Optimizing the Spectral Radius

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Abstract

We suggest an approach for finding the maximal and the minimal spectral radius of linear operators from a given compact family of operators, which share a common invariant cone (e.g. family of nonnegative matrices). In the case of families with so-called product structure, this leads to efficient algorithms for optimizing the spectral radius and for finding the joint and lower spectral radii of the family. Applications to the theory of difference equations and to problems of optimizing the spectral radius of graphs are considered.

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1 Introduction

In this paper we study a problem of finding the minimal and the maximal spectral radius of finite-dimensional linear operators from some compact family $\mathcal{M}$. The main assumption is that all these operators share a common invariant cone K, which is supposed to be convex, closed, solid and pointed. Such operators inherit most properties of operators with nonnegative matrices and have found many applications in functional and convex analysis, large networks, graphs, etc. (see [7, 15, 13] and references therein). In Section 2 we present a general approach for finding the maximal/minimal spectral radius of a family of operators with special structure. Our main attention is paid to so-called *product families*, for which the problem of optimizing

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the spectral radius can be efficiently solved (Sections 3 and 4). A special case of such families was considered in the recent paper [1]. It was shown that, for families of nonnegative matrices with independent column uncertainty, the maximal spectral radius is actually equal to the joint spectral radius. This makes the problem of computing the joint spectral radius, which is extremely hard in general, efficiently solvable. In this paper, we extend this construction onto a wide class of families with convex invariant cones (product families) and prove similar results not only for the maximal, but also for the minimal spectral radius. A new construction in the proof makes it possible to come up with two efficient methods of maximizing and minimizing the spectral radius of product families: the spectral simplex method and the two-level optimization method. They are presented and analyzed in Section 5. Finally, in Section 6, we give two examples of applications. The first application is the optimizing the rate of growth of solutions for a non-stationary difference equation. The second one is the optimizing the spectrum of a graph. Some other applications are also mentioned.

Let us start with some notations. Denote by E a finite-dimensional linear vector space and by E^* its dual space composed by all linear functions s on E. The value of s at x is denoted by $\langle s, x\rangle$. This *scalar product* allows to define the *adjoint operator* A^* of a linear operator $A : E \to E$ by identity

$$\langle s, Ax\rangle \;=\; \langle A^*s, x\rangle, \quad x \in E,\ s \in E^*.$$

Consider a linear operator $A : E \to E$ with an *invariant* closed convex solid pointed cone $K \subset E$:

$$AK \subseteq K. \tag{1.1}$$

By the Krein-Rutman theorem [7] there is an eigenvector $x(A)$ of the operator A such that

$$Ax(A) = \rho(A)\, x(A) \quad \text{and} \quad x(A) \in K, \tag{1.2}$$

where $\rho(A)$ is the *spectral radius* of A (the largest modulus of its eigenvalues). This vector will be referred as a *dominant eigenvector* of A, or a *Perron-Frobenius eigenvector*. Usually we normalize $x(A)$ so that $\langle e_*, x(A)\rangle = 1$, where the normalizing vector e_* belongs to interior of K^*, the cone *dual* to K:

$$K^* \;=\; \{s \in E^* :\ \langle s, x\rangle \geq 0,\ x \in K\} \;\subset\; E^*.$$

Under our assumptions on K, this cone is also closed convex solid and pointed.

Note that $x(A)$ is, in general, not unique (in this case we attribute this notation to any of them). Thus, the spectral radius satisfies $\rho(A) = \langle e_*, Ax(A)\rangle$. If $x(A) \in \operatorname{int} K$, then the operator A is called *irreducible*.

Condition (1.1) is satisfied if and only if $A^*K^* \subseteq K^*$. Hence, using an arbitrary vector $e \in \operatorname{int} K$, we can define in a similar way the dominant eigenvector $s(A) \in K^*$ of the adjoint operator: $A^*s(A) = \rho(A)\, s(A)$, $\langle s(A), e\rangle = 1$.

In the sequel, we often use the following simple observations.

Lemma 1 *Let A satisfy condition (1.1) and $\lambda \geq 0$.*

1. *If* $x \in \operatorname{int} K$, *and* $\lambda x - Ax \in K$, *then* $\lambda \geq \rho(A)$.

2. *If* $x \in \operatorname{int} K$, *and* $\lambda x - Ax \in \operatorname{int} K$, *then* $\lambda > \rho(A)$.

3. *If* $x \in K \setminus \{0\}$, *and* $Ax - \lambda x \in K$, *then* $\lambda \leq \rho(A)$.

4. *If* $x \in K$, *and* $Ax - \lambda x \in \operatorname{int} K$, *then* $\lambda < \rho(A)$.

Proof:
For proving Items 1 and 2, note that

$$0 \;\leq\; \langle s(A), \lambda x - Ax \rangle = \lambda \langle s(A), x \rangle - \langle A^* s(A), x \rangle \;=\; (\lambda - \rho(A)) \, \langle s(A), x \rangle.$$

Since $x \in \operatorname{int} K$, we have $\langle s(A), x \rangle > 0$. Moreover, under conditions of Item 2, the first inequality in the latter chain is strict.

In order to justify Item 3, consider the sequence $x_k \stackrel{\text{def}}{=} A^k x$ with $x \in K \setminus \{0\}$. Let us fix some $s \in K^*$ with $\langle s, x \rangle = 1$. By assumption of Item 3, we have $x_1 - \lambda x_0 \in K$. Therefore, by (1.1), we have $x_{k+1} - \lambda x_k \in K$, $k \geq 0$. Thus, $\langle s, A^k x \rangle = \langle s, x_k \rangle \geq \lambda^k$. However, $\rho(A)$ is an upper bound for the asymptotic growth of the norm of operator A^k. Hence, we obtain the desired inequality. Clearly, under conditions of Item 4, a small increase of λ does not violate conditions of Item 3. □

2 Optimization problems with spectral radius

Let $\mathcal{M}$ be a compact set of linear operators $A : E \to E$. The main goal of this paper is to develop efficient algorithmic tools for approximating the following values:

$$\rho^*(\mathcal{M}) \;=\; \max_{A \in \mathcal{M}} \rho(A), \tag{2.1}$$

$$\rho_*(\mathcal{M}) \;=\; \min_{A \in \mathcal{M}} \rho(A). \tag{2.2}$$

Of course, very often these problems are computationally intractable. Nevertheless, let us write down for them some necessary and sufficient optimality conditions based on the statements of Lemma 1. We consider the case, when our family satisfies the following assumption.

