Global Error bounds for systems of convex polynomials over polyhedral constraints

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Abstract

This paper is devoted to study the Lipschitzian/Holderian type global error bound for systems of many finitely convex polynomial inequalities over a polyhedral constraint. Firstly, for systems of this type, we show that under a suitable asymtotic qualification condition, the Lipschitzian type global error bound property is equivalent to the Abadie qualification condition, in particular, the Lipschitzian type global error bound is satisfied under the Slater condition. Secondly, without regularity conditions, the Hölderian global error bound with an implicit exponent is invertigated.

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1 Introduction

Let $f_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ $(i = 1, ..., p)$ be a finite family of extended-real-valued functions defined on $\mathbb{R}^n$. Denote by S the solution set of the inequality system:

$$\text{Find} \quad x \in \mathbb{R}^n \quad \text{such that} \quad f_i(x) \le 0 \quad \text{for all} \quad i = 1, ..., p. \tag{1}$$

We shall say that the system (1) admits *an error bound* with exponent γ, if there exists a real $\tau > 0$ such that

$$d(x, S) \le \tau([f(x)]_+ + [f(x)]_+^\gamma) \qquad \text{for all} \quad x \in \mathbb{R}^n. \tag{2}$$

Where, $d(x, S)$ denotes the (Euclidean) distance from a point $x \in \mathbb{R}^n$ to the set S; the function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$f(x) := \max\{f_1(x), f_2(x), ..., f_p(x)\}, \quad x \in \mathbb{R}^n;$$

and the symbol $[f(x)]_+$ denotes $\max(f_i(x), 0)$.

It is now well established that error bounds have a large range of applications in different areas such as for example, sensitivity analysis, implementation of numerical methods and for solving optimization problems and the convevergence analysis of these numerical methods, penalty functions methods in mathematical programming.

To the best of our knowledge, it was Hoffman who first drew the attention on error bounds for systems of affine functions. He has established that a global error bound with exponent $\gamma = 1$ (that

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is, a global Lipschitz type error bound) holds for systems of affine equalities and/or inequalities. This work was extended by Robinson to a system of convex inequalities which defines a bounded feasible region with a nonemty interior. Then, Mangasarian derived a global error bound for differentiable convex inequalities which satisfy Slater's condition and an asymtotic constraint qualification (ACQ) instead of the bounded assumption on the feasible region. Later on, Auslender and Crouzeix made important improvements of Mangasarian's work for systems which are not necessarily differentiable. For a more detailed account on the recent development of the theory applications of error bounds, the reader is refered to the works [5, 7, 8, 6, 9, 18, 23, 24, 26, 28, 10, 14, 12, 13, 31, 32, 33], and especially to the survey papers by Azé ([3], Lewis and Pang ([18]), Pang ([28]). For systems of convex quadratic inequalities, Luo and Luo ([22]) have shown that a global Lipschitzian error bound holds assuming only the Slater condition. In [21], Li completed the work by Luo & Luo, proving that a global Lipschitzian error bound holds for convex quadratic inequalities if and only if the Abadie qualification condition is satisfied for all points belonging to S. Without the Slater condition, Wang & Pang established that any system of convex quadratic inequalities admits a global error bound of Holderian type with an exponent $\gamma \leq 2^{-p}$. In [25], Ngai & Théra have generalized these results for systems of convex quadratic inequalities in general Banach spaces. Very recently, global error bounds for convex polynomial functions has been investigated by Li [19] and by Yang [34]. In [19], Li has established that under the Slater condition, the system (1) admits a Lipschitzian type global error bound if $p = 1$ or $\inf f_i > -\infty$ for all $i = 1, ..., p$. Moreover, the Holderian type error bound results for a piecewise convex polynomial function f have been investigated in [20].

The purpose of this paper is to study the Lipschitzian/Holderian type global error bound for systems of many finitely convex polynomial inequalities over a polyhedral constraint. In Section 3, we establish the equivalence between the Lipschitzian type global error bound and the Abadie qualification condition. In particular, we obtain a Lischitzian global error bound result under the Slater condition. In Section 4, we study the Hölderian global error bound for these convex polynomial systems.

2 Preliminaries

Throughout this work, the Euclidean space $\mathbb{R}^n$ is equipped with the canonical inner product $\langle \cdot \rangle$ and the corresponding norm $\|x\| := \langle x, x\rangle^{1/2}$, $x \in \mathbb{R}^n$. The open ball with the center $x \in \mathbb{R}^n$ and radius $\varepsilon > 0$ is denoted by $B(x, \varepsilon)$. Let C be a nonempty closed subset of $\mathbb{R}^n$. Denote by $\delta_C(x)$ the indicator function of C, that is, $\delta_C(x) = 0$ if $x \in C$, otherwise $\delta_C(x) = +\infty$.

Let f be a proper lower semicontinuous convex functions on $\mathbb{R}^n$. Recall that ([1], [29]):

The subdifferential of f at $x \in \text{Dom } f$ is defined by

$$\partial f(x) = \{x^* \in \mathbb{R}^n : \quad \langle x^*, y - x\rangle \leq f(y) - f(x) \quad \forall y \in \mathbb{R}^n\}. \tag{3}$$

For $x \notin \text{Dom } f$, one sets $\partial f(x) = \emptyset$. For a nonempty closed convex subset C of $\mathbb{R}^n$, the normal cone of C at a point $x \in C$ is defined by

$$N(C, x) = \partial \delta_C(x) = \{x^* \in \mathbb{R}^n : \quad \langle x^*, y - x\rangle \leq 0 \ \forall y \in C\}. \tag{4}$$

The recession cone of a convex set C in $\mathbb{R}^n$ is defined by

$$C^\infty = \{v \in \mathbb{R}^n : \quad x + tv \in C \ \ \forall x \in C, \ \forall t > 0\}.$$

For a proper function f, its recession function f^∞ is defined by

$$f^\infty(v) = \liminf_{t\to+\infty, u\to v} \frac{f(tu)}{t},\ v \in \mathbb{R}^n. \tag{5}$$

When f is a lower semicontinous convex function, then (see, e.g., [1]) for any $x \in \mathrm{Dom}\, f$,

$$f^\infty(v) = \lim_{t\to+\infty} \frac{f(x+tv)-f(x)}{t} = \sup_{t>0} \frac{f(x+tv)-f(x)}{t}. \tag{6}$$

Recall that a function f is a (real) polynomial with degree $d \in \mathbb{N}$ if

$$f(x) := \sum_{|\alpha|\le d} \lambda_\alpha x^\alpha,$$

where, $\lambda_\alpha \in \mathbb{R}$, $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$, $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $|\alpha| := \sum_{i=1}^n \alpha_i$.

