A LINEAR TIME ALGORITHM FOR THE KOOPMANS-BECKMANN QAP LINEARIZATION AND RELATED PROBLEMS

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Abstract. An instance of the quadratic assignment problem (QAP) with cost matrix Q is said to be linearizable if there exists an instance of the linear assignment problem (LAP) with cost matrix C such that for each assignment, the QAP and LAP objective function values are identical. The QAP linearization problem can be solved in $O(n^4)$ time. However, for the special cases of Koopmans-Beckmann QAP and the multiplicative assignment problem the input size is of $O(n^2)$. We show that the QAP linearization problem for these special cases can be solved in $O(n^2)$ time. For symmetric Koopmans-Beckmann QAP, Bookhold [4] gave a sufficient condition for linearizability and raised the question if the condition is necessary. We show that Bookhold's condition is also necessary for linearizability of symmetric Koopmans-Beckmann QAP.

1. Introduction

Let $\mathcal{P}_n$ be the family of all permutations of $N = \{1, 2, \ldots, n\}$ and $Q = (q_{ijkl})$ be an $n^2 \times n^2$ matrix where rows and columns of Q are identified by ordered pairs $(i, j) \in N \times N$. Then the *quadratic assignment problem* (QAP) [5] is to

$$\text{Minimize} \sum_{i \in N} \sum_{j \in N} q_{i\pi(i)j\pi(j)}$$
$$\text{Subject to } \pi \in \mathcal{P}_n.$$

This general model of QAP was introduced by Lawler [5] as early as 1963. However its special case, known as *Koopmans-Beckmann model* (QAP-KB), introduced in 1957 is continued to remain the most popular and thoroughly investigated version of the QAP. Let $A = (a_{ij})_{n \times n}$, $B = (b_{ij})_{n \times n}$, $D = (d_{ij})_{n \times n}$ be three prescribed square matrices and $N_i = N \setminus \{i\}$. Then the QAP-KB is defined as

$$\text{Minimize} \sum_{i \in N} \sum_{j \in N_i} a_{ij} b_{\pi(i)\pi(j)} + \sum_{i \in N} d_{i\pi(i)}$$
$$\text{Subject to } \pi \in \mathcal{P}_n.$$

It can be verified that QAP-KB is a special case of QAP when

$$q_{ijkl} = \begin{cases} a_{ik} b_{jl} & \text{if } i, j, k, l \in N; i \neq k \text{ or } j \neq l \\ d_{ij} & \text{if } i = k, j = l \end{cases} \tag{1}$$

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Another special case of QAP studied in literature is the *multiplicative assignment problem* (MAP) [6] where the objective function involves product of two linear functions. Let $A, B, D \in \mathbb{M}^n$. Then the MAP can be stated as

$$\begin{aligned} &\text{Minimize } \left(\sum_{i\in N} a_{i\pi(i)}\right)\left(\sum_{i\in N} b_{i\pi(i)}\right) + \sum_{i\in N} d_{i\pi(i)} \\ &\text{Subject to } \pi \in \mathcal{P}_n. \end{aligned}$$

MAP can also be represented by

$$\begin{aligned} &\text{Minimize } \sum_{i\in N}\sum_{j\in N} a_{i\pi(i)} b_{j\pi(j)} + \sum_{i\in N} d_{i\pi(i)} \\ &\text{Subject to } \pi \in \mathcal{P}_n. \end{aligned}$$

By choosing

$$q_{ijkl} = \begin{cases} a_{ij}b_{kl} & \text{if } i,j,k,l \in N; i \neq j \text{ or } k \neq l \\ d_{ij} + a_{ii}b_{jj} & \text{if } i = j, k = l, \end{cases} \tag{2}$$

it can be seen that the MAP is also a special case of QAP.

QAP, QAP-KB, and MAP are all known to be NP-hard. Shani and Gonzalez [17] showed that existence of a polynomial time ϵ-approximation algorithm for QAP-KB for $\epsilon > 0$ implies $P = NP$. Queyranne [16] strengthened this result by establishing that unless $P = NP$ no polynomial time heuristic exists for QAP-KB satisfying triangle inequality with a bounded asymptotic performance ratio. QAP is also known to be PLS-complete with respect to various neighborhoods [7]. We refer to [5, 7] for detailed complexity results on the problem. On the positive side, various special cases of QAP have been shown to be solvable in polynomial time [12, 7, 8, 9, 10].

Let $C = (c_{ij})_{n\times n}$ be a given matrix. Then the *linear assignment problem* (LAP) with cost matrix C is defined as

$$\begin{aligned} &\text{Minimize } \sum_{i\in N} c_{i\pi(i)} \\ &\text{Subject to } \pi \in \mathcal{P}_n. \end{aligned}$$

Unlike QAP and its special cases discussed above, the LAP can be solved efficiently in $O(n^3)$ time [2].

For any $\pi \in \mathcal{P}_n$, let $Q[\pi] = \sum_{i\in N}\sum_{j\in N} q_{i\pi(i)j\pi(j)}$ be the *quadratic cost* of π with respect to Q and $C(\pi) = \sum_{i\in N} c_{i\pi(i)}$ be the *linear cost* of π with respect to C. We say that Q is *linearizable* if there exists a matrix $C = (c_{ij})_{n\times n}$ such that $Q[\pi] = C(\pi)$ for all $\pi \in \mathcal{P}_n$. Such a cost matrix C is called a *linearization* of Q. An instance of QAP is said to be *linearizable* if its cost matrix Q is linearizable. A linearizable instance of QAP can be solved in polynomial time if a linearization C of its cost matrix Q can be identified in polynomial time.

The *QAP linearization problem* can be stated as follows: "Given an instance of QAP with cost matrix Q, check if it is linearizable and if yes, compute a linearization C of Q"

The terminology "linearization" is used in the QAP literature for the reduction of a integer quadratic programming formulation to a integer linear programming formulation, possibly by introducing additional variables [1]. It may be noted that the linearization problem problem we consider in this paper is different.

Polynomially testable sufficiency conditions are given in [4, 5, 7, 10] for the QAP linearization problem. Recently, Kabadi and Punnen [12] obtained necessary and sufficient conditions for a QAP to be linearizable and proposed an $O(n^4)$ algorithm to solve the QAP linearization problem. This algorithm is the best possible since the data for QAP is of size $O(n^4)$. For QAP-KB and MAP the input size however is $O(n^2)$. This raises an interesting question. Is it possible to solve the linearization problem for QAP-KB and MAP in $O(n^2)$ time? In this paper we show that the linearization problem associated with QAP-KB and MAP can indeed be solved in $O(n^2)$ time. For the special case of QAP-KB where A and B are symmetric, Bookhold [4] gave a simple sufficiency condition for linearizability. He further showed that his condition is necessary for $n = 3, 4$ and stated that "the necessity could not be proved but no counterexample could be found either" for $n \geq 5$. We prove that Bookhold's condition is also necessary for symmetric QAP-KB, resolving his question.