Assumption 1 *All operators of the family* $\mathcal{M}$ *share the same invariant cone* K.

Some conditions for a family of operators to have a common invariant cone can be found in [15, 13]. Let us look first at the maximization problem (2.1). Denote $\bar{\mathcal{M}} \stackrel{\text{def}}{=} \operatorname{Conv}(\mathcal{M})$,

$$L_0(\bar{A}) \;\stackrel{\text{def}}{=}\; \{A : \; (\bar{A} - A)\, x(\bar{A}) \in \operatorname{int} K\}, \quad L(\bar{A}) \;\stackrel{\text{def}}{=}\; \operatorname{cl}(L_0(A)),$$

$$U_0(\bar{A}) \;\stackrel{\text{def}}{=}\; \{A : \; (A - \bar{A})\, x(\bar{A}) \in \operatorname{int} K\}, \quad U(\bar{A}) \;\stackrel{\text{def}}{=}\; \operatorname{cl}(U_0(A)).$$

Assume that for some matrix $\bar{A} \in \mathcal{M}$ we have

$$x(\bar{A}) \in \operatorname{int} K, \quad L(\bar{A}) \supseteq \mathcal{M}. \tag{2.3}$$

This means that $\rho(\bar{A})\, x(\bar{A}) - A\, x(\bar{A}) \in K$ for any $A \in \mathcal{M}$. Hence, by Item 1 of Lemma 1, $\rho(\bar{A}) = \rho^*(\mathcal{M})$. Thus, condition (2.3) is a *sufficient* characterization of the optimal solution of problem (2.1).

Assume now that for two matrices A and $\bar{A}$ from $\mathcal{M}$ we have

$$(A - \bar{A})\, x(\bar{A}) \in \operatorname{int} K.$$

In view of Item 4 of Lemma 1 this means that $\bar{A}$ *cannot* be an optimal solution of the problem (2.1). Hence, the following condition is a *necessary* characterization of the optimal solution of problem (2.1)

$$U_0(\bar{A}) \;\bigcap\; \bar{\mathcal{M}} \;=\; \emptyset. \tag{2.4}$$

The later condition naturally leads to a simple local search procedure, which we call the *Spectral Analytic Center Method*. Let us fix for the cone K a convex barrier function $F(x)$, and let $A_0 \in \mathcal{M}$. Then we iterate the following operation:

$$A_{k+1} \;=\; \arg\min_{A \in \bar{\mathcal{M}}} F((A - A_k)\, x(A_k)). \tag{2.5}$$

If $U_0(A_k) \bigcap \bar{\mathcal{M}} \neq \emptyset$, then in view of Item 4 of Lemma 1 we have $\rho(A_{k+1}) > \rho(A_k)$. In general we cannot say more. However, in Section 5.1 we will show that for some special families of operators a similar scheme finds solution of problem (2.1) in a finite number of steps.

For minimization problem (2.2), the necessary and sufficient optimality conditions can be derived in a similar way. If for $\bar{A} \in \mathcal{M}$ we have

$$U(\hat{A}) \;\supseteq\; \bar{\mathcal{M}}, \tag{2.6}$$

then $\rho(\hat{A}) = \rho_*(\mathcal{M})$ (see Item 3 of Lemma 1). If $\hat{A}$ is an optimal solution to (2.2) and $x(\hat{A}) \in \operatorname{int} K$, then

$$L_0(\hat{A}) \;\bigcap\; \bar{\mathcal{M}} = \emptyset, \tag{2.7}$$

(see Item 2 of Lemma 1). Spectral Analytic Center Method for minimization problem (2.2) looks as follows:

$$A_{k+1} \;=\; \arg\min_{A \in \bar{\mathcal{M}}} F((A_k - A)\, x(A_k)). \tag{2.8}$$

However, it is justified only if we can ensure that $x(A_k) \in \operatorname{int} K$ for all $k \geq 0$.

Since problems (2.1), (2.2) are often difficult, let us define for them some *convex relaxations*. Denote

$$\begin{aligned} \lambda^*(\mathcal{M}) \;&=\; \inf_{\substack{\lambda \in R, \\ x \in \operatorname{int} K}} \{\lambda :\; \lambda x - Ax \in K \;\forall A \in \mathcal{M}\}, \\ \lambda_*(\mathcal{M}) \;&=\; \sup_{\substack{\lambda \in R, \\ x \in K}} \{\lambda :\; Ax - \lambda x \in K \;\forall A \in \mathcal{M}\}. \end{aligned} \tag{2.9}$$

Clearly,

$$\lambda_*(\mathcal{M}) \le \rho_*(\mathcal{M}) \le \rho^*(\mathcal{M}) \le \lambda^*(\mathcal{M}). \tag{2.10}$$

At the same time, characteristics (2.9) are computable. Choose some $e \in \operatorname{int} K, e_* \in \operatorname{int} K^*$ and denote $\Delta = \Delta_{e_*} = \{x \in K |\ \langle e_*, x\rangle = 1\}$ and $\Delta^* = \Delta_e^* = \{s \in K^* |\ \langle s, e\rangle = 1\}$. Since both cones K and K^* are solid and pointed, both sets Δ and Δ^* are compact. Consider the function

$$\psi^*(x, \lambda) = \max_{A\in\mathcal{M}} \max_{s\in\Delta^*} \langle s, Ax - \lambda x\rangle.$$

For fixed λ, $\psi^*(x, \lambda)$ is *convex* in x as a pointwise maximum over $\mathcal{M}$ the convex support functions $\psi_A^*(x) = \max\limits_{s\in\Delta^*} \langle s, Ax - \lambda x\rangle$.