For an extended real-valued function $f : \mathbb{R}^n \to \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$, the *error bound* property is defined by the inequality

$$d(x, S_f) \le c[f(x)]_+, \tag{7}$$

where S_f denotes the lower level set of f:

$$S_f := \{x \in X :\ f(x) \le 0\}, \tag{8}$$

$c \ge 0$, and the notation $\alpha_+ := \max(\alpha, 0)$ is used.

Given an $\bar{x} \in \mathbb{R}^n$ with $f(\bar{x}) = 0$ we say that f admits an (local) *error bound* at $\bar{x}$ if there exist reals $c \ge 0$ and $\delta > 0$ such that (7) holds for all $x \in B_\delta(\bar{x})$. The *best bound* – the exact lower bound of all such c – coincides with $[\mathrm{Er}\, f(\bar{x})]^{-1}$, where

$$\mathrm{Er}\, f(\bar{x}) := \liminf_{\substack{x\to\bar{x} \\ f(x)>0}} \frac{f(x)}{d(x, S(f))} \tag{9}$$

is the *error bound modulus* [10]) of f at $\bar{x}$. Thus, f admits an error bound at $\bar{x}$ if and only if $\mathrm{Er}\, f(\bar{x}) > 0$.

If (7) holds for some $c \ge 0$, and all $x \in X$ then we say that f admits a *global error bound.* In this case, the *best bound* – the exact lower bound of all such c – coincides with $[\mathrm{Er}\, f]^{-1}$, where

$$\mathrm{Er}\, f := \inf_{f(x)>0} \frac{f(x)}{d(x, S(f))} \tag{10}$$

is the global *error bound modulus.*

The convex case has attracted a special attention starting with the pioneering work by Hoffman [15] on error bounds for systems of affine functions, see [4, 7, 8, 9, 12, 18, 25].

The following characterizations of the global and local error bounds for lower semicontinous convex functions is well known (see, for instance, [4], [27]), which are needed in the sequel.

Theorem 1 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous proper convex function. Then one has* *(i) $S(f)$ admits an global error bound if and only if*

$$\tau(f) := \inf\{d(0, \partial f(x)) : \ x \in \mathbb{R}^n, \ f(x) > 0\} > 0.$$

Moreover, the infimum of all Hoffman constants $c(f)$ (the best bound) is given by $c_{\min}(f) := \tau(f)^{-1}$.

(ii) Let $\bar{x} \in bdryS(f)$. $S(f)$ admits a (local) error bound at $\bar{x}$, i.e., there exist $c(f, \bar{x})$, $\varepsilon > 0$ such that

$$d(x, S(f)) \leq c(f, \bar{x})[f(x]_+ \ \textit{for all}\, x \in B(\bar{x}, \varepsilon),$$

if and only if

$$\tau(f, \bar{x}) := \liminf_{x \to x_0,\, f(x) > 0} d(0, \partial f(x)) > 0.$$

Moreover $\tau(f, \bar{x})^{-1}$ is the best bound $c_{\min}(f, \bar{x})$ at $\bar{x}$.

(iii) (the relation between the global error bound and the local error bounds) The following equality holds

$$c_{\min}(f) = \sup_{x \in bdryS(f)} c_{\min}(f, x).$$

The following lemma which can be immediately implied from Theorem 1, list some simple sufficient conditions for the Lipschitzian type global error bound that will be used thereafter (see, e.g., [9][20]).

Lemma 2 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous proper convex function. Then one of the following conditions ensures a Lipschitzian type error bound for $S_f := \{x \in \mathbb{R}^n : \ f(x) \leq 0\}$.*

(i) S_f is bounded and the Slater condition is satisfied: There exists $x_0 \in \mathbb{R}^n$ such that $f(x_0) < 0$.

(ii) There exists $v \in \mathbb{R}^n$ such that $f^\infty(v) < 0$.

3 Lipschitzian type global error bound for systems of convex polynomial functions over polyhedral constraints

Let $f_i : \mathbb{R}^n \to \mathbb{R}$ $(i = 1, ..., p)$ be convex polynomial functions and let K be a polyhedral convex set in $\mathbb{R}^n$. Let us consider the following system

$$\text{Find} \quad x \in K \quad \text{such that} \quad f_i(x) \leq 0 \quad \text{for all} \ \ i = 1, ..., p. \tag{11}$$

Denote by S the solution set of (11). Set

$$f(x) = \max\{f_1(x), ..., f_p(x)\}$$

and

$$I(x) = \{i \in \{1, ..., p\} : \quad f_i(x) = f(x)\}.$$

Then, the solution S can be written as

$$S = \{x \in K : \quad f(x) \leq 0\}. \tag{12}$$

In the case of only one convex polynomial, that is, $p = 1$, it was established by Li ([20], Proposition 3.1) that the system (11) admits a Lipschitzian type global error bound under the Slater condition.

As shown by Exemple 4.1 in [20], without added condition, this result does not hold in general for $p > 1$. In the sequel, we make of use the following assumption:

$$(\mathcal{A}_\infty) \qquad \forall v \in K^\infty, \max_{i=1,\dots,p}\{f_i^\infty(v)\} = 0 \implies f_i^\infty(v) = 0 \text{ for all } i = 1, \dots, p.$$

Obviously, assumption $(\mathcal{A}_\infty)$ is satisfied if either $p = 1$ (that is, the system of only one convex polynomial) or the solution set is bounded, or more general, there exist scalars $\beta_i > 0$ $(i = 1, \dots, p)$ such that

$$\inf_{x \in K} \sum_{i=1}^{p} \beta_i f_i(x) > -\infty.$$

In particular, $(\mathcal{H})$ holds if all f_i $(i = 1, \dots, p$) are bounded from below on K.

We need the following lemma whose proof is similar to the one of Lemma 2.3 in [19].

