

SUBDIFFERENTIALS OF NONCONVEX SUPREMUM FUNCTIONS AND THEIR APPLICATIONS TO SEMI-INFINITE AND INFINITE PROGRAMS WITH LIPSCHITZIAN DATA¹

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Abstract. The paper is devoted to the subdifferential study and applications of the supremum of uniformly Lipschitzian functions over arbitrary index sets with no topology. Based on advanced techniques of variational analysis, we evaluate major subdifferentials of the supremum functions in the general framework of Asplund (in particular, reflexive) spaces with no convexity or relaxation assumptions. The results obtained are applied to deriving new necessary optimality conditions for nonsmooth and nonconvex problems of semi-infinite and infinite programming.

1 Introduction

This paper is motivated by remarkable classes of optimization problems formalized as

$$\begin{cases} \text{minimize } \varphi(x) \text{ subject to} \\ f_t(x) \leq 0 \text{ with } t \in T, \end{cases} \quad (1.1)$$

where T is an infinite index set and the decision space X is Banach. Optimization problems of type (1.1) are known as *semi-infinite problems* (SIP) when $\dim X < \infty$ and as *infinite programs* when $\dim X = \infty$. Problems of this type, in both SIP and infinite programming frameworks, are important for various applications in optimization, equilibria, systems control, approximation theory as well as their practical implementations; see, e.g., [7, 10, 11, 15, 16, 25, 26, 31] and the references therein. However, till recent years the vast majority of publications on SIP and infinite programs have concerned problems (1.1) with compact index sets under certain continuity assumptions imposed on f_t with respect to the index variable; both these requirements seem to be very essential for the methods employed in the aforementioned publications. Such compactness and continuity assumptions are not imposed in [4, 5, 8, 12, 17, 20, 21, 30] among other recent publications.

It is obvious to observe that problem (1.1) is equivalent to the following optimization problem with *just one* inequality constraint given by the *supremum function*:

$$\begin{cases} \text{minimize } \varphi(x) \text{ subject to} \\ f(x) := \sup \{f_t(x) \mid t \in T\} \leq 0. \end{cases} \quad (1.2)$$

However, the price to pay for such a “simplification” in deriving, say, optimality conditions for SIP and infinite programs is the intrinsic *nonsmoothness* of the supremum function f

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in (1.2), which requires evaluating appropriate subdifferentials of f at the reference point. A number of results are obtained in various publications. In what follows we briefly review the major ones of them; see also the references in the mentioned publications.

The vast majority of publications on subdifferentiation of the supremum/maximum function f concern the case when all f_t and hence f are *convex* in x . The precise subdifferential formula of convex analysis (with “co” and “cl*” standing, respectively, for the convex hull and the weak* closure of sets in the dual space X^*)

$$\partial f(\bar{x}) = \text{cl}^* \text{co} \left[\bigcup \left\{ \partial f_t(\bar{x}) \mid t \in T(\bar{x}) \right\} \right] \quad \text{with } T(\bar{x}) := \{t \in T \mid f_t(\bar{x}) = f(\bar{x})\} \quad (1.3)$$

is derived in [14, Theorem 4.2.3] provided that T is a Hausdorff compact, that the mapping $t \mapsto f_t(x)$ is upper semicontinuous (u.s.c.) for each x , and that the functions f_t are continuous at \bar{x} . This result is known as the Ioffe-Tikhomirov theorem (see, e.g., [28, Theorem 2.4.18]); its integral form has been obtained earlier in [13] under the additional requirements that either X is a separable Banach space or $T(\bar{x})$ is a metrizable compact.

Several counterparts of (1.3) in the inclusion and equality forms are obtained for convex functions without any assumptions imposed on the topological structure of the index set T and on the behavior of f_t with respect to t by using the perturbation

$$T_\varepsilon(\bar{x}) := \{t \in T \mid f_t(\bar{x}) \geq f(\bar{x}) - \varepsilon\}, \quad \varepsilon \geq 0, \quad (1.4)$$

of the active index set first used in [27]. To the best of our knowledge, the most powerful results in this direction are obtained in [12, 17] via the approximate subdifferentials of convex analysis for the functions f_t at \bar{x} with no (semi)continuity requirements on $f_t(\cdot)$. The functions $f_t(\cdot)$ are not even assumed to be convex in [17] but the situation is actually reduced to convexity under the relaxation assumption

$$f^{**}(x) = \sup_{t \in T} f_t^{**}(x)$$

via the biconjugate functions imposed in both papers [12, 17].

When the functions f_t are nonconvex but *uniformly Lipschitzian* around \bar{x} , the inclusion

$$\bar{\partial} f(\bar{x}) \subset \text{cl}^* \text{co} \left[\bigcup \left\{ \bar{\partial}^{[T]} f_t(\bar{x}) \mid t \in T(\bar{x}) \right\} \right] \quad (1.5)$$

for the generalized gradient (or convexified subdifferential) by Clarke of the supremum function f over the metrizable compact T under the u.s.c. assumption on $t \mapsto f_t(x)$ is derived in [6, Theorem 2.8.2] by reducing it to the convex case of [13]. The construction $\bar{\partial}^{[T]} f_t(\bar{x})$ in (1.5) is defined by

$$\bar{\partial}^{[T]} f_t(\bar{x}) := \text{cl}^* \text{co} \left\{ x^* \in X^* \mid \begin{array}{l} \text{there exist } t_k \xrightarrow{T_\varepsilon(\bar{x})} t, x_k \rightarrow \bar{x}, \text{ and } x_k^* \in \bar{\partial} f_{t_k}(x_k) \\ \text{such that } x^* \text{ is a weak}^* \text{ cluster point of } x_k^* \end{array} \right\}. \quad (1.6)$$

The upper estimate (1.5) has been widely applied to various problems in SIP, control theory, etc.; see, e.g., [6, 30, 31] and the references therein. However, we are not familiar with any results in the literature concerning counterparts of (1.5) with no topological requirements on T . This paper provides, in particular, results of this type for arbitrary index sets.

In the other lines of developments, a “fuzzy” upper estimate of the regular/Fréchet subdifferential $\hat{\partial} f(\bar{x})$ of the supremum function f at points of its local minima is derived in

[3] in the case of reflexive spaces. The approach of [3] reduces the situation to a minimization problem with finitely many constraints. We mention also the upper estimate

$$\partial f(\bar{x}) \subset \bigcap_{\varepsilon > 0} \text{clco} \left[\bigcup \left\{ \widehat{\partial} f_t(x) \mid \|x - \bar{x}\| \leq \varepsilon, t \in T_\varepsilon(\bar{x}) \right\} \right] \quad (1.7)$$

of the limiting/Mordukhovich subdifferential of the supremum function obtained in [9] in the case of reflexive spaces by implementing the extended Farkas lemma from [7] for infinite linear constraints. Observe that the set on the right-hand side of (1.7) provides also an upper estimate of the generalized gradient $\bar{\partial} f(\bar{x})$ in the reflexive space setting while it does not exploit the nonconvexity of the limiting subdifferential $\partial f(\bar{x})$.

In this paper we derive new upper estimates for the regular and limiting subdifferentials of the supremum of uniformly Lipschitzian functions defined on Asplund spaces (see Section 2) taken over arbitrary index sets. These results incorporate the nonconvexity of the limiting subdifferential and imply new estimates for the convexified one. Then we employ the obtained subdifferential estimates to establish new necessary optimality conditions for nonsmooth and nonconvex problems of semi-infinite and infinite programming.

It is worth mentioning that the class of *cone-constrained* optimization problems, well recognized in optimization theory and applications (see, e.g., [2, 24]), can be reduced to form (1.2). It allows us to apply the results of this paper to deriving optimality conditions for cone-constrained programs, which will be considered in detail in a separate publication.

The rest of the paper is organized as follows. Section 2 presents some preliminary material from variational analysis and generalized differentiation widely used in formulations and proofs of the main results below. The major goal of Section 3 is to constructively evaluate the limiting subdifferential of supremum functions, which is done via deriving new upper estimates of the regular subdifferential and then by passing to the limit. The most efficient results are obtained for the class of *equicontinuously subdifferentiable* functions introduced in this section. Based on these crucial developments, we derive in Section 4 new upper estimates of the generalized gradient of supremum functions in both cases of arbitrary index sets as well as under some topological structures imposed on them.

Section 5 is devoted to applications of the subdifferential results for supremum functions to deriving necessary optimality conditions for SIP and infinite programs with uniformly Lipschitzian inequality constraints. In this way we obtain new results of both Fritz John and KKT (Karush-Kuhn-Tucker) types; the latter requires an appropriate extension of the Mangasarian-Fromovitz constraint qualification introduced in this paper. The final Section 6 contains two technical lemmas used in many proofs of the paper.

Our notation and terminology are basically standard and conventional in the area of variational analysis and generalized differentiation. As usual, $\|\cdot\|$ stands for the norm of the space in question, $\langle \cdot, \cdot \rangle$ signifies the canonical pairing between a Banach space X and its topologically dual X^* , and the symbol $\xrightarrow{w^*}$ indicates the convergence in the weak* topology of X^* . For any $x \in X$ and $r > 0$ we denote by $\mathcal{B}_r(x)$ the closed ball centered at x with radius r , while \mathcal{B} and \mathcal{B}^* stand for the closed unit balls in X and X^* , respectively.

Given a set $\Omega \subset X$, the notation $\mathcal{R}_+\Omega$ signifies the *conic hull* of Ω , while the symbol $\text{cone } \Omega$ is used for the *convex conic hull* of Ω . Given a set-valued mapping $F: Z \rightrightarrows X^*$ between a Banach space Z and the dual space X^* to some Banach space X , recall that

$$\text{Lim sup}_{z \rightarrow \bar{z}} F(z) := \left\{ x^* \in X^* \mid \exists z_n \rightarrow \bar{z}, \exists x_n^* \xrightarrow{w^*} x^* \text{ with } x_n^* \in F(z_n), n \in \mathbb{N} \right\} \quad (1.8)$$

is the *sequential Painlevé-Kuratowski outer limit* of F as $z \rightarrow \bar{z}$, where $\mathbb{N} := \{1, 2, \dots\}$. We say that F is *weak* outer stable* at \bar{z} if $\text{Lim sup}_{z \rightarrow \bar{z}} F(z) \subset \text{cl}^* F(\bar{z})$. The standard notion of *weak* outer semicontinuity* of F at \bar{z} (not used in this paper) corresponds to the last expression when the weak* operation is omitted on the right-hand side.

2 Basic Definitions and Preliminaries

Unless otherwise stated in this paper, X is an *Asplund space*, i.e., a Banach space where every separable subspace has a separable dual. This is a broad class of space including all reflexive spaces, etc.; see [18] for more details and references. Given an extended-real-valued function $\varphi: X \rightarrow \overline{\mathbb{R}} := (-\infty, \infty]$, we always assume that it is *proper* (i.e., $\varphi \not\equiv \infty$) and use the notation $\text{dom } \varphi := \{x \in X \mid \varphi(x) < \infty\}$ for its *domain*. The *regular/Fréchet subdifferential* of φ at $\bar{x} \in \text{dom } \varphi$ is defined by

$$\widehat{\partial}\varphi(\bar{x}) := \left\{ x^* \in X^* \mid \liminf_{x \rightarrow \bar{x}} \frac{\varphi(x) - \varphi(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq 0 \right\}, \quad (2.1)$$

with $\widehat{\partial}\varphi(\bar{x}) := \emptyset$ if $\bar{x} \notin \text{dom } \varphi$ for convenience. When φ is lower semicontinuous (l.s.c.) around \bar{x} , the sequential outer limit

$$\partial\varphi(\bar{x}) := \text{Lim sup}_{x \xrightarrow{\varphi} \bar{x}} \widehat{\partial}\varphi(x), \quad (2.2)$$

where the symbol $x \xrightarrow{\varphi} \bar{x}$ signifies that $x \rightarrow \bar{x}$ with $\varphi(x) \rightarrow \varphi(\bar{x})$, is known as the *limiting/Mordukhovich subdifferential* of φ at \bar{x} . It is worth mentioning that $\partial\varphi(\bar{x}) \neq \emptyset$ if φ is locally Lipschitzian around \bar{x} . The function φ is *lower regular* at \bar{x} if $\widehat{\partial}\varphi(\bar{x}) = \partial\varphi(\bar{x})$. This property is satisfied for various classes of “nice” functions including convex or smooth ones. Furthermore, for convex functions φ both regular and limiting subdifferentials reduce to the classical subdifferentials of convex analysis. The reader is referred to [18, 19] for a full account regarding the constructions $\widehat{\partial}\varphi(\bar{x})$ and $\partial\varphi(\bar{x})$.

In this paper we also use the *normal cone* to a nonempty set $\Omega \subset X$ that is generated by the limiting subdifferential (2.2) and is defined as

$$N(\bar{x}; \Omega) := \partial\delta(\bar{x}; \Omega), \quad \bar{x} \in \Omega,$$

where $\delta(x; \Omega) := 0$ if $x \in \Omega$ and $\delta(x; \Omega) := \infty$ otherwise; see [18] for more details.

