DC APPROACH TO REGULARITY OF CONVEX MULTIFUNCTIONS WITH APPLICATIONS TO INFINITE SYSTEMS[1]

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Abstract. The paper develops a new approach to the study of metric regularity and related well-posedness properties of convex set-valued mappings between general Banach spaces by reducing them to unconstrained minimization problems with objectives given as the difference of convex (DC) functions. In this way we establish new formulas for calculating the exact regularity bound of closed and convex multifunctions and apply them to deriving explicit conditions ensuring well-posedness of infinite convex systems described by inequality and equality constraints.

Key Words: variational analysis, optimization, convex multifunctions, metric regularity

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1 Introduction

Let X and Y be Banach spaces with norms generically denoted by $\|\cdot\|$. For any $x \in X$ and $r > 0$ the symbol $I\!B_r(x)$ stands for the closed ball centered at x with radius r, while the unit closed ball and the unit sphere in X are denoted by $I\!B_X$ and S_X, respectively. Recall that a set-valued mapping/multifunction $F : X \rightrightarrows Y$ is *metrically regular* around $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ with modulus $K > 0$ if there exist neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$d\big(x; F^{-1}(y)\big) \le K d\big(y; F(x)\big) \quad \text{for any } \; x \in U, \; y \in V, \tag{1.1}$$

where $d(x;\Omega)$ stands for the distance from x to Ω in X. The infimum of all the moduli K over (K, U, V) in (1.1), denoted by $\operatorname{reg} F(\bar{x}, \bar{y})$, is called the *exact regularity bound*.

It has been well recognized that the concept of metric regularity is fundamental in nonlinear analysis and optimization and is used not only in theoretical studies but also in numerical methods. For closed and convex multifunctions (i.e., those with the closed and convex graph $\operatorname{gph} F := \{(x, y) \in X \times Y |\; y \in F(x)\}$), this notion is characterized by the *Robinson-Ursescu Theorem*, which says that F is metrically regular around $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if and only if $\bar{y}$ belongs to the interior of the range $\operatorname{rge} F := \{y \in Y |\; y \in F(x),\; x \in X\}$; see, e.g., [1, 2]. This is a natural and far-going extension of the classical *Open Mapping Theorem* for linear operators to convex set-valued mappings.

It is well known that metric regularity can be equivalently described in other forms; in particular, as the *local covering property* of $F\colon X \rightrightarrows Y$ around $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ (also known as linear openness as well as local surjection), which means that there is $C > 0$ such that

$$F(x) \cap V + Cr I\!B_Y \subset F\big(I\!B_r(x)\big) \quad \text{whenever} \quad I\!B_r(x) \subset U \;\text{ as }\; r > 0$$

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for some neighborhoods U of $\bar{x}$ and V of $\bar{y}$. The supremum of all such C is denoted by $\operatorname{cov} F(\bar{x},\bar{y})$, the *exact covering bound*. Furthermore, the reciprocal of $\operatorname{cov} F(\bar{x},\bar{y})$ is the exact regularity bound $\operatorname{reg} F(\bar{x},\bar{y})$; see, e.g., [3] and the bibliographies therein.

A highly important issue for many aspects of nonlinear analysis, especially for its variational frameworks, is the calculation of the exact regularity bound and related quantities; see [3] for results, discussions, and references. When X is an Asplund space (i.e., each of its separable subspaces has a separable dual) and Y is finite-dimensional, this constant is calculated by Mordukhovich [3, Theorem 4.21] via the *coderivative norm* (see Section 2):

$$\operatorname{reg} F(\bar{x},\bar{y}) = \|D^*F^{-1}(\bar{y},\bar{x})\| := \sup\big\{\|y^*\| \;\big|\; y^* \in D^*F^{-1}(\bar{y},\bar{x})(x^*),\ x^* \in I\!B_{X^*}\big\}, \tag{1.2}$$

extending his previous coderivative criterion for mappings between finite-dimensional spaces.