Lemma 3 *Let f_i $(i = 1, \dots, p)$ be convex polynomial functions and let K be a polyhedral convex set in $\mathbb{R}^n$ such that assumption $(\mathcal{A}_\infty)$ is satisfied and set $f = \max_{1 \le i \le p} f_i$. If for $v \in K^\infty$, $f^\infty(v) = 0$ then $f(x + tv) = f(x)$ for all $x \in \mathbb{R}^n$ and all $t \in R$.*

Recall that, the system (12) satisfies *Abadie qualification condition* (AQC) at $x_0 \in S$ if

$$N(S, x_0) = \left\{ \sum_{i=1}^{p} \lambda_i \partial f_i(x_0) + N(K, x_0) : \quad \lambda_i \ge 0, \ \lambda_i f_i(x_0) = 0 \right\}. \tag{13}$$

It is well known that for convex inequality systems, (AQC) holds if the Lipschitzian local error bound property holds, in particular, if the Slater condition is satisfied. As was pointed out by Jourani ([17]), (AQC) is verified for all Affine inequality/equality systems, and it is equivalent to the global error bound property. The equivalence between the local error bound and (AQC) for differentiable convex inequality systems was established in [21], [25] as follows.

Theorem 4 *(Theorem 3, [25]) Let f_i $(i = 1, \dots, p)$ be convex functions which are continuous in some neighborhood of a given point $x_0 \in S$.*

(i). If there exist reals $\tau, \varepsilon > 0$ such that

$$d(x, S) \le \tau [f(x)]_+ \quad \text{for all } \ x \in B(x_0, \varepsilon) \cap K,$$

then the (AQC) is satisfied at all $x \in B(x_0, \delta) \cap S$ for some $\delta > 0$.

(ii). In addition, if we suppose that for all $i = 1, \dots, p$, f_i is differentiable in some neighborhood of x_0, then the converse of the part (i) is true.

The following theorem shows that for convex polynomial systems over polyhedral constraints, the equivalence between the global error bound and (AQC) holds when the hypothesis $(\mathcal{A}_\infty)$ is satisfied.

Theorem 5 *Consider the system of convex polynomials over polyhedral constraints (12), that satisfies $(\mathcal{A}_\infty)$. The following two statements are equivalent:*

(i). There exists $\tau > 0$ such that

$$d(x, S) \leq \tau[f(x)]_+ \quad \textit{for all} \;\; x \in K. \tag{14}$$

(ii). The Abadie qualification condition (AQC) is satisfied at all points of S.

Proof. The implication $(i) \Rightarrow (ii)$ is well-known (see, e.g. [35]). For the sake of completness, we give a sort proof. Suppose that (i) holds. Let $x_0 \in S$ and $r > 0$ be given. Denote by L the Lipschitz constant of f on $B(x_0, r)$. For any $x \in B(x_0, r)$, let $z \in K$ be the projection of x into K, that is, $\|x - z\| = d(x, K)$. Then one has

$$d(x, S) \leq d(z, S) + \|x - z\| \leq \tau[f(z)]_+ d(x, K) \leq \tau[f(x)]_+ + (\tau L + 1)d(x, K). \tag{15}$$

The statement (ii) follows directly from the last relation and the following standard relations in convex analysis

$$N(S, x_0) = \cup_{\lambda > 0} \partial d(x_0, S);$$

$$\partial[f(\cdot)]_+(x_0) = \left\{ \sum_{i=1}^{p} \lambda_i \partial f_i(x_0) : \;\; \lambda_i \geq 0, \;\; \lambda_i f_i(x_0) = 0 \right\}.$$

Let us prove the inverse implication by induction on the dimension d of K.. Let $K := \{x \in \mathbb{R}^n : \langle a_j, x\rangle \leq b_j, \; j = 1, ..., m\}$. When $d = 0$, then K is a single point, therefore the conclusion holds trivially. Let s be an integer, and suppose that the implication $(ii) \Rightarrow (i)$ holds for all $d \leq s$. Consider now the case of $d = s + 1$. Suppose that (ii) is satisfied for p convex polynomials $f_1, ..., f_p$ verifiying $(\mathcal{A}_\infty)$. By translation if necessary, without loss of generality, we can assume that $0 \in K$. Assume to contrary that (14) does not hold. According to Theorem 1, there exist sequences $\{x_k\}_{k \in \mathbb{N}} \subseteq K$ $\{x_k^*\}$ with

$$x_k^* \in \partial(f + \delta_K)(x_k) = \partial f(x_k) + N(K, x_k), \;\; k \in \mathbb{N}$$

such that

$$f(x_k) > 0; \; f(x_k) \to 0 \quad \text{and} \quad \|x_k^*\| \to 0 \;\text{ as }\; k \to \infty. \tag{16}$$

Since (AQC) is satisfied for the system under consideration, then by Theorem 4, this system admits a (Lipschitzian type) local error bound at all $x \in S$. Hence, $\|x_k\| \to \infty$ as $k \to \infty$. By passing to a subsequence if necessary, we can assume that

$$\lim_{k \to \infty} \frac{x_k}{\|x_k\|} := v \in \mathbb{R}^{s+1} \quad \text{with} \quad \|v\| = 1.$$

Then, obviously,

$$v \in K^\infty = \{d \in \mathbb{R}^n : \;\; \langle a_j, d\rangle \leq 0, \; j = 1, ..., m\} \quad \text{and} \quad f^\infty(v) \leq 0.$$

Since $(f + \delta_K)^\infty(v) = f^\infty(v)$ for $v \in K^\infty$, then it follows from Lemma 2 (ii) that $f^\infty(v) = 0$. By Lemma 3, one has

$$f(x + tv) = f(x) \quad \forall x \in \mathbb{R}^{s+1}, \, \forall t \in \mathbb{R}. \tag{17}$$

Denote by $\langle v\rangle := \{tv : \;\; t \in \mathbb{R}\}$ and $L = v^\perp := \{u \in \mathbb{R}^n : \;\; \langle u, v\rangle = 0\}$. Then, $\dim L = d - 1 = s$ and therefore, there exists a matrix $Q \in \mathbb{R}^{(s+1) \times s}$ with $\text{rank} Q = s$ such that $\{Qz : \; z \in \mathbb{R}^s\} = L$. Moreover, for any $x \in \mathbb{R}^n$, we have the following unique representation

$$x = u + tv \quad \text{for} \quad u \in L, \; t \in \mathbb{R}.$$

Let $g_i, g : \mathbb{R}^s \to \mathbb{R}$ $(i = 1, ..., p)$ be defined by

$$g_i(z) := f_i(Qz),\ z \in \mathbb{R}^s,\ i = 1, ..., p;\ g(z) := f(Qz) = \max_{i=1,...,p} h_i(z).$$

Then, g_i, $i = 1, ..., p$ are convex polynomials defined on $\mathbb{R}^s$. Denote by

$$J := \{j \in \{1, ..., m\} : \quad \langle a_j, v\rangle = 0\},$$

and define the convex polyhedral K_s in $\mathbb{R}^s$ by

$$K_s := \{z \in \mathbb{R}^s : \quad \langle a_j, Qz\rangle \le b_j \ \forall j \in J\}$$