If $\psi^*(x, \lambda) \le 0$ for some $x \in \operatorname{int} K$, then

$$\lambda x - Ax \in K \quad \forall A \in \mathcal{M},$$

and therefore $\lambda \ge \lambda^*(\mathcal{M})$. Note that we can justify validity of this upper bound by checking non-positivity of the following decreasing function:

$$\xi^*(\lambda) = \inf_{v\in\operatorname{int}\Delta} \psi^*(v, \lambda). \tag{2.11}$$

Since $\xi^*(\lambda)$ is continuous, the value $\lambda^*(\mathcal{M})$ coincides with its root. Later, in Section 5.2, we will discuss an efficient computational strategy for approximating this root. Here we just mention that the value $\xi^*(\lambda)$ is an optimal value of the convex minimization problem (2.11). As far as the function $\psi^*(x, \lambda)$ is computable, problem (2.11) can be solved up to a reasonable accuracy in polynomial time. We will consider important application examples in Section 6.

Similarly, the bound $\lambda_*(\mathcal{M})$ can be characterized by the following functions:

$$\begin{aligned} \psi_*(x, \lambda) &= \max_{A\in\mathcal{M}} \max_{s\in\Delta^*} \langle s, \lambda x - Ax\rangle, \\ \xi_*(\lambda) &= \inf_{x\in\Delta} \psi_*(x, \lambda). \end{aligned} \tag{2.12}$$

If $\xi_*(\lambda) \le 0$, then $\lambda \le \lambda_*(\mathcal{M})$. Thus, $\lambda_*(\mathcal{M})$ is a root of the increasing function $\xi_*(\lambda)$.

Finally, let us extend our considerations onto the *joint spectral radius* (JSR) and the *lower spectral radius* (LSR) of families of operators. The joint spectral radius $\sigma^*(\mathcal{M}^*)$, is the exact upper bound on the asymptotic growth of the discrete-time switching linear system

$$s_0 \in E^*, \quad s_{k+1} = A_k^* s_k,\ A_k \in \mathcal{M},\ k \ge 0. \tag{2.13}$$

Formally it is defined as $\sigma^*(\mathcal{M}^*) = \lim\limits_{k\to\infty} \max_{B\in(\mathcal{M}^*)^k} \|B\|^{1/k}$, where $(\mathcal{M}^*)^k$ is the set of all products (with repetition permitted) of length k of operators from $\mathcal{M}^*$. JSR has numerous applications and a rich history, see [1, 14] and the bibliography there. Clearly, for JSR we have a trivial lower bound $\rho^*(\mathcal{M}) \le \sigma^*(\mathcal{M}^*)$. In order to get an upper bound, without loss of generality we can consider $s_0 \in K^*$ (otherwise, we represent s_0 as a difference of two vectors in K^*). If $s_0 \in K^*$, then all $s_k \in K^*$. Let us choose some vector $x \in \operatorname{int} K$ and $\lambda > 0$ such that

$$\lambda x - Ax \in K \quad \forall A \in \mathcal{M}.$$

Then

$$\langle s_{k+1}, x\rangle = \langle A_k^* s_k, x\rangle = \langle s_k, A_k x\rangle \le \lambda \langle s_k, x\rangle.$$

Hence, $\lambda \ge \sigma^*(\mathcal{M}^*)$, and we obtain the following sandwich inequality:

$$\rho^*(\mathcal{M}) \le \sigma^*(\mathcal{M}^*) \le \lambda^*(\mathcal{M}). \tag{2.14}$$

In general, among these values, only $\lambda^*(\mathcal{M})$ is computationally tractable. The two other values usually need extraordinary high volume of computations (see the bibliography in [1]). Nevertheless, in the next sections we point out some nontrivial families of matrices, for which the left-hand side of inequality (2.14) coincides with its right-hand side, and therefore all the three values are computable. It is interesting that the sufficient condition (2.3) guarantees that $\rho(\bar{A}) \ge \lambda^*(\mathcal{M})$, and therefore all the inequalities in (2.14) become equalities.

Consider now the *lower spectral radius* (LSR) $\sigma_*(\mathcal{M}^*)$, which is an exact lower bound on the asymptotic maximal growth of the discrete-time switching linear system (2.13). It is defined in a similar way as JSR, but with minimum instead of maximum: $\sigma_*(\mathcal{M}^*) = \lim\limits_{k\to\infty} \min_{B\in(\mathcal{M}^*)^k} \|B\|^{1/k}$. Let us remark that the lower spectral radius is usually harder to compute than JSR [14]. Clearly, $\rho_*(\mathcal{M}) \ge \sigma_*(\mathcal{M}^*)$. For obtaining a lower bound for this value, we can assume again that $s_0 \in K^*$; then all $s_k \in K^*$. Let us choose some vector $x \in \operatorname{int} K$ and $\lambda > 0$ such that

$$Ax - \lambda x \in K \quad \forall A \in \mathcal{M}.$$

Then

$$\langle s_{k+1}, x\rangle = \langle A_k^* s_k, x\rangle = \langle s_k, A_k x\rangle \ge \lambda \langle s_k, x\rangle.$$

Hence, $\sigma_*(\mathcal{M}^*) \ge \lambda$, and we obtain another sandwich inequality:

$$\rho_*(\mathcal{M}) \ge \sigma_*(\mathcal{M}^*) \ge \lambda_*(\mathcal{M}). \tag{2.15}$$

Again, the sufficient condition (2.6) guarantees that $\rho(\hat{A}) \le \lambda_*(\mathcal{M})$, and therefore all inequalities in (2.15) become equalities.

In the subsequent sections we will show that for some special families of operators one can efficiently solve optimization problems (2.1) and (2.2) even if the uncertainty set $\mathcal{M}$ has a combinatorial structure.

3 Product families

Consider now some families of operators with special structure. Let us introduce an intermediate space E_1 with partial ordering defined by a pointed convex cone K_1. We fix a basic operator $B : E_1 \to E$, such that $BK_1 \subseteq K$. Consider now a family $\mathcal{F}$ composed by operators $F : E \to E_1$. Then we can define our main family of operators as

$$\mathcal{M} = \{A = BF,\, F \in \mathcal{F}\}.$$

Assuming $FK \subseteq K_1$ for any $F \in \mathcal{F}$, we guarantee that all operators from family $\mathcal{M}$ share the same invariant cone K.

