Time-inconsistent multistage stochastic programs: martingale bounds

Georg Ch. Pflug[*], Alois Pichler[†]

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Abstract

It is well known that multistage programs, which maximize expectation or expected utility, allow a dynamic programming formulation, and that other objectives destroy the dynamic programming character of the problem.

This paper considers a risk measure at the final stage of a multistage stochastic program. Although these problems are not time consistent, it is shown that optimal decisions evolve as a martingale. A verification theorem is provided, which characterizes optimal decisions by enveloping sub- and supermartingales. To obtain these characterizations the idea of a constant risk profile has to be given up and instead a risk profile, which varies over time, has to be accepted.

The basis of the analysis is a new decomposition theorem for risk measures, which is able to recover the genuine risk measure by measuring risk conditionally.

Keywords: Stochastic optimization, risk measure, Average Value-at-Risk, dynamic programming, time consistency

Classification: 90C15, 60G42, 90C47

1 Introduction

Risk measures have been introduced by Artzner et al. in the pioneering papers [3] and [1]. The set of axioms, which are proposed there, is widely accepted nowadays. Approximately ten years later the same group of authors considered risk measures again in a multistage framework in [2]. An initial example is given there to study the multiperiodic character of the Average Value-at-Risk. Their example reveals that the simple Average Value-at-Risk is not time consistent, that is, decisions and observations at a later stage may contradict decisions, which have been made at an earlier stage of time. Moreover they address the fundamental dynamic programming principle, known as Bellman's, or Pontryagin's principle (cf. Fleming and Soner [11]).

Significant efforts and investigations have been initiated in order to identify classes of multistage stochastic programs which allow stagewise decompositions. Important publications on the subject

[*]University of Vienna. Department of Statistics and Operations Research and International Institute for Applied Systems Analysis (IIASA), Laxenburg, Austria

[†]Norwegian University of Science and Technololgy, NTNU

Contact: alois.pi@ntnu.no

include publications by Shapiro and Ruszczyński [23, 22, 21] (cf. as well [24, 25]) and the recent paper by Carpentier et al., [5].

To manage this conceptual difficulty very exclusive risk measures have been considered, which are designed to anticipate the multistage structure right from the beginning. These measures are nested compositions of conditional risk functionals. However, this is often not what the modeler wants: nested compositions of risk functionals are rather artefacts, they do not offer an immediate interpretation and they lack a natural justification in terms of decision theory.

A further, important topic in a multistage optimization framework is Bellman's principle. It turned out that it is possible in some situations to write down dynamic programming equations which satisfy a generalized Bellman principle, as was elaborated by Shapiro in [23] for the Average Value-at-Risk. These dynamic programming equations necessarily depend on previous decisions, whenever a risk measures is employed, which is not time consistent itself.

This paper is based on a new decomposition of an important risk measure. The decomposition measures risk on conditional level only, and it recovers the initial risk measure by collecting the conditional risk measures via an expectation. To this end the risk profile has to be adapted conditionally, such that the conditional risk profile is not static any more.

Additional information changes the perception of risk. The adaptive choice of appropriate measures of risk complies with the course of action of a risk manager who adjusts the preferences, whenever additional information is available. The decision maker is less reluctant, if an observation reveals that the future will be bright, but conversely she or he will be more strict if losses at the end become more likely. This gives rise to defining an extended notion of a conditional risk measure, which is not just the same risk measure applied to conditional distributions, but which may be a different functional for different conditional distributions, depending on its respective history.

By involving adapted conditional risk measures it is possible to recover dynamic programming principles for multistage stochastic programs. Moreover verification theorems, which are central in dynamic control, are established here for multistage stochastic programs.

The dynamic programming equations presented are based on the dual representation of a risk measure, and different to those provided by Shapiro in [23]. The dynamic programming approach presented enables a characterization of optimal solutions of a multistage stochastic program in terms of enveloping sub- and supermartingales. It is shown that a solution of a multistage problem evolves as a martingale over time, where different risk measures are encountered at each stage.

Dynamic programming equations notably cannot remove the time inconsistency, which is inherent to these problems. But these equations come along with verification theorems, and it is their purpose to enable checking, if a given policy is optimal. By assessing the enveloping sub- and supermartingales it is moreover possible to provide upper and lower bounds, such that the quality of a given multistage policy can be assessed with these sub- and supermartingales as well.

The paper is organized as follows. The following Section 2 provides the setting for the Average Value-at-Risk, as this risk measure is basic for the presentation. Next, the conditional version is considered. The decomposition theorem, the central statement of this paper, is contained in Section 4. Section 5 introduces the multistage optimization problem. Section 6 exposes the dynamic programming formulation, while the subsequent Section 7 introduces the martingale representations, which are in line with dynamic programming.

2 Representations of the genuine risk measure

The conceptual description of the problem is reduced to the Average Value-at-Risk, $\mathsf{AV@R}$. This reduction is justified, as more general coherent risk measures—distortion risk measures—are composed in a linear way of Average Value-at-Risks at different levels. Further, Kusuoka's theorem provides all version independent (also known as law invariant) risk measures via distortion risk measures (cf., for example, Pflug and Römisch [15]), such that this reduction is without loss of generality.

The Average Value-at-Risk is considered in its *concave* variant involving lower quantiles,

$$\mathsf{AV@R}_\alpha(Y) := \frac{1}{\alpha}\int_0^\alpha F_Y^{-1}(u)\,\mathrm{d}u \qquad (0<\alpha\le 1),$$

where α is called *level*. In this setting $\mathsf{AV@R}$ accounts for profits, which are subject to maximization. Throughout this paper we shall assume that $Y\in L^\infty$, but this is for the convenience of the presentation only (L^1 could be chosen in many, but not in all situations).

The *dual representation* of the Average Value-at-Risk at level α is

$$\mathsf{AV@R}_\alpha(Y) = \inf\left\{\mathbb{E}(YZ):\ 0\le Z,\ \alpha Z\le 1 \text{ and } \mathbb{E}(Z)=1\right\}, \tag{1}$$

where the infimum in (1) is among all positive random variables $Z\ge 0$ with expectation $\mathbb{E}(Z)=1$ (densities), satisfying the additional truncation constraint $\alpha Z\le 1$, as indicated. The infimum is attained if $\alpha>0$, and in this case the optimal random variable Z in (1) is coupled in a anti-monotone way with Y (cf. Nelsen [14]).

The equivalent relation[1]

$$\mathsf{AV@R}_\alpha(Y) = \max_{q\in\mathbb{R}}\ q-\frac{1}{\alpha}\mathbb{E}\left(q-Y\right)_+ \tag{2}$$

was elaborated by Rockafellar and Uryasev in [16]. This representation replaces the infimum in (1) by a maximum, which is very helpful in the context of profit maximization. The maximum in (2) is attained at some $q^*\in\mathbb{R}$ satisfying the quantile condition $P(Y<q^*)\le\alpha\le P(Y\le q^*)$. For the sake of completeness the Average Value-at-Risk at level $\alpha=0$ is

$$\mathsf{AV@R}_0(Y) = \lim_{\alpha\to 0}\mathsf{AV@R}_\alpha(Y) = \operatorname{ess\,inf}(Y).$$

Concave–convexity, or the saddle point property

We mention finally that the mapping

$$Y\mapsto \mathsf{AV@R}_\alpha(Y)$$

is concave in the present setting, while the Lorentz curve

$$\alpha\mapsto\alpha\cdot\mathsf{AV@R}_\alpha(Y) = \int_0^\alpha \mathsf{V@R}_q(Y)\,\mathrm{d}q \tag{3}$$

is convex in its parameter α. These properties will be exploited to characterize optimal solutions via saddle points.

[1] x_+ is the positive part of x, that is $x_+=x$ if $x\ge 0$, and $x_+=0$ else.

3 The conditional Average Value-at-Risk at random level

The Average Value-at-Risk, as defined above, is a real-valued function on $L^\infty(\mathcal{F}_T)$, where $\mathcal{F}_T$ is the final sigma algebra. The Average Value-at-Risk thus quantifies the entire future risk associated with Y at stage 0 in the single number $\mathsf{AV@R}(Y)$. Having multistage stochastic optimization in mind it is desirable to have an idea of the accumulated risk at an intermediate stage t as well ($0 < t < T$). To describe this evolution sigma algebras $\mathcal{F}_t \subset \mathcal{F}_T$ are considered, which assemble the information up to time t.

The conditional Average Value-at-Risk. First attempts to define a conditional Average Value-at-Risk for a smaller sigma algebra $\mathcal{F}_t \subset \mathcal{F}_T$ are contained in Pflug and Römisch [15]. An extension to this initial definition considers a level parameter α, which depends on the history only (we are grateful to Werner Römisch for pointing out the references Cheridito and Kupper [7, Section 2.3.1] and [8, 6]). This setting is broad enough to allow a decomposition.

The level α is not considered fixed and constant any longer, but $\mathcal{F}_t$-measurable instead. We write $\alpha \lhd \mathcal{F}_t$ to express that α is measurable with respect to $\mathcal{F}_t$. For the trivial sigma algebra $\mathcal{F}_0 = \{\emptyset, \Omega\}$ the statement $\alpha \lhd \mathcal{F}_0$ notably expresses that α is deterministic, a constant.

Definition 1 (Conditional Average Value-at-Risk at random level)**.** The *conditional* Average Value-at-Risk *at random level* $\alpha \lhd \mathcal{F}_t$ ($0 \le \alpha \le \mathbb{1}$) is the $\mathcal{F}_t$-random variable

$$\mathsf{AV@R}_\alpha(Y|\mathcal{F}_t) := \operatorname{ess\,inf}\left\{\mathbb{E}(YZ|\mathcal{F}_t) : \mathbb{E}(Z|\mathcal{F}_t) = \mathbb{1},\, 0 \le Z,\, \alpha Z \le \mathbb{1}\right\}, \tag{4}$$

where $\mathbb{1}$ is the random variable being identically 1.