2. Notations, definitions, and past results

Throughout this paper, we use the following conventions to represent matrices and vectors. All matrices will be denoted by capital letters, sometimes with superscripts, over-bars etc, and the elements of the matrix will be represented by corresponding small letters with subscripts representing row and column indices. When rows and columns are indexed by elements of the set N, the (i, j)th element of matrix C is c_{ij}, of matrix A^R is a^R_{ij}, of matrix C^{uv} is c^{uv}_{ij} etc. When rows and columns are indexed by elements of $N \times N$, the $((i, j), (k, l))$th element of matrix Q is represented by q_{ijkl}, of matrix Q^R by q^R_{ijkl} etc. Vectors in $\mathbb{R}^n$ are represented by small letters in bold form, sometimes with superscripts, over-bars etc. The ith component of vector $\mathbf{a}$ is a_i, of vector $\bar{\mathbf{b}}$ is $\bar{b}_i$ etc. Rows and columns of all matrices of size $n \times n$ and $(n-1) \times (n-1)$ are indexed by N and N_i, (for suitable i), respectively, whereas rows and columns of all $n^2 \times n^2$ matrices and $(n-1)^2 \times (n-1)^2$ matrices are indexed by $N \times N$ and $N_i \times N_i$ respectively. The vector space of all $n \times n$ matrices over $\mathbb{R}$ with standard matrix addition and scalar multiplication is denoted by $\mathbb{M}^n$. Thus $\mathbb{M}^{n^2}$ is the vector space of all $n^2 \times n^2$ matrices.

Two matrices $C^1,\ C^2 \in \mathbb{M}^n$ are *linear permutation equivalent (LP-equivalent)* if $C^1(\pi) = C^2(\pi)\ \forall\ \pi \in \mathcal{P}_n$. A matrix $C \in \mathbb{M}^n$ is *linear permutation constant (LP-constant)* if there exists a constant K such that $C(\pi) = K$ for all $\pi \in \mathcal{P}_n$.

For any $C \in \mathbb{M}^n$ and $p \in N$, let $a_i = c_{ip} - \frac{1}{2}c_{pp}$ and $b_i = c_{pi} - \frac{1}{2}c_{pp}$ for all $i \in N_p$; and $a_p = b_p = \frac{1}{2}c_{pp}$. The matrix $\hat{C} = (\hat{c}_{ij})$ defined by $\hat{c}_{ij} = c_{ij} - a_i - b_j$ for all $i, j \in N$ is the *p-linear reduced form* of C and the ordered pair of vectors $(\mathbf{a}, \mathbf{b})$ is called a *reduction vector pair* of C. When specifying a value of p is unnecessary, we simply call $\hat{C}$ a *linear reduced form* of C. It is easy to see that all the elements of row p and column p of the p-linear reduced form $\hat{C}$ of C are zeros.

Lemma 1. [12] *For any $C \in \mathbb{M}^n$, the following statements are equivalent.*

(1) C is an LP-constant matrix.

(2) The p-linear reduced form of C is the zero matrix for any $1 \leq p \leq n$.

(3) There exist vectors $\mathbf{a}$ *and* $\boldsymbol{b}$ *in* $\mathbb{R}^n$ *such that* $c_{ij} = a_i + b_j$ *for all* $i, j \in N$.

Corollary 2. [12] *A matrix* $C \in \mathbb{M}^n$ *is LP-constant if and only if* $C(\pi) = \sum_{i \in N}(a_i + b_i)$ *for all* $\pi \in \mathcal{P}_n$ *where* $(\mathbf{a}, \boldsymbol{b})$ *is any reduction vector pair of* C.

Corollary 3. [12] *The p-linear reduced matrix of* C *has all zero entries for some* p*, then the* p*-linear reduced matrix of* C *has all zero entries for all* p.

Two matrices $Q^1, Q^2 \in \mathbb{M}^{n^2}$ are *quadratic permutation equivalent (QP-equivalent)* if $Q^1[\pi] = Q^2[\pi]$ $\forall \pi \in \mathcal{P}_n$. A matrix $Q \in \mathbb{M}^{n^2}$ is *QP-constant* if there exists a constant K such that $Q[\pi] = K$ for all $\pi \in \mathcal{P}_n$. If, in addition, $K = 0$, then Q is a *QP-null matrix.*

A matrix $C \in \mathbb{M}^n$ is a *linearization* of $Q \in \mathbb{M}^{n^2}$ if $Q[\pi] = C(\pi)$ for all $\pi \in \mathcal{P}_n$. In this case we say that Q is a *quadratic form* of C and that the matrix Q is *linearizable.* The collection of all linearizable matrices in $\mathbb{M}^{n^2}$ forms a subspace of $\mathbb{M}^{n^2}$.

Suppose $Q^1, Q^2 \in \mathbb{M}^{n^2}$ are QP-equivalent. Then Q^1 is linearizable if and only if Q^2 is linearizable and the two matrices have the same set of linearizations. In particular, Q^1 and Q^{1^T} are QP-equivalent and hence Q^1 is linearizable if and only if Q^{1^T} is linearizable.

It may be noted that the elements $\{q_{ijkl} : i = k \text{ or } j = l\}$ of Q do not contribute to the quadratic cost of any permutation. We call these elements *redundant.* The redundant elements may be assigned any value without affecting linearizability of Q. A matrix $Q^R \in \mathbb{M}^{n^2}$ is said to be in *quadratic reduced form* if all elements in its rows and columns indexed by $\{(n,p),(p,n) : p \in N\}$ are zeros, except possibly q^R_{npnp} and q^R_{pnpn} for $p \in N$.

Theorem 4. [12] *For any* $Q \in \mathbb{M}^{n^2}$ *there exists a* $Q^R \in \mathbb{M}^{n^2}$ *such that* Q^R *is in quadratic reduced form and is QP-equivalent to* Q.

By Theorem 4 we can restrict our attention to cost matrices that are in quadratic reduced form. Let $Q^R \in \mathbb{M}^{n^2}$ be in quadratic reduced form. Delete rows and columns of Q^R indexed by elements of the set $\{(n,p),(p,n) : p \in N\}$ to get matrix $\bar{Q} \in \mathbb{M}^{(n-1)^2}$. For any $i, j \in N_n$, let $Z^{ij} \in \mathbb{M}^{n-1}$ be defined as

$$z^{ij}_{uv} = \begin{cases} \bar{q}_{ijuv} & \text{if } (i,j) = (u,v) \\ \bar{q}_{ijuv} + \bar{q}_{uvij} & \text{if } (i,j) \neq (u,v). \end{cases} \tag{3}$$

Let $Q' \in \mathbb{M}^{(n-1)^2}$ be the matrix obtained from $\bar{Q}$ by setting all its diagonal elements to zero.

Theorem 5. [12] *Let* $Q^R \in \mathbb{M}^{n^2}$ *be in quadratic reduced form. Then* Q^R *is linearizable if and only if the following two conditions hold.*

(1) For all $(i,j) \in N_n \times N_n$*, the submatrix of* Z^{ij} *obtained by deleting its* i^{th} *row and* j^{th} *column is an LP-constant matrix.*

(2) Q' *is a QP-constant matrix.*

Suppose Q^R is linearizable. Then conditions (1) and (2) of Theorem 5 are satisfied. Let f_{ij} be the constant value of permutations on the submatrix $W^{ij} \in \mathbb{M}^{n-2}$ such that W^{ij} is obtained from Z^{ij} by deleting row i and columns j of Z^{ij}. (Note that the index set of rows of W^{ij} is $N_n - \{i\}$ and that of columns of W^{ij} is $N_n - \{j\}$. These could be renumbered appropriately to get identical index set, say $\theta = \{\theta_1, \theta_2, \ldots, \theta_{n-2}\}$, for rows and columns and hence the permutations under consideration are in fact be viewed as permutations of

θ. Alternatively, assume that we are considering permutations $\sigma \in \mathcal{P}_{n-1}$ with the property that $\sigma(i) = j$.) In [12], it is observed that if condition (1) of Theorem 5 is satisfied then

$$f_{ij} = \sum_{u \in N_n \setminus \{i,p\}} z_{up}^{ij} + \sum_{v \in N_n \setminus \{j,p\}} z_{pv}^{ij} - (n-4) z_{pp}^{ij} \tag{4}$$

where p is an arbitrary but fixed element of $N_n \setminus \{i, j\}$.