A natural question arises about the equality in (1.2) when Y is infinite-dimensional. This has been intensively studied in the recent years; see, e.g., [4, 5, 6, 7, 8]. A simple counterexample to the equality in (1.2) with $\dim Y = \infty$ can be found in [8], while there are a number of positive results in this direction discussed in what follows. In particular, the answer is positive in the framework of $F\colon I\!R^n \rightrightarrows I\!R^S \times \mathcal{C}(T)$ given by

$$F(x) := \big\{(z,p)\big|\ \langle a(t),x\rangle - p(t) \le 0,\ t\in T;\ \langle b(s),x\rangle - z(s) = 0,\ s \in S\big\}, \tag{1.3}$$

where $\mathcal{C}(T)$ is the space of continuous functions on a Hausdorff compact T, $I\!R^S$ is a product space with $|S| < n$ for the cardinality of S, $a : T \to I\!R^n$ is continuous, and $b : S \to I\!R^n$ is a function over discrete domain. The equality in (1.2) in the setting of (1.3) can be derived from [4, Theorem 3.1] and the explicit calculations provided in Corollary 3.2 therein.

A different setting of the equality in (1.2) is identified in [5] for the multifunction $F\colon X \rightrightarrows Y$ from an arbitrary Banach space X to $Y = l_\infty(T)$ with the inverse

$$F^{-1}(y) = \big\{x \in X\big|\ \langle a_t^*, x\rangle - b_t \le y_t,\ t \in T\big\} \quad \text{for all}\ \ y \in l_\infty(T), \tag{1.4}$$

where T is an arbitrary index set and $(a_t^*, b_t) \in X^* \times I\!R$ for all $t \in T$. Besides different spaces X and Y in (1.3) and (1.4), the latter system addresses general infinite constraints (in particular, countable ones) in contrast to the compact index set T in (1.3).

Another approach to the equality in (1.2) for closed and convex multifunctions $F\colon X \rightrightarrows Y$ is developed in [8] under the condition

$$\operatorname{cl}^*\big\{y^* \in S_{Y^*}\big|\ \sigma_\Omega(x^*,y^*) < \infty,\ x^* \in X^*\big\} \subset S_{Y^*} \tag{1.5}$$

imposed on the Banach spaces, where σ_Ω is the support function of the set $\Omega := \operatorname{gph} F - (\bar{x},\bar{y})$. Condition (1.5) holds automatically when Y is finite-dimensional; otherwise, it seems to be restrictive, since the dual unit sphere is never weak* closed in infinite dimensions. However, a careful analysis verifies the validity of (1.5) in both cases (1.3) and (1.4); see below along with the discussion of the recent developments in [6, 7].

The primary goal of this paper is to derive precise formulas of type (1.2) for calculating the exact regularity bound for convex multifunctions between general Banach spaces with no imposing condition (1.5) while involving *ε-coderivatives*. Our approach is based on a simple observation that F is metrically regular around $(\bar{x},\bar{y})$ if and only if $(\bar{x},\bar{y})$ is a local solution to the following unconstrained DC (*difference of convex* functions) program:

$$\text{minimize} \quad Kd\big(y;F(x)\big) - d\big(x;F^{-1}(y)\big) \quad \text{subject to} \quad (x,y) \in X \times Y$$

for some $K > 0$. To the best of our knowledge, this approach is new in literature.

The rest of our paper is organized as follows. In Section 2 we present some basic definitions and preliminaries, which are widely used in the sequel. Section 3 contains the main results of this paper that provide relationships between the exact regularity and the coderivative norms for closed and convex multifunctions. In Section 4 we apply the results obtained in the preceding section to the convex infinite systems containing linear equality and infinitely many convex inequality constraints indexed by an arbitrary set T. A significant part in this section is to show that when adding linear equality constraints to system (1.4), the problem becomes essentially more complicated. Moreover, the results in this section also extend some known ones established recently in [6, 7]. Several instructive examples are constructed to make comparisons of our results with those in [4, 5, 6, 7].