Further, for a function $\varphi: X \rightarrow \overline{\mathbb{R}}$ locally Lipschitzian around $\bar{x} \in \text{dom } \varphi$, the *generalized directional derivative* of φ at \bar{x} in the direction $v \in X$ is defined by

$$\varphi^\circ(\bar{x}; v) := \limsup_{\substack{x \rightarrow \bar{x} \\ t \downarrow 0}} \frac{\varphi(x + tv) - \varphi(x)}{t}. \quad (2.3)$$

Then the *Clarke generalized gradient* (or convexified subdifferential) of φ at \bar{x} is

$$\overline{\partial}\varphi(\bar{x}) := \left\{ x^* \in X^* \mid \langle x^*, v \rangle \leq \varphi^\circ(\bar{x}; v) \text{ for any } v \in X \right\}, \quad (2.4)$$

while $\varphi^\circ(\bar{x}; v) = \max\{\langle x^*, v \rangle \mid x^* \in \overline{\partial}\varphi(\bar{x})\}$ for all $v \in X$; see [6] for more details on these constructions. Recall that φ is *directionally regular* at \bar{x} if

$$\varphi^\circ(\bar{x}; v) = \varphi'(\bar{x}; v) := \lim_{t \downarrow 0} \frac{\varphi(\bar{x} + tv) - \varphi(\bar{x})}{t} \quad \text{for all } v \in X. \quad (2.5)$$

The following relationship [18, Theorem 3.57] between the subdifferentials (2.2) and (2.4) plays an important role in our further considerations:

$$\bar{\partial}\varphi(\bar{x}) = \text{cl}^* \text{co } \partial\varphi(\bar{x}) \quad (2.6)$$

for every locally Lipschitzian functions on Asplund spaces.

Finally in this section, recall some notation concerning the product space \mathbb{R}^T of multipliers $\lambda = (\lambda_t \mid t \in T)$, where T is an arbitrary index set. Let $\tilde{\mathbb{R}}^T$ be the collection of $\lambda \in \mathbb{R}^T$ with $\lambda_t \neq 0$ for *finitely many* $t \in T$. The symbol $\text{supp } \lambda$ stands for the *support* of $\lambda \in \tilde{\mathbb{R}}^T$, which is the set of all t such that $\lambda_t \neq 0$. The *positive cone* and the *generalized simplex* in $\tilde{\mathbb{R}}^T$ are defined, respectively, by

$$\tilde{\mathbb{R}}_+^T := \{ \lambda \in \tilde{\mathbb{R}}^T \mid \lambda_t \geq 0 \text{ for all } t \in T \} \text{ and } \Delta(T) := \left\{ \lambda \in \tilde{\mathbb{R}}_+^T \mid \sum_{t \in T} \lambda_t = 1 \right\}. \quad (2.7)$$

3 The Limiting Subdifferential of Supremum Functions

Given an Asplund space X and an arbitrary nonempty index set T , the main goal of this section is to evaluate the limiting subdifferential (2.2) of the supremum function

$$f(x) = \sup \{ f_t(x) \mid t \in T \} \quad (3.1)$$

over the family of proper functions $f_t : X \rightarrow \overline{\mathbb{R}}$, $t \in T$, that are assumed in what follows to be *uniformly locally Lipschitzian* around $\bar{x} \in \text{dom } f$ with some rank $K > 0$. This means the existence of a positive number δ such that

$$|f_t(x) - f_t(y)| \leq K \|x - y\| \quad \text{for all } x, y \in \mathbb{B}_\delta(\bar{x}), \quad t \in T. \quad (3.2)$$

It is easy to check that (3.2) implies that the supremum function (3.1) is locally Lipschitzian around \bar{x} with rank K . As in Section 1, we consider the set $T_\varepsilon(\bar{x})$ of ε -active indices (1.4) at \bar{x} with $T(\bar{x}) := T_0(\bar{x})$ and observe that $T_\varepsilon(\bar{x}) \neq \emptyset$ for $\varepsilon > 0$.

Prior to pointwise evaluating the limiting subdifferential of the supremum function f in (3.1), we establish some “fuzzy” estimates of regular subgradients of f , which are of a certain independent interest while playing a preliminary role in our consideration. The proof of the next theorem follows the main idea of [3, Theorem 3.18], where it is used for the case of the supremum function f attaining its local minimum at \bar{x} in reflexive spaces.

Theorem 3.1 (fuzzy estimates of regular subgradients of supremum functions).

Let V^* be a weak* neighborhood of the origin in X^* . Then the following assertions hold:

(i) For each $x^* \in \hat{\partial}f(\bar{x})$ and $\varepsilon > 0$ there exist $\hat{x} \in \mathbb{B}_\varepsilon(\bar{x})$ and $\lambda \in \Lambda_\varepsilon(\hat{x}) := \{ \lambda \in \Delta(T_\varepsilon(\bar{x})) \mid f_t(\hat{x}) = f_s(\hat{x}) \text{ for all } t, s \in \text{supp}(\lambda) \}$ such that

$$x^* \in \sum_{t \in T_\varepsilon(\bar{x})} \lambda_t \partial f_t(\hat{x}) + V^*. \quad (3.3)$$

(ii) For each $x^* \in \hat{\partial}f(\bar{x})$ and $\varepsilon > 0$ there exist $\lambda \in \Delta(T_\varepsilon(\bar{x}))$ and $\hat{x}_t \in \mathbb{B}_\varepsilon(\bar{x})$ for all $t \in T_\varepsilon(\bar{x})$ such that

$$x^* \in \sum_{t \in T_\varepsilon(\bar{x})} \lambda_t \hat{\partial}f_t(\hat{x}_t) + V^*. \quad (3.4)$$

Proof. To justify (i), fix arbitrary $x^* \in \widehat{\partial}f(\bar{x})$ and then find $n \in \mathbb{N}$, $\gamma > 0$, and $x_k \in X$ as $k = 1, \dots, n$ satisfying the inclusion

$$\gamma \mathbb{B}^* + \text{span} \{x_1, \dots, x_n\}^\perp \subset \frac{1}{2} V^*. \quad (3.5)$$

Without loss of generality we assume that V^* is convex and that $2\varepsilon \leq \gamma$ and then define $L := \text{span} \{x_1, \dots, x_n\}$. Since $x^* \in \widehat{\partial}f(\bar{x})$, there is $\delta > 0$ with

$$\delta < \frac{1}{2}, \quad 2(K+1)\delta \leq \varepsilon, \quad \text{and} \quad \frac{\varepsilon + 2(K+1)\delta}{1-2\delta} \leq \gamma \quad (3.6)$$

such that the functions f_t are uniformly Lipschitzian with rank K in $\mathbb{B}_{2\delta}(\bar{x})$ and that

$$f(x) - f(\bar{x}) - \langle x^*, x - \bar{x} \rangle \geq -\varepsilon \|x - \bar{x}\| \quad \text{for all } x \in \mathbb{B}_\delta(\bar{x}). \quad (3.7)$$

Consider now the following constrained optimization problem:

$$\begin{cases} \text{minimize} & y - \langle x^*, x - \bar{x} \rangle + \varepsilon \|x - \bar{x}\| - f(\bar{x}) \\ \text{subject to} & f_t(x) - y \leq 0, \quad t \in T, \quad (x, y) \in \mathbb{B}_\delta(\bar{x}) \times \mathbb{R}. \end{cases} \quad (3.8)$$

It follows from (3.7) that $(\bar{x}, f(\bar{x}))$ is a local minimizer of (3.8). Define $g : X \times \mathbb{R} \rightarrow \overline{\mathbb{R}}$ by

$$g(x, y) := y - \langle x^*, x - \bar{x} \rangle + \varepsilon \|x - \bar{x}\| - f(\bar{x}) + \delta((x, y); \Omega)$$

with $\Omega := (L \cap \mathbb{B}_\delta(\bar{x})) \times [f(\bar{x}) - 1, f(\bar{x}) + 1]$ and then a family of functions $g_t : X \times \mathbb{R} \rightarrow \overline{\mathbb{R}}$ by $g_t(x, y) := f_t(x) - y$ for all $t \in T$. Due to (3.7) we have the inclusion

$$\{(x, y) \in X \times \mathbb{R} \mid g(x, y) + \delta^2 \leq 0\} \subset \bigcup_{t \in T} \{(x, y) \in \text{int } \mathbb{B}_{2\delta}(\bar{x}) \times \mathbb{R} \mid g_t(x, y) > 0\}.$$

Since the left-hand side of the latter inclusion is closed and bounded in the finite-dimensional space $L \times \mathbb{R}$, it is compact. Furthermore, each set $\{(x, y) \in \text{int } \mathbb{B}_{2\delta}(\bar{x}) \times \mathbb{R} \mid g_t(x, y) > 0\}$ is open due to the Lipschitz continuity of the functions f_t on $\mathbb{B}_{2\delta}(\bar{x})$. Thus there exists a *finite subset* S of T such that

$$\{(x, y) \in X \times \mathbb{R} \mid g(x, y) + \delta^2 \leq 0\} \subset \bigcup_{s \in S} \{(x, y) \in \text{int } \mathbb{B}_{2\delta}(\bar{x}) \times \mathbb{R} \mid g_s(x, y) > 0\},$$

which implies that $(\bar{x}, f(\bar{x}))$ is a δ^2 -*optimal solution* to the following optimization problem with *finite many* inequality constraints:

$$\begin{cases} \text{minimize} & g(x, y) \\ \text{subject to} & g_s(x, y) \leq 0, \quad s \in S, \quad (x, y) \in \mathbb{B}_\delta(\bar{x}) \times \mathbb{R}. \end{cases} \quad (3.9)$$

Note further that $\partial g_s(x, y) \subset X^* \times \{-1\}$ for all $s \in S$ and that $N((x, y); \mathbb{B}_\delta(\bar{x}) \times \mathbb{R}) \subset N(x; \mathbb{B}_\delta(\bar{x})) \times \{0\}$. This ensures the implication

$$\left[0 \in \sum_{s \in S(x, y)} \lambda_s \partial g_s(x, y) + N((x, y); \mathbb{B}_\delta(\bar{x}) \times \mathbb{R}) \right] \implies [\lambda_s = 0, s \in S(x, y)]$$

whenever $\lambda_s \geq 0$ for all $s \in S(x, y) := \{s \in S \mid g_s(x, y) = 0\} = \{s \in S \mid f_s(x) = y\}$. Applying now the *strong suboptimality conditions* from [18, Theorem 5.30] to problem (3.9),

in terms of the limiting/basic generalized differential constructions, gives us $(\hat{x}, \hat{y}) \in X \times \mathbb{R}$, $(\hat{x}^*, 1) \in \partial g(\hat{x}, \hat{y})$, $(x_s^*, -1) \in \partial g_s(\hat{x}, \hat{y})$ as $s \in S$, $(u^*, 0) \in N((\hat{x}, \hat{y}); \mathcal{B}_\delta(\bar{x}) \times \mathbb{R})$, and $\lambda \in \mathbb{R}_+^S$ such that $\|\hat{x} - \bar{x}\| + |\hat{y} - f(\bar{x})| \leq \frac{\delta}{2}$ and

$$\left\| (\hat{x}^*, 1) + \sum_{s \in S(\hat{x}, \hat{y})} \lambda_s (x_s^*, -1) + (u^*, 0) \right\| \leq 2\delta. \quad (3.10)$$

Since $\hat{x} \in \mathcal{B}_{\frac{\delta}{2}}(\bar{x}) \subset \text{int } \mathcal{B}_\delta(\bar{x})$, we have $u^* \in N(\hat{x}; \mathcal{B}_\delta(\bar{x})) = \{0\}$. Moreover, it follows from the convexity of the function g that

$$\begin{aligned} \hat{x}^* &\in -x^* + \varepsilon \mathcal{B}^* + N(\hat{x}; L \cap \mathcal{B}_\delta(\bar{x})) \subset -x^* + \varepsilon \mathcal{B}^* + N(\hat{x}; L) + N(\hat{x}; \mathcal{B}_\delta(\bar{x})) \\ &\subset -x^* + \varepsilon \mathcal{B}^* + L^\perp. \end{aligned}$$

Thus inequality (3.10) implies that $\|\sum_{s \in S(\hat{x}, \hat{y})} \lambda_s - 1\| \leq 2\delta$ and that

$$x^* \in \sum_{s \in S(\hat{x}, \hat{y})} \lambda_s x_s^* + (\varepsilon + 2\delta) \mathcal{B}^* + L^\perp. \quad (3.11)$$