Theorem 6. [12] *If Q^R is linearizable, then its linearization C is given by*

$$c_{ij} = \begin{cases} q_{inin}^R & \text{for } j = n \text{ and } i \in N_n \\ q_{njnj}^R & \text{for } i = n \text{ and } j \in N_n \\ \frac{1}{(n-2)} K + q_{nnnn}^R & \text{for } i = j = n \\ f_{ij} + z_{ij}^{ij} - \frac{1}{(n-2)} K & \text{for } i, j \in N_n \end{cases} \tag{5}$$

where $K = \frac{1}{2}\left(\sum_{i \in N_n \setminus \{n-1\}} f_{i(n-1)} + \sum_{j \in N_n \setminus \{n-1\}} f_{(n-1)j} - (n-3) f_{(n-1)(n-1)}\right)$.

3. QAP-KB Linearization problem

Note that an instance of QAP-KB is completely represented by the triplet (A, B, D). The instance (A, B, D) of QAP-KB is equivalent to the instance of QAP with cost matrix Q given by equation (1). We call such a Q the *general form* of the QAP-KB instance (A, B, D) and this relationship is denoted by $Q = Q(A, B, D)$. Thus the linearization problem for QAP-KB can be solved in $O(n^4)$ time using the algorithm given in [12]. However, unlike QAP, the input size of QAP-KB is $O(n^2)$. We now show that QAP-KB linearization problem can be solved in $O(n^2)$ time.

Let (A^1, B^1, D^1) and (A^2, B^2, D^2) be two instances of QAP-KB and let $Q^1 = Q(A^1, B^1, D^1)$ and $Q^2 = Q(A^2, B^2, D^2)$ be the corresponding general form matrices. We say that (A^1, B^1, D^1) and (A^2, B^2, D^2) are *QP-equivalent* if Q^1 and Q^2 are QP-equivalent.

To solve the QAP-KB linearization problem in $O(n^2)$ time, we use the same approach as described in [12] which in turn verifies the conditions of Theorem 5 and then compute the linearization C given by Theorem 6. As observed in [12] this can be done in $O(n^4)$ time by converting QAP-KB into general form using equation (1). However, to achieve improved complexity results, we do not compute the matrix $Q(A, B, D)$ and a QP-equivalent matrix Q^R in quadratic reduced form, explicitly. We first establish that given an instance (A, B, D) of QAP-KB, a QP-equivalent instance (A^R, B^R, D^R) can be obtained in $O(n^2)$ time such that $Q(A^R, B^R, D^R)$ is in quadratic reduced form.

Let (A, B, D) be a given instance of the QAP-KB. For each $i \in N$, define

$$\sum_{j \in N_i} a_{ij} = \alpha_i, \quad \sum_{j \in N_i} a_{ji} = \beta_i, \quad \sum_{j \in N_i} b_{ij} = \gamma_i, \text{ and } \sum_{j \in N_i} b_{ji} = \mu_i$$

For any $i \in N$ and any real number ω, consider the following operations on the triplet (A, B, D):

$g_A^r(i,\omega)$: For all $j \in N$, subtract ω from a_{ij} and add $\omega\gamma_j$ to d_{ij} to get new matrices $\tilde{A}$, $\tilde{D}$ and set $\tilde{B} = B$.

$g_A^c(i,\omega)$: For all $j \in N$, subtract ω from a_{ji} and add $\omega\mu_j$ to d_{ji} to get new matrices $\tilde{A}$, $\tilde{D}$ and set $\tilde{B} = B$.

$g_B^r(i,\omega)$: For all $j \in N$, subtract ω from b_{ij} and add $\omega\alpha_j$ to d_{ij} to get new matrices $\tilde{B}$, $\tilde{D}$ and set $\tilde{A} = A$.

$g_B^c(i,\omega)$: For all $j \in N$, subtract ω from b_{ji} and add $\omega\beta_j$ to d_{ji} to get new matrices $\tilde{B}$, $\tilde{D}$ and set $\tilde{A} = A$.

Lemma 7. *Suppose $(\tilde{A}, \tilde{B}, \tilde{D})$ is obtained from (A, B, D) using one of the operations $g_A^r(i,\omega)$, $g_A^c(i,\omega)$, $g_B^r(i,\omega)$ and $g_B^c(i,\omega)$. Then the QAP-KB instances (A, B, D) and $(\tilde{A}, \tilde{B}, \tilde{D})$ are QP-equivalent.*

Proof. Let $Q, \tilde{Q} \in \mathbb{M}^{n^2}$ be the general form matrices of (A, B, D) and $(\tilde{A}, \tilde{B}, \tilde{D})$ respectively. Suppose $(\tilde{A}, \tilde{B}, \tilde{D})$ is obtained from (A, B, D) using the operation $g_A^r(i,\omega)$, for some i and ω. Note that for any $\pi \in \mathcal{P}_n$, $Q[\pi] = \sum_{i \in N} \sum_{j \in N_i} a_{ij} b_{\pi(i)\pi(j)}$. Thus

$$Q[\pi] - \tilde{Q}[\pi] = \omega \sum_{j \in N_i} b_{\pi(i)\pi(j)} - \omega\gamma_{\pi(i)} = 0.$$

This proves the lemma for the operation $g_A^r(i,\omega)$. The proof for the remaining operations $g_A^c(i,\omega)$, $g_B^r(i,\omega)$ and $g_B^c(i,\omega)$ can be obtained analogously. □

Reduction Algorithm 1: Let (A^0, B^0, D^0) be obtained from (A, B, D) using the reduction operations $g_A^r(i, a_{in})$ for all $i \in N_n$ followed by the reduction operations $g_B^r(i, b_{in})$ for all $i \in N_n$. Now apply $g_{A^0}^c(i, a_{ni}^0)$ for all $i \in N_n$ followed by $g_{B^0}^c(i, b_{ni}^0)$ for all $i \in N_n$ to (A^0, B^0, D^0) to get (A^R, B^R, D^R). We call (A^R, B^R, D^R) the *reduced KB-form* of (A, B, D).

By repeated applications of Lemma 7 it can be established that the QAP-KB instances (A, B, D) and (A^R, B^R, D^R) are QP-equivalent. It may be noted that all elements of the nth row and column of A^R and B^R are zeros except possibly a_{nn}^R and b_{nn}^R.