(Note that J may be empty, in this case, $K = \mathbb{R}^s$). Consider the following system of convex polynomial inequalities with polyhedral constraint in $\mathbb{R}^s$:

$$S_s := \{z \in K_s : \quad g(x) \le 0\}\,. \tag{18}$$

We see that the system (18) also satisfies the assumption $(\mathcal{A}_\infty)$. Indeed, let $w \in K_s^\infty$ such that $\max_{i=1,...,p} g_i^\infty(w) = 0$. Since $\langle a_j, v\rangle < 0$ for all $j \notin J$, then there exists $\alpha > 0$ (sufficiently large) such that

$$\langle a_j, Qw + \alpha v\rangle < 0 \quad \text{for all } j \notin J.$$

For $j \in J$, then $\langle a_j, Qw + \alpha v\rangle = \langle a_j, Qw\rangle \le 0$. Thus $Qw + \alpha v \in K^\infty$, and furthemore, one has

$$\max_{i=1,...,p} f_i^\infty(Qw + \alpha v) = \max_{i=1,...,p} f_i^\infty(Qw) = \max_{i=1,...,p} g_i^\infty(w) = 0.$$

Since assumption $(\mathcal{A}_\infty)$ holds for the system (11), then

$$g_i^\infty(w) = f_i^\infty(Qw) = f_i^\infty(Qw + \alpha v) = 0 \ \text{ for all } i = 1, ..., p.$$

We prove next that the system (18) satisfies the (AQC). Let $z \in S_s$ and $z^* \in N(S_s, z)$ be given. One can take $t_0 > 0$ sufficiently large such that $x := Qz + t_0 v \in S$. Let $u^* \in L$ such that $z^* = Q^T u^*$. For any $y \in S$, there are $z' \in \mathbb{R}^s$ and $t > 0$ such that $y = Qz' + tv$. Then, obviously $z' \in S_s$ and therefore one has

$$\langle u^*, y - x\rangle = \langle u^*, Q(z' - z)\rangle = \langle Q^T u^* z' - z\rangle = \langle z^*, z' - z\rangle \le 0 \ \text{ for all } \ y \in S.$$

Equivalently, $u^* \in N(S, x)$. Hence,

$$N(S_s, z) \subseteq \left\{\lambda Q^T \partial f(x) + Q^T N(K, x) : \quad \lambda \ge 0\right\}. \tag{19}$$

On the other hand, since $f(x + tv) = f(x)$ for all $x \in \mathbb{R}^n$ and all $t \in \mathbb{R}$, then $\partial f(x) = \partial f(Qz)$. Thus

$$\partial g(z) = Q^T \partial f(Qz) = Q^T \partial f(x). \tag{20}$$

Note that for $x \in K$,

$$N(K, x) = \left\{\sum_{j \in J(x)} \beta_j a_j : \quad \beta_j \ge 0 \ \forall j \in J(x)\right\} \tag{21}$$

where $J(x) := \{j \in \{1, ..., m\} : \ \langle a_j, x\rangle = b_j\}$. Then, when t_0 is sufficiently large, one has

$$N(K_s, z) = Q^T N(K, Qz) = Q^T N(K, x). \tag{22}$$

Combining relations (19), (20), and (22), one obtains

$$N(S_s, z) \subseteq \{\lambda \partial g(z) + N(K_s, z) : \quad \lambda \geq 0\}. \tag{23}$$

Noticing that the inverse inclusion

$$N(S_s, z) \supseteq \{\lambda \partial g(z) + N(K_s, z) : \quad \lambda \geq 0\}$$

is always true, thus (AQC) is verified for system (18).

By the induction hypothesis, the function $g + \delta_{K_s}$ admits a Lipschitzian type global error bound. According to Theorem 1, one has

$$\inf\{d(0, \partial g(z) + N(K_s, z)) : \ z \in \mathbb{R}^s, \ g(z) > 0\} > 0. \tag{24}$$

For each $k \in \mathbb{N}$, there exists $z_k \in \mathbb{R}^s$, $t_k \in \mathbb{R}$ such that $x_k = Qz_k + t_k v$. As relation (20), one has

$$\partial g(z) = Q^T \partial f(Qz_k) = Q^T \partial f(x_k). \tag{25}$$

Since $\frac{x_k}{\|x_k\|} \to v$, then obviously, $\|Qz_k\|/t_k \to 0$ as $k \to \infty$. Hence, for all $j \in \{1, ..., m\} \setminus J$, one has

$$\langle a_j, x_k \rangle = t_k(\langle a_j, Qz_k/t_k \rangle + \langle a_j, v \rangle) \to -\infty \ \text{ as } k \to \infty.$$

It follows that $J(x_k) \subseteq J$. Since

$$\langle a_j, Qz + tv \rangle = \langle a_j, Qz \rangle \quad \forall z \in \mathbb{R}^s, \ \forall t \in \mathbb{R}, \forall j \in J,$$

then $z_k \in K_s$, and by (22), one obtains

$$N(K_s, z_k) = Q^T N(K, Qz_k) = Q^T N(K, x_k) \ \text{ for all } k \in \mathbb{N}. \tag{26}$$

This relation and (25) yield

$$Q^T x_k^* \in Q^T(\partial f(x_k) + N(K, x_k)) = \partial g(z_k) + N(K_s, z_k).$$

Relation (16) shows that $\|Q^T x_k^*\| \to 0$, which contradicts (24). The proof is completed. □

The preceding theorem yields directly the following Lipschitzian global error bound result under assumption ($\mathcal{A}_\infty$) and the Slater condition, which generalizes the one of Li in [20].