By $\delta < \frac{1}{2}$ we have $\sum_{s \in S(\hat{x}, \hat{y})} \lambda_s > 0$. Let us further define

$$\lambda'_s := \lambda_s \left[\sum_{t \in S(\hat{x}, \hat{y})} \lambda_t \right]^{-1} \text{ for all } s \in S(\hat{x}, \hat{y})$$

and get $\sum_{s \in S(\hat{x}, \hat{y})} \lambda'_s = 1$. Since $\|x^*\| \leq K$, it follows from (3.5), (3.6), and (3.11) that

$$\begin{aligned} x^* &\in \frac{1}{\sum_{s \in S(\hat{x}, \hat{y})} \lambda_s} x^* + \left| 1 - \frac{1}{\sum_{s \in S(\hat{x}, \hat{y})} \lambda_s} \right| \|x^*\| \mathcal{B}^* \\ &\subset \sum_{s \in S(\hat{x}, \hat{y})} \lambda'_s x_s^* + \frac{\varepsilon + 2\delta}{\sum_{s \in S(\hat{x}, \hat{y})} \lambda_s} \mathcal{B}^* + L^\perp + \frac{\left| \sum_{s \in S(\hat{x}, \hat{y})} \lambda_s - 1 \right|}{\sum_{s \in S(\hat{x}, \hat{y})} \lambda_s} K \mathcal{B}^* \\ &\subset \sum_{s \in S(\hat{x}, \hat{y})} \lambda'_s x_s^* + \frac{\varepsilon + 2\delta}{1 - 2\delta} \mathcal{B}^* + L^\perp + \frac{2\delta}{1 - 2\delta} K \mathcal{B}^* \\ &\subset \sum_{s \in S(\hat{x}, \hat{y})} \lambda'_s x_s^* + \frac{\varepsilon + 2(K + 1)\delta}{1 - 2\delta} \mathcal{B}^* + L^\perp \\ &\subset \sum_{s \in S(\hat{x}, \hat{y})} \lambda'_s x_s^* + \gamma \mathcal{B}^* + L^\perp \subset \sum_{s \in S(\hat{x}, \hat{y})} \lambda'_s x_s^* + V^*. \end{aligned} \quad (3.12)$$

Now we claim that $S(\hat{x}, \hat{y}) \subset T_\varepsilon(\bar{x})$. Indeed, it follows from (3.2) that

$$f_s(\bar{x}) \geq f_s(\hat{x}) - K \|\bar{x} - \hat{x}\| \geq \hat{y} - K \frac{\delta}{2} \geq f(\bar{x}) - \frac{\delta}{2} - K \frac{\delta}{2} \geq f(\bar{x}) - \varepsilon$$

for each $s \in S(\hat{x}, \hat{y})$, which implies that $s \in T_\varepsilon(\bar{x})$. It gives $S(\hat{x}, \hat{y}) \subset T_\varepsilon(\bar{x})$ and ensures by (3.12) that (3.3) holds and thus completes the proof of assertion (i) of the theorem.

To justify (ii), we get $x_s^* \in \partial f_s(\hat{x})$ for $s \in S(\hat{x}, \hat{y})$ from the proof of (i) and then by (2.2) find $x_s \in X$ and $\hat{x}_s^* \in \hat{\partial} f_s(x_s)$ such that $\|x_s - \hat{x}\| \leq \delta$ and $x_s^* \in \hat{x}_s^* + V^*$. This gives

$$\begin{aligned} x^* &\in \sum_{s \in S(\hat{x}, \hat{y})} \lambda'_s x_s^* + V^* \subset \sum_{s \in S(\hat{x}, \hat{y})} \lambda'_s \hat{x}_s^* + \sum_{s \in S(\hat{x}, \hat{y})} \lambda'_s V^* + V^* \\ &\subset \sum_{s \in S(\hat{x}, \hat{y})} \lambda'_s \hat{x}_s^* + V^* + V^* \subset \sum_{s \in S(\hat{x}, \hat{y})} \lambda'_s \hat{x}_s^* + 2V^* \end{aligned}$$

by the convexity of V^* . Since $\|x_s - \bar{x}\| \leq \|x_s - \hat{x}\| + \|\hat{x} - \bar{x}\| \leq \delta + \delta \leq \varepsilon$ and $S(\hat{x}, \hat{y}) \subset T_\varepsilon(\bar{x})$, we arrive at (3.4) and complete the proof of the theorem. \triangle

Note that, in contrast to [3, Theorem 3.18], our analysis works for the general class of supremum functions over arbitrary many uniformly Lipschitzian ones. Furthermore, utilizing the well-developed calculus of limiting subgradients in the proof of Theorem 3.1 helps us get information about the active set, while the aforementioned result of [3] does not. This fact is of great significance, since we can restrict the index T to the smaller subset $T_\varepsilon(\bar{x})$ in estimating regular subgradients of supremum functions. This allows us to efficiently evaluate the limiting subdifferential of (3.1) in the next theorem.

Theorem 3.2 (pointwise estimates of limiting subgradients of supremum functions). *Given f_t in (3.1), define $C : \mathbb{R}_+ \rightrightarrows X^*$ by*

$$C(\varepsilon) := \bigcup \left\{ \sum_{t \in T_\varepsilon(\bar{x})} \lambda_t \partial f_t(x) \mid x \in \mathcal{B}_\varepsilon(\bar{x}), \lambda \in \Lambda_\varepsilon(x) \right\} \text{ for all } \varepsilon \geq 0, \quad (3.13)$$

where $\Lambda_\varepsilon(x)$ is defined in (i) of Theorem 3.1 with $\varepsilon > 0$ and $\Lambda_0(x) := \Delta(T(x))$. The following assertions hold:

(i) *The limiting subdifferential of the supremum function f in (3.1) at \bar{x} is estimated by*

$$\partial f(\bar{x}) \subset \bigcap_{\varepsilon > 0} \text{cl}^* C(\varepsilon). \quad (3.14)$$

(ii) *If the mapping (3.13) is weak* outer stable at zero, then*

$$\partial f(\bar{x}) \subset \text{cl}^* \left[\bigcup \left\{ \sum_{t \in T(\bar{x})} \lambda_t \partial f_t(\bar{x}) \mid \lambda \in \Delta(T(\bar{x})) \right\} \right]. \quad (3.15)$$

(iii) *If in addition the space X is reflexive and all the functions f_t , $t \in T(\bar{x})$, are lower regular at \bar{x} , then f is also lower regular at \bar{x} and (3.15) holds as equality.*

Proof. To justify (i), let V^* be an arbitrary weak* neighborhood of the origin in X^* and pick any $x^* \in \partial f(\bar{x})$. By definition (2.2) of the limiting subdifferential there are sequences $x_n \rightarrow \bar{x}$ and $x_n^* \in \widehat{\partial} f(x_n)$ satisfying $x_n^* \xrightarrow{w^*} x^*$. Let us find U^* , another weak* neighborhood of the origin in X^* , with $\text{cl}^* U^* \subset V^*$. Choose a sequence $\delta_n \downarrow 0$ satisfying $\delta_n > \|x_n - \bar{x}\|$ for all $n \in \mathbb{N}$. It follows from Theorem 3.1 that there exist $\hat{x}_n \in \mathcal{B}_{\delta_n}(x_n)$ and $\lambda_n \in \Delta(T_{\delta_n}(x_n))$ with $f_t(\hat{x}_n) = f_s(\hat{x}_n)$ for all $t, s \in \text{supp}(\lambda_n)$ such that

$$x_n^* \in \sum_{t \in T_{\delta_n}(x_n)} \lambda_{nt} \partial f_t(\hat{x}_n) + U^*. \quad (3.16)$$

Note further that for large n we have

$$\begin{aligned} f_t(\bar{x}) &\geq f_t(x_n) - K\|x_n - \bar{x}\| \geq f(x_n) - \delta_n - K\|x_n - \bar{x}\| \\ &\geq f(\bar{x}) - 2K\|x_n - \bar{x}\| - \delta_n \geq f(\bar{x}) - (2K + 1)\delta_n \end{aligned}$$

whenever $t \in T_{\delta_n}(x_n)$. Defining $\varepsilon_n := \max\{2\delta_n, (2K + 1)\delta_n\}$ and using the above inequalities give us the inclusions $\hat{x}_n \in \mathcal{B}_{\varepsilon_n}(\bar{x})$ and $T_{\delta_n}(x_n) \subset T_{\varepsilon_n}(\bar{x})$. This implies that $\lambda_n \in \Lambda_{\varepsilon_n}(\hat{x}_n)$ and $x_n^* \in C(\varepsilon_n) + U^*$ by (3.16). It follows that there are $\hat{x}_n^* \in C(\varepsilon_n)$ and $u_n^* \in U^*$ satisfying

$x_n^* = \widehat{x}_n^* + u_n^*$. Observe further that $C(\varepsilon_n)$ is contained in KB^* when n is sufficiently large. Then the sequence $\{\widehat{x}_n^*\}$ is bounded in X^* and hence contains a weak* convergent subsequence, since X is Asplund. Assuming without loss generality of it itself converges to some $x^* \in X^*$, we get $u_n^* \xrightarrow{w^*} x^* - \widehat{x}^* \in \text{cl}^*U^*$ and therefore

$$x^* = \widehat{x}^* + (x^* - \widehat{x}^*) \in [\text{Lim sup}_{\varepsilon \downarrow 0} C(\varepsilon)] + \text{cl}^*U^* \subset [\text{Lim sup}_{\varepsilon \downarrow 0} C(\varepsilon)] + V^*$$

for any V^* , which implies that x^* belongs to the set $\text{cl}^*[\text{Lim sup}_{\varepsilon \downarrow 0} C(\varepsilon)]$. By the auxiliary result of Lemma 6.2, proved in Section 6, the latter conclusion ensures (3.14) and thus completes the proof of (i). Assertion (ii) follows from (3.14) by the assumed weak* outer stability of mapping (3.13).

It remains to prove (iii) under the additional assumptions made therein. Take any $x^* \in C(0)$ and find $\lambda \in \Delta(T(\bar{x}))$ such that

$$x^* \in \sum_{t \in T(\bar{x})} \lambda_t \partial f_t(\bar{x}) = \sum_{t \in T(\bar{x})} \lambda_t \widehat{\partial} f_t(\bar{x}).$$

Observe the obvious inclusions

$$\sum_{t \in T(\bar{x})} \lambda_t \widehat{\partial} f_t(\bar{x}) \subset \widehat{\partial} \left(\sum_{t \in T(\bar{x})} \lambda_t f_t \right) (\bar{x}) \quad \text{and} \quad \widehat{\partial} \left(\sum_{t \in T(\bar{x})} \lambda_t f_t \right) (\bar{x}) \subset \widehat{\partial} f(\bar{x})$$

following from the facts that $\sum_{t \in T(\bar{x})} \lambda_t f_t(\bar{x}) = f(\bar{x})$ and $\sum_{t \in T(\bar{x})} \lambda_t f_t(x) \leq f(x)$ for all $x \in X$. Thus we get $C(0) \subset \widehat{\partial} f(\bar{x})$, and it follows from (3.15) that

$$\widehat{\partial} f(\bar{x}) \subset \partial f(\bar{x}) \subset \text{cl}^*C(0) \subset \text{cl}^*\widehat{\partial} f(\bar{x}) = \widehat{\partial} f(\bar{x}),$$

where the last equality holds due to the reflexivity of the space X . This justifies the equality in (3.15) and completes the proof of the theorem. \triangle

Now let us construct a finite-dimensional example showing that the set on the right-hand side of (3.14) is generally *nonconvex*. Furthermore, in this example the *equality* holds in (3.14) and the usage of the perturbed set $\Lambda_\varepsilon(x)$ in (3.13) is essential.

Example 3.3 (nonconvex estimate of the limiting subdifferential). Let $X = \mathbb{R}^2$ and $T = (0, 1) \subset \mathbb{R}$, and let the supremum function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by

$$f(x) := \sup \left\{ tx_1^3 - \frac{1}{(t+1)^2} |x_2| + t^3 - 1 \mid t \in T \right\}.$$

Denote $f_t(x) := tx_1^3 - \frac{1}{(t+1)^2} |x_2| + t^3 - 1$ for all $t \in T$ and let $\bar{x} = (0, 0)$. It is easy to check that the functions f_t are uniformly Lipschitz continuous around \bar{x} , that $T(\bar{x}) = \emptyset$, and that $T_\varepsilon(\bar{x}) = \{t \in T \mid t \geq \sqrt[3]{1 - \varepsilon}\}$ for all $\varepsilon > 0$. Pick any $x^* \in C(\varepsilon)$ and find by (3.13) elements $x \in \mathbb{B}_\varepsilon(\bar{x})$ and $\lambda \in \Delta(T_\varepsilon(\bar{x}))$ such that

$$x^* \in \sum_{t \in T_\varepsilon(\bar{x})} \lambda_t \partial f_t(x) \quad \text{and} \quad f_t(x) = f_s(x) \quad \text{for all } t, s \in \text{supp}(\lambda).$$

If $f_t(x) = f_s(x)$ and $t \neq s$, then we get from the above that

$$tx_1^3 - \frac{1}{(t+1)^2} |x_2| + t^3 - 1 = sx_1^3 - \frac{1}{(s+1)^2} |x_2| + s^3 - 1,$$

which implies in turn the equation

$$x_1^3 - \frac{t+s+2}{(t+1)^2(s+1)^2}|x_2| = -t^2 + ts - s^2.$$

When ε is sufficiently small, this equation has no solution, since its left-hand side is close to 0 while the other side is close to -1 for $x \in \mathcal{B}_\varepsilon(\bar{x})$ and $t \in T_\varepsilon(\bar{x})$. It follows therefore that $C(\varepsilon) = \bigcup\{\partial f_t(x) \mid t \in T_\varepsilon(\bar{x}), x \in \mathcal{B}_\varepsilon(\bar{x})\}$ for $\varepsilon > 0$ sufficiently small. Note further that $\partial f_t(x) \subset \{(3tx_1^2, \frac{1}{(t+1)^2}), (3tx_1^2, -\frac{1}{(t+1)^2})\}$, where the equality holds for $x_2 = 0$. Applying again Lemma 6.2 gives us the expression

$$\bigcap_{\varepsilon>0} \text{cl } C(\varepsilon) = \text{cl} [\text{Lim sup}_{\varepsilon \downarrow 0} C(\varepsilon)] = \left\{ \left(0, \frac{1}{4}\right), \left(0, -\frac{1}{4}\right) \right\},$$

which is not a convex set. We have furthermore that $f(x) = x_1^3 - \frac{1}{4}|x_2|$ for all x around \bar{x} . Hence the equality holds in (3.14), and the subgradient set $\partial f(\bar{x})$ is nonconvex as well.