Lemma 8. *$Q^R = Q(A^R, B^R, D^R)$ is in quadratic reduced form and is QP-equivalent to $Q(A, B, D)$. Further, if A (B) is symmetric, the corresponding matrix A^R (B^R) is also symmetric.*

Proof. All the non-diagonal elements of the nth row and column of A^R and B^R are zeros. Thus, for all $p \in N$, all the non-redundant elements in the rows and columns of Q^R indexed by elements of the set $\{(n,p),(p,n) : p \in N\}$ are zeros, except possibly the diagonal elements. Thus Q^R is in quadratic reduced form. From Lemma 7, Q^R and $Q(A, B, D)$ are QP-equivalent. The reduction from A to A^R (B to B^R) preserves symmetry follows from definition. □

Let $\bar{A}, \bar{B}, \bar{D} \in \mathbb{M}^{n-1}$ be obtained from A^R, B^R, D^R, respectively, by deleting their nth rows and columns and let $\bar{Q} = Q(\bar{A}, \bar{B}, \bar{D})$. The matrix Q' of Theorem 5 is thus given by $Q' = Q(\bar{A}, \bar{B}, O)$, where $O \in \mathbb{M}^{n-1}$ with all elements zero. Thus Q' is also of the QAP-KB type. Now, consider condition (1) of Theorem 5. Since $\bar{Q} = Q(\bar{A}, \bar{B}, \bar{D})$, for any $i, j \in N_n$, the elements of the matrix $Z^{ij} \in \mathbb{M}^{n-1}$ of equation (3) become

$$z_{uv}^{ij} = \begin{cases} \bar{d}_{ij} & \text{if } (i,j) = (u,v) \\ \bar{a}_{iu}\bar{b}_{jv} + \bar{a}_{ui}\bar{b}_{vj} & \text{otherwise} \end{cases} \tag{6}$$

Let $W^{ij} \in \mathbb{M}^{n-2}$ be obtained by deleting row i and column j of Z^{ij}. We have to check if W^{ij} is an LP-constant matrix. Let us now prove two general results to establish that this verification can be done in $O(n)$ time.

Lemma 9. *Let $C \in \mathbb{M}^n$ be defined by $c_{ij} = x_i y_j + g_i h_j$ where $\mathbf{x}, \mathbf{y}, \mathbf{g}, \mathbf{h} \in \mathbb{R}^{\mathbf{n}}$. Then its p-linear reduced matrix $\hat{C}$ satisfies $\hat{c}_{ij} = \hat{x}_i \hat{y}_j + \hat{g}_i \hat{h}_j$ for all $i, j \in N$ where the vectors $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{g}}, \hat{\mathbf{h}} \in \mathbb{R}^n$ are given by $\hat{x}_j = x_j - x_p, \hat{y}_j = y_j - y_p$, $\hat{g}_j = g_j - g_p$, $\hat{h}_j = h_j - h_p$ for and $j \in N$.*

The proof of the above lemma follows from simple algebra and hence omitted.

Lemma 10. *For $\mathbf{x}, \mathbf{y}, \mathbf{g}, \mathbf{h} \in \mathbb{R}^{\mathbf{n}}$, let $C \in \mathbb{M}^n$ be defined by $c_{ij} = x_i y_j + g_i h_j$ for all $i, j \in N$. Let $S^1 = \{i : x_i \neq 0, g_i \neq 0\}$ and $S^2 = \{i : y_i \neq 0, h_i \neq 0\}$. Then $c_{ij} = 0$ for all i, j if and only if at least one vector from each of the pairs $\{\mathbf{x}, \mathbf{y}\}, \{\mathbf{g}, \mathbf{h}\}$ is zero or there exists α such that $\frac{x_i}{g_i} = \alpha$ for all $i \in S^1$ and $\frac{h_i}{y_i} = -\alpha$ for all $i \in S^2$.*

Proof. Suppose $c_{ij} = 0$ for all i, j. If $x_i = 0, g_i \neq 0$ for some $i \in N$, then $h_j = 0$ for all $j \in N$ and either $y_j = 0$ for all j or $x_j = 0$ for all j. Similarly, If $y_j = 0, h_j \neq 0$ for some $j \in N$, then $g_i = 0$ for all $i \in N$ and either $x_i = 0$ for all i or $y_i = 0$ for all i. By symmetry, the cases $g_i = 0, x_i \neq 0$ for some $i \in N$ and $h_j = 0, x_j \neq 0$ for some $j \in N$ also yields at least one vector from each of the pairs $\{\mathbf{x}, \mathbf{y}\}, \{\mathbf{g}, \mathbf{h}\}$ is zero. If $S^1 = \emptyset$ then $\mathbf{x} = \mathbf{g} = \mathbf{0}$ and if $S^2 = \emptyset$ then $\mathbf{y} = \mathbf{h} = \mathbf{0}$. Thus we are left with the case $S^1 \neq \emptyset$ and $S^2 \neq \emptyset$ and $x_i \neq 0$ if and only if $g_i \neq 0$ and $y_i \neq 0$ if and only if $h_i \neq 0$. Since $c_{ij} = 0$, we have for all $i \in S^1, j \in S^2$ $c_{ij} = x_i y_j + g_i h_j = 0$ which implies $\frac{x_i}{g_i} = -\frac{h_j}{y_j}$. The converse can be verified easily. ☐

Let us now come back to the question of testing if W^{ij} is LP-constant. From Lemma 1 we have W^{ij} is LP-constant if and only if its p-linear reduced matrix has all zero entries, for some $p \neq i, j$. But $w_{uv}^{ij} = x_u y_v + g_u h_v$ where $\mathbf{x}, \mathbf{y}, \mathbf{g}, \mathbf{h} \in \mathbb{R}^{\mathbf{n-2}}$ and $\mathbf{x}(\mathbf{g})$ is the row (column) i of $\bar{A}$ with $\bar{a_{ii}}$ deleted and $\mathbf{y}(\mathbf{h})$ is row (column) j of $\bar{B}$ with $\bar{b}_{jj}$ deleted. (Note that the index set of vectors $\mathbf{x}, \mathbf{y}$ and the index set of rows of W^{ij} is $N_n - \{i\}$ and the index set of vectors $\mathbf{g}, \mathbf{h}$ and the index set of columns of W^{ij} is $N_n - \{j\}$.) It now follows from lemmas 9 and 10 that this can be checked in $O(n)$ time. Thus we can verify condition (1) of Theorem 5 for all $i, j \in N$ in $O(n^3)$ time. If condition (1) of the theorem is satisfied, then it is shown in [12] for the general case of QAP that condition (2) of theorem 5 can be verified in $O(n^3)$ time. This yields an overall complexity of $O(n^3)$ for solving the linearization problem for QAP-KB, which improves the $O(n^4)$ bound for the general QAP given in [12]. Let us now discuss how to reduce this complexity by a factor of $O(n)$.

