int}(X_k^n))$ and $\mathrm{int}(X_k^n)$ minus a finite collection of closed split sets, if nonempty, is an open set and thus has nonzero volume.

When $n = 1$, the result is obviously true for any $k \geq 1$ as X_k^1 is simply the interval $[0, k] \subseteq \mathbb{R}$, and any closed split set in $\mathbb{R}$ is contained in a closed interval $[l, l+1]$ for some integer l, and therefore $k-1$ closed split sets do not cover the interior of X_k^1.

Now assume the result is true for X_k^m for all $m < n$ and all $k \geq 1$ and consider X_k^n for some $k \geq 1$. Consider an arbitrary collection of $k-1$ closed split sets

$$\bar{S}(\pi_i, \gamma_i) \text{ for } i = 1, \dots, k-1. \tag{3}$$

Without loss of generality, we can assume that the coefficients of π_i are co-prime: if $\pi_i = t\pi_i'$ for some integer $t \geq 2$ and integer vector π_i', then $S(\pi_i, \gamma_i)$ is strictly contained in $S(\pi_i', \lfloor \gamma_i/t \rfloor)$. Using the closure of this split set instead of $\bar{S}(\pi_i, \gamma_i)$, if the new closed split set collection does not cover $\mathrm{int}(X_k^n)$, then the original collection does not either.

Some of these sets would have $\pi_i = e_n$ and therefore would be of the form $S_i = \{x \in \mathbb{R}^n : \gamma_i \leq x_n \leq \gamma_i + 1\}$ (recall that $\pi_j \neq te_n$ for an integer $t \geq 2$ for any j, as we only consider non-dominated split sets). Let V be the collection of indices of the sets in (3) with $\pi_i = e_n$, and let V' be the set of remaining indices. Define $l = |V|$ and note that $0 \leq l \leq k-1$. Let

$$S' = X_k^n \setminus \cup_{i \in V} \bar{S}(\pi_i, \gamma_i)$$

and notice that S' is non-empty as X_k^n contains points with $x_n = 0$ and $x_n = k$. Therefore, S' must contain an open set of the form $\{x \in X_k^n : \alpha < x_n < \alpha + 1\}$ for some integer α where $0 \leq \alpha \leq l$. We will show that this latter set is not covered by the remaining split sets.

As $\alpha \leq l \leq k-1$, it follows that $k - \alpha \geq 1$ and $|V'| = k - 1 - l \leq k - \alpha - 1$. Notice that the set $X_k^n \cap \{x : x_n = \alpha\}$ is a translate (by the vector $(0, \dots, 0, \alpha)$) of the $(n-1)$-dimensional simplex $X_{k-\alpha}^{n-1}$ embedded in n-space. Also, note that for any $i \in V'$ the set $\bar{S}(\pi_i, \gamma_i) \cap \{x : x_n = \alpha\}$ is the translate of a closed split set in $\mathbb{R}^{n-1}$. By the induction hypothesis

$$\mathrm{relint}(X_k^n \cap \{x : x_n = \alpha\}) \setminus \cup_{i \in V'}(\bar{S}(\pi_i, \gamma_i) \cap \{x : x_n = \alpha\}) \neq \emptyset,$$

and must contain a point of the form $p = (\hat{x}, \alpha)$. Therefore $p \notin \cup_{i \in V'} \bar{S}(\pi_i, \gamma_i)$, and as the set V' is finite, and the closed split sets are convex, the distance between p and the sets $\bar{S}(\pi_i, \gamma_i)$ is greater than some $\varepsilon > 0$. In other words, the ball $B(p, \varepsilon)$ does not intersect any of the closed split sets $\bar{S}(\pi_i, \gamma_i)$ for $i \in V'$. There is a point in this ball which intersects $\{x \in X_k^n : \alpha < x_n < \alpha + 1\}$ and is contained in the interior of X_k^n; such a point is not contained in $\bar{S}(\pi_i, \gamma_i)$ for $i = 1, \ldots, k-1$ and the result follows. ∎

We will next strengthen the previous result by showing a lower bound on the volume of X_n^n not covered by any collection of $n-1$ split sets. We will later use this lower bound to move from the result that $y \leq 0$ cannot be obtained as a $(n-1)$-branch split cut for P_n to the result that $y \leq 0$ is not valid for $(P_n)_{n-1}^{[1]}$, i.e., the first $(n-1)$-branch split closure of P_n.