Although the level α is random it should be noted that it is measurable with respect to $\mathcal{F}_t$, but *not* with respect to $\mathcal{F}_T$. $\alpha \lhd \mathcal{F}_t$ depends on he history only, and this is an important limitation in comparison with $Y \lhd \mathcal{F}_T$.

Remark 2. $\mathbb{E}(YZ|\mathcal{F}_t)$ is a random variable, the essential infimum in (4) thus is an infimum over a family of random variables, and the resulting $\mathsf{AV@R}_\alpha(Y|\mathcal{F}_t)$ is an $\mathcal{F}_t$-random variable itself; for the definition of the essential infimum of random variables we refer—for example—to Karatzas and Shreve [12, Appendix] or the classical reference Dunford and Schwartz [9].

The next theorem elaborates that the *conditional* Average Value-at-Risk at random level basically preserves all properties of the usual Average Value-at-Risk.

Theorem 3. *For the conditional Average Value-at-Risk at random level $\alpha \lhd \mathcal{F}_t$ ($0 \le \alpha \le \mathbb{1}$) the following hold true:*

(i) PREDICTABILITY: $\mathsf{AV@R}_\alpha(Y|\mathcal{F}_t) = Y$ *if* $Y \lhd \mathcal{F}_t$*;*

(ii) TRANSLATION EQUIVARIANCE[2] : $\mathsf{AV@R}_\alpha(Y + c\,|\mathcal{F}_t) = \mathsf{AV@R}_\alpha(Y|\mathcal{F}_t) + c$ *if* $c \lhd \mathcal{F}_t$*;*

(iii) POSITIVE HOMOGENEITY: $\mathsf{AV@R}_\alpha(\lambda Y|\mathcal{F}_t) = \lambda\,\mathsf{AV@R}_\alpha(Y|\mathcal{F}_t)$ *whenever λ is nonnegative, bounded and $\mathcal{F}_t$-measurable, $\lambda \ge 0$ and $\lambda \lhd \mathcal{F}_t$;*

(iv) MONOTONICITY: $\mathsf{AV@R}_{\alpha_1}(Y_1|\mathcal{F}_t) \le \mathsf{AV@R}_{\alpha_2}(Y_2|\mathcal{F}_t)$ *whenever $Y_1 \le Y_2$ and $\alpha_1 \le \alpha_2$ almost surely;*

[2] In an economic or monetary environment this is often called CASH INVARIANCE instead.

(v) Concavity: $\mathsf{AV@R}_\alpha\left((\mathbb{1}-\lambda)\,Y_0+\lambda Y_1|\mathcal{F}_t\right)\geq(\mathbb{1}-\lambda)\,\mathsf{AV@R}_\alpha\left(Y_0|\mathcal{F}_t\right)+\lambda\,\mathsf{AV@R}_\alpha\left(Y_1|\mathcal{F}_t\right)$ *for* $\lambda\lhd\mathcal{F}_t$ *and* $0\leq\lambda\leq\mathbb{1}$ *almost surely;*

(vi) Lower and Upper bounds: $\mathsf{AV@R}_0\left(Y\right)\leq\mathsf{AV@R}_0\left(Y|\mathcal{F}_t\right)\leq\mathsf{AV@R}_\alpha\left(Y|\mathcal{F}_t\right)\leq\mathbb{E}\left(Y|\mathcal{F}_t\right)$.

Proof. As for the Predictability just observe that

$$\begin{aligned}\mathsf{AV@R}_\alpha\left(Y|\mathcal{F}_t\right)&=\operatorname{ess\,inf}\left\{Y\cdot\mathbb{E}\left(Z|\mathcal{F}_t\right):\mathbb{E}\left(Z|\mathcal{F}_t\right)=\mathbb{1},\,0\leq Z,\,\alpha Z\leq\mathbb{1}\right\},\\&=\operatorname{ess\,inf}\left\{Y\cdot\mathbb{1}:\mathbb{E}\left(Z|\mathcal{F}_t\right)=\mathbb{1},\,0\leq Z,\,\alpha Z\leq\mathbb{1}\right\}=Y\end{aligned}$$

whenever $Y\lhd\mathcal{F}_t$, and Translation equivariance follows from

$$\begin{aligned}\mathsf{AV@R}_\alpha\left(Y+c|\mathcal{F}_t\right)&=\operatorname{ess\,inf}\left\{\mathbb{E}\left(YZ|\mathcal{F}_t\right)+c\,\mathbb{E}\left(Z|\mathcal{F}_t\right):\mathbb{E}\left(Z|\mathcal{F}_t\right)=\mathbb{1},\,0\leq Z,\,\alpha Z\leq\mathbb{1}\right\}\\&=\operatorname{ess\,inf}\left\{\mathbb{E}\left(YZ|\mathcal{F}_t\right)+c\colon\ \mathbb{E}\left(Z|\mathcal{F}_t\right)=\mathbb{1},\,0\leq Z,\,\alpha Z\leq\mathbb{1}\right\}\\&=\mathsf{AV@R}_\alpha\left(Y|\mathcal{F}_t\right)+c.\end{aligned}$$

To accept that the conditional Average Value-at-Risk is positively homogeneous observe that the assertion is correct for $\lambda=\mathbb{1}_A$ ($A\in\mathcal{F}_t$); by passing to the limit one gets the assertion for simple functions (step-functions) first, then for any nonnegative function $\lambda\in L^\infty\left(\mathcal{F}_t\right)$.

To prove Concavity as stated observe that

$$(\mathbb{1}-\lambda)\,\mathbb{E}\left(Y_0Z|\mathcal{F}_t\right)+\lambda\,\mathbb{E}\left(Y_1Z|\mathcal{F}_t\right)=\mathbb{E}\left(\left((\mathbb{1}-\lambda)\,Y_0+\lambda Y_1\right)Z|\mathcal{F}_t\right)$$

by the measurability assumption $\lambda\lhd\mathcal{F}_t$, hence

$$\begin{aligned}\mathsf{AV@R}_\alpha\left((\mathbb{1}-\lambda)\,Y_0+\lambda Y_1|\mathcal{F}_t\right)&=\operatorname*{ess\,inf}_Z\,(\mathbb{1}-\lambda)\,\mathbb{E}\left(Y_0Z|\mathcal{F}_t\right)+\lambda\,\mathbb{E}\left(Y_1Z|\mathcal{F}_t\right)\\&\geq\operatorname*{ess\,inf}_{Z_0,Z_1}\,(\mathbb{1}-\lambda)\,\mathbb{E}\left(Y_0Z_0|\mathcal{F}_t\right)+\lambda\,\mathbb{E}\left(Y_1Z_1|\mathcal{F}_t\right)\\&\geq(\mathbb{1}-\lambda)\operatorname*{ess\,inf}_{Z_0}\mathbb{E}\left(Y_0Z_0|\mathcal{F}_t\right)+\lambda\operatorname*{ess\,inf}_{Z_1}\mathbb{E}\left(Y_1Z_1|\mathcal{F}_t\right)\\&=(\mathbb{1}-\lambda)\,\mathsf{AV@R}_\alpha\left(Y_0|\mathcal{F}_t\right)+\lambda\mathsf{AV@R}_\alpha\left(Y_1|\mathcal{F}_t\right),\end{aligned}$$

where $Z_0\geq0$ and $Z_1\geq0$ are chosen to satisfy $\mathbb{E}\left(Z_i|\mathcal{F}_t\right)=\mathbb{1}$ and $\alpha Z_i\leq\mathbb{1}$ each.

To observe the monotonicity property recall that $\alpha_1\leq\alpha_2$, hence

$$\begin{aligned}\mathsf{AV@R}_{\alpha_1}\left(Y_1|\mathcal{F}_t\right)&=\operatorname*{ess\,inf}_Z\left\{\mathbb{E}\left(ZY_1|\mathcal{F}_t\right):Z\geq0,\,\alpha_1Z\leq\mathbb{1},\,\mathbb{E}\left(Z|\mathcal{F}_t\right)=\mathbb{1}\right\}\\&\leq\operatorname*{ess\,inf}_Z\left\{\mathbb{E}\left(ZY_2|\mathcal{F}_t\right):Z\geq0,\,\alpha_1Z\leq\mathbb{1},\,\mathbb{E}\left(Z|\mathcal{F}_t\right)=\mathbb{1}\right\}\\&\leq\operatorname*{ess\,inf}_Z\left\{\mathbb{E}\left(ZY_2|\mathcal{F}_t\right):Z\geq0,\,\alpha_2Z\leq\mathbb{1},\,\mathbb{E}\left(Z|\mathcal{F}_t\right)=\mathbb{1}\right\}\\&=\mathsf{AV@R}_{\alpha_2}\left(Y_2|\mathcal{F}_t\right).\end{aligned}$$

The Upper bound finally becomes evident because $Z=\mathbb{1}$ is feasible for (4), the lower bounds already have been used. □

The next characterization, used in Pflug and Römisch [15] to define the conditional Average Value-at-Risk in a simpler context, extends to the situation $\alpha\lhd\mathcal{F}_t$, but replaces the essential infimum by a usual infimum.

Proposition 4 (Characterization of the Average Value-at-Risk)**.** *Suppose that* $\alpha\lhd\mathcal{F}_t$.