Lemma 2 *Assume that for $\bar{A} = B\bar{F} \in \mathcal{M}$ with $x(\bar{A}) \in \operatorname{int} K$ we have*

$$(\bar{F} - F)\, x(\bar{A}) \in K_1 \quad \forall F \in \mathcal{F}. \tag{3.1}$$

Then $\rho(\bar{A}) = \rho^*(\mathcal{M}) = \sigma^*(\mathcal{M}) = \lambda^*(\mathcal{M})$.
If $x(\bar{A}) \in K$, and $(F - \bar{F})\bar{x} \in K_1$ for all $F \in \mathcal{F}$, then $\rho(\bar{A}) = \rho_*(\mathcal{M}) = \sigma_*(\mathcal{M}) = \lambda_*(\mathcal{M})$.

Proof:
Indeed, in view of conditions (3.1), for any $A \in \mathcal{M}$ we have

$$\rho(\bar{A})\, x(\bar{A}) - Ax(\bar{A}) = (\bar{A} - A)\, x(\bar{A}) = B\,(\bar{F} - F)\, x(\bar{A}) \overset{(3.1)}{\in} K.$$

Thus, $\rho(\bar{A}) \overset{(2.9)}{\geq} \lambda^*(\mathcal{M})$. It remains to use (2.14). For the second statement, note that:

$$Ax(\bar{A}) - \rho(\bar{A})\, x(\bar{A}) = (A - \bar{A})\, x(\bar{A}) = B\,(F - \bar{F})\, x(\bar{A}) \overset{(3.1)}{\in} K.$$

Hence, $\lambda_*(\mathcal{M}) \geq \rho(\bar{A}) \geq \rho_*(\mathcal{M})$. It remains to use (2.15). □

Recall that the operator A is called *positive* on K if $Ax \in \operatorname{int} K$ for all $x \in K \setminus \{0\}$.

Lemma 3 *For matrix $A = BF$, assume that there exists $\tilde{F}$ such that*

$$d \overset{\text{def}}{=} (\tilde{F} - F)\, x(A) \in K_1, \quad d \neq 0.$$

If $\tilde{A} = B\tilde{F}$ is positive on K, then $\rho(\tilde{A}) > \rho(A)$. If $(F - \tilde{F})\, x(A) \in K_1 \setminus \{0\}$, and $\tilde{A} = B\tilde{F}$ is positive on K, then $\rho(\tilde{A}) < \rho(A)$.

Proof:
Denote $\hat{x} \overset{\text{def}}{=} \tilde{A}\, x(A) = B\,(F + (\tilde{F} - F))\, x(A) = \rho(A)x(A) + Bd$. Then

$$\tilde{A}\hat{x} = \rho(A)\tilde{A}x(A) + \tilde{A}Bd = \rho(A)\hat{x} + \tilde{A}Bd.$$

Since $\tilde{A}$ is positive on K, we have $\tilde{A}Bd \in \operatorname{int} K$. Hence, by Item 3 of Lemma 1, we obtain $\rho(\tilde{A}) > \rho(A)$. For proving the second part of the lemma, we use Item 2 of Lemma 1. □

In the remainder of this section we assume that

$$Fx \in \operatorname{int} K_1 \quad \forall x \in K \setminus \{0\}.$$

Thus, all operators $A \in \mathcal{M}$ are *positive on K*: $Ax \in \operatorname{int} K$ for all $x \in K \setminus \{0\}$.

In order to prove the main result of this section, we need to introduce additional structural assumptions.

Assumption 2 *Cone K_1 is a positive orthant: $K_1 \equiv R^N_+$.*

Under this assumption, the operators B and F can be associated with the matrices:

$$\begin{aligned} B &= (B_1, \ldots, B_N), \quad F^T = (F_1, \ldots, F_N), \\ B_j &\in K, \quad F_j \in K^*, \quad j = 1, \ldots, N. \end{aligned} \tag{3.2}$$

Then the matrix of operator A is represented as $A = BF$. From now on, we use the same notation for operators and for corresponding matrices.

In this situation, the statement of Lemma 3 can be rewritten as follows.

Theorem 1 *Let $x = x(A) \in K$ be a dominant eigenvector of a positive operator $A \in \mathcal{M}$. Assume that for some index j and some $\tilde{f} \in \operatorname{int} K^*$ we have $\langle \tilde{f}, x \rangle > \langle F_j, x \rangle$. Then for*

$$\tilde{A} = A + B_j(\tilde{f} - F_j)^T$$

we have $\rho(\tilde{A}) > \rho(A)$. If $\langle \tilde{f}, x \rangle < \langle F_j, x \rangle$, then $\rho(\tilde{A}) < \rho(A)$.

Proof:
Indeed, denoting by $\tilde{F}$ the matrix F with jth row replaced by $\tilde{f}^T$, we get $(\tilde{F} - F)x(A) \in R^N_+$. It remains to use Lemma 3. The proof of the second statement is similar. □

For given compact sets $\mathcal{F}_j \subset K^*$, $j = 1, \ldots, N$, we define a family of operators

$$\mathcal{M} = \mathcal{M}[\mathcal{F}_1, \ldots, \mathcal{F}_N] = \{A = BF,\ F_j \in \mathcal{F}_j,\ j = 1, \ldots, N\} \tag{3.3}$$

Such families with *independent* column uncertainty sets are called *product families*. A product family is positive on K if $\mathcal{F}_j \subset \operatorname{int} K^*$ for all $j = 1, \ldots, N$.