We need the following definition and subsequent lemma from [2].

Definition 2.2. *Let K be a bounded set in $\mathbb{R}^n$ and let $\varepsilon > 0$ be given. We define $\mathcal{L}(K, \varepsilon)$ as the collection of vectors $a \in \mathbb{Z}^n$ such that, for some $b \in \mathbb{Z}$, the volume of $K \cap S(a, b)$ is at least ε.*

Note that $\mathcal{L}(K, \varepsilon)$ can be empty, for example if ε is greater than the volume of K. We use the following basic result from [2] which can be proved by simply observing that as the norm of the vector a increases, the set $S(a, b)$ becomes thinner and consequently, its intersection with the bounded set K becomes smaller.

Lemma 2.3 ([2]). *For any bounded set $K \subset \mathbb{R}^n$ and any number $\varepsilon > 0$, the list $\mathcal{L}(K, \varepsilon)$ is finite.*

Lemma 2.4. *Let K be a bounded set in $\mathbb{R}^n$ with nonzero volume. Let t be a positive integer such that K cannot be weakly covered by any collection of t split sets. Then there exists a constant $\varepsilon > 0$ such that the volume of K covered by any collection of t split sets is at most $\text{vol}(K) - \varepsilon$.*

Proof. Note that we are dealing with an infinite subset $\mathcal{C} = \{C_1, C_2 \ldots\}$ whose members are t split sets of the form $C_i = \{S_i^1, S_i^2, \ldots, S_i^t\}$ for $i \geq 1$. Even though the volume of K left uncovered by a given C_i, denoted by ε_i, is strictly positive by assumption, $\mathcal{C}$ can potentially contain a sequence of C_is such that ε_i goes to zero in the limit. We will show that this is not possible.

The proof is by induction on t. To show the base case, let $t = 1$, i.e., assume $\text{vol}(K \setminus S) > 0$ for any split set S. Let $\delta = \text{vol}(K)/2$ and consider $\mathcal{L}(K, \delta)$. If $\mathcal{L}(K, \delta) = \emptyset$ then the volume of K contained in any single split set is at most $\text{vol}(K)/2$. On the other hand, if $\mathcal{L}(K, \delta) \neq \emptyset$, then let $\delta' < \text{vol}(K)$ be the maximum volume of intersection of a split set defined by a vector in $\mathcal{L}(K, \delta)$. This maximum exists as $\mathcal{L}(K, \delta)$ is finite and for every vector in $\mathcal{L}(K, \delta)$, there are only finitely many associated split sets which intersect K. As $\delta' \geq \delta$ and the volume of intersection of a split set defined by a vector not in $\mathcal{L}(K, \delta)$ is less than δ, we conclude that the volume of $K \setminus S$ is at least $\text{vol}(K) - \delta'$ for any split set S.

Now assume that the claim holds for all integers less than $t > 1$ but not for t. In other words, assume that there exists a set K that cannot be weakly covered by any collection of t split sets but for any fixed $\epsilon > 0$, there exists a collection of t split sets C that cover $\text{vol}(K) - \epsilon$ of its volume. Consider the sequence of numbers $\{\epsilon_1, \epsilon_2, \ldots\}$ where $\epsilon_i = (1/2)^i \text{vol}(K)$ and the corresponding sequence of collections of t split sets $\{C_1, C_2, \ldots\}$ where the collection $C_i = \{S_i^1, S_i^2, \ldots, S_i^t\}$ covers at least $\text{vol}(K) - \epsilon_i$ of the volume of K. Notice that not all split sets $S_i^j \in \cup_i C_i$ can come from a finite collection Π of split sets because if they do, it is possible to compute a lower bound on the volume left uncovered by any collection of t split sets belonging to Π.