We first show that testing if the matrix W^{ij} is LP-constant and computing the corresponding constant value of permutations, for all $i, j \in N_n$ can be done more efficiently in $O(n^2)$ time. First, let us establish some general results. Consider $n^2 + 2$ matrices $A, B \in \mathbb{M}^n$ and $C^{rs} \in \mathbb{M}^n$ for $(r, s) \in N \times N$ such that $c_{ij}^{rs} = a_{ri} b_{sj} + a_{ir} b_{js}$. We give necessary and sufficient conditions for $c_{ij}^{rs} = 0$ for all $(i, j), (r, s) \in N \times N$, $i \neq r$, $j \neq s$ and establish that the validity of these conditions can be verified in $O(n^2)$ time, even though there are $O(n^4)$ such

c_{ij}^{rs} values. If A or B is a diagonal matrix, then clearly $c_{ij}^{rs} = 0$ for all $(i,j),(r,s) \in N \times N$, $i \neq r,\ j \neq s$. Let

$$S^a = \{(i,j) \in N \times N :\ \text{exactly one of } \{a_{ij}, a_{ji}\} \text{ is zero}\}$$

and

$$S^b = \{(i,j) \in N \times N :\ \text{exactly one of } \{b_{ij}, b_{ji}\} \text{ is zero}\}.$$

Lemma 11. *Suppose A and B are not diagonal matrices. If $S^a \cup S^b \neq \emptyset$ then there exists $(i,j),(r,s) \in N \times N$, $i \neq r,\ j \neq s$, such that $c_{ij}^{rs} \neq 0$.*

Proof. By symmetry in the definition of S^a and S^b, let us assume without loss of generality that $S^a \neq \emptyset$. Choose an $(i,j) \in S^a$. Again, by symmetry, we assume that $a_{ij} \neq 0$ and $a_{ji} = 0$. Let b_{pq} be a non-zero, non-diagonal element of B. By hypothesis, such an element exists. Then $c_{jq}^{ip} = a_{ij} b_{pq} \neq 0$. □

Lemma 12. *Suppose A and B are not diagonal matrices and $c_{ij}^{rs} = a_{ri} b_{sj} + a_{ir} b_{js}$. Then $c_{ij}^{rs} = 0$ for all $(i,j),(r,s) \in N \times N$, $i \neq r,\ j \neq s$, if and only if A is symmetric and B is skew-symmetric or B is symmetric and A is skew-symmetric.*

Proof. Suppose $c_{ij}^{rs} = 0$ for all $(i,j),(r,s) \in N \times N$, $i \neq r,\ j \neq s$. By Lemma 11, $Z^a \cup Z^b = \emptyset$. Let $\Omega^a = \{(i,j) \in N \times N :, i \neq j, a_{ij} \neq 0, a_{ji} \neq 0\}$ and $\Omega^b = \{(i,j) \in N \times N : i \neq j, b_{ij} \neq 0, b_{ji} \neq 0\}$. Since A and B are not diagonal matrices and $Z^a \cup Z^b = \emptyset$, $\Omega^a \neq \emptyset$ and $\Omega^b \neq \emptyset$ and $c_{ij}^{rs} = a_{ri} b_{sj} + a_{ir} b_{js} = 0$ for all $(r,i) \in \Omega^a$ and $(s,j) \in \Omega^b$. Thus

$$\frac{a_{ri}}{a_{ir}} = -\frac{b_{js}}{b_{sj}} = \alpha(\text{say})\ \forall (r,i) \in \Omega^a \text{ and } (s,j) \in \Omega^b \tag{7}$$

Note that if $(r,i) \in \Omega^a$ then $(i,r) \in \Omega^a$. Likewise, if $(j,s) \in \Omega^b$ then $(s,j) \in \Omega^b$. We thus have

$$\frac{a_{ri}}{a_{ir}} = \frac{a_{ir}}{a_{ri}} = -\frac{b_{js}}{b_{sj}} = -\frac{b_{sj}}{b_{js}} = \alpha\ \forall (r,i) \in \Omega^a \text{ and } (s,j) \in \Omega^b \tag{8}$$

From (7) and (8), we have $\alpha = 1$ and hence either A is symmetric and B is skew-symmetric or B is symmetric and A is skew symmetric.

Conversely suppose A and B are not diagonal matrices and either A is symmetric and B is skew-symmetric or B is symmetric and A is skew-symmetric. Then, it can be verified from the definition that $c_{ij}^{rs} = 0$ for all $(i,j),(r,s) \in N \times N$, $i \neq r,\ j \neq s$. □

Recall the definition of $Z^{ij} \in \mathbb{M}^{n-1}$ given in equation (6) and the fact that $W^{ij} \in \mathbb{M}^{n-2}$ is obtained from Z^{ij} by deleting row i and column j. To verify conditions of Theorem 5, we first have to check if W^{ij} is LP-constant for all $(i,j) \in N_n \times N_n$. By Lemma 1, W^{ij} is LP-constant, if and only if its p-linear reduced matrix $\bar{W}^{ij}$ has all zero entries for some $p \neq i,j$. We want to verify this condition for all (i,j) in $O(n^2)$ time.

Case 1: $1 \leq i,j \leq n-2$**:** Let $\hat{W}^{ij}$ be the $(n-1)$-linear reduced matrix of W^{ij}. From Lemma 9, it can be verified that $\hat{w}_{uv}^{ij} = \hat{a}_{iu}\hat{b}_{jv} + \bar{a}_{ui}\bar{b}_{vj}$ for $(i,j) \neq (u,v)$ where $\hat{a}_{iu} = \bar{a}_{iu} - \bar{a}_{i,n-1}$ and $\hat{b}_{jv} = \bar{b}_{jv} - \bar{b}_{j,n-1}$. Then by Lemma 12, $\hat{w}_{uv}^{ij} = 0$ for $i,j \in \{1,2,\ldots n-2\}$, $(i,j) \neq (u,v)$, $u,v \in \{1,2,\ldots n-2\}$ if and only if one of the matrices $\hat{A}, \bar{B}$ is symmetric and the other is skew symmetric. This can be done in $O(n^2)$ time.

Case 2: $i = n-1, 2 \leq j \leq n-1$ **or** $j = n-1, 2 \leq i \leq n-1$**:** In this case choose $\hat{W}^{ij}$ as the 1-linear reduced matrix of W^{ij}. From Lemma 9, it can be verified that $\hat{w}^{ij}_{uv} = \hat{a}_{iu}\hat{b}_{jv} + \hat{a}_{ui}\hat{b}_{vj}$ for $(i,j) \neq (u,v)$ where $\hat{a}_{iu} = \bar{a}_{iu} - \bar{a}_{i1}$ and $\hat{b}_{jv} = \bar{b}_{jv} - \bar{b}_{j1}$. Let $\tilde{w}^{ij}_{uv} = \tilde{a}_{iu}\tilde{b}_{jv} + \tilde{b}_{ui}\tilde{b}_{vj}$ where $\tilde{A}$ and $\tilde{B}$ be obtained from $\hat{A}$ and $\hat{B}$ by replacing diagonal entries by zero and deleting row $n-1$ and column $n-1$. Then for each i,j satisfying conditions of this case, $\hat{w}^{ij}_{uv} = 0$ for $i,j \in \{1,2,\ldots n-2\}$, $(i,j) \neq (u,v)$, $u,v \in \{1,2,\ldots n-2\}$ if and only if $\tilde{w}^{ij}_{uv} = 0$ for all $u,v \in \{1,2,\ldots n-2\}$. For each i,j, using Lemma 10 on matrix $\tilde{W}^{ij}$, this can be verified in $O(n)$ time. Since there are only $O(n)$ choices of (i,j) pairs in this case, the overall complexity is $O(n^2)$ for this case.

Case 3: $i = 1, j = n-1$ **or** $j = 1, i = n-1$**:** In this case let $\hat{W}^{ij}$ be the 2-linear reduced matrix of W^{ij}. There are only two matrices to consider here and by direct computation, $\hat{W}^{ij}$ is LP-constant or not can be verified in $O(n^2)$ time.