Corollary 6 *Under Assumption ($\mathcal{A}_\infty$), if system (11) verifies the Slater condition :*

$$\exists x_0 \in K : \ f_i(x_0) < 0, \ \textit{ for all } i = 1, ..., p,$$

then there exists $\tau > 0$ such that

$$d(x, S) \leq \tau [f(x)]_+ \quad \forall x \in K.$$

Next, we establish a global error bound result for system (11) under the Slater condition but *without assumption ($\mathcal{A}_\infty$).* Let us denote by

$$d := \max\{\deg f_1, ..., \deg f_p\},$$

$$|\nabla^j| f(x) := \max\{\|\nabla^j f_i(x)\| : \quad i = 1, ..., p\}, \ j = 1, ..., d.$$

Theorem 7 *Consider system (11) of convex polynomials over a polyhedral convex set K. If the Slater condition is satisfied then there exists $\tau > 0$ such that*

$$d(x, S) \leq \tau \left([f(x)]_+ \sum_{j=1}^{d} |\nabla^j| f(x) [f(x)]_+^j \right) \quad \forall x \in K.$$

Proof. Denote by

$$C_\infty := \{v \in K^\infty : \;\; f^\infty(v) = 0\},$$

and for each $v \in C_\infty$, set

$$I(v) := \{i \in \{1, ..., p\} : \;\; f_i^\infty(v) < 0\} \;\text{ and }\; J(v) := \{1, ..., p\} \setminus I(v).$$

Let us pick a direction $\bar{v} \in C_\infty$ such that

$$I(\bar{v}) := \max\{I(v) : \;\; v \in C_\infty\}.$$

Consider the following inequality system:

$$\bar{S} := \{x \in K : \;\; f_j(x) \leq 0, \; j \in J(\bar{v})\} \tag{27}$$

We claim that this system verifies assumption $(\mathcal{A}_\infty)$. Indeed, assume to contrary that this is not to be hold, i.e., there exists $v \in K^\infty$ such that $\max_{j \in J(\bar{v})}\{f_j^\infty(v)\} = 0$ but $f_{j_0}^\infty(v) < 0$ for some $j_0 \in J(\bar{v})$. Then, for a sufficiently small positive α, one has

$$f_i^\infty(\bar{v} + \alpha v) \leq f_i^\infty(\bar{v}) + \alpha f_i^\infty(v) < 0 \;\; \forall i \in I(\bar{v});$$

$$f_j^\infty(\bar{v} + \alpha v) \leq f_j^\infty(\bar{v}) + \alpha f_i^\infty(v) \leq 0 \;\; \forall j \in J(\bar{v});$$

and,

$$f_{j_0}^\infty(\bar{v} + \alpha v) \leq f_{j_0}^\infty(\bar{v}) + \alpha f_{j_0}^\infty(v) = \alpha f_{j_0}^\infty(v) < 0.$$

Thus, $\bar{v} + \alpha v \in C_\infty$ and $I(\bar{v} + \alpha v) \geq I(\bar{v}) + 1$, which contradicts the definition of $I(\bar{v})$.

According to Corollary 6, there exists $\tau_1 > 0$ such that

$$d(x, \bar{S}) \leq \tau_1 [\max_{j \in J(\bar{v})} f_j(x)]_+ \leq \tau_1 [f(x)]_+ \;\text{ for all }\; x \in K. \tag{28}$$

If $J(\bar{v}) = p$ then $\bar{S} = S$ the proof ends. Otherwise, $0 < J(\bar{v}) < p$, the solution set S of the system (11) under consideration can be written as

$$S = \left\{ x \in \mathbb{R}^n : \;\; g(x) := \max_{i \in I(\bar{v})} f_i(x) + \delta_{\bar{S}}(x) \leq 0 \right\}. \tag{29}$$

For any $x \in \bar{S}$, $x^* \in \partial g(x) = \partial(\max_{i \in I(\bar{v})} f_i)(x) + N(\bar{S}, x)$, since $\bar{v} \in \bar{S}^\infty$, one has

$$\langle x^*, \bar{v} \rangle \leq \frac{g(x + t\bar{v}) - g(x)}{t} = \frac{\max_{i \in I(\bar{v})} f_i(x + t\bar{v}) - \max_{i \in I(\bar{v})} f_i(x)}{t} \;\text{ for all } t > 0,$$

consequently (note that $\bar{v} \neq 0$), Therefore, thanks to Theorem 1, one obtains

$$d(x, S) \leq \tau_2 [\max_{i \in I(\bar{v})} f_i(x)]_+ \leq \tau_2 [f(x)]_+ \;\text{ for all }\; x \in \bar{S}. \tag{30}$$

Let now $x \in K$ be given and let $y \in \bar{S}$ be the projection of x on $\bar{S}$. By relations (28) and (30), one has

$$d(x, S) \leq d(x, \bar{S}) + d(y, S) \leq \tau_1[f(x)]_+ + \tau_2[f(y)]_+.$$

On the other hand, by using the Taylor tranformation of the functions f_i $(i = 1, ...p)$ at x, one derives that

$$[f(y)]_+ \leq [f(x)]_+ + \sum_{j=1}^{d} |\nabla^j|f(x)\|y - x\|^j \leq [f(x)]_+ + \sum_{j=1}^{d} |\nabla^j|f(x)\tau_1^j[f(x)]_+^j.$$

By combining the last inequalities, we complete the proof of the theorem. □

4 Hölderian global error bound

As in Section 2, Consider the inequality system (11) of convex polynomials over a convex polyhedral set, with the solution set S. Denote by $\deg f_i$ the degree of f_i and by

$$f(x) := \max\{f_1(x), ..., f_p(x)\}; \quad I(x) := \{i \in \{1, ..., p\} : \; f_i(x) = f(x)\}.$$

It is shown by Theorem 5 that, under Asumption($\mathcal{A}_\infty$), the system (11) admits a Lispchitz type global error bound if the Abadie qualification condition is satified at all points belong to its solution set. Consequently, the Lipschitz type global error bound holds when the system satifies the Slater condition (and Asumption($\mathcal{A}_\infty$)). Without the Slater condition, in the case of $p = 1$, it was established in [20] that a Hölderian global error bound holds with an exponent $\gamma = ((d-1)^n + 1)^{-1}$. In this final section, we consider the Hölderian error bound for the general systems of many finitely convex polynomials. We shall show a Hölderian global error bound with an explicit exponent holds for these systems with/without asumption($\mathcal{A}_\infty$).

Let us define the following quantity, which will be served below as a lower bound for exponents γ.

$$\gamma(n, d) := \frac{2}{(2d-1)^n + 1}, \quad \text{where } \; d := \max\{\deg f_1, ..., \deg f_p\}.$$

Theorem 8 *Let $f_i : \mathbb{R}^n \to \mathbb{R}$ $(i = 1, ..., p)$ be convex polynomials. Suppose that* $\inf_{x \in K} f(x) = 0$. *Then for any $\bar{x} \in S$, there exist $\varepsilon, \tau > 0$ and $\gamma(n, d) \leq \gamma \leq 1$ such that*

$$d(x, S) \leq \tau f(x)^\gamma \quad \textit{for all } \; x \in B(\bar{x}, \varepsilon) \cap K.$$

We need the following result from ([11]) on an error bound property for a polynomial around its strict minimum point.