Our notation and terminology are basically standard and conventional in the area of variational analysis and semi-infinite/infinite programming; see, e.g., [3, 9]. As usual, $\langle\cdot,\cdot\rangle$ signifies the canonical pairing between X and its dual X^* with the symbol $\overset{w^*}{\to}$ indicating the convergence in the weak* topology of X^* and the cl^* standing for the weak* topological closure of a set. Given an arbitrary index set T, we consider the product space of multipliers $I\!R^T := \{\lambda = (\lambda_t)|\ t \in T\}$ with $\lambda_t \in I\!R$ for $t \in T$ and denote by $\widetilde{I\!R}^T$ the collection of $\lambda \in I\!R^T$ such that $\lambda_t \neq 0$ for only finitely many $t \in T$. The *positive cone* in $\widetilde{I\!R}^T$ is defined by

$$\widetilde{I\!R}^T_+ := \big\{\lambda \in \widetilde{I\!R}^T \big|\ \lambda_t \geq 0 \ \text{ for all } \ t \in T\big\}.$$

2 Preliminaries

Let X and Y be Banach spaces, and let $\varphi\colon X \to \overline{I\!R} := I\!R \cup \{\infty\}$ be an extended-real-valued function with the *domain* $\mathrm{dom}\,\varphi := \{x \in X|\ \varphi(x) < \infty\}$. We always assume that φ is *proper* (i.e., $\mathrm{dom}\,\varphi \neq \emptyset$) and *convex*. Its *conjugate function* $\varphi^*\colon X^* \to \overline{I\!R}$ is defined by

$$\varphi^*(x^*) := \sup\big\{\langle x^*, x\rangle - \varphi(x)\big|\ x \in X\big\} = \sup\big\{\langle x^*, x\rangle - \varphi(x)\big|\ x \in \mathrm{dom}\,\varphi\big\}. \tag{2.1}$$

The ε-*subdifferential* of $\varphi\colon X \to \overline{I\!R}$ at $\bar{x} \in \mathrm{dom}\,\varphi$ is given by

$$\partial_\varepsilon\varphi(\bar{x}) := \big\{x^* \in X^*\big|\ \langle x^*, x - \bar{x}\rangle \leq \varphi(x) - \varphi(\bar{x}) + \varepsilon \ \text{ for all } \ x \in X\big\}, \quad \varepsilon \geq 0. \tag{2.2}$$

When $\varepsilon = 0$ in (2.2), the set $\partial\varphi(\bar{x}) := \partial_0\varphi(\bar{x})$ is the classical *subdifferential of convex analysis*. Taking (2.1) into account, the ε-subdifferential (2.2) can be written as the form

$$\partial_\varepsilon\varphi(\bar{x}) = \big\{x^* \in X^*\big|\ \varphi^*(x^*) \leq \langle x^*, \bar{x}\rangle - \varphi(\bar{x}) + \varepsilon\big\}. \tag{2.3}$$

The following *sum rule* is well-known in convex analysis (see, e.g., [2, Theorem 2.8.7]):

$$\partial_\varepsilon(\varphi_1 + \varphi_2)(\bar{x}) = \bigcup_{\substack{\varepsilon_1+\varepsilon_2=\varepsilon\\ \varepsilon_1,\varepsilon_2\geq 0}} \Big[\partial_{\varepsilon_1}\varphi_1(\bar{x}) + \partial_{\varepsilon_2}\varphi_2(\bar{x})\Big] \tag{2.4}$$

provided that one of the functions φ_i is continuous at $\bar{x} \in \mathrm{dom}\,\varphi_1 \cap \mathrm{dom}\,\varphi_2$.