(i) The conditional Average Value-at-Risk at random level α is a $\mathcal{F}_t$-random variable satisfying

$$\mathbb{E}\left[\mathbb{1}_B \cdot \mathsf{AV@R}_\alpha\left(Y|\mathcal{F}_t\right)\right] = \inf\left\{\mathbb{E}\left(YZ\right):\ 0 \le Z,\, \alpha Z \le \mathbb{1}_B,\, \mathbb{E}\left(Z|\mathcal{F}_t\right) = \mathbb{1}_B\right\}$$

for every set $B \in \mathcal{F}_t$.

(ii) Moreover the conjugate duality relation

$$\mathsf{AV@R}_\alpha\left(Y|\mathcal{F}_t\right) = \operatorname*{ess\,inf}_Z \mathbb{E}\left(YZ|\mathcal{F}_t\right) - \mathsf{AV@R}_\alpha^*\left(Z|\mathcal{F}_t\right)$$

with

$$\mathsf{AV@R}_\alpha^*\left(Z|\mathcal{F}_t\right) = \begin{cases} 0 & \text{if } \mathbb{E}\left(Z|\mathcal{F}_t\right) = \mathbb{1},\, 0 \le Z \text{ and } \alpha Z \le \mathbb{1} \\ -\infty & \text{else} \end{cases} \tag{5}$$

holds true.

Remark 5. Notably α, Z and $\mathbb{E}\left(Z|\mathcal{F}_t\right)$ may have various versions. The defining equation (5) is understood to provide a version of $\mathsf{AV@R}_\alpha^*$ for any such version and $\mathsf{AV@R}_\alpha^*$ thus is well-defined.

Proof. The essential infimum ess inf, by the characterizing theorem (Appendix A in Karatzas and Shreve [12] or [9]), is a density provided by the Radon–Nikodým theorem satisfying

$$\int_B \mathsf{AV@R}_\alpha\left(Y|\mathcal{F}_t\right)\,\mathrm{d}P = \inf\left\{\mathbb{E}\left[\sum_{k=1}^K \mathbb{1}_{B_k}\mathbb{E}\left(YZ_k|F_t\right)\right]:\ 0 \le Z_k,\, \alpha Z_k \le \mathbb{1},\, \mathbb{E}\left(Z_k|\mathcal{F}_t\right) = \mathbb{1}\right\},$$

where the infimum is among feasible Z_k and pairwise disjoint sets $B_k \in \mathcal{F}_t$ with $B = \bigcup_{k=1}^K B_k$. The random variable $Z := \sum_{k=1}^K \mathbb{1}_{B_k} Z_k$ satisfies $Z = \mathbb{1}_B \cdot Z$, and the equation in the latter display thus rewrites as

$$\mathbb{E}\left[\mathbb{1}_B \mathsf{AV@R}_\alpha\left(Y|\mathcal{F}_t\right)\right] = \inf\left\{\mathbb{E}\,YZ:\ 0 \le Z,\, \alpha Z \le \mathbb{1}_B,\, \mathbb{E}\left[Z|\mathcal{F}_t\right] = \mathbb{1}_B\right\},$$

which is the desired assertion.

The second assertion is the conditional equivalent to (1). For this recall the Fenchel-Moreau-Rockafellar duality theorem, which states that

$$\mathsf{AV@R}_\alpha\left(Y|\mathcal{F}_t\right) = \operatorname*{ess\,inf}_Z \mathbb{E}\left(YZ|\mathcal{F}_t\right) - \mathsf{AV@R}_\alpha^*\left(Z|\mathcal{F}_t\right),$$

where

$$\mathsf{AV@R}_\alpha^*\left(Z|\mathcal{F}_t\right) = \operatorname*{ess\,inf}_Y \mathbb{E}\left(YZ|\mathcal{F}_t\right) - \mathsf{AV@R}_\alpha\left(Y|\mathcal{F}_t\right).$$

Thus

$$\begin{aligned} \mathsf{AV@R}_\alpha^*\left(Z|\mathcal{F}_t\right) &\le \operatorname*{ess\,inf}_{\gamma\in\mathbb{R}} \mathbb{E}\left[\left(\gamma\mathbb{1}\right)Z|\mathcal{F}_t\right] - \mathsf{AV@R}_\alpha\left(\gamma\mathbb{1}|\mathcal{F}_t\right) \\ &= \operatorname*{ess\,inf}_{\gamma\in\mathbb{R}} \gamma\left(\mathbb{E}\left[Z|\mathcal{F}_t\right] - \mathbb{1}\right) \end{aligned}$$

and whence $\mathsf{AV@R}_\alpha^*\left(Z|\mathcal{F}_t\right) = -\infty$ on the $\mathcal{F}_t$-set $\left\{\mathbb{E}\left(Z|\mathcal{F}_t\right) \ne \mathbb{1}\right\}$.

Next suppose that $B := \{Z < 0\}$ has positive measure, then $\mathbb{E}\left(Z\,\mathbb{1}_B|\mathcal{F}_t\right) < 0$ on B. Thus

$$\begin{aligned}\mathsf{AV@R}_\alpha^*\left(Z|\mathcal{F}_t\right) &\leq \operatorname*{ess\,inf}_{\gamma>0} \mathbb{E}\left(\gamma\mathbb{1}_B Z|\mathcal{F}_t\right) - \mathsf{AV@R}_\alpha\left(\gamma\mathbb{1}_B|\mathcal{F}_t\right)\\ &\leq \operatorname*{ess\,inf}_{\gamma>0} \gamma\mathbb{E}\left(Z\mathbb{1}_B|\mathcal{F}_t\right) = -\infty\end{aligned}$$

on B. Finally suppose that $C := \{\alpha Z > \mathbb{1}\}$ has positive measure, so

$$\begin{aligned}\mathsf{AV@R}_\alpha^*\left(Z|\mathcal{F}_t\right) &\leq \operatorname*{ess\,inf}_{\gamma>0} \mathbb{E}\left(-\gamma\alpha\mathbb{1}_C Z|\mathcal{F}_t\right) - \mathsf{AV@R}_\alpha\left(-\gamma\alpha\mathbb{1}_C|\mathcal{F}_t\right)\\ &\leq \operatorname*{ess\,inf}_{\gamma>0} -\gamma\mathbb{E}\left(\alpha Z\mathbb{1}_C|\mathcal{F}_t\right) + \gamma\mathbb{E}\left(\mathbb{1}_C|\mathcal{F}_t\right)\\ &= \operatorname*{ess\,inf}_{\gamma>0} -\gamma\left(\mathbb{E}\left[\left(\alpha Z - \mathbb{1}\right)\mathbb{1}_C|\mathcal{F}_t\right]\right) = -\infty\end{aligned}$$

on C by the same reasoning. Combining all three ingredients gives the statement, as they constitute all conditions for the Average Value-at-Risk in (4). □

Concave–convexity, or the saddle point property

The Average Value-at-Risk, in its respective variable, is convex *and* concave:

(i) Concavity of the Average Value-at-Risk

$$Y \mapsto \mathsf{AV@R}_\alpha\left(Y|\mathcal{F}_t\right)$$

was elaborated in Theorem 3.

(ii) Convexity of the Average Value-at-Risk, for the level parameter α, is mentioned in (3). The following Theorem 6 extends this observation for measurable level parameters.

Theorem 6 (Convexity of the AV@R in its level parameter)**.** *Let α, Z_0, Z_1 and $\lambda \lhd \mathcal{F}_t$ with $0 \leq \lambda \leq \mathbb{1}$ be fixed, then*

$$Z_\lambda \cdot \mathsf{AV@R}_{\alpha\cdot Z_\lambda}\left(Y|\mathcal{F}_t\right) \leq \left(\mathbb{1}-\lambda\right) Z_0 \cdot \mathsf{AV@R}_{\alpha\cdot Z_0}\left(Y|\mathcal{F}_t\right) + \lambda Z_1 \cdot \mathsf{AV@R}_{\alpha\cdot Z_1}\left(Y|\mathcal{F}_t\right),$$

where Z_λ is the convex combination $Z_\lambda = \left(\mathbb{1}-\lambda\right) Z_0 + \lambda Z_1$.

Proof. Recall that by the definition of the conditional Average Value-at-Risk we have that

$$\begin{aligned}&\left(\mathbb{1}-\lambda\right) Z_0 \cdot \mathsf{AV@R}_{\alpha\cdot Z_0}\left(Y|\mathcal{F}_t\right) + \lambda Z_1 \cdot \mathsf{AV@R}_{\alpha\cdot Z_1}\left(Y|\mathcal{F}_t\right)\\ &\quad = \operatorname{ess\,inf} \mathbb{E}\left(Y\left(\mathbb{1}-\lambda\right) Z_0 f_0|\mathcal{F}_t\right) + \operatorname{ess\,inf} \mathbb{E}\left(Y\lambda Z_1 f_1|\mathcal{F}_t\right)\\ &\quad = \operatorname{ess\,inf} \mathbb{E}\left(Y\left(\left(\mathbb{1}-\lambda\right) Z_0 f_0 + \lambda Z_1 f_1\right)|\mathcal{F}_t\right)\end{aligned}$$

where $f_0 \geq 0$, $f_1 \geq 0$, $\mathbb{E}\left(f_0|\mathcal{F}_t\right) = \mathbb{1}$, $\mathbb{E}\left(f_1|\mathcal{F}_t\right) = \mathbb{1}$, and moreover $\alpha Z_0 f_0 \leq \mathbb{1}$ and $\alpha Z_1 f_1 \leq \mathbb{1}$. It follows that $\alpha\left(\left(\mathbb{1}-\lambda\right) Z_0 f_0 + \lambda Z_1 f_1\right) \leq \mathbb{1}$ and hence $\alpha Z_\lambda f \leq \mathbb{1}$ for $f := \frac{(\mathbb{1}-\lambda)Z_0 f_0 + \lambda Z_1 f_1}{Z_\lambda}$. Notice that f is nonnegativ as well, and $\mathbb{E}\left(f|\mathcal{F}_t\right) = \mathbb{E}\left(\frac{(\mathbb{1}-\lambda)Z_0 f_0 + \lambda Z_1 f_1}{Z_\lambda}\middle|\mathcal{F}_t\right) = \frac{(\mathbb{1}-\lambda)Z_0 + \lambda Z_1}{Z_\lambda} = \mathbb{1}$. The latter display continues thus as

$$\begin{aligned}&\left(\mathbb{1}-\lambda\right) Z_0 \cdot \mathsf{AV@R}_{\alpha\cdot Z_0}\left(Y|\mathcal{F}_t\right) + \lambda Z_1 \cdot \mathsf{AV@R}_{\alpha\cdot Z_1}\left(Y|\mathcal{F}_t\right)\\ &\quad \geq \operatorname{ess\,inf} \mathbb{E}\left(Y Z_\lambda f|\mathcal{F}_t\right),\end{aligned}$$