Lemma 9 *([11], Theorem 3) Let f be a polynomial with degree $d \in \mathbb{N}^*$. Assume that there exists $\delta > 0$ such that $f(x) > f(0) = 0$ for all $x \in B(0, \delta) \setminus \{0\}$. Then there exist $\tau, \varepsilon > 0$ such that*

$$\|x\| \leq \tau f(x)^{((d-1)^n+1)^{-1}} \quad \textit{for all } x \in B(0, \varepsilon).$$

Proof of Theorem 8. We prove the theorem by induction on p. Obviously, the conclusion is true when $p = 0$. Suppose that the conclusion holds for any p convex polynomials and any convex polyhedral K

with $\inf_{x\in K} f(x) = 0$ whenever $p \leq s$. Let us show that the conclusion is true when $p = s+1$. By translation if necessary, it suffices to prove the theorem for the case of $\bar{x} = 0 \in S$. Let

$$K := \{x \in \mathbb{R}^n : \ \langle a_j, x\rangle \leq b_j, \ j = 1, ..., m\}.$$

Since $\bar{x} = 0$ is a global minimizer of f on K, then according to the standard Kuln-Taker condition in Convex Programming, there exist $\alpha_i \geq 0$, $\sum_{i=1}^p \alpha_i = 1$; $\beta_j \geq 0$, $j = 1, ..., m$ such that $\alpha_i f_i(0) = 0$; $\beta_j b_j = 0$ and

$$\sum_{i=1}^p \alpha_i f_i(x) + \sum_{j=1}^m \beta_j(\langle a_j, x\rangle - b_j) \geq 0 \quad \text{for all } x \in \mathbb{R}^n. \tag{31}$$

Denote by

$$I := \{i \in \{1, ..., p\} : \ \alpha_i > 0\} \quad \text{and} \quad J := \{j \in \{1, ..., m\} : \ \beta_j > 0\}.$$

Then,

$$0 \in A := \{v \in \mathbb{R}^n : \ \ f_i(v) = 0, \ i \in I; \ \langle a_j, v\rangle - b_j = \langle a_j, v\rangle = 0, \ j \in J\}. \tag{32}$$

Let $v_1, v_2 \in A$ and $v \in [v_1, v_2]$. Then, $f_i(v) \leq 0$ and $\langle a_j, v\rangle - b_j \leq 0$ for all $i \in I$, all $j \in J$. In virtue of relation (31), one has $f_i(v) \leq 0$ and $\langle a_j, v\rangle - b_j \leq 0$ for all $i \in I$, all $j \in J$, that is f_i; $\langle a_j, \cdot\rangle - b_j$ $(i \in I, \ j \in J)$ are constant 0 on $[v_1, v_2]$. Since f_i are polynomials, then $\text{aff}\{v_1, v_2\} \subseteq A$. Therefore A is a linear subspace of $\mathbb{R}^n$, and according to Lemma 3,

$$f(x + tv) = f(x) \quad \text{for all } x \in \mathbb{R}^n, \ v \in A, \ t \in \mathbb{R}.$$

Denote by $A^\perp := \{u \in \mathbb{R}^n : \ \langle u, v\rangle = 0 \ \text{ for all } v \in A\}$ the orthogonal subspace with A. Set $\dim A := k$, then $\dim A^\perp = n - k$.

For any $x := u + v \in S$ with $u \in A^\perp$; $v \in A$, one has

$$f_i(u) = f_i(u + v) = f_i(x) \leq 0, \ \ \langle a_j, u\rangle - b_j \leq 0 \quad \text{for all } i \in I, \ j \in J.$$

By relation (31), it implies $u \in A$, thus, $u = 0$. Hence,

$$S = \{v \in A : \ \ f_i(v) \leq 0, \ \langle a_j, v\rangle - b_j \leq 0, \ \ \forall i \notin I, \ \forall j \notin J\} \tag{33}$$

By the induction assumption, we can find $\tau_1, \delta_1 > 0$ such that

$$d(v, S) \leq \tau_1 \left([f(v)]_+ \max_{j\notin J}(\langle a_j, v\rangle - b_j)_+\right)^{\gamma(k,d)} \quad \text{for all } v \in B(0, \delta_1) \cap A. \tag{34}$$

On the other hand, one has

$$\sum_{i\in I} f_i(u)^2 + \sum_{j\in J}(\langle a_j, u\rangle - b_j)^2 > 0 \quad \text{for all } u \in A^\perp \setminus \{0\}.$$

By Lemma 9, there exist $\tau, \delta_2 > 0$ such that

$$\|u\| \leq \tau_2 \left(\sum_{i\in I} f_i(u)^2 + \sum_{j\in J}(\langle a_j, u\rangle - b_j)^2\right)^{((2d-1)^{n-k}+1)^{-1}} \quad \text{for all } u \in B(0, \delta_1) \cap A^\perp. \tag{35}$$

Define $\varepsilon := \min\{\delta_1, \delta_2\}$ and denote by L the Lipschitz constant of f on $B(0, \delta_2)$. Let $x \in B(0, \varepsilon) \cap K$ be given. Then, $x = u + v$ with $u \in B(0, \delta_1) \cap A^{\perp}$ as well as $v \in B(0, \delta_1) \cap A$. By (34), one has

$$f(x) \geq f_i(x) \geq -\frac{\sum_{r \in I \setminus \{i\}} \alpha_r f_r(x)}{\alpha_i}; \;\; 0 \geq \langle a_j, x\rangle - b_j \geq -\frac{\sum_{r\in I} \alpha_r f_r(x)}{\beta_j}, \; \forall i \in I, \; j \in J.$$

By setting

$$M := \max\left\{1, \frac{\sum_{r\in I\setminus\{i\}} \alpha_r}{\alpha_i}, \frac{1}{\beta_j}, \; i \in I, \; j \in J\right\},$$

one has

$$|f_(x)| \leq M|f(x)|, \;\; |\langle a_j, x\rangle - b_j| \leq M|f(x)|, \; \forall i \in I, \; \forall j \in J.$$

Since $f_i(u) = f_i(x)$ and $\langle a_j, u\rangle - b_j = \langle a_j, x\rangle - b_j$ for all $i \in I$, all $j \in J$, then , by relation (35), one obtains

$$\|u\| \leq \tau_3 f(x)^{2((2d-1)^{n-k}+1)^{-1}}, \;\; \tau_3 := \tau_2(M(I| + |J|))^{((2d-1)^{n-k}+1)^{-1}}. \tag{36}$$