Next we introduce a new property for infinite families of functions, which makes them behave similarly to collections of finitely many Lipschitzian functions. Then we show that it allows us to simplify the evaluations of the limiting subdifferential in Theorem 3.2.

Definition 3.4 (equicontinuous subdifferentiability). *The functions $f_t: X \rightarrow \overline{\mathbb{R}}$ as $t \in T$ are called EQUICONTINUOUSLY SUBDIFFERENTIABLE at \bar{x} if for any weak* neighborhood V^* of the origin in X^* there is $\varepsilon > 0$ such that*

$$\partial f_t(x) \subset \partial f_t(\bar{x}) + V^* \text{ for all } t \in T_\varepsilon(\bar{x}), x \in \mathcal{B}_\varepsilon(\bar{x}). \quad (3.17)$$

The following proposition shows that property (3.17) is automatic for Lipschitzian functions when either the index set T is finite, or f_t are *uniformly strictly differentiable* at \bar{x} .

Proposition 3.5 (sufficient conditions for equicontinuous subdifferentiability). *The functions $f_t(x)$, $t \in T$, are equicontinuously subdifferentiable at \bar{x} if:*

- (i) *either the index set T is finite and f_t are locally Lipschitzian around \bar{x} for all $t \in T$,*
- (ii) *or the functions f_t are uniformly strictly differentiable at \bar{x} , i.e., they are Fréchet differentiable at this point and*

$$r(\eta) := \sup_{t \in T} \sup_{\substack{x, x' \in \mathcal{B}_\eta(\bar{x}) \\ x \neq x'}} \frac{|f_t(x) - f_t(x') - \langle \nabla f_t(\bar{x}), x - x' \rangle|}{\|x - x'\|} \rightarrow 0 \text{ as } \eta \downarrow 0. \quad (3.18)$$

Proof. To justify assertion (i), consider finitely many functions f_t locally Lipschitzian around \bar{x} ; they are obviously uniformly Lipschitzian around \bar{x} with some rank K . Take any weak* neighborhood V^* of the origin and assume that V^* is convex. Suppose on the contrary that f_t are not equicontinuously subdifferentiable at \bar{x} . This together with definition (2.2) implies that there are sequences $\varepsilon_n \downarrow 0$, $x_n \in \mathcal{B}_{\varepsilon_n}(\bar{x})$, $u_n \in \mathcal{B}_{\varepsilon_n}(x_n)$, $t_n \in T_{\varepsilon_n}(\bar{x})$, $x_n^* \in \partial f_{t_n}(x_n)$, and $u_n^* \in \hat{\partial} f_{t_n}(u_n)$ such that $x_n^* \notin \partial f_{t_n}(\bar{x}) + V^*$ and $x_n^* \in u_n^* + \frac{V^*}{2}$. Since T is finite, we find a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ whose elements are constant, say \bar{t} . When k is sufficiently large, the norms $\|u_{n_k}^*\|$ are bounded by K . Hence there is a subsequence (not relabeling) of $\{u_{n_k}^*\}$ weak* converging to u^* . By definition (2.2) of the limiting subdifferential we get that $u^* \in \partial f_{\bar{t}}(\bar{x})$. This shows that

$$x_{n_k}^* \in u_{n_k}^* + \frac{V^*}{2} \subset u^* + \frac{V^*}{2} + \frac{V^*}{2} \subset \partial f_{\bar{t}}(\bar{x}) + V^* = \partial f_{t_{n_k}}(\bar{x}) + V^*$$

as k is sufficiently large, which leads us to a contradiction and thus justifies (i).

To prove (ii), fix any $\delta > 0$ such that $\delta\mathcal{B}^* \subset V^*$, where V^* is supposed to be convex. Observe from (3.18) that each function f_t is strictly differentiable at \bar{x} . Thus $\partial f_t(\bar{x}) = \{\nabla f_t(\bar{x})\}$ for each $t \in T$. Moreover, (3.18) allows us to find $\eta > 0$ such that $r(\eta) < \frac{\delta}{2}$. Define $\varepsilon := \frac{\eta}{2}$ and take any $x \in \mathcal{B}_\varepsilon(\bar{x})$ and $x_t^* \in \partial f_t(x)$ for some $t \in T_\varepsilon(\bar{x})$. Then there are $x_t \in \mathcal{B}_\varepsilon(x)$, $\hat{x}_t^* \in \hat{\partial} f_t(x_t)$, and $\varepsilon_t \in (0, \varepsilon)$ such that $x_t^* \in \hat{x}_t^* + V^*$ and that

$$f_t(u) - f_t(x_t) \geq \langle \hat{x}_t^*, u - x_t \rangle - \frac{\delta}{2} \|u - x_t\| \quad \text{for all } u \in \mathcal{B}_{\varepsilon_t}(x_t).$$

Employing (3.18) again, we get the relationship

$$f_t(u) - f_t(x_t) \leq \langle \nabla f_t(\bar{x}), u - x_t \rangle + r(\eta) \|u - x_t\| \quad \text{for all } u \in \mathcal{B}_{\varepsilon_t}(x_t) \subset \mathcal{B}_{\varepsilon+\varepsilon_t}(\bar{x}) \subset \mathcal{B}_\eta(\bar{x}).$$

Putting all the above inequalities together gives us

$$\langle \hat{x}_t^* - \nabla f_t(\bar{x}), u - x_t \rangle \leq \left(r(\eta) + \frac{\delta}{2} \right) \|u - x_t\| \leq \delta \|u - x_t\|$$

for all $u \in \mathcal{B}_{\varepsilon_t}(x_t)$, which implies in turn that $\|\hat{x}_t^* - \nabla f_t(\bar{x})\| \leq \delta$. It follows that

$$x_t^* \in \hat{x}_t^* + V^* \subset \nabla f_t(\bar{x}) + \delta\mathcal{B}^* + V^* \subset \partial f_t(\bar{x}) + V^* + V^*.$$

By the convexity of V^* we conclude that $\partial f_t(x) \subset \partial f(\bar{x}) + 2V^*$ for all $x \in \mathcal{B}_\varepsilon(\bar{x})$ and thus complete the proof of the proposition. \triangle

The uniform strict differentiability property of infinite families of functions f_t is introduced in our paper [20] to derive necessary optimality conditions for nonlinear semi-infinite and infinite programs. This property can be treated as a natural extension of the strict differentiability of finitely many functions at the reference point. It is more general than the equicontinuity of the gradients $\nabla f_t(x)$ introduced in [23]; see [20] for more discussions.

Corollary 3.6 (enhanced estimates of limiting subgradients of supremum functions under equicontinuous subdifferentiability). *Assuming the equicontinuous subdifferentiability of the functions f_t at \bar{x} in the setting of Theorem 3.2, we have*

$$\partial f(\bar{x}) \subset \bigcap_{\varepsilon > 0} \text{cl}^* D(\varepsilon), \quad (3.19)$$

where the mapping $D : \mathbb{R}_+ \rightrightarrows X^*$ is defined by

$$D(\varepsilon) := \bigcup \left\{ \sum_{t \in T_\varepsilon(\bar{x})} \lambda_t \partial f_t(\bar{x}) \mid \lambda \in \Delta(T_\varepsilon(\bar{x})) \right\} \quad \text{for all } \varepsilon \geq 0. \quad (3.20)$$

If in addition the mapping D in (3.20) is weak* outer stable at zero, then

$$\partial f(\bar{x}) \subset \text{cl}^* \left[\bigcup \left\{ \sum_{t \in T(\bar{x})} \lambda_t \partial f_t(\bar{x}) \mid \lambda \in \Delta(T(\bar{x})) \right\} \right]. \quad (3.21)$$

Proof. Let V^* be an arbitrary convex weak* neighborhood of $0 \in X^*$. Since f_t are equicontinuously subdifferentiable at \bar{x} , there is $\bar{\varepsilon} > 0$ such that inclusion (3.17) holds for all $\varepsilon < \bar{\varepsilon}$. Employing Theorem 3.2 and Lemma 6.2, we get inclusion (3.19) by showing that

$$\text{Lim sup}_{\varepsilon \downarrow 0} C(\varepsilon) \subset \text{cl}^* \text{Lim sup}_{\varepsilon \downarrow 0} D(\varepsilon). \quad (3.22)$$

To proceed with proving (3.22), pick any x^* from the left-hand side of (3.22) and take $\delta > 0$. Then there are $\varepsilon_n \downarrow 0$, $x_n \in B_{\varepsilon_n}(\bar{x})$, $\lambda_n \in \Lambda_{\varepsilon_n}(x_n)$, and $x_n^* \xrightarrow{w^*} x^*$ such that

$$x_n^* \in \sum_{t \in T_{\varepsilon_n}(\bar{x})} \lambda_{n_t} \partial f_t(x_n).$$

Since f_t are equicontinuously subdifferentiable at \bar{x} and V^* is convex, the latter implies that

$$x_n^* \in \sum_{t \in T_{\varepsilon_n}(\bar{x})} \lambda_{n_t} (\partial f_t(\bar{x}) + V^*) \subset \sum_{t \in T_{\varepsilon_n}(\bar{x})} \lambda_{n_t} \partial f_t(\bar{x}) + \lambda_{n_t} V^* \subset D(\varepsilon_n) + V^*$$

for all $n \in \mathbb{N}$ sufficiently large. Thus there is $u_n^* \in D(\varepsilon_n)$ such that $x_n^* \in u_n^* + V^*$. Since the sets $D(\varepsilon_n)$ are uniformly bounded in X^* , we have by passing to a subsequence that $u_n^* \xrightarrow{w^*} u^* \in \text{Lim sup}_{\varepsilon \downarrow 0} D(\varepsilon)$. It ensures therefore the inclusions

$$x^* \in x_n^* + V^* \subset u_n^* + V^* + V^* \subset u^* + V^* + V^* + V^* \subset \text{Lim sup}_{\varepsilon \downarrow 0} D(\varepsilon) + 3V^*.$$

Since V^* was chosen arbitrarily, this means that x^* belongs to the right-hand side of (3.22), which justifies inclusion (3.19). The rest of the proof is similar to Theorem 3.2. \triangle

The next corollary provides a verifiable sufficient condition, which ensures the weak* outer stability of mapping (3.20) at zero and allows us to eliminate the weak* closure in the subdifferential upper estimate (3.21).