Thus, combining cases (1), (2) and (3), condition (1) of Theorem 5 can be verified in $O(n^2)$ time. Once W^{ij} is verified to be LP-constant for all (i,j), we have to find the constant value f_{ij} of all the permutations on W^{ij} for each $(i,j) \in N_n \times N_n$. For each $i \in N_n$, let us define $\bar{\alpha}_i, \bar{\beta}_i, \bar{\gamma}_i$ and $\bar{\mu}_i$ as follows:

$$\sum_{\substack{j=1\\ j\neq i}}^{n-1} \bar{a}_{ij} = \bar{\alpha}_i, \quad \sum_{\substack{j=1\\ j\neq i}}^{n-1} \bar{a}_{ji} = \bar{\beta}_i, \quad \sum_{\substack{j=1\\ j\neq i}}^{n-1} \bar{b}_{ij} = \bar{\gamma}_i, \text{ and } \sum_{\substack{j=1\\ j\neq i}}^{n-1} \bar{b}_{ji} = \bar{\mu}_i \tag{9}$$

Then for each $(i,j) \in N_n \times N_n$ and any $p \in N_n \setminus \{i,j\}$,

$$\begin{aligned} f_{ij} &= \sum_{u\in N_n\setminus\{i,p\}} z^{ij}_{up} + \sum_{v\in N_n\setminus\{j,p\}} z^{ij}_{pv} - (n-4)z^{ij}_{pp} \\ &= \sum_{u\in N_n\setminus\{i,p\}} \left(\bar{a}_{iu}\bar{b}_{jp} + \bar{a}_{ui}\bar{b}_{pj}\right) + \sum_{v\in N_n\setminus\{j,p\}} \left(\bar{a}_{ip}\bar{b}_{jv} + \bar{a}_{pi}\bar{b}_{vj}\right) - (n-4)(\bar{a}_{ip}\bar{b}_{jp} + \bar{a}_{pi}\bar{b}_{pj}) \\ &= \bar{b}_{jp}\bar{\alpha}_i + \bar{b}_{pj}\bar{\beta}_i + \bar{a}_{ip}\bar{\gamma}_j + \bar{a}_{pi}\bar{\mu}_j - (n-4)(\bar{a}_{ip}\bar{b}_{jp} + \bar{a}_{pi}\bar{b}_{pj}) \end{aligned} \tag{10}$$

The values $\{\bar{\alpha}_i, \bar{\beta}_i, \bar{\gamma}_i, \bar{\mu}_i : i \in N_n\}$ can all be computed in $O(n^2)$ time. Using these values and equation (10) each f_{ij} value can be obtained in constant time; and thus all the f_{ij} values can be computed in $O(n^2)$ time. In [12] it is shown that if condition (1) of theorem 5 is satisfied, then condition (2) is satisfied if and only if the matrix $F = (f_{ij})$ is LP-constant. Hence, condition (2) of Theorem 5 can be verified in $O(n^2)$ time. Now, the linearization C can be identified using Theorem 6.

3.1. **Symmetric QAP-KB.** An interesting special case of QAP-KB is the *Bookhold* QAP where A is upper-triangular, B is symmetric and $d_{ij} = a_{ii}b_{jj}$. Bookhold [4] investigated this case and proved the following sufficient condition.

Theorem 13. [4] *The Bookhold QAP (A,B,D) with $n \geq 3$ is linearizable if $a_{ij} = a_{1i} + a_{1j} + a_{(n-1)n} - a_{1(n-1)} - a_{1n}$ for $i = 2,3,\ldots n-2$ and $j = i+1,\ldots,n$ or $b_{ij} = b_{1i} + b_{1j} + b_{(n-1)n} - b_{1(n-1)} - b_{1n}$ for $i = 2,3,\ldots n-2$ and $j = i+1,\ldots,n$.*

Bookhold [4] showed that the condition of Theorem 13 is also necessary for $n = 3, 4$ and for $n \geq 5$ he states that the "the necessity could not be proved but no counter examples could be found either". We now show that the condition of Theorem 13 is indeed necessary for all $n \geq 3$. Consider an instance (A, B, D) of Bookhold QAP and let $A' = (A + A^T)/2$. It is easy to see that A' is symmetric and the Bookhold QAP (A, B, D) satisfies the conditions of Theorem 13 if and only if (A', B, D) satisfies the conditions of Theorem 13. Thus we restrict our attention to *symmetric* QAP-KB where A and B are symmetric.

Note that in the reduced form (A^R, B^R, D^R) of (A, B, D), all the elements in the nth row and nth column of A^R and B^R are zero except possibly the diagonal entries a^R_{nn} and b^R_{nn}; and $\bar{A}$, $\bar{B}$ and $\bar{D}$ are obtained from A^R, B^R and D^R, respectively, by deleting their nth rows and columns. Thus, for each pair $i, j \in N_n$,$\bar{a}_{i,j} = a_{ij} - a_{in} - a_{nj}$ and $\bar{b}_{i,j} = b_{ij} - b_{in} - b_{nj}$ and since A and B are symmetric, $\bar{A}$ and $\bar{B}$ are symmetric as well. It now follows from Lemma 12 and its application in our $O(n^2)$ scheme for testing linearization of QAP-KB that W^{ij} is LP-constant if and only if either all the non-diagonal elements of $\bar{A}$ have equal value or all the non-diagonal elements of $\bar{B}$ have equal value. Thus, either for all $i, j \in N_n, i < j$, $\bar{a}_{ij} = a_{ij} - a_{in} - a_{nj} = \bar{a}_{1(n-1)} = a_{1(n-1)} - a_{1n} - a_{n(n-1)}$ or for all $i, j \in N_n, i < j$, $\bar{b}_{ij} = b_{ij} - b_{in} - b_{nj} = \bar{b}_{1(n-1)} = b_{1(n-1)} - b_{1n} - b_{n(n-1)}$. Hence,

$$a_{ij} = a_{in} + a_{nj} + a_{1(n-1)} - a_{1n} - a_{n(n-1)} \text{ for all } i, j \in N_n, i < j \text{ or} \tag{11}$$

$$b_{ij} = b_{in} - b_{nj} + b_{1(n-1)} - b_{1n} - b_{n(n-1)} \text{ for all } i, j \in N_n, i < j \tag{12}$$

It may be noted that if we change the labeling of the elements of the set N and permute the rows and columns of matrices A, B and D accordingly, we get an equivalent instance of QAP. Interchanging the indices 1 and n in equations (11) and (12) and using the fact that A and B are symmetric matrices, we get precisely the Bookhold conditions. We thus have the following theorem, which resolves the question raised in [4].

Theorem 14. *For the symmetric QAP-KB (A, B, D) the Bookhold conditions of Theorem 13 are necessary and sufficient for linearizability.*

4. The MAP Linearization problem

An instance of MAP is completely represented by the triple (A, B, D). As observed in the introduction section, MAP is equivalent to QAP with cost matrix Q defined in equation (2). We represent this relationship by $Q = H(A, B, D)$ and we call Q the general form of the MAP instance (A, B, D). Thus by Theorem 5, the linearization problem for MAP can be solved in $O(n^4)$ time. As in the case of QAP-KB, the data for MAP is $O(n^2)$. We now show that the linearization problem for MAP can also be solved in $O(n^2)$ time.