Noticing that

$$[f(v)]_+ \leq f(x) + L\|u\|, \;\; \max_{j\notin J}(\langle a_j, v\rangle - b_j)_+ \leq \max_{j\notin J}(\langle a_j, v\rangle - b_j)_+ + \max_{j\notin J}\|a_j\|\|u\| = \max_{j\notin J}\|a_j\|\|u\|,$$

by relation (34), one derives that

$$d(v, S) \leq \tau_1\left(f(x) + (L + \max_{j\notin J}\|a_j\|)\|u\|\right)^{\gamma(k,d)}. \tag{37}$$

Combining relations (36) and (37), one obtains

$$d(x, S) \leq \|u\| + d(v, S) \leq \tau_3 f(x)^{2((2d-1)^{n-k}+1)^{-1}} + \tau_1\left(f(x) + (L + \max_{j\notin J}\|a_j\|)\tau_3 f(x)^{2((2d-1)^{n-k}+1)^{-1}}\right)^{\gamma(k,d)}. \tag{38}$$

By noticing that

$$((2d-1)^{n-k} + 1)((2d-1)^k + 1) \leq 2((2d-1)^n + 1), \;\; \text{for } 0 \leq k \leq n,$$

relation (38) implies that the conclusion is true with $p = s + 1$. □

Next, we establish a Hölder global error bound result for the system (11) when the solution set is assumed to be compact.

Theorem 10 *Let $f_i : \mathbb{R}^n \to \mathbb{R}$ $(i = 1, ..., p)$ be convex polynomials and let K be a convex polyhedral such that*

$$S := \{x \in K : \;\; f(x) := \max\{f_1(x), ..., f_p(x)\} \leq 0\}$$

is a nonempty compact set. Then there exist $\tau > 0$ and $\gamma(n, d) \leq \gamma \leq 1$ such that

$$d(x, S) \leq \tau([f(x)]_+ + [f(x)]_+^{\gamma}) \quad \textit{for all } \; x \in K.$$

Proof. Define $C := \{x \in K : \ d(x,S) \leq 1\}$. Since S is compact, then so C is a compact set. Thank to Theorem 8, we can find $\gamma(n,d) \leq \gamma \leq 1$ such that for each $z \in S$, there exist $0 < \epsilon(z) < 1$, $\tau(z) > 0$ such that

$$d(x,S) \leq \tau(z) f(x)^{\gamma} \quad \text{for all } \ x \in B(\bar{x}, \varepsilon(z)) \cap K. \tag{39}$$

By the compactness, there exist $z_1, ..., z_m$ in S such that

$$S \subset \bigcup_{i=1}^{m} B(z_i, \epsilon(z_i)/2).$$

Set $\epsilon = \min\{\epsilon(z_i) : i = 1, ..., m\}$; $\tau = \max\{\tau(z_i) : i = 1, ..., m\}$. Let $x \in C$ such that $d(x,S) \leq \epsilon/2$. Then we can find $z \in S$ such that $\|x - z\| \leq \epsilon/2$. Therefore, there is an index $i \in \{1, ..., m\}$ such that $z \in B(z_i, \epsilon(z_i)/2)$. It follows that

$$\|x - z_i\| \leq \|x - z\| + \|z - z_i\| \leq \epsilon/2 + \epsilon(z_i)/2 \leq \epsilon(z_i).$$

Hence, we obtain $d(x,S) \leq \tau[f(x)]_+^{\gamma}$. Now let $x \in C$ with $d(x,S) > \epsilon/2$. We shall show that there is a $\eta > 0$ such that

$$f(x) \geq \eta \ \text{ for all } x \in C, \ d(x,S) > \epsilon/2.$$

Indeed, if this is not the case, one can select a sequence $\{x_k\} \subset C$ such that $h(x_k) \leq \eta_k$ and $d(x_k, S) > \epsilon/2$ for all k, where $\{\eta_k\}$ is a sequence of positive numbers with $\lim_{k \to +\infty} \eta_k = 0$. By the compactness, without loss of generality, assume that $\{x_k\}$ converges to some $x^* \in C$. Then $f(x^*) \leq 0$, that is, $x^* \in S$, which contradicts $d(x_k, S) \leq \|x_k - x^*\| \to 0$.

Hence, for all $x \in C$ with $d(x,S) > \epsilon/2$, one has

$$d(x,S) \leq 1 \leq \frac{1}{\eta^{\gamma}} f(x)^{\gamma}.$$

By taking $\tau^* = \max\{\tau, 1/\eta^{\gamma}\}$, we obtain

$$d(x,S) \leq \tau^*[f(x)]_+^{\gamma} \quad \text{for all } x \in C. \tag{40}$$

Let now $x \in K$ be given with $d(x,S) > 1$. Let $z \in S$ such that $\|x - z\| = d(x,S)$. Then, $f(z) = 0$ and for $t := 1/\|x - z\|$, one has $y := (1-t)z + tx \in [z,x] \cap C$. Therefore, by (40),

$$d(y,S) = t\|x - z\| = 1 \leq \tau^*[f(y)]_+^{\gamma} \leq \tau^*[tf(x)]_+^{\gamma}.$$

Consequently, $d(x,S) \leq \tau^*[f(x)]_+$. The proof is complete. □

When the solution set is not neccessarily compact, we obtain the Hölder global error bound result under the assumption $(\mathcal{A}_\infty)$.

Theorem 11 *Let $f_i : \mathbb{R}^n \to \mathbb{R}$ $(i = 1, ..., p)$ be convex polynomials and let K be a convex polyhedral such that S is nonempty. Suppose that the assumption $(\mathcal{A}_\infty)$ is verified. Then there exist $\tau > 0$ and $\gamma(n,d) \leq \gamma \leq 1$ such that*

$$d(x,S) \leq \tau([f(x)]_+ + [f(x)]_+^{\gamma}) \quad \textit{for all } \ x \in K. \tag{41}$$

Proof. We prove the theorem by induction on the dimension n of $\mathbb{R}^n$. Obviously, the conclusion holds trivially when $n = 0$. Suppose that the conclusion holds for all $n \leq s$. Let us prove that the conclusion holds for $n = s + 1$. Assume to contrary that (45) is not true. That is, there is a sequence $\{x_k\} \subseteq K$ with $f(x_k) > 0 (\forall k)$ such that

$$\lim_{k\to\infty} \frac{f(x_k) + f(x_k)^{\gamma(n,d)}}{d(x_k, S)} = 0. \tag{42}$$