Corollary 3.7 (subdifferential estimate with no weak* closure). *Let the functions f_t in (3.1) are equicontinuously subdifferentiable at \bar{x} , and let the set*

$$\bigcup \left\{ \sum_{t \in T} \lambda_t (\partial f_t(\bar{x}), f_t(\bar{x})) \mid \lambda \in \Delta(T) \right\} \quad (3.23)$$

be weak closed in $X^* \times \mathbb{R}$. Then we have the estimate*

$$\partial f(\bar{x}) \subset \bigcup \left\{ \sum_{t \in T(\bar{x})} \lambda_t \partial f_t(\bar{x}) \mid \lambda \in \Delta(T(\bar{x})) \right\}. \quad (3.24)$$

Proof. To justify (3.24), it is sufficient to prove by inclusion (3.21) of Corollary 3.6 that mapping (3.20) is weak* outer stable at zero and that the set $D(0)$ is weak* closed under the assumption made. Pick any $x^* \in \text{cl}^*[\text{Lim sup}_{\varepsilon \downarrow 0} D(\varepsilon)]$ and employ Lemma 6.2. Given $\varepsilon > 0$, this allows us to find a net $(\lambda_\nu)_{\nu \in \mathcal{N}} \subset \Delta(T_\varepsilon(\bar{x}))$ and subgradients $x_{\nu_t}^* \in \partial f_t(\bar{x})$ for each $\nu \in \mathcal{N}$ and $t \in T$ such that

$$x^* = w^* - \lim_{\nu} \sum_{t \in T_\varepsilon(\bar{x})} \lambda_{\nu_t} x_{\nu_t}^*. \quad (3.25)$$

Since $\text{supp}(\lambda_\nu) \subset T_\varepsilon(\bar{x})$, we observe that

$$f(\bar{x}) \geq \sum_{t \in T_\varepsilon(\bar{x})} \lambda_{\nu_t} f_t(\bar{x}) \geq \sum_{t \in T_\varepsilon(\bar{x})} \lambda_{\nu_t} (f(\bar{x}) - \varepsilon) = f(\bar{x}) - \varepsilon.$$

The latter implies by (3.25) that

$$(x^*, f(\bar{x})) \in \text{cl}^* \left[\bigcup \left\{ \sum_{t \in T} \lambda_t (\partial f_t(\bar{x}), f_t(\bar{x})) \mid \lambda \in \Delta(T) \right\} \right] + \{0\} \times [0, \varepsilon].$$

Letting above $\varepsilon \downarrow 0$, we obtain the relationships

$$(x^*, f(\bar{x})) \in \text{cl}^* \left[\bigcup \left\{ \sum_{t \in T} \lambda_t (\partial f_t(\bar{x}), f_t(\bar{x})) \mid \lambda \in \Delta(T) \right\} \right] = \bigcup \left\{ \sum_{t \in T} \lambda_t (\partial f_t(\bar{x}), f_t(\bar{x})) \mid \lambda \in \Delta(T) \right\}.$$

This ensures the existence of $\lambda \in \Delta(T)$ such that $(x^*, f(\bar{x})) \in \sum_{t \in T} \lambda (\partial f_t(\bar{x}), f_t(\bar{x}))$, which obviously implies that $0 = \sum_{t \in T} \lambda_t (f_t(\bar{x}) - f(\bar{x}))$. Thus we have that $\text{supp}(\lambda) \subset T(\bar{x})$ and that $x^* \in \sum_{t \in T(\bar{x})} \lambda_t \partial f_t(\bar{x}) \subset D(0)$, which justify the weak* outer stability of mapping (3.20) at zero. To prove finally that the set $D(0)$ is weak* closed in X^* , take any $u^* \in \text{cl}^* D(0)$ and show similarly to the above that

$$(u^*, f(\bar{x})) \in \bigcup \left\{ \sum_{t \in T} \lambda_t (\partial f_t(\bar{x}), f_t(\bar{x})) \mid \lambda \in \Delta(T) \right\}.$$

Therefor we get $u^* \in D(0)$, which shows that the set $D(0)$ is weak* closed in X^* and thus completes the proof of the corollary. \triangle

We conclude this section with yet another consequence of Theorem 3.2 that provides a *precise calculation* of the limiting subdifferential of the supremum function (3.1) in the case of functions f_t uniformly strictly differentiable at the reference point.

Corollary 3.8 (calculating limiting subgradients for suprema of uniformly strictly differentiable functions). *Let the functions f_t in (3.1) be uniformly strictly differentiable at \bar{x} , and let their gradient set $\{\nabla f_t(\bar{x})\}$ be bounded in X^* . Then the supremum function (3.1) is lower regular at \bar{x} and its limiting subdifferential $\partial f(\bar{x})$ at this point is calculated by*

$$\partial f(\bar{x}) = \bigcap_{\varepsilon > 0} \text{cl}^* \text{co} \left\{ \nabla f_t(\bar{x}) \mid t \in T_\varepsilon(\bar{x}) \right\}. \quad (3.26)$$

If in addition the set $\text{co} \{(\nabla f_t(\bar{x}), f_t(\bar{x})) \mid t \in T\}$ is weak* closed in $X^* \times \mathbb{R}$, then we have

$$\partial f(\bar{x}) = \text{co} \{ \nabla f_t(\bar{x}) \mid t \in T(\bar{x}) \}. \quad (3.27)$$

Proof. The inclusion “ \subset ” in (3.26) follows from Proposition 3.5 and Corollary 3.6. To justify the converse inclusion, take any $\delta > 0$ and pick x^* from the right-hand side of (3.26). Then for each $\varepsilon > 0$ we find a net $(\lambda_\nu)_{\nu \in \mathcal{N}} \in \Delta(T_\varepsilon(\bar{x}))$ satisfying

$$x^* = w^* - \lim_{\nu} \sum_{t \in T_\varepsilon(\bar{x})} \lambda_{\nu_t} \nabla f_t(\bar{x}). \quad (3.28)$$

It follows from the uniform strict differentiability (3.18) that there is $\eta > 0$ such that

$$f_t(x) - f_t(\bar{x}) \geq \langle \nabla f_t(\bar{x}), x - \bar{x} \rangle - \delta \|x - \bar{x}\| \quad \text{for all } x \in \mathbb{B}_\eta(\bar{x}), t \in T.$$

Applying the latter to (3.28), we get

$$\begin{aligned} f(x) - f(\bar{x}) + \varepsilon &\geq \limsup_{\nu} \sum_{t \in T_\varepsilon(\bar{x})} \lambda_{\nu_t} (f_t(x) - f_t(\bar{x})) \\ &\geq \limsup_{\nu} \sum_{t \in T_\varepsilon(\bar{x})} \lambda_{\nu_t} (\langle \nabla f_t(\bar{x}), x - \bar{x} \rangle - \delta \|x - \bar{x}\|) \\ &\geq \langle x^*, x - \bar{x} \rangle - \delta \|x - \bar{x}\| \end{aligned}$$

whenever $x \in \mathcal{B}_\eta(\bar{x})$. Letting now $\varepsilon \rightarrow 0$ gives us

$$f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle - \delta \|x - \bar{x}\| \quad \text{for all } x \in \mathcal{B}_\eta(\bar{x}),$$

which means that $x^* \in \widehat{\partial}f(\bar{x})$. Thus we obtain the inclusions

$$\partial f(\bar{x}) \subset \bigcap_{\varepsilon > 0} \text{cl}^* \text{co} \left\{ \nabla f_t(\bar{x}) \mid t \in T_\varepsilon(\bar{x}) \right\} \subset \widehat{\partial}f(\bar{x}).$$

Since $\widehat{\partial}f(\bar{x}) \subset \partial f(\bar{x})$, this shows that f is lower regular at \bar{x} and equality (3.26) holds. Formula (3.27) follows directly from Corollary 3.7 due to lower regularity of f . \triangle

4 The Generalized Gradient of Supremum Functions

The underlying property of the limiting subdifferential (2.2) is its nonconvexity. Taking into account relationship (2.6) between the limiting subdifferential and the generalized gradient (2.4) of locally Lipschitzian functions on Asplund spaces, we can get upper estimates and precise formulas for evaluating the generalized gradient of supremum functions by using the convex weak* closure of those obtained for the limiting subdifferential in Section 3. This approach allows us to derive in what follows new results for the generalized gradient of the supremum (3.1) of uniformly Lipschitzian functions f_t over an arbitrary index set T as well as in the case of T endowed with some topological structure.

The first result concerns supremum functions over arbitrary index sets.

Theorem 4.1 (generalized subgradients of supremum functions over arbitrary index sets). *In the setting of (3.1), defined $E : \mathbb{R}_+ \rightrightarrows X^*$ by*

$$E(\varepsilon) := \bigcup \{ \bar{\partial}f_t(x) \mid \|x - \bar{x}\| \leq \varepsilon, t \in T_\varepsilon(\bar{x}) \} \quad \text{for all } \varepsilon \geq 0. \quad (4.1)$$

Then the generalized gradient of the supremum function f in (3.1) is estimated by

$$\bar{\partial}f(\bar{x}) \subset \bigcap_{\varepsilon > 0} \text{cl}^* \text{co} E(\varepsilon). \quad (4.2)$$

If in addition the mapping E in (4.1) is weak outer stable at zero, we have*

$$\bar{\partial}f(\bar{x}) \subset \text{cl}^* \text{co} \left[\bigcup \{ \bar{\partial}f_t(\bar{x}) \mid t \in T(\bar{x}) \} \right]. \quad (4.3)$$

The equality holds in (4.3) when the functions $f_t, t \in T(\bar{x})$, are directionally regular at \bar{x} .

Proof. Thanks to (3.14), (2.6), and Lemma 6.2 we have that

$$\partial f(\bar{x}) \subset \text{cl}^* \text{Lim sup}_{\varepsilon \downarrow 0} \text{co} E(\varepsilon) = \text{cl}^* \text{co} \left[\text{Lim sup}_{\varepsilon \downarrow 0} E(\varepsilon) \right] = \bigcap_{\varepsilon > 0} \text{cl}^* \text{co} E(\varepsilon),$$

which yields (4.2). If furthermore the set-valued mapping E is weak* outer stable at 0, then (4.2) leads us to $\bar{\partial}f(\bar{x}) \subset \text{cl}^* [\text{co cl}^* E(0)]$. Moreover, it is easy to check that $\text{cl}^* [\text{co cl}^* E(0)] = \text{cl}^* \text{co} E(0)$. Thus we get $\bar{\partial}f(\bar{x}) \subset \text{cl}^* \text{co} E(0)$ and justify (4.3).

Finally, let us prove the equality in (4.3) under the directional regularity assumption on f_t at \bar{x} for all $t \in T(\bar{x})$. Pick any $x^* \in \text{co} \left[\bigcup \{ \bar{\partial}f_t(\bar{x}) \mid t \in T(\bar{x}) \} \right]$ and find $\lambda \in \Delta(T(\bar{x}))$ and

$x_t^* \in \bar{\partial}f_t(\bar{x})$ for $t \in T(\bar{x})$ with $x^* = \sum_{t \in T(\bar{x})} \lambda_t x_t^*$. Since the set $\text{supp}(\lambda)$ contains finitely many elements, for each $v \in X$, $\delta > 0$ there is $\varepsilon > 0$ with

$$\langle x_t^*, v \rangle \leq f_t^\circ(\bar{x}; v) = f_t'(\bar{x}; v) \leq \frac{f_t(\bar{x} + rv) - f_t(\bar{x})}{r} + \delta \leq \frac{f(\bar{x} + rv) - f(\bar{x})}{r} + \delta$$

for all $0 < r < \varepsilon$ and $t \in \text{supp}(\lambda)$. It yields that

$$\langle x^*, v \rangle \leq \frac{f(\bar{x} + sv) - f(\bar{x})}{s} + \delta \quad \text{whenever } 0 < s < \varepsilon \text{ and } v \in X,$$

which implies in turn that $x^* \in \bar{\partial}f(\bar{x})$. Combining this with (4.3), we get

$$\bar{\partial}f(\bar{x}) \subset \text{cl}^* \text{co} \left[\bigcup \{ \bar{\partial}f_t(\bar{x}) \mid t \in T(\bar{x}) \} \right] \subset \text{cl}^* \bar{\partial}f(\bar{x}) = \bar{\partial}f(\bar{x}).$$

This ensures the equality in (4.3) and completes the proof of the theorem. \triangle

Observe that inclusion (4.2) looks similar to the original Valadier's formula [27] obtained for convex functions. Furthermore, Theorem 4.1 seems to be the first result in the literature that extends [6, Theorem 2.8.2] to the case when the index set T is not compact and even not endowed with any topological structure and when the functions $t \mapsto f_t(x)$ do not enjoy any (semi)continuous property. Let us present a consequence of Theorem 4.1 in the form of [6, Theorem 2.8.2] showing that some of assumptions of the latter theorem are not needed in the case of Asplund spaces X .

Corollary 4.2 (generalized subgradients of supremum functions over metrizable index sets). *Let T be a metrizable space, and let $T_\varepsilon(\bar{x})$ be a compact subset of T for some $\varepsilon > 0$ sufficiently small. Assume also the function $t \mapsto f_t(\bar{x})$ is u.s.c. on $T_\varepsilon(\bar{x})$. Then the generalized gradient of the supremum function f at \bar{x} is estimated by*

$$\bar{\partial}f(\bar{x}) \subset \text{cl}^* \text{co} \left[\bigcup \{ \bar{\partial}^{[T]} f_t(\bar{x}) \mid t \in T(\bar{x}) \} \right], \quad (4.4)$$

where the subdifferential construction $\bar{\partial}^{[T]} f_t(\bar{x})$ is defined by

$$\bar{\partial}^{[T]} f_t(\bar{x}) := \text{cl}^* \text{co} \left[\text{Lim sup}_{\substack{T_\varepsilon(\bar{x}) \\ s \xrightarrow{x \rightarrow \bar{x}} t}} \{ \bar{\partial} f_s(x) \} \right]. \quad (4.5)$$

Furthermore, the equality holds in (4.4) provided that the functions f_t are directionally regular at \bar{x} and that

$$\bar{\partial}^{[T]} f_t(\bar{x}) \subset \bar{\partial}f_t(\bar{x}) \quad \text{for all } t \in T(\bar{x}). \quad (4.6)$$

Proof. To derive (4.4) from Theorem 4.1, fix $\varepsilon > 0$, take any $x^* \in \text{Lim sup}_{\varepsilon \downarrow 0} E(\varepsilon)$, and find sequences $x_n \rightarrow \bar{x}$, $\varepsilon_n \downarrow 0$, $t_n \in T_{\varepsilon_n}(\bar{x})$, and $x_n^* \in \bar{\partial}f_{t_n}(x_n)$ such that $x_n^* \xrightarrow{w^*} x^*$. Since $T_{\varepsilon_n}(\bar{x})$ are subsets of $T_\varepsilon(\bar{x})$ when ε_n is sufficiently small, the sequence $\{t_n\}$ contains a subsequence (without relabeling), which converges to some $\bar{t} \in T_\varepsilon(\bar{x})$. Moreover, since the mapping $t \mapsto f_t(\bar{x})$ is u.s.c. on $T_\varepsilon(\bar{x})$, we have

$$f_t(\bar{x}) \geq \limsup_{n \rightarrow \infty} f_{t_n}(\bar{x}) \geq \limsup_{n \rightarrow \infty} f(\bar{x}) - \varepsilon_n = f(\bar{x}), \quad (4.7)$$

which implies that $\bar{t} \in T(\bar{x})$. Using further (4.5) gives us $x^* \in \bar{\partial}^{[T]} f_{\bar{t}}(\bar{x})$ and thus yields

$$\text{Lim sup}_{\varepsilon \downarrow 0} E(\varepsilon) \subset \bigcup \{ \bar{\partial}^{[T]} f_t(\bar{x}) \mid t \in T(\bar{x}) \}.$$

Combining the latter inclusion with Theorem 4.1 and Lemma 6.2, we arrive at (4.4).