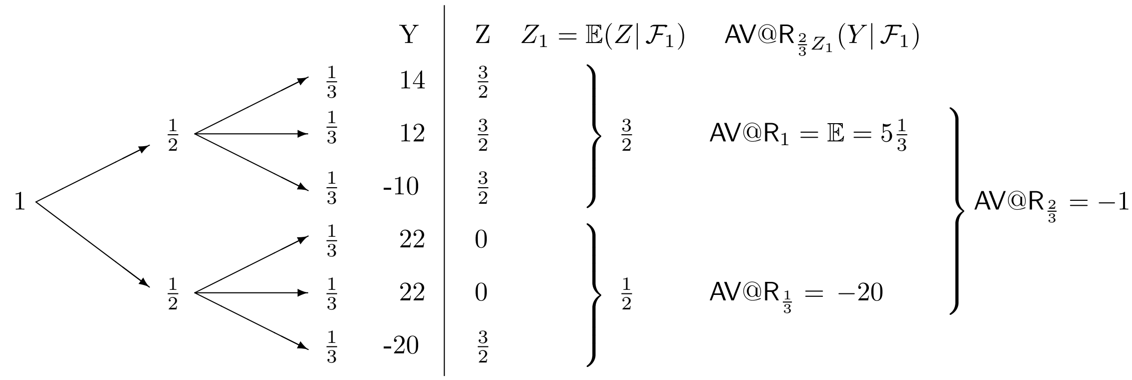

Figure 1: (cf. Artzner et al [2] and Example 11) Nested computation of $\mathsf{AV@R}_\alpha(Y)$ with $\alpha = \frac{2}{3}$ and outcomes with equal probabilities. The intriguing and misleading fact is that the *conditional* Average Value-at-Risk, computed with the *initial* constant $\alpha = \frac{2}{3}$, is $+1$ at each in-between node ($\mathsf{AV@R}_{\frac{2}{3}\mathbb{1}}(Y|\mathcal{F}_t) = +\mathbb{1}$), but $\mathsf{AV@R}_{\frac{2}{3}}(Y) = -1$.

where the essential infimum is among all random variables $f \geq 0$ with $\mathbb{E}(f|\mathcal{F}_t) = \mathbb{1}$ and $\alpha Z_\lambda f \leq \mathbb{1}$. Hence

$$\begin{aligned}(\mathbb{1} - \lambda) Z_0 \cdot \mathsf{AV@R}_{\alpha\cdot Z_0}(Y|\mathcal{F}_t) &+ \lambda Z_1 \cdot \mathsf{AV@R}_{\alpha\cdot Z_1}(Y|\mathcal{F}_t) \\ &\geq \mathsf{AV@R}_{\alpha Z_\lambda}(Y Z_\lambda|\mathcal{F}_t) \\ &= Z_\lambda \mathsf{AV@R}_{\alpha Z_\lambda}(Y|\mathcal{F}_t)\end{aligned}$$

by positive homogeneity, and this is the desired assertion. □

Representation as a maximum

In addition to the defining equation (4) for the Average Value-at-Risk at random level there is a further representation, which follows the classical equivalence of (1) and (2). It is stated without proof. The proof is along the lines of the classical equivalence and rather technical.

Theorem 7. *The Average Value-at-Risk at strictly positive random level $\alpha > 0$ has the additional representation*

$$\mathsf{AV@R}_\alpha(Y|\mathcal{F}_t) = \operatorname{ess\,sup}\left\{Q - \frac{1}{\alpha}\mathbb{E}\left[(Q-Y)_+|\mathcal{F}_t\right] : Q \lhd \mathcal{F}_t\right\}$$

where the essential supremum is among all bounded random variables $Q \in L^\infty(\mathcal{F}_t)$ ($Q \lhd \mathcal{F}_t$).

4 The decomposition theorem

Given the Average Value-at-Risk conditionally on $\mathcal{F}_t$, how can one reassemble the Average Value-at-Risk at time 0? This is the content of the next theorem, which contains a central result on the Average Value-at-Risk in multistage situations. It is the basis for the martingale representation and the verification theorems in multistage stochastic optimization, which are presented in Section 7.

Theorem 8 (Decomposition of the Average Value-at-Risk)**.** *Let $Y \in L^\infty(\mathcal{F}_T)$ and $\mathcal{F}_t \subset \mathcal{F}_T$.*

(i) For $\alpha \in [0,1]$ the Average Value-at-Risk obeys the decomposition

$$\mathsf{AV@R}_\alpha(Y) = \inf \mathbb{E}\left[Z_t \cdot \mathsf{AV@R}_{\alpha \cdot Z_t}(Y|\mathcal{F}_t)\right], \tag{6}$$

where the infimum is among all densities $Z_t \lhd \mathcal{F}_t$ with $0 \le Z_t$, $\alpha Z_t \le \mathbb{1}$ and $\mathbb{E}(Z_t) = 1$. For $\alpha > 0$ the infimum in (6) is attained.

(ii) Moreover, if Z is the optimal dual density for (1), then

$$Z_t = \mathbb{E}(Z|\mathcal{F}_t) \tag{7}$$

is the best choice in (6).

(iii) Let $\mathcal{F}_t \subset \mathcal{F}_\tau \subset \mathcal{F}_T$ be nested sigma algebras. The conditional *Average Value-at-Risk at random level $\alpha \lhd \mathcal{F}_t$ ($0 \le \alpha \le \mathbb{1}$) has the recursive (nested) representation*

$$\mathsf{AV@R}_\alpha(Y|\mathcal{F}_t) = \operatorname{ess\,inf} \mathbb{E}\left[\, Z_\tau \cdot \mathsf{AV@R}_{\alpha \cdot Z_\tau}(Y|\mathcal{F}_\tau) |\, \mathcal{F}_t\right], \tag{8}$$

where the infimum is among all densities $Z_\tau \lhd \mathcal{F}_\tau$ with $0 \le Z_\tau$, $\alpha Z_\tau \le \mathbb{1}$ and $\mathbb{E}[Z_\tau|\mathcal{F}_t] = \mathbb{1}$.

Remark 9. Note that $\alpha \cdot Z_t$ in the index of the inner $\mathsf{AV@R}_{\alpha \cdot Z_t}$ is a $\mathcal{F}_t$ random variable satisfying $0 \le \alpha Z_t \le \mathbb{1}$, which means that $\mathsf{AV@R}_{\alpha \cdot Z_t}(Y|\mathcal{F}_t)$ is indeed available and almost everywhere well-defined.

Remark 10. One might think that a nested decomposition of the AV@R might be a consequence of the fact that it can be written in terms of utility functions (having in mind that expected utility allows always a dynamic decomposition). By introducing the family of concave utility functions $U_q^\alpha(y) := q - \frac{1}{\alpha}(q-y)_+$, the $\mathsf{AV@R}_\alpha$ can be written, following (2), as

$$\mathsf{AV@R}_\alpha(Y) = \max_{q \in \mathbb{R}} \mathbb{E}\, U_q^\alpha(Y).$$

Obviously, for the conditional distributions $(Y|\mathcal{F}_t)$ the maximizing q depends on $\mathcal{F}_t$, but this is not the crucial point: in fact, as is the content of the decomposition theorem (Theorem 8), only the *extension* of the class $\left(U_q^\alpha\right)_{q \in \mathbb{R}}$ to the much larger class $\left(U_q^\alpha\right)_{q \in \mathbb{R},\, \alpha \in [0,1]}$ of utility functions with *different* q's and α's *on every atom* of $\mathcal{F}_t$ allows a decomposition.

Example 11. Both, Figure 1 and 2, depict a typical, simple situation with two stages in time, the increasing sigma algebras are visualized via the tree structure.

The example in Figure 1 is due to Artzner et al., [2]. The Average Value-at-Risk of the random variable Y in Figure 1 is $\mathsf{AV@R}_{\frac{2}{3}}(Y) = -1$. The intriguing fact here is that the *conditional* Average Value-at-Risk, computed with the *initial* $\alpha = \frac{2}{3}$, is $\mathsf{AV@R}_{\frac{2}{3}\mathbb{1}}(Y|\mathcal{F}_t) = +\mathbb{1}$, which is in conflicting contrast to $\mathsf{AV@R}_{\frac{2}{3}}(Y) = -1$.

However, the decomposition Theorem 8 eliminates the discrepancy by involving the conditional Average Value-at-Risk at random level $\alpha \lhd \mathcal{F}_t$.

Both figures display the optimal variables Z and Z_1: Z is the optimal dual for (1), and $Z_1 = \mathbb{E}(Z|\,\mathcal{F}_t)$ is the optimal dual for the decomposition at $t = 1$ according (7).