Let us show that we can select a sequence verifying (42) such that $\lim_{k\to\infty} f(x_k) = 0$. Indeed, for this, assume that there is $\alpha > 0$ such that $f(x_k) > \alpha > 0$ for all $k = 1, 2, ...$ Set $t_k := \frac{\alpha}{k^{\gamma(n,d)} f(x_k)}$ and let $z_k \in S$ such that $\|x_k - z_k\| = d(x_k, S)$ $(k = 1, 2, ...)$. Define $y_k := (1-t_k)z_k + t_k x_k$ $(k = 1, 2, ...)$. Then, $d(y_k, S) = t_k d(x_k, S)$ and by the convexity of f,

$$0 < f(y_k) \leq (1 - t_k)f(z_k) + t_k f(x_k) = t_k f(x_k) = \alpha/k^{\gamma(n,d)}.$$

Consequently, $\lim_{k\to\infty} f(y_k) = 0$, moreover,

$$\frac{f(y_k) + f(y_k)^{\gamma(n,d)}}{d(y_k, S)} \leq \frac{\alpha/k + (\alpha/k)^{\gamma(n,d)}}{t_k d(x_k, S)} = \frac{f(x_k)(1/k^{1-\gamma(k,d)} + \alpha^{\gamma(n,d)-1})}{d(x_k, S)}.$$

By (42), $\lim_{k\to\infty} \frac{f(x_k)}{d(x_k,S)} = 0$, therefore one obtains

$$\lim_{k\to\infty} \frac{f(y_k) + f(y_k)^{\gamma(n,d)}}{d(y_k, S)} = 0.$$

Thus, we can assume that $\lim_{k\to\infty} f(x_k) = 0$.

According to Theorem 8, the system admits a Hölder local error bound with exponent $\gamma(n, d)$ at all $x \in S$, that implies that $\|x_k\| \to \infty$ as $k \to \infty$. By passing to a subsequence if necessary, we can assume that

$$\lim_{k\to\infty} \frac{x_k}{\|x_k\|} := v \in \mathbb{R}^n \quad \text{with} \quad \|v\| = 1.$$

Then, obviously,

$$v \in K^\infty = \{d \in \mathbb{R}^n : \ \langle a_j, d\rangle \leq 0, \ j = 1, ..., m\} \quad \text{and} \quad f^\infty(v) \leq 0.$$

Since $(f + \delta_K)^\infty(v) = f^\infty(v)$ for $v \in K^\infty$, then it follows from Lemma 2 (ii) that $f^\infty(v) = 0$. By Lemma 3, one has

$$f(x + tv) = f(x) \quad \forall x \in \mathbb{R}^{s+1}, \ \forall t \in \mathbb{R}. \tag{43}$$

Denote by $\langle v\rangle := \{tv : \ t \in \mathbb{R}\}$ and $L = v^\perp := \{u \in \mathbb{R}^n : \ \langle u, v\rangle = 0\}$. Then, $\dim L = n - 1 = s$

Denote by

$$J := \{j \in \{1, ..., m\} : \quad \langle a_j, v\rangle = 0\},$$

and define the convex polyhedral K_s and the following inequality system of convex polynomials in L by

$$K_s := \{u \in L : \quad \langle a_j, u\rangle \leq b_j \ \forall j \in J\};$$

$$S_s := \{u \in K_s : \ \ f(u) \leq 0\}.$$

Similarly as in the proof of Theorem 5, the system defining S_s satisfies the assumption $(\mathcal{A}_\infty)$. Hence, by the induction assumption, there exists $\tau > 0$ such that

$$d(u, S_s) \leq \tau([f(u)]_+ + [f(u)]_+^{\gamma(s,d)}) \quad \forall u \in K_s. \tag{44}$$

For each $k \in \mathbb{N}$, let $u_k \in L$, and $t_k \in \mathbb{R}$ such that $x_k = u_k + t_k v$. Since

$$\langle a_j, u + tv\rangle = \langle a_j, u\rangle \quad \forall u \in L,\ \forall t \in \mathbb{R}, \forall j \in J,$$

then $u_k \in K_s$. Since $\lim_{k\to\infty} f(u_k) = \lim_{k\to\infty} f(x_k) = 0$, then by relation (44), one has $\lim_{k\to\infty} d(u_k, S_s) = 0$. Let $w_k \in S_s$ such that $\|u_k - w_k\| = d(u_k, S_s)$ and $z_k := w_k + t_k v$. By $\frac{x_k}{\|x_k\|} \to v$, then, $\lim_{k\to\infty} t_k = +\infty$ and $\lim_{k\to\infty} \|u_k\|/t_k = 0$. Hence, for all $j \in \{1, ..., m\} \setminus J$, one has

$$\langle a_j, z_k\rangle = t_k(\langle a_j, u_k/t_k\rangle + \langle a_j, w_k - u_k\rangle/t_k + \langle a_j, v\rangle) \to -\infty \text{ as } k \to \infty.$$

On the other hand, $\langle a_j, z_k\rangle = \langle a_j, w_k\rangle \leq 0$ for all $j \in J$, and $f(z_k) = f(w_k) \leq 0$, which follows that $z_k \in S$. Hence, from (44), one has

$$\lim_{k\to\infty} \frac{f(x_k) + f(x_k)^{\gamma(n,d)}}{d(x_k, S)} \geq \liminf_{k\to\infty} \frac{f(x_k) + f(x_k)^{\gamma(n,d)}}{\|x_k - z_k\|} = \liminf_{k\to\infty} \frac{f(u_k) + f(u_k)^{\gamma(n,d)}}{d(u_k, S)} \geq \tau > 0,$$

which contradicts (42) and completes the proof. □

Finally, without assumption $(\mathcal{A}_\infty)$, one obtains the following error bound result.

Theorem 12 *Let $f_i : \mathbb{R}^n \to \mathbb{R}$ $(i = 1, ..., p)$ be convex polynomials and let K be a convex polyhedral such that the solution set S is nonempty. Then there exist $\tau > 0$ and $\gamma(n, d) \leq \gamma \leq 1$ such that*

$$d(x, S) \leq \tau \left([f(x)]_+ + [f(x)]_+^\gamma + \sum_{j=1}^{d} |\nabla^j| f(x)([f(x)]_+ + [f(x)]_+^\gamma)^j \right) \quad \textit{for all } x \in K.$$

Proof. The proof is very similar to the one of Theorem 7. Here, instead of using Corollary 6, we use Theorem 11. □

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