Let (A^1, B^1, D^1) and (A^2, B^2, D^2) be two instances of MAP and let $Q^1 = H(A^1, B^1, D^1)$ and $Q^2 = H(A^2, B^2, D^2)$ be the corresponding general form matrices. We say that (A^1, B^1, D^1) and (A^2, B^2, D^2) are *QP-equivalent* if and only if Q^1 and Q^2 are QP-equivalent.

For any $i \in N$ and real number ω, define the following operations on the triplet (A, B, D):

$h_A^r(i, \omega)$: Subtract ω from row i of A to obtain matrix A^i and generate the triplet $(\hat{A}, \hat{B}, \hat{D}) = (A^i, B, D + \omega B)$

$h_A^c(i, \omega)$: Subtract ω from column i of A to obtain matrix A^i, generate the triplet $(\hat{A}, \hat{B}, \hat{D}) = (A^i, B, D + \omega B)$

$h_B^r(i, \omega)$: Subtract ω from row i of B to obtain matrix B^i and generate the triplet $(\hat{A}, \hat{B}, \hat{D}) = (A, B^i, D + \omega A)$

$h_B^c(i, \omega)$: Subtract ω from column i of B to obtain matrix B^i, generate the triplet $(\hat{A}, \hat{B}, \hat{D}) = (A, B^i, D + \omega A)$

Lemma 15. *The MAP instance $(\hat{A}, \hat{B}, \hat{C})$ obtained by any of the operations $h_A^r(i, \omega)$, $h_A^c(i, \omega)$, $h_B^r(i, \omega)$, and $h_B^c(i, \omega)$ is QP-equivalent to the MAP instance (A, B, C).*

Proof. Let $Q = H(A, B, C)$ and $\hat{Q} = H(\hat{A}, \hat{B}, \hat{C})$ for the operation $h_A^r(i, \omega)$. For any $\pi \in \mathcal{P}_n$

$$\begin{aligned}\hat{Q}[\pi] &= \sum_{k \in N} \sum_{j \in N} \hat{a}_{k\pi(k)} \hat{b}_{j\pi(j)} + \sum_{j \in N} \hat{d}_{j\pi(j)} \\ &= \sum_{k \in N} \sum_{j \in N} a_{k\pi(k)} b_{j\pi(j)} - \omega \sum_{j \in N} b_{j\pi(j)} + \sum_{j \in N} d_{j\pi(j)} + \omega \sum_{j \in N} b_{j\pi(j)} \\ &= Q[\pi]\end{aligned}$$

Proof for the remaining operations follows analogously. □

Definition 1. Let (A^0, B^0, D^0) be obtained from (A, B, D) using the reduction operations $h_A^r(i, a_{in})$ for all $i \in N_n$ followed by the reduction operations $h_B^r(i, b_{in})$ for all $i \in N_n$. Now apply to (A^0, B^0, D^0), operations $h_{A^0}^c(i, a'_{ni})$ for all $i \in N_n$ followed by operations $h_{B^0}^c(i, b_{ni}^0)$ for all $i \in N_n$ to get (A^R, B^R, D^R). We call (A^R, B^R, D^R) the *reduced product form* of (A, B, D).

Note that all the non-diagonal elements in the nth row and column of A^R and B^R are zeros.

Lemma 16. *(A^R, B^R, D^R) and (A, B, D) are QP-equivalent. Further, $Q^R = H(A^R, B^R, D^R)$ is the quadratic reduced form of $Q = H(A, B, D)$.*

Proof. The QP-equivalence of (A^R, B^R, D^R) and (A, B, D) follows from repeated applications of Lemma 15. Since $a_{in}^R = b_{in}^R = a_{ni}^R = b_{ni}^R = 0$ for $i \in N_n$, all the non-redundant elements in the rows and columns of Q^R indexed by elements of the set $\{(n, p), (p, n) : p \in N\}$ are zeros, except possibly the diagonal elements. □

Let $\bar{A}$, $\bar{B}$ and $\bar{D}$ be obtained by deleting the nth row and column from A^R, B^R and D^R, respectively.

To obtain an $O(n^2)$ algorithm for the MAP linearization problem, it is enough to show that condition (1) of Theorem 5 can be verified in $O(n^2)$ time and the f_{ij} values of equation (4) can be obtained in $O(n^2)$ time. From equation (2), the matrix Z^{ij} for $i, j \in N_n$ of Theorem 5 is given by,

$$z_{uv}^{ij} = \begin{cases} \bar{d}_{ij} + \bar{a}_{ii}\bar{b}_{jj} & \text{if } (i,j) = (u,v) \\ \bar{a}_{ij}\bar{b}_{uv} + \bar{a}_{uv}\bar{b}_{ij} & \text{otherwise} \end{cases} \tag{13}$$

It may be noted that the matrix W^{ij} is obtained from C^{ij} by deleting its ith row and jth column. Let us index the rows and columns of W^{ij} by $\bar{N}_i = N_n - \{i\}$ and $\bar{N}_j = N_n - \{j\}$, respectively. Thus, $w_{uv}^{ij} = \bar{a}_{ij}\bar{b}_{uv} + \bar{a}_{uv}\bar{b}_{ij}$ for all $u \in \bar{N}_i$ and $v \in \bar{N}_j$. By Lemma 1 and Corollary 2, W^{ij} is LP-constant for all $i, j \in N_n$ if and only if the p-linear reduced form of W^{ij} is the zero matrix for some $p \in N_n \setminus \{i, j\}$ (and hence for any $p \in N_n \setminus \{i, j\}$).

Let us now prove some general results. Let $\tau_{uv}^{ij} = x_{ij}\bar{y}_{uv} + y_{ij}\bar{x}_{uv}$ for some $X, Y, \bar{X}, \bar{Y} \in \mathbb{M}^n$. Consider the sets

$$S^1 = \{(i,j) : x_{ij} \neq 0, y_{ij} \neq 0\} \text{ and } S^2 = \{(i,j) : \bar{x}_{ij} \neq 0, \bar{y}_{ij} \neq 0\}$$

Lemma 17. *Let $X, Y, \bar{X}, \bar{Y} \in \mathbb{M}^n$ be such that each of them contains at least two rows or at least two columns with non-zero entries and let $\tau_{uv}^{ij} = x_{ij}\bar{y}_{uv} + y_{ij}\bar{x}_{uv}$. Then $\tau_{uv}^{ij} = 0$ for all $u, v \in N, u \neq i, v \neq j$ if and only if there exists some $\alpha \in \mathbb{R}$ such that $\frac{x_{ij}}{y_{ij}} = \alpha$ for all $(i,j) \in S^1$ and $\frac{\bar{x}_{ij}}{\bar{y}_{ij}} = -\alpha$ for all $(i,j) \in S^2$.*