Corollary 12 (Bounds)**.** *For any level* $0 \le \alpha \le 1$ *it holds that*

$$\mathsf{AV@R}_\alpha\left(Y\right) \le \mathbb{E}\left[\mathsf{AV@R}_\alpha\left(Y|\mathcal{F}_t\right)\right] \le \mathbb{E}\left(Y\right);$$

for any $\alpha \lhd \mathcal{F}_t$ $(0 \le \alpha \le \mathbb{1})$ *moreover*

$$\mathsf{AV@R}_\alpha\left(Y|\mathcal{F}_t\right) \le \mathbb{E}\left[\mathsf{AV@R}_\alpha\left(Y|\mathcal{F}_\tau\right)|\,\mathcal{F}_t\right] \le \mathbb{E}\left(Y|\mathcal{F}_t\right).$$

Proof. The first inequality is immediate by choosing the feasible random variable $Z = \mathbb{1}$ in Theorem 8. The second inequality follows from the monotonicity property in Theorem 3, as

$$\mathbb{E}\,\mathsf{AV@R}_\alpha\left(Y|\mathcal{F}_t\right) \le \mathbb{E}\,\mathsf{AV@R}_1\left(Y|\mathcal{F}_t\right) = \mathbb{E}\left(\mathbb{E}\left(Y|\mathcal{F}_t\right)\right) = \mathbb{E}\left(Y\right)$$

and $\mathsf{AV@R}_1$ has just one feasible dual variable, the dual variable $Z = \mathbb{1}$. □

Proof of the decomposition theorem, Theorem 8. We shall assume first that $\alpha > 0$.

Let $Z \lhd \mathcal{F}_t$ be a simple function (i.e, a step function with finitely many outcomes) with $Z \ge 0$ and $\mathbb{E}Z = 1$, i.e., $Z = \sum_i b_i \mathbb{1}_{B_i}$ where $b_i \ge 0$, $B_i \in \mathcal{F}_t$ and $B_i \cap B_j = \emptyset$ whenever $i \ne j$. Then, by the characterization (Theorem 4),

$$\begin{aligned}\mathbb{E}\left[Z\,\mathsf{AV@R}_{\alpha\cdot Z}\left(Y|\mathcal{F}_t\right)\right] = \sum_i b_i \mathbb{E}\left[\mathbb{1}_{B_i}\mathsf{AV@R}_{\alpha\cdot Z}\left(Y|\mathcal{F}_t\right)\right] =&\\ = \sum_i b_i \inf\left\{\mathbb{E}\left[YX_i\right]:\ 0 \le X_i,\, \alpha b_i \mathbb{1}_{B_i} X_i \le \mathbb{1}_{B_i},\, \mathbb{E}\left[X_i|\mathcal{F}_t\right] = \mathbb{1}_{B_i}\right\}&\\ = \inf\left\{\sum_i b_i \mathbb{E}\left[YX_i\right]:\ 0 \le X_i,\, \alpha b_i \mathbb{1}_{B_i} X_i \le \mathbb{1}_{B_i},\, \mathbb{E}\left[X_i|\mathcal{F}_t\right] = \mathbb{1}_{B_i}\right\}.&\end{aligned}$$

As $\mathbb{E}\left[X_i|\mathcal{F}_t\right] = \mathbb{1}_{B_i}$, together with the additional constraint $X_i \ge 0$, one infers that $X_i = 0$ on the complement of B_i, that is to say $X_i \mathbb{1}_{B_i} = X_i$.

Define $X := \sum_i \mathbb{1}_{B_i} X_i$, thus

$$Z\,X = \sum_{i,j} b_i \mathbb{1}_{B_i} \mathbb{1}_{B_j} X_j = \sum_i b_i \mathbb{1}_{B_i} X_i = \sum_i b_i X_i$$

and

$$\mathbb{E}\left[XYZ\right] = \sum_i b_i \mathbb{E}\left[YX_i\right],$$

such that we further obtain by assembling on the mutually disjoint sets B_i

$$\mathbb{E}\left[Z \cdot \mathsf{AV@R}_{\alpha\cdot Z}\left(Y|\mathcal{F}_t\right)\right] = \inf\left\{\mathbb{E}\left[YZX\right]:\ 0 \le X,\, \alpha Z X \le \mathbb{1},\, \mathbb{E}\left[X|\mathcal{F}_t\right] = \mathbb{1}\right\}. \tag{9}$$

Note next that $\mathbb{E}\left[XZ\right] = \mathbb{E}\left[Z \cdot \mathbb{E}\left[X|\mathcal{F}_t\right]\right] = \mathbb{E}\left[Z \cdot \mathbb{1}\right] = 1$, and hence (associate $\tilde{Z}$ with XZ)

$$\begin{aligned}\mathbb{E}\left[Z \cdot \mathsf{AV@R}_{\alpha\cdot Z}\left(Y|\mathcal{F}_t\right)\right] &\ge \inf\left\{\mathbb{E}\left[Y\tilde{Z}\right]:\ 0 \le \tilde{Z},\, \alpha\tilde{Z} \le \mathbb{1},\, \mathbb{E}\left[\tilde{Z}\right] = 1\right\}\\ &= \mathsf{AV@R}_\alpha\left(Y\right).\end{aligned}$$

It follows by semi-continuity that $\mathbb{E}\left[Z \cdot \mathsf{AV@R}_{\alpha\cdot Z}\left(Y|\mathcal{F}_t\right)\right] \ge \mathsf{AV@R}_\alpha\left(Y\right)$ for all $Z \ge 0$ with $\mathbb{E}Z = 1$ and $\alpha Z \le \mathbb{1}$.

To obtain equality it remains to be shown that there is a $Z_t \lhd \mathcal{F}_t$ such that $\mathsf{AV@R}_\alpha(Y) = \mathbb{E}\left[Z_t\, \mathsf{AV@R}_{\alpha Z_t}(Y|\mathcal{F}_t)\right]$. For this let Z be the optimal dual variable in equation (1), that is $\mathsf{AV@R}_\alpha(Y) = \mathbb{E}\, YZ$ with $Z \geq 0$, $\alpha Z \leq \mathbb{1}$ and $\mathbb{E}Z = 1$, and define

$$Z_t := \mathbb{E}\left[Z|\mathcal{F}_t\right].$$

$Z_t \lhd \mathcal{F}_t$ is feasible, as $0 \leq Z_t$, $\alpha Z_t \leq \mathbb{1}$ and $\mathbb{E}Z_t = 1$. From the fact that $X := \begin{cases} \frac{Z}{Z_t} & \text{if } Z_t > 0 \\ 1 & \text{if } Z_t = 0 \end{cases}$ is P-a.e. well-defined and feasible for (9) one deduces further that

$$\begin{aligned} \mathbb{E}\left[Z_t \cdot \mathsf{AV@R}_{\alpha\cdot Z_t}(Y|\mathcal{F}_t)\right] &= \inf\left\{\mathbb{E}\left[Y Z_t X\right] :\ 0 \leq X,\, \alpha Z_t X \leq \mathbb{1},\, \mathbb{E}\left[X|\mathcal{F}_t\right] = \mathbb{1}\right\} \\ &\leq \mathbb{E}\, Y Z_t \frac{Z}{Z_t} = \mathbb{E} YZ = \mathsf{AV@R}_\alpha(Y). \end{aligned}$$

This is the converse inequality such that assertion (6) follows. The minimum is thus indeed attained for $Z_t = \mathbb{E}\left[Z|\mathcal{F}_t\right]$, where Z is the optimal dual variable for the $\mathsf{AV@R}_\alpha$, which exists for $\alpha > 0$.

As for $\alpha = 0$ recall that $\mathsf{AV@R}_0(Y) = \operatorname{ess\,inf} Y$ and $\mathsf{AV@R}_0(Y) \leq \mathsf{AV@R}_0(Y|\mathcal{F}_t)$, and thus

$$\mathsf{AV@R}_0(Y) = \mathbb{E}\, Z_t \mathsf{AV@R}_0(Y) \leq \mathbb{E}\, Z_t \mathsf{AV@R}_0(Y|\mathcal{F}_t) = \mathbb{E}\, Z_t \mathsf{AV@R}_{0\cdot Z_t}(Y|\mathcal{F}_t).$$

For the converse inequality choose $Z^\varepsilon \geq 0$ with $\mathbb{E}Z^\varepsilon Y \leq \mathsf{AV@R}_0(Y) + \varepsilon$. By the conditional $L^1 - L^\infty$-Hölder inequality it holds that

$$\begin{aligned} \mathsf{AV@R}_0(Y) + \varepsilon \geq \mathbb{E}\, Z_\varepsilon Y &\geq \mathbb{E}\left(\mathbb{E}\left[Z^\varepsilon|\mathcal{F}_t\right] \mathsf{AV@R}_0(Y|\mathcal{F}_t)\right) \\ &\geq \mathbb{E}\left(\mathbb{E}\left[Z^\varepsilon|\mathcal{F}_t\right] \mathsf{AV@R}_0(Y)\right) = \mathsf{AV@R}_0(Y), \end{aligned}$$

hence

$$\mathsf{AV@R}_0(Y) \geq \mathbb{E}\, Z_t^\varepsilon \mathsf{AV@R}_{0\cdot Z_t^\varepsilon}(Y|\mathcal{F}_t) - \varepsilon$$

for $Z_t^\varepsilon := \mathbb{E}\left[Z^\varepsilon|\mathcal{F}_t\right]$.