Let $S_i^j = S(\pi^{j,i}, \gamma_i^j)$. For convenience, assume that for any given i, $S_i^t = S(\pi^{t,i}, \gamma_i^t)$ satisfies $||\pi^{t,i}|| \geq ||\pi^{j,i}||$ for $j = 1, \ldots, t-1$. That is, the last split set in the collection C_i is defined by a vector with maximum norm among all vectors defining split sets in C_i. Clearly,

$$||\pi^{t,i}|| \to \infty \text{ as } i \to \infty$$

and therefore the volume of K contained in S_i^t tends to zero as i tends to infinity. Then

$$\text{vol}(K \setminus \cup_{j=1}^{t-1} S_i^j) \to 0 \text{ as } i \to \infty.$$

But any collection of $t-1$ split sets cannot weakly cover B, and by the induction hypothesis, a fixed constant volume of K is left uncovered by any collection of $t-1$ split sets which contradicts the above expression. ∎

Consequently, for every integer $n > 0$, there exists a constant $\varepsilon_n > 0$ such that the volume of X_n^n not covered by any collection of $n-1$ split sets is at least $\varepsilon_n > 0$. In other words, if S_i for some i is a split set and $\bar{S}_i$ is its closure, then

$$\text{vol}(X_n^n \setminus \cup_{i=1}^{n-1} S_i) = \text{vol}(X_n^n \setminus \cup_{i=1}^{n-1} \bar{S}_i) \geq \varepsilon_n.$$

Let δ_n denote ε_n divided by the *surface area* of X_n^n (defined as the $(n-1)-$dimensional volume of the boundary of X_n^n) and note that the set of points that are at most δ_n away from its boundary (denoted by $\text{bnd}(X_n^n)$) has a volume of at most ε_n as

$$\text{vol}\Big(\{x \in X_n^n : \text{dist}(x, \text{bnd}(X_n^n)) \leq \delta_n\}\Big) \leq \text{ (surface area of } X_n^n) \times \delta_n = \varepsilon_n.$$

We therefore make the following observation.

Corollary 2.5. *For every integer $n > 0$, there exists a constant $\delta_n > 0$, such that for any collection of $n-1$ split sets $S_1, \ldots, S_{n-1}$, the set $X_n^n \setminus \cup_{i=1}^{n-1} S_i$ contains a point that is at a distance of at least δ_n from any facet of X_n^n.*

Let $v_0, v_1, \ldots, v_n$ stand for the $n+1$ vertices of X_n^n, with $v_0 = \mathbf{0}$, and $v_i = ne_i$; the vertices are affinely independent, and thus X_n^n is full-dimensional, but the convex hull of any n vertices defines a facet of X_n^n. We also need the following simple fact about X_n^n in our subsequent result.

Lemma 2.6. *Let x and y be any two points in the interior of X_n^n, and let $y = \sum_{i=0}^n \beta_i v_i$ with $\sum_{i=0}^n \beta_i = 1$ and $\beta_i > 0$ for $i = 0, \ldots, n$. Then y can be expressed as a convex combination of x and some n out of the $n+1$ vertices of X_n^n; further, if v_k is the missing vertex in this convex combination, then the coefficient of x is greater than β_k.*

Proof. There exist positive numbers $\alpha_0, \ldots, \alpha_n$ and $\beta_0, \ldots, \beta_n$ such that $x = \sum_{i=0}^n \alpha_i v_i, \sum_{i=0}^n \alpha_i = 1$, and $y = \sum_{i=0}^n \beta_i v_i, \sum_{i=0}^n \beta_i = 1$; the coefficient of any vertex cannot be zero in either expression as the convex combination of any n vertices lies on a facet and not in the interior. Multiply the equation $0 = x - \sum_{i=0}^n \alpha_i v_i$ by some $\tau > 0$ such that the sum of the scaled equation and $y = \sum_{i=0}^n \beta_i v_i$ has a nonnegative coefficient for each v_i, and a zero coefficient for v_k for some k. This can be done by choosing τ such that $\tau = \min_{0 \leq i \leq n}\{\beta_i/\alpha_i\}$; assume that $\tau = \beta_k/\alpha_k$. Then y is a convex combination of x and $\{v_0, \ldots, v_n\} \setminus \{v_k\}$. Further, $\tau = \beta_k/\alpha_k > \beta_k$ as $\alpha_k < 1$ when $n \geq 1$. ∎