To justify the equality in (4.4) under the additional assumptions made, we observe that that of (4.6) implies the weak* outer stability of mapping (4.1) at zero, and thus the claimed equality in (4.4) follows from the corresponding part of Theorem 4.1. This completes the proof of the corollary. \triangle

Note that the construction $\bar{\partial}^{[T]} f_t(\bar{x})$ defined in (1.8) reduces to that of (4.5) for Asplund spaces due to the weak* sequential compactness of the dual unit ball in this case. Inclusion (4.4) was first established in [6, Theorem 2.8.2] under the additional assumptions that T is a metrizable compact and the mappings $t \mapsto f_t(x)$ are u.s.c. on T for all $x \in X$. Our assumptions are essentially less restrictive, since we merely require the u.s.c. property for only one mapping $t \mapsto f_t(\bar{x})$ on the compact set $T_\varepsilon(\bar{x})$, which is a small subset of T .

The next corollary provides verifiable sufficient conditions that allow us to remove the closure operation in (4.4).

Corollary 4.3 (evaluation of generalized gradients of supremum functions with no closure operation). *In addition to the assumptions ensuring (4.4) in Corollary 4.2, suppose that $\dim X < \infty$ and that (4.6) is satisfied. Then we have the inclusion*

$$\bar{\partial} f(\bar{x}) \subset \text{co} \left[\bigcup \{ \bar{\partial} f_t(\bar{x}) \mid t \in T(\bar{x}) \} \right], \quad (4.8)$$

which holds as equality if the functions f_t , $t \in T(\bar{x})$, are directionally regular at \bar{x} .

Proof. Having $X = \mathbb{R}^n$, inclusion (4.4) now reads as

$$\bar{\partial} f(\bar{x}) \subset \text{clco} \left[\bigcup \{ \bar{\partial} f_t(\bar{x}) \mid t \in T(\bar{x}) \} \right].$$

It remains to prove that the set $A := \text{co} \left[\bigcup \{ \bar{\partial} f_t(\bar{x}) \mid t \in T(\bar{x}) \} \right]$ is closed in \mathbb{R}^n . To proceed, take any sequence $\{x_k^*\} \subset A$ converging to some $x^* \in \mathbb{R}^n$ and, by the classical Carathéodory theorem for convex hulls, find $\lambda_k \in \mathbb{R}_+^{n+1}$, $(t_{k_1}, \dots, t_{k_{n+1}}) \in T(\bar{x})^{n+1}$, and $u_k^* \in \mathbb{R}^{n(n+1)}$ with $u_{k_1}^* \in \bar{\partial} f_{t_{k_1}}(\bar{x}), \dots, u_{k_{n+1}}^* \in \bar{\partial} f_{t_{k_{n+1}}}(\bar{x})$ such that $\sum_{i=1}^{n+1} \lambda_{k_i} = 1$ and

$$x_k^* = \sum_{i=1}^{n+1} \lambda_{k_i} x_{k_i}^* \quad \text{for all } k \in \mathbb{N}.$$

By the compactness of $T(\bar{x})$ we suppose with no loss of generality that the sequences t_{k_i} converge to some s_i as $k \rightarrow \infty$ for all $i = 1, \dots, n+1$. The same is true for the bounded sequences $\{\lambda_k\} \subset \mathbb{R}^{n+1}$ and $\{u_k^*\} \subset \mathbb{R}^{n(n+1)}$ with the corresponding limits $\mu \in \mathbb{R}_+^{n+1}$ and $v^* \in \mathbb{R}^{n(n+1)}$. It gives therefore that

$$v_i^* = \lim_{k \rightarrow \infty} u_{k_i}^* \in \text{Lim sup}_{t_{k_i} \rightarrow s_i} \bar{\partial} f_{t_{k_i}}(\bar{x}) \subset \bar{\partial}^{[T]} f_{s_i}(\bar{x}) \subset \bar{\partial} f_{s_i}(\bar{x})$$

and allows us to conclude that

$$x^* = \lim_{k \rightarrow \infty} x_k^* = \lim_{k \rightarrow \infty} \sum_{i=1}^{n+1} \lambda_{k_i} x_{k_i}^* = \sum_{i=1}^{n+1} \mu_i v_i^* \in \sum_{i=1}^{n+1} \mu_i \bar{\partial} f_{s_i}(\bar{x}).$$

Thus we get $x^* \in A$, and ensures that the set A is closed in \mathbb{R}^n , and hence inclusion (4.8) holds. The equality in (4.8) under the directional regularity of f_t at \bar{x} for all $t \in T(\bar{x})$ follows from (4.4) and the proof of Corollary 4.2. \triangle

The next corollary of Theorem 4.1, inspired by the corresponding result of [30], provides precise calculations of the generalized gradient of the supremum of semismooth functions as a direct consequence of Corollary 4.2 and Corollary 4.3. Subsmooth sets were introduced and comprehensively studied in [1]. This notion was modified for functions, in the way used in what follows, in [29]; see also the references therein as well as [19, Commentaries 5.5.4]. Following [29], we say that $\varphi : X \rightarrow \overline{\mathbb{R}}$ is *subsmooth* at \bar{x} if for any $\varepsilon > 0$ there is $\delta > 0$ with

$$\varphi(x) - \varphi(u) \geq \langle u^*, x - u \rangle - \varepsilon \|x - u\| \quad \text{for all } x, u \in \mathbb{B}_\delta(\bar{x}) \text{ and } u^* \in \overline{\partial}\varphi(u).$$

Corollary 4.4 (calculating generalized subgradients of suprema of uniformly semismooth functions). *Let T be a metrizable space, let $T_\varepsilon(\bar{x})$ be a compact subset of T for some $\varepsilon > 0$ sufficiently small, and let the function $t \mapsto f_t(x)$ is u.s.c. on $T_\varepsilon(\bar{x})$ for each x sufficiently closed to \bar{x} . Suppose further that the functions f_t , $t \in T_\varepsilon(\bar{x})$, are uniformly subsmooth at $\bar{x} \in X$, i.e., for all $\tilde{\varepsilon} > 0$ there is $\delta > 0$ such that*

$$f_t(x) - f_t(u) \geq \langle u^*, x - u \rangle - \tilde{\varepsilon} \|x - u\| \quad (4.9)$$

whenever $x, u \in \mathbb{B}_\delta(\bar{x})$ and $u^* \in \overline{\partial}f_t(u)$. Then the generalized gradient of the supremum function (3.1) at \bar{x} is calculated by

$$\overline{\partial}f(\bar{x}) = \text{cl}^* \text{co} \left[\bigcup \{ \overline{\partial}f_t(\bar{x}) \mid t \in T(\bar{x}) \} \right]. \quad (4.10)$$

If in addition X is a finite-dimensional space, then

$$\overline{\partial}f(\bar{x}) = \text{co} \left[\bigcup \{ \overline{\partial}f_t(\bar{x}) \mid t \in T(\bar{x}) \} \right]. \quad (4.11)$$

Proof. By the results of Corollary 4.2 and Corollary 4.3, all we need is to verify that the functions f_t are directionally regular at \bar{x} for each $t \in T(\bar{x})$ and that condition (4.6) is satisfied. To check the former, observe that the uniform subsmoothness (4.9) yields that for each $t \in T(\bar{x})$ and $v \in X$ we have

$$\liminf_{r \downarrow 0} \frac{f_t(\bar{x} + rv) - f_t(\bar{x})}{r} \geq \langle x^*, v \rangle \quad \text{for all } x^* \in \overline{\partial}f_t(\bar{x}),$$

which implies together with (2.4) the relationships

$$\liminf_{r \downarrow 0} \frac{f_t(\bar{x} + rv) - f_t(\bar{x})}{r} \geq \sup \{ \langle x^*, v \rangle \mid x^* \in \overline{\partial}f_t(\bar{x}) \} = f_t^\circ(\bar{x}; v).$$

Invoking now definition (2.3) of the generalized directional derivative, we get from the latter that $f_t'(\bar{x}; v) = f_t^\circ(\bar{x}; v)$ for any $t \in T(\bar{x})$ and $v \in X$. It means by (2.5) that all the functions f_t , $t \in T(\bar{x})$, are directionally regular at \bar{x} .

Since the generalized gradient set (2.4) is weak* closed and convex in X^* for any locally Lipschitzian function and due the validity of (4.7), inclusion (4.6) follows from the fact that the existence of sequences $x_n \rightarrow \bar{x}$, $t_n \rightarrow t \in T(\bar{x})$, and $x_n^* \in \overline{\partial}f_{t_n}(x_n)$ with $f_{t_n}(\bar{x}) \rightarrow f_t(\bar{x})$ and $x_n^* \xrightarrow{w^*} x^*$ for some $x^* \in X^*$ implies that $x^* \in \overline{\partial}f_t(\bar{x})$. To justify this, we derive from the

uniform semismoothness of f_t in (4.9) and the assumed upper semicontinuity of $t \mapsto f_t(x)$ on $T_\varepsilon(\bar{x})$ for all x sufficiently close to \bar{x} that

$$\begin{aligned} f_t(x) - f_t(\bar{x}) &\geq \limsup_{n \rightarrow \infty} (f_{t_n}(x) - f_{t_n}(\bar{x}) + K\|x_n - \bar{x}\|) \geq \limsup_{n \rightarrow \infty} (f_{t_n}(x) - f_{t_n}(x_n)) \\ &\geq \limsup_{n \rightarrow \infty} (\langle x_n^*, x - x_n \rangle - \tilde{\varepsilon}\|x - x_n\|) \geq \langle x^*, x - \bar{x} \rangle - \tilde{\varepsilon}\|x - \bar{x}\| \end{aligned}$$

whenever $x \in \mathbb{B}_\delta(\bar{x})$. Since the number $\tilde{\varepsilon} > 0$ was chosen arbitrarily, it gives therefore that $x^* \in \widehat{\partial}f_t(\bar{x}) \subset \bar{\partial}f_t(\bar{x})$. Thus we get $\bar{\partial}^{[T]}f_t(\bar{x}) \subset \bar{\partial}f_t(\bar{x})$ for $t \in T(\bar{x})$, which justifies (4.10) and completes the proof of the corollary. \triangle

The results of Corollary 4.4 have been recently established in [30, Theorem 3.1 and Theorem 3.2] under additional assumptions that all the functions f_t as $t \in T$ are uniformly subsmooth at \bar{x} and the function $(x, t) \mapsto f_t(x)$ is continuous on the whole space $X \times T$. The approach developed above allows us to shrink T to the smaller set $T_\varepsilon(\bar{x})$, to impose the uniform semismoothness of f_t only on the latter set, and to assume merely the upper semicontinuity of the mapping $t \mapsto f_t(x)$ when x is closed to \bar{x} .

The next proposition concerns the class of equicontinuously subdifferentiable functions introduced in Definition 3.4 and shows that the generalized gradient for this class of functions satisfies the condition similar to (3.17) for arbitrary index sets.

Proposition 4.5 (generalized subgradients of equicontinuously subdifferentiable functions). *Let the functions f_t , $t \in T$, be equicontinuously subdifferentiable at $\bar{x} \in \text{dom } f$ in the setting of (3.1). Then for any weak* neighborhood V^* of the origin in X^* there is $\varepsilon > 0$ such that*

$$\bar{\partial}f_t(x) \subset \bar{\partial}f_t(\bar{x}) + V^* \quad \text{for all } t \in T_\varepsilon(\bar{x}) \text{ and } x \in \mathbb{B}_\varepsilon(\bar{x}). \quad (4.12)$$

Proof. Take any weak* neighborhoods U^* and V^* of the origin in X^* such that U^* is convex and that $\text{cl}^*U^* \subset V^*$. Since the functions f_t are equicontinuously subdifferentiable at $\bar{x} \in \text{dom } f$, we find a number $\varepsilon > 0$ for which

$$\partial f_t(x) \subset \partial f_t(\bar{x}) + U^* \quad \text{whenever } t \in T_\varepsilon(\bar{x}) \text{ and } x \in \mathbb{B}_\varepsilon(\bar{x}),$$

Taking into account that U^* is convex, this implies the inclusions

$$\text{co } \partial f_t(x) \subset \text{co}(\partial f_t(\bar{x}) + U^*) \subset \text{co } \partial f_t(\bar{x}) + U^* \quad \text{for all } t \in T_\varepsilon(\bar{x}) \text{ and } x \in \mathbb{B}_\varepsilon(\bar{x}).$$

Since the set $\text{cl}^*\text{co } \partial f_t(\bar{x}) = \bar{\partial}f_t(\bar{x})$ is weak* compact in X^* , we get that

$$\bar{\partial}f_t(x) = \text{cl}^*\text{co } \partial f_t(x) \subset \bar{\partial}f_t(\bar{x}) + \text{cl}^*U^* \subset \bar{\partial}f_t(\bar{x}) + V^*$$

for $t \in T_\varepsilon(\bar{x})$ and $x \in \mathbb{B}_\varepsilon(\bar{x})$. This justifies the claimed inclusion (4.12). \triangle

We conclude this section deriving an efficient estimate of the type given in Corollary 3.6 for the generalized gradient of the supremum of a particular class of functions, including equicontinuously subdifferentiable ones, over arbitrary index sets.