The proof for the remaining statement (iii) of the theorem reads along the same lines as above, but conditioned on $\mathcal{F}_t$. □

5 Multistage optimization: problem formulation

It is the aim of this and the following sections to make time inconsistent stochastic optimization problems, which involve the Average Value-at-Risk or an acceptability functional, available for dynamic programming. The problem we consider here incorporates—in the sense of integrated risk management—the acceptability functional in the objective, such as

$$\begin{array}{ll} \text{maximize} & \mathbb{E}\, Y + \gamma \cdot \mathsf{AV@R}_\alpha(Y) \\ \text{subject to} & Y \in \mathscr{Y}. \end{array} \tag{10}$$

γ is a positive parameter to account for the emphasis that should be given to risk: γ is the risk appetite, the degree of uncertainty the investor is willing to accept in respect of negative changes to its assets.

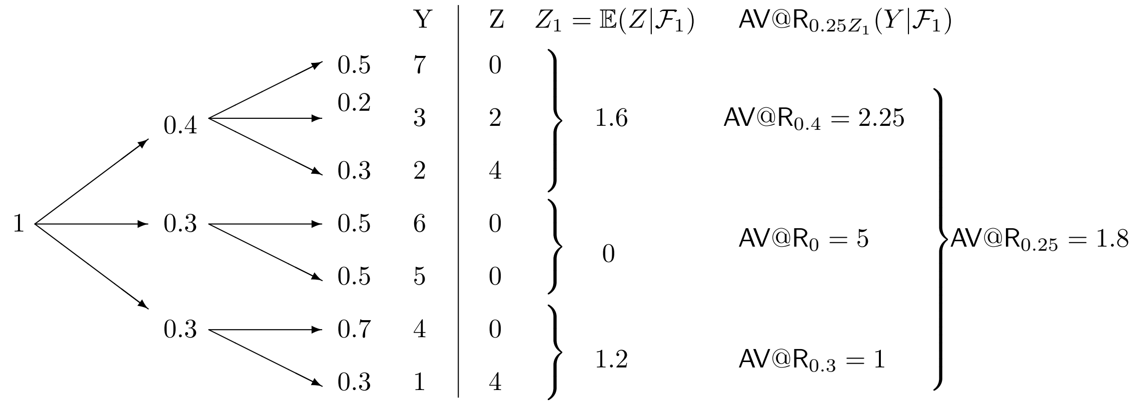

Figure 2: Nested computation of $\mathsf{AV@R}_\alpha(Y)$ with $\alpha = 0.25$; in this tree-example with 11 nodes and 7 leaves the transitional probabilities are indicated. It holds true that $\mathsf{AV@R}_{0.25}(Y) = 1.8 = \mathbb{E}\, YZ = \mathbb{E}\left[Z_1 \cdot \mathsf{AV@R}_{\alpha Z_1}(Y|F_t)\right]$.

The problem formulation (10) applies for optimal investment problems and it can be found in multistage decision models for electricity management as well (cf. Eichhorn [10]). Multistage stochastic problems thus naturally can be formulated as

$$\begin{array}{rl} \text{maximize} & \mathbb{E}\, H(x,\xi) + \gamma \cdot \mathsf{AV@R}_\alpha\left(H(x,\xi)\right) \\ \text{subject to} & x \lhd \mathcal{F}, \\ & x \in \mathscr{X}, \end{array} \tag{11}$$

where $H(x,\xi)$ is a short notation for $H(\xi_0, x_0, \xi_1, \dots, x_{T-1}, \xi_T, x_T)$. The constraint $x \lhd \mathcal{F}$ is the *nonanticipativity* constraint, which is $x_t \lhd \mathcal{F}_t$ for all $t \in \mathbf{T} := \{0, 1, \dots T\}$).

Note that $x \lhd \mathcal{F}$ forces x_t to be a function of the tree process, $x_t = x_t(\nu_t)$ (cf. Shiryaev [27, Theorem II.4.3] for the respective measurability), which reflects the fact that the decisions x_t have to be fixed *without* knowledge of the future.

We shall require the real-valued function H to be concave in x, for x in a convex set $\mathscr{X}$, such that

$$H\left((1-\lambda)\, x' + \lambda x'', \xi\right) \geq (1-\lambda)\, H\left(x', \xi\right) + \lambda H\left(x'', \xi\right)$$

for any fixed state ξ.

By the monotonicity property and concavity of the acceptability functional it holds thus that

$$\begin{aligned} \mathsf{AV@R}\left(H\left((1-\lambda)x' + \lambda x'', \xi\right)\right) &\geq \mathsf{AV@R}\left((1-\lambda)H\left(x', \xi\right) + \lambda H\left(x'', \xi\right)\right) \\ &\geq (1-\lambda)\mathsf{AV@R}\left(H\left(x', \xi\right)\right) + \lambda \mathsf{AV@R}\left(H\left(x'', \xi\right)\right), \end{aligned} \tag{12}$$

which means that the mapping $x \mapsto \mathsf{AV@R}\left(H(x,\xi)\right)$ is concave as well. Notably concavity and (12) hold for distortion functionals and their conditional variants, in particular for the Average Value-at-Risk and the conditional Average Value-at-Risk at random level.

Remark 13 (Notational convention). We shall write $H(x)$ for the random variable $H(x) := H(x,\xi) = H(\xi_0, x_0, \dots \xi_T, x_T)$. For notational convenience we shall use the straight forward abbreviation $\xi_{i:j}$ for the substring $\xi_{i:j} = (\xi_i, \xi_{i+1}, \dots \xi_j)$; in particular $\xi_{i:i} = (\xi_i)$, and $x_{i:i-1} = ()$, the empty string.

Remark 14 (Multiperiod acceptability functionals). Some papers exclusively treat functions of the form $H(x) = \sum_{t=0}^{T} H_t(x_{0:t})$ in the present setting. This particular setting is just a special case and included in our general formulation and framework of problem (11).

6 Dynamic programming formulation

The *dynamic programming principle* is the basis of the solution technique developed by Bellman [4] in the 1950's for deterministic optimal control problems. They have been extended later to account for stochastic problems as well, where typically

(i) the objective is an expectation and

(ii) the transition does not depend on the history, but just on the current state of the system—that is to say for Markov chains.

The decomposition of the Average Value-at-Risk elaborated in Theorem 8 is the key which allows to define—in line with the classical dynamic programming principle—a value function with properties analogous to the classical theory. The new value function overcomes the restrictions (i) and (ii), as we shall employ a risk measure in the objective, and the value function explicitly depends on the history. The value function is then used in the following Section 7 to state the verification theorems.

The theory developed below applies to more general acceptability functionals (risk functions), other than the Average Value-at-Risk, it includes in particular all approximations of law invariant acceptability functionals by Kusuoka's theorem of the form

$$\mathcal{A} = \sum_k \gamma_k \mathsf{AV@R}_{\alpha_k},$$

and acceptability functionals of the type

$$\mathcal{A}(Y) = \sum_k \mathbb{E}\, \gamma_k\, \mathsf{AV@R}_{\alpha_k}(Y|\mathcal{F}_{t_k}) \tag{13}$$

for some $\mathcal{F}_{t_k}$-measurable α_t and γ_t $(0 \leq \alpha_t, \gamma_t \lhd \mathcal{F}_{t_k})$. However, as these more general acceptability functionals are to be treated analogously we may continue with the simple Average Value-at-Risk in lieu of the more general setting (13).

For Markov processes the value function is—in a natural way—a function of time and the current status of the system. In order to derive dynamic programming equations for the general multistage problem it is necessary to carry the entire history of earlier decisions. This is respected by the following definition.

Definition 15 (Value function)**.** The *value function* at stage t is

$$\mathscr{V}_t(x_{0:t-1}, \alpha, \gamma) := \operatorname*{ess\,sup}_{(x_{0:t-1}, x_{t:T}) \in \mathscr{X}} \mathbb{E}\left[H(x_{0:T}) \middle| \mathcal{F}_t \right] + \gamma \cdot \mathsf{AV@R}_\alpha \left(H(x_{0:T}) | \mathcal{F}_t \right), \tag{14}$$

where $x_{t:T}$ in (14) is chosen such that the concatenated string $x_{0:T} = (x_{0:t-1},\, x_{t,T})$ satisfies $x_{0:T} \in \mathscr{X}$.

The value function (14) depends on

- the decisions $x_{0:t-1}$ up to time $t-1$ and
- the random model parameters $\alpha \lhd \mathcal{F}_t$ and $\gamma \lhd \mathcal{F}_t$, and

$\mathscr{V}_t$ is measurable with respect to $\mathcal{F}_t$.

The initial stage $t=0$. The initial problem (11) can be expressed by the value function at initial time $t=0$ and assuming that the sigma-algebra $\mathcal{F}_0$ is trivial, as

$$\begin{aligned}\sup_{x_{0:T}} \mathbb{E}\, H\left(x_{0:T}\right) + \gamma \cdot \mathsf{AV@R}_\alpha\left(H\left(x_{0:T}\right)\right) = \\ &= \operatorname*{ess\,sup}_{x_{0:T}} \mathbb{E}\left[H\left(x_{0:T}\right)|\mathcal{F}_0\right] + \gamma \cdot \mathsf{AV@R}_\alpha\left(H\left(x_{0:T}\right)|\mathcal{F}_0\right) \\ &= \mathscr{V}_0\left(\left(\right), \alpha, \gamma\right).\end{aligned} \tag{15}$$

The stags $t = 1 \dots T$. The decomposition theorem (Theorem 8) above allows to relate the value function at different stages. The recursion obtained can be considered as generalized dynamic programming principle.

Theorem 16 (Dynamic programming principle)**.** *Assume that $\alpha > 0$, and H is random upper semi-continuous with respect to x and ξ evaluates in some convex, compact subset of $\mathbb{R}^n$. Then the following hold true.*

(i) The value function evaluates to

$$\mathscr{V}_T\left(x_{0:T-1}, \alpha, \gamma\right) = (1+\gamma) \operatorname*{ess\,sup}_{x_T} H\left(x_{0:T}\right)$$

at terminal time T.