Example 1 *(see [1]) The simplest example of product family is a set of nonnegative $n \times n$-matrices with independent column uncertainty sets. In this case $K = R^n_+$, $N = n$, and B is an identity operator. The family $\mathcal{M}$ consists of all matrices such that for every $j = 1, \ldots, n$ the jth column belongs to $\mathcal{F}_j$. This product family is positive on K if all column uncertainty sets consist of strictly positive vectors. In this case, all matrices in $\mathcal{M}$ are strictly positive.*

Theorem 2 *For any positive product family we have*

$$\lambda^*(\mathcal{M}) = \sigma^*(\mathcal{M}) = \rho^*(\mathcal{M}), \quad \lambda_*(\mathcal{M}) = \sigma_*(\mathcal{M}) = \rho_*(\mathcal{M}). \tag{3.4}$$

Proof:
Denote by $\bar{A} \equiv B\bar{F}$ the solution of optimization problem (2.1), and by $\bar{x}$ its dominant eigenvector. In view of Theorem 1, we have

$$\langle F_j, \bar{x} \rangle \ \leq \ \langle \bar{F}_j, \bar{x} \rangle, \quad \forall F_j \in \mathcal{F}_j,\ j = 1, \ldots, N. \tag{3.5}$$

Therefore,

$$\rho(\bar{A})\, \bar{x} - BF\bar{x} = B(\bar{F} - F)\bar{x} \overset{(3.2)}{\in} K \quad \forall F \in \mathcal{F}.$$

Hence, $\lambda^*(\mathcal{M}) \overset{(2.9)}{\leq} \rho(\bar{A})$ and all inequalities in (2.14) are satisfied as equalities. The second chain of equalities can be proved in the same way. □

Corollary 1 *For product families, the values $\rho^*(\mathcal{M})$ and $\rho_*(\mathcal{M})$ are not changing if we replace the set $\mathcal{F}$ by its convex hull. In any case, the optimal solutions of these problems are extreme points of the feasible sets.*

Proof:
Indeed, in the proof of Theorem 2 we need conditions (3.5) to be satisfied only by extreme points $\bar{F}_j$ of the feasible sets $\mathcal{F}_j$. □

4 Extension to nonnegative product families

Now we are going to extend the results of the previous section to general product families of the type (3.3) without strict positivity assumption $\mathcal{F}_j \subset \operatorname{int} K^*$. For that, we take an arbitrary positive on K operator H and consider a positive family $\mathcal{M}^\epsilon = \{A+\epsilon H, A \in \mathcal{M}\}$. In particular, for product families we take an arbitrary $h \in \operatorname{int} K^*$, and define the following perturbed families $\mathcal{F}_j^\epsilon = \mathcal{F}_j + \epsilon h$, $\epsilon > 0$. The corresponding family $\mathcal{M}^\epsilon$ is positive. Now we are going to study the continuity of our characteristics as $\epsilon \to 0$. We need some auxiliary results.

Lemma 4 *For any compact family $\mathcal{M}$ with an invariant cone, the function $\sigma^*(\mathcal{M}^\epsilon)$ is continuous and the function $\sigma_*(\mathcal{M}^\epsilon)$ is lower semicontinuous in ϵ.*

Proof:
For any m denote $H_m = \sup\limits_{\alpha_j \in \mathcal{A}} \|A_{\alpha_1} \cdots A_{\alpha_m}\|^{1/m}$, $R_m = \sup\limits_{\alpha_j \in \mathcal{A}} \rho^{1/m}(A_{\alpha_1} \cdots A_{\alpha_m})$, and $h_m = \inf\limits_{\alpha_j \in \mathcal{A}} \|A_{\alpha_1} \cdots A_{\alpha_m}\|^{1/m}$. All these functions are continuous in ϵ for each m. Since $\sigma^*(\mathcal{M}^\epsilon) = \inf_{m \in \mathcal{N}} H_m$, and $\sigma_*(\mathcal{M}^\epsilon) = \inf_{m \in \mathcal{N}} h_m$, both of them are lower semicontinuous. Moreover, since $\sigma^*(\mathcal{M}^\epsilon) = \sup_{m \in \mathcal{N}} R_m$, this function is upper semicontinuous, and therefore continuous. □

Corollary 2 *For any family $\mathcal{M}$ with an invariant cone we have*

$$\sigma^*(\mathcal{M}^\epsilon) \to \sigma^*(\mathcal{M}), \quad \sigma_*(\mathcal{M}^\epsilon) \to \sigma_*(\mathcal{M})$$

as $\epsilon \to 0$.

Proof:
For the joint spectral radius, the statement follows directly from Lemma 4. For the lower spectral radius, note that each operator $A^\epsilon \in \mathcal{M}^\epsilon$ is nondecreasing in ϵ. Therefore, the function $\sigma_*(\mathcal{M}^\epsilon)$ is nondecreasing in ε as well. Combining this observation with its lower semicontinuity (Lemma 4), we see that this function is continuous as $\epsilon \to 0$. □

Theorem 3 *For any family $\mathcal{M}$ with invariant cone we have*

$$\lambda^*(\mathcal{M}^\epsilon) \to \lambda^*(\mathcal{M}), \quad \lambda_*(\mathcal{M}^\epsilon) \to \lambda_*(\mathcal{M})$$

as $\epsilon \to 0$.

Proof:
Since the value $\lambda^*(\mathcal{M}^\epsilon)$ is non-decreasing in ϵ, its continuity at zero is a direct consequence of the open domain in the definition (2.9) of $\lambda^*(\mathcal{M})$.

Let us establish the second statement. Note that the function $\lambda_*(\mathcal{M}^\epsilon)$ is non-decreasing in $\epsilon > 0$. Assume it converges to $q > \lambda_*(\mathcal{M})$ as $\epsilon \to 0$. Let us take an arbitrary λ on the interval $(\lambda_*(\mathcal{M}), q)$. For any $\epsilon > 0$ we can find x_ϵ such that $\|x_\epsilon\| = 1$, and $A^\epsilon x_\epsilon - \lambda x_\epsilon \in K$ for all $A^\epsilon \in \mathcal{M}^\epsilon$. By the compactness of $\mathcal{M}$, there exists a sequence $\epsilon^k \to 0$ such that the corresponding vectors $x_k = x_{\epsilon_k}$ converge to some $x \in K$, $\|x\| = 1$. Note that for any $A^{\epsilon_k} \in \mathcal{M}^{\epsilon_k}$, we have $A^{\epsilon_k} x_k - \lambda x_k \in K$. Taking a limit as $k \to \infty$, we get $Ax - \lambda x \in K$ for all $A \in \mathcal{M}$ (we use the fact that x_k does not converge to zero). Thus, $\lambda_*(\mathcal{M}) \geq \lambda$, which contradicts to our assumption. □

Combining this with Theorem 2 we obtain the following result.