Corollary 4.6 (evaluation of generalized subgradients for suprema of equicontinuously subdifferentiable functions over arbitrary index sets). *Let the functions f_t , $t \in T$, satisfy property (4.12) at \bar{x} held, in particular, for equicontinuously subdifferentiable functions. Then we have the estimate*

$$\bar{\partial}f(\bar{x}) \subset \bigcap_{\varepsilon > 0} \text{cl}^*\text{co} \left[\bigcup \{ \bar{\partial}f_t(\bar{x}) \mid t \in T_\varepsilon(\bar{x}) \} \right]. \quad (4.13)$$

Proof. It is similar to that in Corollary 3.8, with taking into account Proposition 4.5. \triangle

5 Optimality Conditions for SIP and Infinite Programs

This section is devoted to deriving necessary optimality conditions for the (semi)infinite programming problems (1.1) by reducing them to the single-constrained form (1.2) and applying then the subdifferential formulas for supremum functions given in the previous sections. In this way we obtain a number of new optimality conditions for nonsmooth infinite and semi-infinite programs of *KKT/qualified* types in both *asymptotic* (i.e., with the weak* closure operation) and *multiplier* (without weak* closure) forms in terms of limiting and generalized subgradients. Define

$$\Omega := \{x \in X \mid f_t(x) \leq 0, t \in T\}.$$

Recall our standing assumption that the functions f_t are uniformly locally Lipschitzian with rank $K > 0$ around the reference point \bar{x} . For simplicity we assume in this section that the cost function φ in (1.1) is locally Lipschitzian around \bar{x} too.

In what follows we assume that $f(\bar{x}) = 0$, since the case of $f(\bar{x}) < 0$ is trivial. Then we have the expressions

$$T_\varepsilon(\bar{x}) = \{t \in T \mid f_t(\bar{x}) \geq -\varepsilon\} \quad \text{and} \quad T(\bar{x}) = \{t \in T \mid f_t(\bar{x}) = 0\}.$$

The first theorem provides several versions of necessary optimality conditions of the KKT type in terms of the limiting subdifferential.

Theorem 5.1 (KKT optimality conditions via limiting subgradients). *Let \bar{x} be a local minimizer for the infinite program (1.1), and let the constraint qualification condition $0 \notin \bigcap_{\varepsilon > 0} \text{cl}^* C(\varepsilon)$ be satisfied. Then we have the inclusion*

$$0 \in \partial\varphi(\bar{x}) + \mathbb{R}_+ \bigcap_{\varepsilon > 0} \text{cl}^* C(\varepsilon). \quad (5.1)$$

If in addition the set-valued mapping $\mathbb{R}_+ C : \mathbb{R}_+ \rightrightarrows X^$ is weak* outer stable at zero, then*

$$0 \in \partial\varphi(\bar{x}) + \text{cl}^* \left\{ \sum_{t \in T(\bar{x})} \lambda_t \partial f_t(\bar{x}) \mid t \in T(\bar{x}) \right\}. \quad (5.2)$$

If furthermore the set $\mathbb{R}_+ C(0)$ is weak closed, then there is a multiplier $\lambda \in \tilde{\mathbb{R}}_+^T$ such that*

$$0 \in \partial\varphi(\bar{x}) + \sum_{t \in T(\bar{x})} \lambda_t \partial f_t(\bar{x}). \quad (5.3)$$

Proof. To justify (5.1), assume that $0 \notin \bigcap_{\varepsilon > 0} \text{cl}^* C(\varepsilon)$. Let us consider the supremum function f in (3.1) and define

$$\psi(x) := \max \{ \varphi(x) - \varphi(\bar{x}), f(x) \}. \quad (5.4)$$

Since \bar{x} is a local minimizer of (1.1), it is also a local minimizer of the unconstrained problem:

$$\text{minimize } \psi(x) \quad \text{subject to } x \in X. \quad (5.5)$$

By the generalized Fermat rule for problem (5.5) we have $0 \in \partial\psi(\bar{x})$. Applying [18, Theorem 3.46] to the function ψ in (5.4) gives us

$$\partial\psi(\bar{x}) \subset \begin{cases} \partial\varphi(\bar{x}) & \text{if } f(\bar{x}) < 0, \\ \bigcup \{ \partial(\mu\varphi + (1-\mu)f)(\bar{x}) \mid \mu \in [0, 1] \} & \text{if } f(\bar{x}) = 0. \end{cases} \quad (5.6)$$

Since $f(\bar{x}) = 0$, we find $\mu \in [0, 1]$ such that

$$0 \in \partial(\mu\varphi + (1-\mu)f)(\bar{x}) \subset \partial(\mu\varphi)(\bar{x}) + \partial((1-\mu)f)(\bar{x}) \subset \mu\partial\varphi(\bar{x}) + (1-\mu)\partial f(\bar{x}) \quad (5.7)$$

due to sum rule from [18, Theorem 3.36] for limiting subgradients of Lipschitz continuous functions. Applying further Theorem 3.2 to inclusion (5.7) ensures that $\mu \neq 0$ therein. Dividing the latter by $\mu \in (0, 1]$ gives us (5.1). Suppose further that the set-valued mapping \mathbb{R}_+C is weak* outer stable at zero. It follows from Lemma 6.2 and inclusion (5.1) that

$$0 \in \partial\varphi(\bar{x}) + (1-\mu) \bigcap_{\varepsilon > 0} \text{cl}^*C(\varepsilon) = \partial\varphi(\bar{x}) + \text{cl}^*[\text{Lim sup}_{\varepsilon \downarrow 0} \mathbb{R}_+C(\varepsilon)] \subset \partial\varphi(\bar{x}) + \text{cl}^*[\mathbb{R}_+C(0)],$$

which justifies (5.2). Moreover, inclusion (5.3) clearly follows from (5.2) provided that the cone $\mathbb{R}_+C(0)$ is weak* closed. This completes the proof of the theorem. \triangle

When the constraint functions in (3.1) are equicontinuously subdifferentiable at the reference point, the results of Theorem 5.1 can be simplified by replacing the set-valued mapping C with that of D defined in (3.20).

Corollary 5.2 (simplified KKT conditions for equicontinuously subdifferentiable functions). *Let the constraint functions f_t be equicontinuously subdifferentiable at the local minimizer \bar{x} for (1.1), and let $D: \mathbb{R}_+ \rightrightarrows X^*$ be defined in (3.20). Then the qualification condition $0 \notin \bigcap_{\varepsilon > 0} \text{cl}^*D(\varepsilon)$ implies that*

$$0 \in \partial\varphi(\bar{x}) + \mathbb{R}_+ \bigcap_{\varepsilon > 0} \text{cl}^*D(\varepsilon).$$

If we assume in addition the set

$$\bigcup \left\{ \sum_{t \in T} \lambda_t (\partial f_t(\bar{x}), f_t(\bar{x})) \mid \lambda \in \tilde{\mathbb{R}}_+^T \right\} \quad (5.8)$$

is weak* closed in $X^* \times \mathbb{R}$, then there is a multiplier $\lambda \in \tilde{\mathbb{R}}_+^T$ such that

$$0 \in \partial\varphi(\bar{x}) + \sum_{t \in T(\bar{x})} \lambda_t \partial f_t(\bar{x}).$$

Proof. Following the lines in the proof of Corollary 3.7, we get that the set-valued mapping $\mathbb{R}_+C: \mathbb{R}_+ \rightrightarrows X^*$ is weak* outer stable at zero and the $\mathbb{R}_+C(0)$ is weak* closed in X^* provided that the set (5.8) is weak* closed in $X^* \times \mathbb{R}$. This together with Theorem 5.1 justifies the results in this corollary. \triangle

It is worth noting that when the constraint functions f_t are linear, i.e.,

$$f_t(x) = \langle a_t^*, x \rangle - b_t \quad \text{with } (a_t^*, b_t) \in X^* \times \mathbb{R}, \quad t \in T,$$

the closedness condition for (5.8) is equivalent to the classical *Farkas-Minkowski Constraint Qualification* meaning that the convex cone $\text{cone} \{(a_t^*, b_t) \mid t \in T\}$ is weak* closed in $X^* \times \mathbb{R}$; see, e.g., [5, 7, 10]. More generally, for uniformly strictly differentiable function f_t the closedness condition for set (5.8) in Corollary 5.2 reduces to the so-called *Nonlinear Farkas-Minkowski Constraint Qualification* (NFMCQ) introduced recently in our paper [20], i.e., the set $\text{cone} \{(\nabla f_t(\bar{x}), f_t(\bar{x})) \mid t \in T\}$ is weak* closed in $X^* \times \mathbb{R}$.

Next we define and employ an extension of another constraint qualification for infinite programs introduced in [20] for smooth functions under the name of *Perturbed Mangasarian-Fromovitz Constraint Qualification* (PMFCQ). Our new condition concerns Lipschitzian constraints and is formulated in terms of the generalized direction derivative (2.3).

Definition 5.3 (Generalized PMFCQ). *We say that the infinite program (1.1) satisfies the GENERALIZED PERTURBED MANGASARIAN-FROMOVITZ CONSTRAINT QUALIFICATION (GENERALIZED PMFCQ) at \bar{x} if there is a direction $d \in X$ such that*

$$\inf_{\varepsilon > 0} \sup_{t \in T_\varepsilon(\bar{x})} f_t^\circ(\bar{x}; d) < 0. \quad (5.9)$$

When f_t are uniformly strictly differentiable at \bar{x} , the Generalized MFCQ (5.9) reduces to

$$\inf_{\varepsilon > 0} \sup_{t \in T_\varepsilon(\bar{x})} \langle \nabla f_t(\bar{x}), d \rangle < 0 \text{ for some } d \in \mathbb{R},$$

which is the PMFCQ employed in [20] to derive qualified necessary optimality conditions for smooth infinite programs with arbitrary index sets. Prior to the application of the Generalized PMFCQ, we give its dual characterization in terms of the generalized gradient.

Proposition 5.4 (dual characterization of the Generalized PMFCQ). *The Generalized PMFCQ condition (5.9) holds at \bar{x} if and only if we have*

$$0 \notin \bigcap_{\varepsilon > 0} \text{cl}^* \text{co} \left[\bigcup \left\{ \bar{\partial} f_t(\bar{x}) \mid t \in T_\varepsilon(\bar{x}) \right\} \right]. \quad (5.10)$$

Proof. Assume that the Generalized PMFCQ holds at \bar{x} and find numbers $\varepsilon, \delta > 0$ with

$$\sup_{t \in T_\varepsilon(\bar{x})} f_t^\circ(x; d) < -\delta.$$

Arguing by contradiction, suppose that $0 \in \text{cl}^* \text{co} \left[\bigcup \left\{ \bar{\partial} f_t(\bar{x}) \mid t \in T_\varepsilon(\bar{x}) \right\} \right]$ for some $\tilde{\varepsilon} \in (0, \varepsilon)$. Then there are nets $(\lambda_\nu)_{\nu \in \mathcal{N}} \in \Delta(T_{\tilde{\varepsilon}}(\bar{x}))$ and $x_{\nu_t}^* \in \bar{\partial} f_t(\bar{x})$ as $t \in T_{\tilde{\varepsilon}}(\bar{x})$ and $\nu \in \mathcal{N}$ such that

$$0 = w^* - \lim_{\nu \in \mathcal{N}} \sum_{t \in T_{\tilde{\varepsilon}}(\bar{x})} \lambda_{\nu_t} x_t^*.$$

It follows from the duality between the generalized directional derivative and the generalized gradient (2.4) that

$$0 = \lim_{\nu \in \mathcal{N}} \sum_{t \in T_{\tilde{\varepsilon}}(\bar{x})} \lambda_{\nu_t} \langle x_t^*, d \rangle \leq \limsup_{\nu \in \mathcal{N}} \sum_{t \in T_{\tilde{\varepsilon}}(\bar{x})} \lambda_{\nu_t} f_t^\circ(x; d) \leq \limsup_{\nu \in \mathcal{N}} \sum_{t \in T_{\tilde{\varepsilon}}(\bar{x})} \lambda_{\nu_t} (-\delta) = -\delta < 0,$$

which is a contradiction that justifies the implication (5.9) \implies (5.10).