(ii) For any $t < \tau$ ($t, \tau \in \mathbf{T}$) the recursive relation

$$\mathscr{V}_t\left(x_{0:t-1}, \alpha, \gamma\right) = \operatorname*{ess\,sup}_{x_{t:\tau-1}} \operatorname*{ess\,inf}_{Z_{t:\tau}} \mathbb{E}\left[\mathscr{V}_\tau\left(x_{0:\tau-1}, \alpha \cdot Z_{t:\tau}, \gamma \cdot Z_{t:\tau}\right) \middle| \mathcal{F}_t\right], \tag{16}$$

holds true, where $Z_{t:\tau} \lhd \mathcal{F}_\tau$, $0 \le Z_{t:\tau}$, $\alpha Z_{t:\tau} \le \mathbb{1}$ and $\mathbb{E}\left[Z_{t:\tau}|F_t\right] = \mathbb{1}$.

Proof. A direct evaluation at terminal time $t = T$ gives

$$\begin{aligned}\mathscr{V}_T\left(x_{0:T-1}, \alpha, \gamma\right) &= \operatorname*{ess\,sup}_{x_{T:T}} \mathbb{E}\left[H\left(x_{0:T}\right)|\mathcal{F}_T\right] + \gamma \cdot \mathsf{AV@R}_\alpha\left(H\left(x_{0:T}\right)|\mathcal{F}_T\right) \\ &= \operatorname*{ess\,sup}_{x_T} H\left(x_{0:T}\right) + \gamma \cdot H\left(x_{0:T}\right) \\ &= (1+\gamma) \operatorname*{ess\,sup}_{x_T} H\left(x_{0:T}\right),\end{aligned}$$

because the random variables, conditionally on the entire observations $\xi_{0:T}$, are constant. The final maximizations over $x_{T:T} = x_T\left(\xi_{0:T}\right)$ are deterministic, because all stochastic observations are available at this final stage.

As for the recursion at an intermediate time ($t < T$) observe that

$$\begin{aligned}\mathscr{V}_t\left(x_{0:t-1},\alpha,\gamma\right) &= \operatorname*{ess\,sup}_{x_{t:T}} \mathbb{E}\left[H\left(x_{0:T}\right)|\mathcal{F}_t\right] + \gamma\cdot\mathsf{AV@R}_\alpha\left(H\left(x_{0:T}\right)|\mathcal{F}_t\right)\\ &= \operatorname*{ess\,sup}_{x_{t:T}} \operatorname*{ess\,inf}_{Z_{t:t+1}} \mathbb{E}\left[\begin{array}{l}\mathbb{E}\left[H\left(x_{0:T}\right)|\mathcal{F}_{t+1}\right]\\ +\gamma\cdot Z_{t:t+1}\cdot\mathsf{AV@R}_{\alpha\cdot Z_{t:t+1}}\left(H\left(x_{0:T}\right)|\mathcal{F}_{t+1}\right)\end{array}\middle|\mathcal{F}_t\right]\end{aligned}$$

due to the nested decomposition (8) of the Average Value-at-Risk at random level, elaborated in Theorem 8. The $\operatorname{ess\,inf}$ is among all random variables $Z_{t:t+1} \lhd \mathcal{F}_{t+1}$ satisfying $\alpha Z_{t:t+1} \leq \mathbb{1}$ and $\mathbb{E}\left(Z_{t:t+1}|\mathcal{F}_t\right) = \mathbb{1}$. By the discussions in the preceding sections the inner expression is concave in $x_{0:T}$ and convex in $Z_{t:t+1}$. $Z_{t:t+1}$ is moreover chosen from the $\sigma\left(L^\infty, L^1\right)$ compact set $Z_{t:t+1} \in \left\{Z \in L^\infty : 0 \leq Z \leq \frac{1}{\alpha}\right\}$. By Sion's minimax theorem (cf. Sion [28] and [13]) one may thus interchange the min and max to obtain

$$\mathscr{V}_t\left(x_{0:t-1},\alpha,\gamma\right) = \operatorname*{ess\,sup}_{x_t} \operatorname*{ess\,inf}_{Z_{t:t+1}} \operatorname*{ess\,sup}_{x_{t+1:T}} \mathbb{E}\left[\begin{array}{l}\mathbb{E}\left[H\left(x_{0:T}\right)|\mathcal{F}_{t+1}\right]\\ +\gamma\cdot Z_{t:t+1}\cdot\mathsf{AV@R}_{\alpha\cdot Z_{t:t+1}}\left(H\left(x_{0:T}\right)|\mathcal{F}_{t+1}\right)\end{array}\middle|\mathcal{F}_t\right].$$

As H is upper semi-continuous by assumption one may further apply the interchangeability principle (cf. Wets and Rockafellar [19, Theorem 14.60], or Ruszczyński and Shapiro [26, p. 405]) such that

$$\begin{aligned}\mathscr{V}_t\left(x_{0:t-1},\alpha,\gamma\right) &= \operatorname*{ess\,sup}_{x_t} \operatorname*{ess\,inf}_{Z_{t:t+1}} \mathbb{E}\left[\operatorname*{ess\,sup}_{x_{t+1:T}} \begin{array}{l}\mathbb{E}\left[H\left(x_{0:T}\right)|\mathcal{F}_{t+1}\right]\\ +\gamma\cdot Z_{t:t+1}\cdot\mathsf{AV@R}_{\alpha\cdot Z_{t:t+1}}\left(H\left(x_{0:T}\right)|\mathcal{F}_{t+1}\right)\end{array}\middle|\mathcal{F}_t\right]\\ &= \operatorname*{ess\,sup}_{x_t} \operatorname*{ess\,inf}_{Z_{t:t+1}} \mathbb{E}\left[\mathscr{V}_{t+1}\left(x_{0:t},\alpha\cdot Z_{t:t+1},\gamma\cdot Z_{t:t+1}\right)\middle|\mathcal{F}_t\right],\end{aligned}$$

which is the desired relation for $\tau = t+1$. Repeating the computation from above $t-\tau$ times, or conditioning on $\mathcal{F}_\tau$ instead of $\mathcal{F}_{t+1}$ reveals the general result. □

7 Martingale representation and the verification theorems

The value function $\mathscr{V}_t$ introduced in (14) is a function of some general $\alpha \lhd \mathcal{F}_t$ and $\gamma \lhd \mathcal{F}_t$. To specify for the right and optimal parameters assume that the optimal policy $\mathbf{x} = x_{0:T}$ of problem (11) exists.[3] Theorem 16 then gradually reveals the optimal dual variables $\mathbf{Z}_T, \mathbf{Z}_{T-1}, \dots$ and finally $\mathbf{Z}_0$ (assuming again that the respective argmins of the essential infimum $\operatorname{ess\,inf}$ exist). The conditions $\mathbb{E}\left(\mathbf{Z}_\tau|\mathcal{F}_t\right) = \mathbb{1}$ ($\tau > t$) imposed on the dual variables suggest to compound the densities and to consider the densities $\mathbf{Z}_{t:\tau} := \mathbf{Z}_t \cdot \mathbf{Z}_{t+1} \cdot \ldots \mathbf{Z}_\tau$ such that $\mathbb{E}\left(\mathbf{Z}_{t:\tau}|\mathcal{F}_t\right) = \mathbf{Z}_t$ and $\mathbb{E}\left(\mathbf{Z}_{0:\tau}|\mathcal{F}_t\right) = \mathbf{Z}_{0:t}$. With this setting the process $\mathbf{Z} := \left(\mathbf{Z}_{0:t}\right)_{t\in\mathbf{T}}$ is a martingale, satisfying moreover $0 \leq \mathbf{Z}_t$ and $\alpha\mathbf{Z}_t \leq \mathbb{1}$ during all times $t \in \mathbf{T}$. The optimal pair $(\mathbf{x}, \mathbf{Z})$ is a saddle point for the Lagrangian corresponding to the initial problem (11).

This gives rise for the following definition.

Definition 17. Let $\alpha \in [0,1]$ be a fixed level.

(i) $Z = \left(Z_t\right)_{t\in\mathbf{T}}$ is a feasible (for the nonanticipativity constraints) process of densities if

(ii) Z_t is a martingale with respect to the filtration $\mathcal{F}_t$ and

[3]Optimal decisions $\mathbf{x}$, and the respective optimal dual variables $\mathbf{Z}$ are displayed in bold letters.

(iii) $0 \le Z_t,\ \alpha Z_t \le \mathbb{1}$ and $\mathbb{E}\left(Z_t\right) = 1$ for all $t \in \mathbf{T}$.

(iv) The intermediate densities are

$$Z_{t:\tau} := \begin{cases} \frac{Z_\tau}{Z_{t-1}} & \text{if } Z_{t-1} > 0 \\ 0 & \text{else} \end{cases} \qquad (0 < t < \tau),$$

and $Z_{0:\tau} := Z_\tau$.

For feasible x and Z we consider the stochastic process

$$M_t\left(x, Z\right) := \mathscr{V}_t\left(x_{0:t-1},\ \alpha Z_{0:t},\ \gamma Z_{0:t}\right) \quad (t \in \mathbf{T})$$

where α and γ – in contrast to (14) – are simple real numbers.

Recall from (15) that M_0 is a constant (as $\mathcal{F}_0$ is trivial) solving the original problem (11) if $(\mathbf{x}, \mathbf{Z})$ are optimal. Above that we shall prove in the next theorem that $M_t\left(\mathbf{x}, \mathbf{Z}\right)$ is a martingale in this case (we refer to the papers [17, 18] by Rockafellar and Wets for very early occurrences of martingales in a related context).