Since the above subdifferential constructions and results are given for extended-real-valued functions, they encompass the case of sets $\Omega \subset X$ by considering their *indicator functions* $\delta(x;\Omega)$ equal to 0 when $x \in \Omega$ and ∞ otherwise. In this way, the collection of ε-*normals* to a convex set Ω at $\bar{x} \in \Omega$ is defined by

$$N_\varepsilon(\bar{x};\Omega) := \partial_\varepsilon\delta(\bar{x};\Omega) = \big\{x^* \in X^*\big|\ \langle x^*, x - \bar{x}\rangle \leq \varepsilon \ \text{ for all } \ x \in \Omega\big\}, \quad \varepsilon \geq 0. \tag{2.5}$$

Then it follows from (2.3) that

$$N_\varepsilon(\bar{x};\Omega) = \big\{x^* \in X^* \big|\ \sigma_\Omega(x^*) \le \langle x^*, \bar{x}\rangle + \varepsilon\big\}, \tag{2.6}$$

where σ_Ω stands for the *support function* of Ω defined by $\sigma_\Omega(x^*) := \sup\{\langle x^*, x\rangle |\ x \in \Omega\}$ for $x^* \in X^*$. Furthermore, we use the notation $\operatorname{co}\Omega$ and $\operatorname{cone}\Omega$ to signify the *convex hull* and the *conic convex hull* of the set Ω, respectively.

Given a set-valued mapping $F : X \rightrightarrows Y$, define its *$\varepsilon$-coderivative* $D^*_\varepsilon F(\bar{x},\bar{y})\colon Y^* \rightrightarrows X^*$ at $(\bar{x},\bar{y}) \in \operatorname{gph} F$ in the direction $y^* \in Y^*$ by

$$D^*_\varepsilon F(\bar{x},\bar{y})(y^*) := \big\{x^* \in X^* \big|\ (x^*, -y^*) \in N_\varepsilon\big((\bar{x},\bar{y}); \operatorname{gph} F\big)\big\}, \quad \varepsilon \ge 0, \tag{2.7}$$

with $D^*F(\bar{x},\bar{y}) := D^*_0 F(\bar{x},\bar{y})$ for the case of coderivative. Consider the *ε-coderivative norm*

$$\|D^*_\varepsilon F(\bar{x},\bar{y})\| := \sup\big\{\|x^*\|\ \big|\ x^* \in D^*_\varepsilon F(\bar{x},\bar{y})(y^*),\ y^* \in I\!B_{Y^*}\big\}. \tag{2.8}$$

It follows from [3, Theorem 1.44] that $D^*F^{-1}(\bar{y},\bar{x})(0) = \{0\}$ if F is *metrically regular* around $(\bar{x},\bar{y}) \in \operatorname{gph} F$, and hence we have in this case that

$$\|D^*F^{-1}(\bar{y},\bar{x})\| = \sup\big\{\|y^*\|\ \big|\ y^* \in D^*F^{-1}(\bar{y},\bar{x})(x^*),\ x^* \in S_{X^*}\big\}. \tag{2.9}$$

We conclude this section with two results for DC programs used in this paper.

Lemma 2.1 [10] **(necessary and sufficient conditions for global DC minimizers).** *Let $\varphi_1, \varphi_2 : X \to \overline{I\!R}$ be proper and convex functions on a Banach space X. Then $\bar{x}$ is a global minimizer of the unconstrained DC program:*

$$\textit{minimize}\ \ \varphi_1(x) - \varphi_2(x)\ \ \textit{over}\ \ x \in X \tag{2.10}$$

if and only $\partial_\varepsilon\varphi_2(\bar{x}) \subset \partial_\varepsilon\varphi_1(\bar{x})$ for all $\varepsilon \ge 0$.

For the case of *local* minimizers of (2.10), the following sufficient condition is established in [11] with a counterexample showing that this condition is not necessary.