Theorem 4 *Any product family (3.3) possesses property (3.4).*

5 Computing the optimal spectral radius for product families

Theorems 1 and 4 suggest at least two efficient strategies for finding the optimal spectral radius for product families.

5.1 Spectral simplex method

Let us assume that all sets $\mathcal{F}_j$ are either finite, or they are polyhedral sets defined by the systems of linear inequalities. Assume first that the product family $\mathcal{M}$ is positive. Consider the following strategy for maximizing the spectral radius.[1]

Input: Choose arbitrary $F^0 \in \mathcal{F}$. Define $A_0 = BF^0$. **Iteration** $k \geq 0$**:**

1. Compute the dominant eigenvector x_k of $A_k = BF^k$.
2. Find j_k: $\langle F^k_{j_k}, x_k\rangle < \langle \tilde{f}^k_{j_k}, x_k\rangle$, where $\tilde{f}^k_j \stackrel{\rm def}{=} \arg\max\limits_{F_j \in \mathcal{F}_j} \langle F_j, x_k\rangle$, $j = 1, \ldots, N$.
3. If j_k is not found, then STOP. Otherwise, $F^{k+1} = F^k + e_{j_k}(\tilde{f}^k_{j_k} - F^k_{j_k})^T$.

(5.1)

[1]For minimizing the spectral radius, the scheme must be adapted in a straightforward manner.

In this scheme, e_j is the jth coordinate vector in R^N.

By Theorem 1, we have $\rho(A_{k+1}) > \rho(A_k)$. In the case of polyhedral set $\mathcal{F}$, we can choose F^0 to be its extreme point. Then all other vectors F^k will be also the vertices of $\mathcal{F}$. Since the number of vertices is finite, our algorithm terminates in a finite time. The convergence and the rate of convergence of the spectral simplex method for general $\mathcal{F}$ in not justified yet.

If $\mathcal{M}$ is not necessarily positive, then we can take a perturbed family $\mathcal{M}^\epsilon$ for arbitrarily small $\epsilon > 0$ and run for it the spectral simplex-method. The accuracy of such an approximation can be derived from the standard results on the continuity of eigenvalues.

5.2 Two-level optimization

As we have seen in Section 2, value $\lambda^*(\mathcal{M})$ can be characterized as a unique root of a decreasing function $\xi^*(\lambda)$. The value of the latter function is an optimal value of the convex optimization problem (2.11). Hence, it can be computed with any necessary accuracy. However, we are going to use this information in a localization procedure. Therefore, in order to estimate the complexity of the whole process, we need some bounds on the growth of function $\xi^*(\lambda)$ in the neighborhood of its root.

Let $\bar{A}$ be the optimal solution of the maximization problem (2.1). Denote by $\bar{x} \in \Delta_{e_*} \subset K$ its dominant eigenvector, and by $\bar{s} \in \Delta_e \subset K^*$ the dominant eigenvector of operator $\bar{A}^*$. Thus,

$$\bar{A}\bar{x} = \lambda^* \bar{x}, \quad \bar{A}^* \bar{s} = \lambda^* \bar{s},$$

where $\lambda^* \stackrel{\text{def}}{=} \rho^*(\mathcal{M})$. Denote

$$\gamma_L = \max_{x \in \Delta_{e^*}} \langle \bar{s}, x \rangle, \quad \mu_L = \min_{x \in \Delta_{e^*}} \langle \bar{s}, x \rangle,$$

$$\gamma_R = \max_{s \in \Delta_e} \langle s, \bar{x} \rangle, \quad \mu_R = \min_{s \in \Delta_e} \langle s, \bar{x} \rangle.$$

Let us assume that $\mu_L > 0$ and $\mu_R > 0$.

Theorem 5 *For any $\lambda \in R$ we have*

$$\min\{\gamma_L(\lambda^* - \lambda), \mu_L(\lambda^* - \lambda)\} \leq \xi^*(\lambda) \leq \max\{\gamma_R(\lambda^* - \lambda), \mu_R(\lambda^* - \lambda)\}. \tag{5.2}$$

Proof:
Indeed,

$$\xi^*(\lambda) = \inf_{x \in \Delta_{e_*}} \max_{A \in \mathcal{M}} \max_{s \in \Delta_e} \langle s, Ax - \lambda x \rangle \geq \min_{x \in \Delta_{e_*}} \langle \bar{s}, \bar{A}x - \lambda x \rangle$$

$$= \min_{x \in \Delta_{e_*}} (\lambda^* - \lambda) \langle \bar{s}, x \rangle = \min\{\gamma_L (\lambda^* - \lambda), \mu_L (\lambda^* - \lambda)\}.$$

Further, since λ^* is a root of function $\xi^*(\,)$, we have

$$0 = \psi(\bar{v}, \lambda^*) = \max_{A \in \mathcal{M}} \max_{s \in \Delta_e} \langle s, A\bar{x} - \lambda^* \bar{x} \rangle.$$

This means that for all $s \in K^*$ and $A \in \mathcal{M}$ we have $\langle s, \lambda^* \bar{x} - A\bar{x} \rangle \geq 0$. Therefore,

$$\lambda^* \bar{x} - A\bar{x} \in K \quad \forall A \in \mathcal{M}.$$

Thus, we conclude that

$$\begin{aligned} \xi^*(\lambda) &= \inf_{x \in \Delta_{e_*}} \max_{A \in \mathcal{M}} \max_{s \in \Delta_e} \langle s, Ax - \lambda x \rangle \leq \max_{A \in \mathcal{M}} \max_{s \in \Delta_e} \langle s, (A\bar{x} - \lambda^* \bar{x}) + (\lambda^* - \lambda)\bar{x} \rangle \\ &\leq \max_{s \in \Delta_e} \langle s, (\lambda^* - \lambda)\bar{x} \rangle = \max\{\gamma_R (\lambda^* - \lambda), \mu_R (\lambda^* - \lambda)\}. \end{aligned}$$

□

Now we can use the above bounds for justifying the complexity estimates of the following search procedure based on *Golden Section*. Denote $\tau_1 \stackrel{\text{def}}{=} \frac{3-\sqrt{5}}{2} < \tau_2 \stackrel{\text{def}}{=} \frac{\sqrt{5}-1}{2}$.