Proof. Suppose $\tau_{uv}^{ij} = 0$ for all $u, v \in N, u \neq i, v \neq j$. First, we establish that $x_{ij} \neq 0$ if and only if $y_{ij} \neq 0$. Thus, suppose $x_{ij} = 0$ and $y_{ij} \neq 0$. Then $\tau_{uv}^{ij} = y_{ij}\bar{x}_{uv} = 0$ and hence $\bar{x}_{uv} = 0$ for all u, v except $u = i, v = j$, a contradiction to the assumption that $\bar{X}$ contains at least two non-zero rows or columns. Similarly, if $x_{ij} \neq 0$ and $y_{ij} = 0$, we get a contradiction. Using similar arguments, it can be established that $\bar{x}_{ij} \neq 0$ if and only if $\bar{y}_{ij} \neq 0$. Thus, $\tau_{uv}^{ij} = 0$ for all $u, v \in N, u \neq i, v \neq j$ if and only if $x_{ij}\bar{y}_{uv} + y_{ij}\bar{x}_{uv} = 0$ for all $(i,j) \in S^1$ and $(u,v) \in Z$, $i \neq u, j \neq v$. This is possible if and only if $\frac{x_{ij}}{y_{ij}} = -\frac{\bar{x}_{uv}}{\bar{y}_{uv}}$ for all $(i,j) \in S^2, (u,v) \in S$ and the result follows. □

Lemma 18. *If one of the matrices $X, Y, \bar{X}, \bar{Y} \in \mathbb{M}^n$ contains at most one row and column with non-zero entries, then we can verify whether $\tau_{uv}^{ij} = x_{ij}\bar{y}_{uv} + y_{ij}\bar{x}_{uv} = 0$ for all $u, v \in N, u \neq i, v \neq j$ in $O(n^2)$ time.*

Proof. Suppose $\bar{X} = 0$. If X is zero, then the result follows. If $X \neq 0$ then there exists $r, s \in N$ such that $x_{rs} \neq 0$. Then $\bar{y}_{uv} = 0$ except for $u = r$ or $v = s$. If row r or column s of $\bar{Y}$ contains a non-zero element, then X contains at most one row, say p and one column, say q, with non-zero entries. Thus $\tau_{uv}^{ij} = 0$ for all $u, v \in N, u \neq i, v \neq j$ can be verified in $O(n^2)$ time by simply verifying the condition $\tau_{uv}^{ij} = 0$ for $i = p, j \in N$; $j = q, i \in N$; $u = r, v \in N$; $v = s, u \in N$; and $u \neq, i, v \neq j$. The other three cases, $X = 0$, $Y = 0$ and $\bar{Y} = 0$ can be verified similarly.

Suppose $\bar{X}$ contains exactly one row and column with non-zero entries. In this case, it can be verified that X and $\bar{Y}$ will have at most two rows and columns with non zero entries and Y will have at most three rows and columns with non-zero entries. Thus we need to consider only $O(n^2)$ combinations of i, j, u, v to verify $\tau_{uv}^{ij} = 0$ for all $u, v \in N, u \neq i, v \neq j$ and hence this can be done in $O(n^2)$ times. The other three cases where X, Y and $\bar{Y}$ have exactly one row and/or column with possible non zero entries can be verified similarly and the result follows. □

We shall use the above results to show that for MAP, condition (1) of Theorem 5 can be verified in $O(n^2)$ time.

Case 1: $1 \leq i, j \leq n-2$**:** Let $\hat{\bar{A}}$ and $\hat{\bar{B}}$ be the $(n-1)$-linear reduced forms of matrices $\bar{A}$ and $\bar{B}$, respectively. Then W^{ij} is LP-constant for all $1 \leq i, j \leq n-2$ if and only if $w^{ij}_{uv} = \bar{a}_{ij}\hat{\bar{b}}_{uv} + \bar{b}_{uv}\hat{\bar{a}}_{uv} = 0$ for all $1 \leq i, j, u, v \leq n-2, u \neq i, v \neq j$. Choosing $X = \bar{A}, \bar{Y} = \hat{\bar{B}}, Y = \bar{B}, \bar{X} = \hat{\bar{A}}$ in lemmas 17 and 18, this condition can be verified in $O(n^2)$ time.

Case 2: $i = n-1, 2 \leq j \leq n-1$ **or** $j = n-1, 2 \leq i \leq n-1$**:** Let $\hat{\bar{A}}$ and $\hat{\bar{B}}$ be the 1-linear reduced matrix of $\bar{A}$ and $\bar{B}$ respectively. Then W^{ij} is LP-constant for all $i = n-1, 2 \leq j \leq n-1$ or $j = n-1, 2 \leq i \leq n-1$ if and only if $w^{ij}_{uv} = \bar{a}_{ij}\hat{\bar{b}}_{uv} + \bar{b}_{uv}\hat{\bar{a}}_{uv} = 0$ for all $2 \leq i, j, u, v \leq n-1, u \neq i, v \neq j$. Choosing $X = \bar{A}, \bar{Y} = \hat{\bar{B}}, Y = \bar{B}, \bar{X} = \hat{\bar{A}}$ in lemmas 17 and 18, this condition can be verified in $O(n^2)$ time.

Case 3: $i = 1, j = n-1$ **or** $j = 1, i = n-1$**:** Let $\hat{\bar{A}}$ and $\hat{\bar{B}}$ be the 2-linear reduced matrix of $\bar{A}$ and $\bar{B}$ respectively. Then W^{ij} is LP-constant for all $i = 1, j = n-1$ or $j = 1, i = n-1$ if and only if $w^{ij}_{uv} = \bar{a}_{ij}\hat{\bar{b}}_{uv} + \bar{b}_{uv}\hat{\bar{a}}_{uv} = 0$ for all $i = 1, j = n-1$ or $j = 1, i = n-1$ $1 \leq u, v \leq n-1, u \neq i, v \neq j$. This can be verified directly in $O(n^2)$ time.

Let us now compute the f_{ij} (given in equation (4)) in $O(n^2)$ time. For any $p \in N_n$, define:

$$\alpha^p = \sum_{k \in N_n \setminus \{p\}} \bar{a}_{kp};\ \beta^p = \sum_{k \in N_n \setminus \{p\}} \bar{a}_{pk};\ \gamma^p = \sum_{k \in N_n \setminus \{p\}} \bar{b}_{kp} \text{ and } \zeta^p = \sum_{k \in N_n \setminus \{p\}} \bar{b}_{pk}.$$

Then it can be verified that for each pair $i, j \in N_n$, and any choice of $p \in N_n \setminus \{i, j\}$,

$$f_{ij} = \bar{a}_{ij}(\gamma^p - \bar{b}_{ip} + \zeta^p - \bar{b}_{pj}) + \bar{b}_{ij}(\alpha^p - \bar{a}_{ip} + \beta^p - \bar{a}_{pj}) - (n-4)(\bar{a}_{ij}\bar{b}_{pp} + \bar{b}_{ij}\bar{a}_{pp})$$

All the values $\{\alpha^p, \beta^p, \gamma^p, \zeta^p : p \in N_n\}$ can be computed in $O(n^2)$ time. Using these and the above expression, each f_{ij} value can be computed in constant time, and hence all the f_{ij} values can be computed in in $O(n^2)$ time. In [12] it is shown that if condition (1) of theorem 5 is satisfied, then condition (2) is satisfied if and only if the matrix $F = (f_{ij})$ is LP-constant. Hence, condition (2) of Theorem 5 can be verified in $O(n^2)$ time. Now, the linearization C can be identified using Theorem 6.

5. Conclusion

We showed that the linearization problem for QAP-KB and MAP can be solved in $O(n^2)$. For these problems, the data is of size $O(n^2)$ and hence these algorithms are the best possible. As a consequence, we have faster algorithms for solving some special cases of QAP. For symmetric QAP-KB, we show that a sufficient condition established by in [4] is both necessary and sufficient, resolving a question of Bookhold [4].

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