Recall that $L(X_n^n, w_n, \epsilon) = \text{conv}\left((X_n^n \times \{0\}) \cup \{(w_n, \epsilon)\}\right)$ and

$$P_n = \Big\{(x, y) \in \mathbb{Z}^n \times \mathbb{R} : (x, y) \in L(X_n^n, w_n, \epsilon)\Big\}$$

where $\epsilon > 0$ is an arbitrary fixed number and $w_n = \frac{n-1}{n}\mathbf{1}_n$ which is clearly contained in P_n's interior. The next result settles the Li-Richard conjecture.

Theorem 2.7. *The inequality $y \leq 0$ is not valid for kth $(n-1)$-branch split closure of P_n for any finite k.*

Proof. Clearly $P_n^{LP} = L(X_n^n, w_n, \epsilon)$. Let $Q^{[k]}$ stand for the kth $(n-1)$-branch split closure of P_n. We will show that there exists a ϵ_1 with $0 < \epsilon_1 < \epsilon$ such that

$$Q^{[1]} \supseteq L(X_n^n, w_n, \epsilon_1). \tag{4}$$

Then the $(n-1)$-branch split closure of $Q^{[1]}$ equals $Q^{[2]}$ and contains the $(n-1)$-branch split closure of $L(X_n^n, w_n, \epsilon_1)$, which by (4) contains $L(X_n^n, w_n, \epsilon_2)$ for some ϵ_2 with $0 < \epsilon_2 < \epsilon_1$. Iteratatively applying this argument, we can obtain the fact that the kth $(n-1)$-branch split closure of P_n, for any positive integer k, contains $L(X_n^n, w_n, \epsilon_k)$ and the point (w_n, ϵ_k) where $0 < \epsilon_k < \epsilon_{k-1} < \cdots < \epsilon$. This point does not satisfy the inequality $y \leq 0$, and therefore we will have shown that $y \leq 0$ has unbounded $(n-1)$-branch split rank.

First note that any valid split disjunction for $\mathbb{Z}^n \times \mathbb{R}$ is defined by a vector with the first n components integral and the last component zero. Let Y^0 denote the set $\{(x, y) \in \mathbb{R}^{n+1} \ : \ y = 0\}$. Consider any collection of $n-1$ valid split sets $S_1, \ldots, S_{n-1}$ for $\mathbb{Z}^n \times \mathbb{R}$ and note that $S_i \cap Y^0$ is congruent to a split set in $\mathbb{R}^n$, and $P_n^{LP} \cap Y^0$ is congruent to X_n^n. Therefore $P_n^{LP} \cap Y^0$ must contain a point $p = (\bar{x}, 0)$ not contained in any of the split sets such that $\bar{x}$ is at a distance of at least δ_n to any facet of X_n^n. By Lemma 2.6, $\bar{x}$ can be expressed as a convex combination of w_n and a proper subset of the vertices $\{v_0, \ldots, v_n\}$; let f stand for the face defined by these vertices. The coefficient of w_n in this convex combination equals the ratio of the (Euclidean) distance of $\bar{x}$ from the face f (which is bounded below by δ_n) to the distance of w_n from f. The latter distance can be bounded above by the maximum distance of w_n to the vertices of X_n^n, say σ_n. Therefore the coefficient of w_n in the above combination is at least δ_n/σ_n. As P_n^{LP} contains $(w_n, 0)$ and (w_n, ϵ), and $(v_0, 0), \ldots, (v_n, 0)$, it must contain $(\bar{x}, \gamma)$ where $\gamma = \frac{\delta_n}{\sigma_n}\epsilon > 0$ and $\gamma < \epsilon$. Therefore $P_n^{LP} \setminus \cup_{i=1}^{n-1} S_i$ contains $(\bar{x}, \gamma)$. Note that γ does not depend on the choice of the split sets $S_1, \ldots, S_{n-1}$.