Theorem 18 (Martingale property)**.** *Given that* $\mathbf{x}$ *and* $\mathbf{Z}$ *are optimal, then the process* $M_t\left(\mathbf{x}, \mathbf{Z}\right)$ *is a martingale with respect to the filtration* $\mathcal{F}_t$*.*

Conversely, if $M_t\left(x, Z\right)$ *is a martingale and the* argmax *sets (for* x*) and* argmin *sets (for* Z*) in* (16) *are nonempty, then* x *and* Z *are optimal.*

Proof. By the dynamic programming equation (16) and the respective maximality of $\mathbf{Z}_{t+1}$ and $\mathbf{x}_{t+1}$ we have that

$$\begin{aligned} M_t\left(\mathbf{x}, \mathbf{Z}\right) &= \mathscr{V}_t\left(\mathbf{x}_{0:t-1},\ \alpha \mathbf{Z}_{0:t},\ \gamma \mathbf{Z}_{0:t}\right) \\ &= \operatorname*{ess\,sup}_{x_{t:t}} \operatorname*{ess\,inf}_{Z_{t+1}} \mathbb{E}\left[\left.\mathscr{V}_{t+1}\left(\left(\mathbf{x}_{0:t-1}, x_t\right),\ \alpha \cdot \mathbf{Z}_{0:t} Z_{t+1},\ \gamma \cdot \mathbf{Z}_{0:t} Z_{t+1}\right)\right| \mathcal{F}_t\right] \\ &= \operatorname*{ess\,sup}_{x_{t:t}} \mathbb{E}\left[\left.\mathscr{V}_{t+1}\left(\left(\mathbf{x}_{0:t-1}, x_t\right),\ \alpha \cdot \mathbf{Z}_{0:t+1},\ \gamma \cdot \mathbf{Z}_{0:t+1}\right)\right| \mathcal{F}_t\right] \\ &= \mathbb{E}\left[\left.\mathscr{V}_{t+1}\left(\mathbf{x}_{0:t},\ \alpha \cdot \mathbf{Z}_{0:t+1},\ \gamma \cdot \mathbf{Z}_{0:t+1}\right)\right| \mathcal{F}_t\right] \\ &= \mathbb{E}\left[\left. M_{t+1}\left(\mathbf{x}, \mathbf{Z}\right)\right| \mathcal{F}_t\right] \end{aligned}$$

again by the interchangeable principle. M_t, hence, is a martingale with respect to the filtration $\mathcal{F}_t$.

The converse follows from the following corollary. □

Verification theorems. Verification theorems characterize optimal decisions in Bellman's principle. For multistage stochastic optimization verification theorems are accessible as well, they are provided by the following corollary.

Corollary 19 (Verification theorem)**.** *Let* x *be feasible for* (11)*, and* Z *be feasible according Definition* 17*.*

(i) Suppose that $\mathscr{W}$ *satisfies*

$$\begin{aligned} &\mathscr{W}_T\left(x_{0:T-1}, \alpha Z_{0:T}, \gamma Z_{0:T}\right) \ge \left(1 + \gamma Z_{0:T}\right) H\left(x_{0:T}\left(\xi_{0:T}\right)\right) \quad \text{and} \\ &\mathscr{W}_t\left(x_{0:t-1}, \alpha Z_{0:t}, \gamma Z_{0:t}\right) \ge \operatorname*{ess\,sup}_{x_t} \mathbb{E}\left[\left.\mathscr{W}_{t+1}\left(x_{0:t}, \alpha \cdot Z_{0:t+1}, \gamma \cdot Z_{0:t+1}\right)\right| \mathcal{F}_t\right], \end{aligned}$$

then the process $\mathscr{W}_t\left(x_{0:t-1}, \alpha Z_{0:t}, \gamma Z_{0:t}\right)$ $(t \in \mathbf{T})$ *is a supermartingale dominating* $\mathscr{V}\left(x_{0:t-1}, \alpha Z_{0:t}, \gamma Z_{0:t}\right)$*,* $\mathscr{V} \le \mathscr{W}$*.*

(ii) Let $\mathscr{U}$ satisfy

$$\mathscr{U}_T\left(x_{0:T-1},\alpha Z_{0:T},\gamma Z_{0:T}\right)\leq\left(1+\gamma Z_{0:T}\right)H\left(x_{0:T}\left(\xi_{0:T}\right)\right)\quad\text{and}$$
$$\mathscr{U}_t\left(x_{0:t-1},\alpha Z_{0:t},\gamma Z_{0:t}\right)\leq\operatorname*{ess\,inf}_{Z_{t+1}}\mathbb{E}\left[\left.\mathscr{U}_{t+1}\left(x_{0:t},\alpha\cdot Z_{0:t+1},\gamma\cdot Z_{0:t+1}\right)\right|\mathcal{F}_t\right],$$

then the process $\mathscr{U}_t\left(x_{0:t-1},\alpha Z_{0:t},\gamma Z_{0:t}\right)$ is a submartingale dominated by $\mathscr{V}\left(x_{0:t-1},\alpha Z_{0:t},\gamma Z_{0:t}\right)$, $\mathscr{U}\leq\mathscr{V}$.

Proof. The proof is by induction on t, starting at the final stage T. Observe first that $\mathscr{U}_T\leq\mathscr{V}_T\leq\mathscr{W}_T$ by assumption and (12). Then

$$\begin{aligned}\mathscr{U}_t\left(x_{0:t-1},\alpha Z_{0:t},\gamma Z_{0:t}\right)&\leq\operatorname*{ess\,inf}_{Z_{t+1}}\mathbb{E}\left[\left.\mathscr{U}_{t+1}\left(x_{0:t},\alpha\cdot Z_{0:t+1},\gamma\cdot Z_{0:t+1}\right)\right|\mathcal{F}_t\right]\\&\leq\operatorname*{ess\,sup}_{x_t}\operatorname*{ess\,inf}_{Z_{t+1}}\mathbb{E}\left[\left.\mathscr{V}_{t+1}\left(x_{0:t},\alpha\cdot Z_{0:t+1},\gamma\cdot Z_{0:t+1}\right)\right|\mathcal{F}_t\right]\\&=\mathscr{V}_t\left(x_{0:t-1},\alpha Z_{0:t},\gamma Z_{0:t}\right),\end{aligned}$$

and thus $\mathscr{U}\leq\mathscr{V}$.

As for $\mathscr{W}_t$ observe that

$$\begin{aligned}\mathscr{W}_t\left(x_{0:t-1},\alpha Z_{0:t},\gamma Z_{0:t}\right)&\geq\operatorname*{ess\,sup}_{x_t}\mathbb{E}\left[\left.\mathscr{W}_{t+1}\left(x_{0:t},\alpha\cdot Z_{0:t+1},\gamma\cdot Z_{0:t+1}\right)\right|\mathcal{F}_t\right]\\&\geq\operatorname*{ess\,inf}_{Z_{t+1}}\operatorname*{ess\,sup}_{x_t}\mathbb{E}\left[\left.\mathscr{V}_{t+1}\left(x_{0:t},\alpha\cdot Z_{0:t+1},\gamma\cdot Z_{0:t+1}\right)\right|\mathcal{F}_t\right]\\&\geq\operatorname*{ess\,sup}_{x_t}\operatorname*{ess\,inf}_{Z_{t+1}}\mathbb{E}\left[\left.\mathscr{V}_{t+1}\left(x_{0:t},\alpha\cdot Z_{0:t+1},\gamma\cdot Z_{0:t+1}\right)\right|\mathcal{F}_t\right]\\&=\mathscr{V}_t\left(x_{0:t-1},\alpha\cdot Z_{0:t},\gamma\cdot Z_{0:t}\right),\end{aligned}$$

because it always holds true that $\inf_z\sup_x L\left(x,z\right)\geq\sup_x\inf_z L\left(x,z\right)$.

$\mathscr{W}$ is a supermartingale, because

$$\begin{aligned}\mathscr{W}_t\left(x_{0:t-1},\alpha Z_{0:t},\gamma Z_{0:t}\right)&\geq\operatorname*{ess\,sup}_{x_t}\mathbb{E}\left[\left.\mathscr{W}_{t+1}\left(x_{0:t},\alpha\cdot Z_{0:t+1},\gamma\cdot Z_{0:t+1}\right)\right|\mathcal{F}_t\right]\\&\geq\mathbb{E}\left[\left.\mathscr{W}_{t+1}\left(x_{0:t},\alpha\cdot Z_{0:t+1},\gamma\cdot Z_{0:t+1}\right)\right|\mathcal{F}_t\right],\end{aligned}$$

which is the characterizing property. The proof that $\mathscr{U}$ is a submartingale is analogous. □

8 Concluding remarks and summary

Among influential papers and attempts to obtain dynamic programming equations for multistage programming are the papers by Shapiro [23] and Römisch and Guigues [20], which address the time consistency aspect. A focus on Bellman's principle is given in Artzner et al. [2].

In this paper we demonstrate by use of an example that a naïve composition of risk measures is not time consistent. We introduce the conditional Average Value-at-Risk *at random risk level*. The central result is a decomposition, which allows to reassemble the Average Value-at-Risk given just the conditional risk observations. For this purpose it is necessary to give up the constant risk level and to accept a random risk level instead. The random risk level is adapted for each partial observation and reflects the fact that risk has to be quantified by adapted means, whenever information already is available.

The risk levels, which have to be applied at different levels, are not known a priori, they come along with the solution of the entire problem. This is of course in line with time inconsistency, which is intrinsic to these types of problems. However, dynamic programming principles can still be derived, they can be stated as verification theorems. Those verification theorems are formulated by employing enveloping super- and submartingales. They can be used to check, if a given policy for a stochastic program is optimal or not. Further, the sub- and supermartingales provide useful lower and upper bounds for the objective of the stochastic program.

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