Input: Choose λ_0, $\xi^*(\lambda_0) > 0$, and $\delta_0 > 0$ such that $\xi^*(\lambda_0 + \delta_0) < 0$.

Iteration $k \geq 0$**.**

1. Define two trial values $\lambda'_k = \lambda_k + \tau_1 \delta_k$, and $\lambda''_k = \lambda_k + \tau_2 \delta_k$.
2. Start *in parallel* two processes for minimizing functions $\psi^*(\cdot, \lambda'_k)$ and $\psi^*(\cdot, \lambda''_k)$. (5.3)
3. Stop both processes after justification of one of the statements:

   Case A: $\xi^*(\lambda'_k) > 0$, Case B: $\xi^*(\lambda''_k) < 0$.
4. Set $\delta_{k+1} = \tau_2 \delta_k$. In Case A set $\lambda_{k+1} = \lambda'_k$. Otherwise $\lambda_{k+1} = \lambda_k$.

The idea of this scheme is very simple. Since $\lambda^* \in [\lambda_k, \lambda_k + \delta_k]$, we always have

$$\max\{|\lambda'_k - \lambda^*|, |\lambda''_k - \lambda^*|\} \geq \max\{\tau_1, \tau_2 - \tau_1, 1 - \tau_2\} \delta_k = \tau_1 \delta_k.$$

Therefore, in view of Theorem 5, the necessary accuracy for minimizing functions $\psi^*(\cdot, \lambda'_k)$ and $\psi^*(\cdot, \lambda''_k)$ does not exceed $O(\delta_k)$. If we solve these problems by a polynomial-time method, which needs $O(p(n) \ln \frac{1}{\epsilon})$ iterations for generating an ϵ-solution to a convex problem, then at the kth iteration of the scheme (5.3) we run this method at most for

$$N_k = O\left(p(n) \ln \tfrac{1}{\delta_k}\right) = O(p(n)\, k)$$

steps. Hence, k steps of the scheme (5.3) need at most $O(p(n)\, k^2)$ iterations of the lower-level method. Since the total number of iterations of the upper-level process is bounded by $O(\ln \frac{1}{\epsilon})$, where ϵ is the target accuracy of approximation for λ^*, we conclude that our scheme (5.3) has polynomial-time complexity.

We can apply similar technique for approximating $\lambda_*(\mathcal{M})$. Since all reasonings are very similar to the maximization case, we omit the proofs.

Let $\hat{A}$ be the optimal solution of the minimization problem (2.2). Denote by $\hat{x} \in \Delta_{e_*} \subset K$ its dominant eigenvector, and by $\hat{s} \in \Delta_e \subset K^*$ the dominant eigenvector of the operator $\hat{A}^*$. Denote $\lambda_* = \rho_*(\mathcal{M})$, and

$$\hat{\gamma}_L \;=\; \max_{x \in \Delta_{e_*}} \langle \hat{s}, x \rangle, \quad \hat{\mu}_L \;=\; \min_{x \in \Delta_{e_*}} \langle \hat{s}, x \rangle,$$

$$\hat{\gamma}_R \;=\; \max_{s \in \Delta_e} \langle s, \hat{x} \rangle, \quad \hat{\mu}_R \;=\; \min_{s \in \Delta_e} \langle s, \hat{x} \rangle.$$

Let us assume that $\hat{\mu}_L > 0$ and $\hat{\mu}_R > 0$.

Theorem 6 *For any $\lambda \in R$ we have*

$$\min\{\hat{\gamma}_L(\lambda - \lambda_*), \hat{\mu}_L(\lambda - \lambda_*)\} \leq \;\; \xi_*(\lambda) \;\; \leq \max\{\hat{\gamma}_R(\lambda - \lambda_*), \hat{\mu}_R(\lambda - \lambda_*)\}. \tag{5.4}$$

Thus, in order to compute an approximation to $\lambda_*(\mathcal{M})$, we can apply a version of method (5.3) subject to appropriate modifications.

6 Application examples

In this section we consider several important applications related to optimization of spectral radius. Of course, this list is far from being complete.

6.1 Nonstationary difference equations

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For the sake of simplicity, we focus on univariate difference equations, keeping in mind that the results below can be easily extended onto multivariate case. Consider a finite difference equation in the form

$$x_k \;=\; \sum_{j=1}^{m} q_j \, x_{k-j} \, , \qquad k \;\geq\; m+1 \, , \tag{6.1}$$

where $\bar{q} \;=\; \{q_j\}_1^m$ is a given vector of coefficients. It is well-known that the general solution $\{x_k\}$ for this equation can be written explicitly in terms of the roots of characteristic polynomial. All solutions form an m-dimensional linear space, and any particular solution is uniquely defined by the initial conditions, i.e., by the values $x_1, \ldots, x_m$.

The largest and the smallest possible rates of asymptotic growth of the sequence $\{x_k\}$ are $|\lambda_1|^k \, k^{\,\mu}$ and $|\lambda_m|^k$ respectively, where λ_1 and λ_m are the largest and the smallest (by modulo) roots of the characteristic polynomial, and μ is the multiplicity of the root λ_1 (e.g. [9], [11]).

Now assume that the sequence of coefficients $\bar{q}$ is not fixed, and at any step k of the recurrence we can chose independently any $\bar{q}$ from a given uncertainty set $\mathcal{Q} \subset \mathbf{R}^m$. The question is as follows:

What are the largest and the smallest possible rates of asymptotic growth of the sequence $\{x_k\}$?

By the rate of growth we understand the value $\limsup_{k\to\infty} \frac{1}{k} \ln |x_k|$. What strategy of choosing the vector coefficients $\bar{q}_k \in \mathcal{Q}$ provides the largest and, respectively, the smallest growth? These questions, which are very difficult in general, have simple solution in the case when all vectors $\bar{q} \in \mathcal{Q}$ are nonnegative.

Let $\mathcal{Q}$ be a given compact subset of $\mathbf{R}^m_+$ and let the initial values $x_1, \ldots, x_m$ be positive. Actually, we assume positivity of the starting values only for the sake of simplicity. Similar analysis can be made for arbitrary initial data.