As w_n is contained in the convex hull of $\bar{x}$ and $\{v_0, \ldots, v_n\} \setminus \{v_j\}$ for some j, we can write

$$w_n = \bar{\alpha}\bar{x} + \sum_{i=0, i\neq j}^{n} \alpha_i v_i \text{ with } \bar{\alpha} + \sum_{i=0, i\neq j}^{n} \alpha_i v_i = 1 \Rightarrow \bar{\alpha} \geq \frac{n-1}{n^2}.$$

The last inequality follows from Lemma 2.6 and the fact that $w_n = v_0/n + \frac{(n-1)}{n^2}\sum_{i=1}^n v_i$; the coefficient of each vertex in the latter expression is at least $(n-1)/n^2$. Therefore, the convex hull of $(\bar{x}, \gamma)$ and $(v_0, 0), \ldots, (v_n, 0)$ contains $(w_n, \bar{\alpha}\gamma)$ and therefore $(w_n, \gamma(n-1)/n^2)$. But $P_n^{LP} \setminus \cup_{i=1}^{n-1} S_i$ contains the points $(\bar{x}, \gamma)$ and $(v_0, 0), \ldots, (v_n, 0)$ and therefore

$$\text{conv}\left(P_n^{LP} \setminus \cup_{i=1}^{n-1} S_i\right) \supseteq L(X_n^n, w_n, \gamma(n-1)/n^2).$$

As $\gamma(n-1)/n^2$ does not depend on the collection of split sets $S_1, \ldots, S_{n-1}$, setting $\epsilon_1 = \gamma(n-1)/n^2$, we get (4) and the proof is complete. ∎

As we noted earlier, the above proof (and thus the above result) applies when w_n is any point in the interior of P_n.

3 Concluding comments

An interesting question is whether this result can be improved in the sense that one can obtain mixed-integer sets in $\mathbb{R}^n$ such that their integer hulls have unbounded split rank with respect to t-branch split sets for t

which grows super-linearly with n. Answering this question with our proof techniques seems difficult. It is known that all lattice-free sets in $\mathbb{R}^n$ have lattice-width at most $O(n^{4/3}\log^c n)$ for some constant $c > 0$ [5], and it is conjectured that such sets have lattice-width at most $c'n$ for some constant $c' > 0$. In other words, any lattice-free set in $\mathbb{R}^n$ can be weakly covered by $O(n^{4/3}\log^c n)$ split sets which are defined by the direction of minimum lattice width. Thus it is not possible to significantly increase the number of split sets in our result which says that S_n^n must have a volume of ε_n left over after removing $n-1$ split sets by replacing S_n^n by another lattice-free set. This is in contrast with the result in [2] which constructs a family of lattice-free sets, one in $\mathbb{R}^n$ for each n, such that an exponential number of split sets are needed to cover the body.

References

[1] W. J. Cook, R. Kannan, and A. Schrijver, Chvátal closures for mixed integer programming problems, *Mathematical Programming* **47** (1990), 155–174.

[2] S. Dash, N. B. Dobbs, O. Günlük, T. J. Nowicki, G. Swirszcz, Disjunctive cuts, lattice-free sets and mixed-integer programming, Manuscript, 2011.

[3] R. E. Gomory, An algorithm for the mixed integer problem, RM-2597, The Rand Corporation, 1960.

[4] Y. Li and J.-P. P. Richard, Cook, Kannan and Schrijver's example revisited, *Discrete Optimization* **5** (2008), 724–734.

[5] M. Rudelson, Distances between non-symmetric convex bodies and the MM* estimate, *Positivity* **4** (2000), 161–178.