Conversely, assume that (5.10) holds, i.e., there is $\varepsilon > 0$ such that

$$0 \notin \text{cl}^* \text{co} \left[\bigcup \{ (\bar{\partial} f_t(\bar{x}) \mid t \in T_\varepsilon(\bar{x})) \} \right].$$

By the classical separation theorem of convex sets we find $d \in X$ such that

$$\sup \{ \langle x^*, d \rangle \mid x^* \in \bar{\partial} f_t(\bar{x}), t \in T_\varepsilon(\bar{x}) \} < 0.$$

Since $\sup \{ \langle x^*, d \rangle \mid x^* \in \bar{\partial} f_t(\bar{x}) \} = f_t^\circ(x; d)$ for any $t \in T_\varepsilon(\bar{x})$, the last inequality ensures that (5.9) is satisfied, which completes the proof of the proposition. \triangle

The final result of this section provides necessary optimality conditions of the KKT type for infinite programs (1.1) with arbitrary index sets that are expressed in terms of the generalized gradient of the cost and constraint functions.

Theorem 5.5 (optimality conditions of the KKT type via generalized subgradients under the Generalized PMFCQ). *Let the constraint functions f_t satisfy (4.12) at the local optimal solution \bar{x} of the infinite program (1.1); this is automatic when f_t are equicontinuously subdifferentiable at \bar{x} . If the Generalized PMFCQ holds at \bar{x} , then we have*

$$0 \in \bar{\partial} \varphi(\bar{x}) + \mathbb{R}_+ \bigcap_{\varepsilon > 0} \text{cl}^* \text{co} \left[\bigcup \{ \bar{\partial} f_t(\bar{x}) \mid t \in T_\varepsilon(\bar{x}) \} \right]. \quad (5.11)$$

If furthermore the convex conic hull cone $\{ (\bar{\partial} f_t(\bar{x}), f_t(\bar{x})) \mid t \in T \}$ is weak closed in $X^* \times \mathbb{R}$, then there is a multiplier $\lambda \in \widetilde{\mathbb{R}}_+^T$ such that*

$$0 \in \bar{\partial} \varphi(\bar{x}) + \sum_{t \in T(\bar{x})} \lambda_t \bar{\partial} f_t(\bar{x}). \quad (5.12)$$

Proof. It follows from the assumed Generalized PMFCQ and Proposition 5.4 that condition (5.10) is satisfied. Then Corollary 4.6 ensures that $0 \notin \bar{\partial} f(\bar{x})$, and thus $0 \notin \partial f(\bar{x})$. Similarly to the proof of Theorem 5.1 we get that there is $\mu \in \mathbb{R}_+$ such that

$$0 \in \partial \varphi(\bar{x}) + \mu \partial f(\bar{x}) \subset \bar{\partial} \varphi(\bar{x}) + \mu \bar{\partial} f(\bar{x}).$$

Combining the latter with the conclusion of Corollary 4.6 gives us (5.11). Furthermore, the weak* closedness of the convex conic hull cone $\{ (\bar{\partial} f_t(\bar{x}), f_t(\bar{x})) \mid t \in T \}$ allows us reducing (5.11) to inclusion (5.12) by arguments similar to those in the proof of Corollary 5.2. \triangle

Somewhat different of necessary optimality conditions for (1.1) via generalized subgradients can be derived from the subdifferential estimates of supremum functions obtained in Theorem 4.1 and its consequences in Section 4. Note also that other qualification conditions ensuring the validity of results similar to (5.12) are derived in [15, 31] in the SIP case when the space X is finite-dimensional and the index set T is compact in (1.1).

6 Some Technical Lemmas

In this final section (actually the appendix to the paper) we present two self-contained technical lemmas, which are often used in the proofs above. The first lemma is an extension of the finite-dimensional result in [22, Corollary 9.6.1] to arbitrary Banach spaces.

Lemma 6.1 (weak* closed conic hulls). *Let X be a Banach space, and let A be a weak* compact in X^* such that $0 \notin A$. Then the conic hull \mathbb{R}_+A is weak* closed in X^* .*

Proof. To show that the cone \mathbb{R}_+A is weak* closed in X^* , take any net $\{x_\nu^*\}_{\nu \in \mathcal{N}} \subset \mathbb{R}_+A$ weak* converging to some $x^* \in X^*$. Hence there exist nets $\{\lambda_\nu\}_{\nu \in \mathcal{N}} \subset \mathbb{R}_+$ and $\{u_\nu^*\}_{\nu \in \mathcal{N}} \subset A$ such that $\lambda_\nu u_\nu^* = x_\nu^* \xrightarrow{w^*} x^*$. Define $\lambda := \text{Lim sup}_\nu \lambda_\nu$. If $\lambda = \infty$, then we find a subnet $\{\lambda_\nu\}$ (without relabeling) converging to ∞ . Since A is weak* compact, assume without loss of generality that $u_\nu^* \xrightarrow{w^*} u^*$. Furthermore, the relationships $\langle \lambda_\nu u_\nu^*, x \rangle \rightarrow \langle x^*, x \rangle$ and $\langle u_\nu^*, x \rangle \rightarrow \langle u^*, x \rangle$ for all $x \in X$ imply that $\langle u_\nu^*, x \rangle \rightarrow 0$ for all $x \in X$ due to $\lambda_\nu \rightarrow \infty$. This gives $0 \in A$, which contradicts the assumption made. Thus $\lambda < \infty$. By similar arguments, we get that $\lambda_\nu \rightarrow \lambda \in \mathbb{R}_+$ and $u_\nu^* \xrightarrow{w^*} u^* \in A$. It follows then that $x^* = \lambda u^* \in \mathbb{R}_+A$, which shows that \mathbb{R}_+A is weak* closed and completes the proof of the lemma. \triangle

The second lemma establishes some relationships for sequential outer limits (1.8) of increasing set-valued mappings in Asplund spaces.

Lemma 6.2 (outer limits of increasing mappings). *Let X be an Asplund space, and let $F : \mathbb{R}_+ \rightrightarrows X^*$ be a set-valued mapping. Assume that there is $\varepsilon > 0$ such that $F(\varepsilon)$ is bounded in X^* and that F is increasing, i.e., $F(\varepsilon_1) \subset F(\varepsilon_2)$ whenever $0 \leq \varepsilon_1 \leq \varepsilon_2$. Then the following assertions hold:*

$$(i) \quad \text{cl}^*[\text{Lim sup}_{\varepsilon \downarrow 0} F(\varepsilon)] = \bigcap_{\varepsilon > 0} \text{cl}^* F(\varepsilon).$$

$$(ii) \quad \text{cl}^* \text{co} [\text{Lim sup}_{\varepsilon \downarrow 0} F(\varepsilon)] = \text{cl}^* [\text{Lim sup}_{\varepsilon \downarrow 0} \text{co} F(\varepsilon)].$$

(iii) *If $0 \notin \text{cl}^* [\text{Lim sup}_{\varepsilon \downarrow 0} F(\varepsilon)]$, then we have*

$$\mathbb{R}_+ \text{cl}^* [\text{Lim sup}_{\varepsilon \downarrow 0} F(\varepsilon)] = \text{cl}^* \text{Lim sup}_{\varepsilon \downarrow 0} [\mathbb{R}_+ F(\varepsilon)], \quad (6.1)$$

Proof. The inclusion “ \subset ” in (i) is straightforward due to (1.8) and the monotonicity property of this set-valued mapping. To justify the converse inclusion, pick any x^* belonging to the set on right-hand side set of (i) and take an arbitrary convex weak* neighborhood V^* of the origin in X^* . Hence we find sequences $\varepsilon_n \downarrow 0$ and $x_n^* \in F(\varepsilon_n)$ such that $x^* \in x_n^* + \frac{V^*}{2}$. Since $F(\varepsilon) \subset KB^*$ for some $\varepsilon, K > 0$, there is a subsequence of $\{x_n^*\}$ (without relabeling) weak* converging to some $u^* \in X^*$. It follows that u^* is an element of $\text{Lim sup}_{\varepsilon \downarrow 0} F(\varepsilon)$. When

n is sufficiently large, we have $x_n^* \in u^* + \frac{V^*}{2}$. This implies that

$$x^* \in u^* + \frac{V^*}{2} + \frac{V^*}{2} = u^* + V^* \subset \text{Lim sup}_{\varepsilon \downarrow 0} F(\varepsilon) + V^*$$

Since V^* was chosen arbitrarily, the latter means that x^* belongs to the set on the left-hand side of (i), which ensures the equality in (i).

To proceed with the proof of (ii), observe from (i) that

$$\text{Lim sup}_{\varepsilon \downarrow 0} F(\varepsilon) \subset \bigcap_{\varepsilon > 0} \text{cl}^* F(\varepsilon) \subset \bigcap_{\varepsilon > 0} \text{cl}^* \text{co} F(\varepsilon) = \text{cl}^* [\text{Lim sup}_{\varepsilon \downarrow 0} \text{co} F(\varepsilon)],$$

where the last expression is weak* closed and convex in X^* . This implies the inclusion “ \subset ” in (ii). To prove the converse inclusion in (ii), it suffices to show that

$$\text{Lim sup}_{\varepsilon \downarrow 0} \text{co } F(\varepsilon) \subset \text{cl}^* \text{co} \left[\text{Lim sup}_{\varepsilon \downarrow 0} F(\varepsilon) \right]. \quad (6.2)$$

Arguing by contradiction, suppose that (6.2) does not hold and then find sequences $\varepsilon_n \downarrow 0$ and $x_n^* \xrightarrow{w^*} x^*$ with $x_n^* \in \text{co } F(\varepsilon_n)$ such that x^* does not belong to the set on the right-hand side of (ii). The classical separation theorem for convex sets gives us $0 \neq v \in X$ and $\alpha, \beta \in \mathbb{R}$ satisfying the inequalities

$$\langle x^*, v \rangle > \alpha > \beta > \langle u^*, v \rangle \quad \text{for all } u^* \in \text{Lim sup}_{\varepsilon \downarrow 0} F(\varepsilon). \quad (6.3)$$

Since $x_n^* \xrightarrow{w^*} x^*$, we assume without loss of generality that $\langle x_n^*, v \rangle > \frac{\alpha + \beta}{2}$ for all $n \in \mathbb{N}$ due to (6.3). By $x_n^* \in \text{co } F(x_n)$ there exist a finite index set S_n , $\lambda_n \in \Delta(S_n)$, and $x_{n_s}^* \in F(\varepsilon_n)$ as $s \in S_n$ satisfying the equalities

$$x_n^* = \sum_{s \in S_n} \lambda_{n_s} x_{n_s}^*, \quad n \in \mathbb{N}.$$

Among elements of the set $\{x_{n_s}^* \mid s \in S_n\}$ for each $n \in \mathbb{N}$ we select $\hat{x}_n^* \in F(\varepsilon_n)$ such that $\langle \hat{x}_n^*, v \rangle = \max\{\langle x_{n_s}^*, v \rangle \mid s \in S_n\}$. It follows therefore that

$$\langle \hat{x}_n^*, v \rangle \geq \sum_{s \in S_n} \lambda_{n_s} \langle x_{n_s}^*, v \rangle = \langle x_n^*, v \rangle > \frac{\alpha + \beta}{2}.$$

Note further that $\{\hat{x}_n^*\}$ is bounded in X^* . Since X is Asplund, we can assume that \hat{x}_n^* weak* converges to some $u^* \in X^*$. This implies that $u^* \in \text{Lim sup}_{\varepsilon \downarrow 0} F(\varepsilon)$ and that $\langle u^*, v \rangle \geq \frac{\alpha + \beta}{2}$, which contradicts to (6.3) and thus justifies (ii).

It remains to prove (iii) under the assumption made therein. The inclusion “ \subset ” in (6.1) is obvious. To justify the converse inclusion, it suffices to show that

$$\text{Lim sup}_{\varepsilon \downarrow 0} [\mathbb{R}_+ F(\varepsilon)] \subset \mathbb{R}_+ \text{cl}^* \left[\text{Lim sup}_{\varepsilon \downarrow 0} F(\varepsilon) \right], \quad (6.4)$$

since the right-hand of the latter inclusion is weak* closed due to Lemma 6.1. To prove (6.4), pick any element $x^* \neq 0$ from the set on the left-hand side of (6.4) and find $\varepsilon_n \downarrow 0$, $\lambda_n \in \mathbb{R}_+$, and $u_n^* \in F(\varepsilon_n)$ as $n \in \mathbb{N}$ such that $\lambda_n u_n^* \xrightarrow{w^*} x^*$. Following the proof of Lemma 6.1, we suppose with no loss of generality that $\lambda_n \rightarrow \lambda \in \mathbb{R}_+$ and $u_n^* \xrightarrow{w^*} u^* \in \text{Lim sup}_{\varepsilon \downarrow 0} F(\varepsilon)$.

It implies that $\lambda_n u_n^* \xrightarrow{w^*} \lambda u^* = x^*$, and the latter elements belongs to the set on the right-hand side of (6.4). This justifies (iii) and completes the the proof of the lemma. \triangle

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