In order to formulate the statement, we need some notation. We say that sequence $\{\bar{q}_i\}$ is stationary, if $\bar{q}_i$ are the same for all i. Furthermore, for any $\bar{q} \in \mathcal{Q}$ we write $A_{\bar{q}}$ for the $m \times m$-matrix, whose first row is $\bar{q}$ and the others are respectively $e_1, \ldots, e_{m-1}$ (the coordinate row vectors):

$$A_{\bar{q}} \quad = \quad \begin{pmatrix} q_1 & q_2 & q_3 & \ldots & q_{m-1} & q_m \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix},$$

Denote $\mathcal{M} = \{A_{\bar{q}} : \bar{q} \in \mathcal{Q}\}$.

Proposition 1 *Under the assumptions above, the largest and the smallest rates of growth of the sequence $\{x_k\}$ are both achieved at stationary sequences.*

For the largest rate of growth we have $x_k \sim \lambda_1^k$, and for the smallest growth we have $x_k \sim \lambda_0^k$, where $\lambda_1 = \rho(A_{\bar{q}_1}) = \max\limits_{\bar{q}\in\mathcal{Q}} \rho(A_{\bar{q}})$ and $\lambda_0 = \rho(A_{\bar{q}_0}) = \min\limits_{\bar{q}\in\mathcal{Q}} \rho(A_{\bar{q}})$.

Proof:
Denote $y_k = (x_k, \ldots, x_{k-m+1})$, $k \geq m$. Then equation (6.1) can be written as $y_k = A_{\bar{q}}\, y_{k-1}$. Therefore $\|y_k\| = \|A_{k-m} \cdots A_1 y_1\|$, where we use a short notation $A_j = A_{\bar{q}_j}$. For any norm in $\mathbf{R}^m$, the largest value $\|A_{k-m} \cdots A_1 y_1\|$ over all matrices $A_1, \ldots, A_{k-m} \in \mathcal{M}$ is asymptotically equivalent to $[\sigma^*(\mathcal{M})]^{k-m}$, and the smallest value is equivalent to $[\sigma_*(\mathcal{M})]^{k-m}$ [12]. Therefore, the largest and the smallest x_k are asymptotically proportional to these values, and hence to the values $[\sigma^*(\mathcal{M})]^k$ and $[\sigma_*(\mathcal{M})]^k$ respectively. On the other hand, the family $\mathcal{M}$ has the product structure with one uncertainty set $\mathcal{F}_1 = \mathcal{Q}$ for the first row (the other rows of the matrices are fixed). Applying now Theorem 4, we conclude the proof. □

Thus, in the case of nonnegative uncertainty set of coefficients for equation (6.1), any mixing strategy, based on different vectors $\bar{q}$, is not optimal. The fastest growth of the sequence $\{x_k\}$ is attained at a stationary sequence, when we take the same vector of coefficients every time. This vector corresponds to the maximal spectral radius of matrices $A \in \mathcal{M}$, and can be found by the technique discussed in Section 5. The rate of growth λ_1 can be found by solving optimization problem (2.1). For the slowest growth, we can apply similar technique.

Note that the Perron-Frobenius eigenvector of matrix $A_{\bar{q}}$ is $v = (\lambda^{m-1}, \ldots, \lambda, 1)$, where $\lambda = \rho(A_{\bar{q}})$ is the largest positive root of the characteristic equation $t^m = \sum_{j=1}^{m} q_j\, t^{m-j}$. This vector is always positive, whenever $\bar{q} \neq 0$.

6.2 Optimizing the spectral radius for graphs

One of the classical problems of spectral graph theory consists of finding the largest (or smallest) possible spectral radius of a graph under some conditions (level of connectivity, bounds for the weighted sums of edges and the largest degrees of some vertices, etc.) The spectral radius of a graph is by definition the spectral radius of its adjacency matrix A ($A_{ij} = 1$ if there is an edge from the vertex j to the vertex i; otherwise $A_{ij} = 0$). Most of the results on optimizing the graph spectral radius are related to undirected graphs, which have symmetric adjacency matrices [3]-[16]. At the same time, there exist well-known results on directed graphs, related to maximization of spectral radius of arbitrary $0-1$-matrices [2, 5]. Since adjacency matrices are nonnegative, we can apply our results both to maximizing and minimizing the spectral radius of the graphs. In both cases, we get an approximate optimal value of our problem. However, if the set of graphs possesses a product structure property (either by rows or by columns), we can approach an exact optimal value.

We give two examples of problems for undirected graphs, which can be solved numerically by applying our technique.

Problem 1. Consider the set of vertices $g_1, \ldots, g_n$. For any vertex g_i we are given a finite family $\mathcal{F}_i$ of subsets of these vertices. Consider the set of all directed graphs such that the set of incoming edges (more precisely, the set the corresponding adjacent vertices) for any vertex g_i belongs to $\mathcal{F}_i,\ i = 1, \ldots, n$. The problem is to compute the largest and the smallest spectral radius for graphs of this set.

Problem 2. We are given a set of vertices $g_1, \ldots, g_n$ and a set of nonnegative integers $d_1, \ldots d_n$. The problem is to find graphs with the largest and the smallest spectral radius among all directed graphs such that the number of incoming edges for any vertex g_i equals to $d_i,\ i = 1, \ldots, n$.

Replacing rows by columns, we obtain the similar problems for the sets of outgoing edges instead of incoming ones. The set of adjacency matrices in both cases has a product structure. For the first problem all uncertainty sets $\mathcal{F}_i$ are finite, for the second one they are polyhedral and given by the systems of inequalities: $\mathcal{F}_i = \{x \in \mathbf{R}^n \mid \sum_{k=1}^{n} x_k \leq d_i\,,\ 0 \leq x_k \leq 1,\ k = 1, \ldots, n\}$. The minimal and maximal spectral radii are both attained at extreme points of the uncertainty sets (Corollary 1), i.e., precisely when the the ith row has d_i ones and all other entries are zeros.

To conclude with examples, let us mention that there is a wide area of applications of our results related to the Markov processes in the graphs.

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