Lemma 2.2 [11] **(sufficient conditions for local DC minimizers).** *Let $\varphi_1, \varphi_2 : X \to \overline{I\!R}$ be proper and convex functions on a Banach space X, and let φ_2 be locally Lipschitzian around the point $\bar{x} \in \operatorname{dom}\varphi_1 \cap \operatorname{dom}\varphi_2$. Then $\bar{x}$ is a local minimizer of* (2.10) *if there is $\varepsilon_0 > 0$ such that $\partial_\varepsilon\varphi_2(\bar{x}) \subset \partial_\varepsilon\varphi_1(\bar{x})$ for all $\varepsilon \in [0, \varepsilon_0]$.*

3 Metric Regularity of Closed and Convex Multifunctions

Throughout this section we assume that $F : X \rightrightarrows Y$ is a *closed* and *convex* multifunction between two Banach spaces with $(\bar{x},\bar{y}) \in \operatorname{gph} F$. Let us present the main result of the paper that establishes relationships between metric regularity and ε-coderivatives.

Theorem 3.1 (calculating the exact regularity bound via ε-coderivatives). *Given $(\bar{x},\bar{y}) \in \operatorname{gph} F$, assume that $\bar{y} \in \operatorname{int}(\operatorname{rge} F)$. Then we have*

$$\operatorname{reg} F(\bar{x},\bar{y}) = \lim_{\varepsilon\downarrow 0} \|D^*_\varepsilon F^{-1}(\bar{y},\bar{x})\|, \tag{3.1}$$

$$\operatorname{reg} F(\bar{x},\bar{y}) = \lim_{\varepsilon\downarrow 0} \Big[\sup\Big\{\|x^*\|^{-1}\Big|\ x^* \in D^*_\varepsilon F(\bar{x},\bar{y})(y^*),\ y^* \in S_{Y^*}\Big\}\Big]. \tag{3.2}$$

Proof. Since $\bar{y} \in \operatorname{int}(\operatorname{rge} F)$, it follows from the Robinson-Ursescu theorem that F is metrically regular around $(\bar{x}, \bar{y})$. We can rewrite (1.1) as: there are $\eta, K > 0$ such that

$$d\big(x; F^{-1}(y)\big) \le K d\big(y; F(x)\big) \quad \text{whenever } (x,y) \in I\!B_\eta(\bar{x},\bar{y}) \subset X \times Y. \tag{3.3}$$

Now define proper and convex functions $\varphi_1, \varphi_2 : X \times Y \to \overline{I\!R}$ by

$$\varphi_1(x,y) := d\big(y; F(x)\big) \text{ and } \varphi_2(x,y) := d\big(x; F^{-1}(y)\big). \tag{3.4}$$

It follows from the covering property in [1, Theorem 1] that there is $r > 0$ such that $I\!B_{2r}(\bar{y}) \subset F(\bar{x} + I\!B_X)$. Combining this with the construction of φ_2 in (3.4) gives us

$$\varphi_2(x,y) \le \|x - \bar{x}\| + 1 \quad \text{for all } y \in I\!B_{2r}(\bar{y}),$$

which implies that φ_2 is bounded above around $(\bar{x},\bar{y})$, and thus it is locally Lipschitzian around this point due to [2, Corollary 2.2.13]. We have from metric regularity (3.3) that $(\bar{x},\bar{y})$ is a local minimizer of the DC program:

$$\text{minimize} \quad K\varphi_1(x,y) - \varphi_2(x,y) \quad \text{subject to } (x,y) \in X \times Y. \tag{3.5}$$

Consequently, $(\bar{x},\bar{y})$ is a global minimizer of the following DC program:

$$\text{minimize} \quad \big(K\varphi_1 + \delta(\cdot; I\!B_\eta(\bar{x},\bar{y}))\big)(x,y) - \varphi_2(x,y) \quad \text{subject to } (x,y) \in X \times Y. \tag{3.6}$$

Applying Lemma 2.1 to problem (3.6), we get the inclusion

$$\partial_\varepsilon \varphi_2(\bar{x},\bar{y}) \subset \partial_\varepsilon\big(K\varphi_1 + \delta(\cdot; I\!B_\eta(\bar{x},\bar{y}))\big)(\bar{x},\bar{y}) \qquad \text{for all } \varepsilon \ge 0.$$

Since $\delta((\bar{x},\bar{y}); I\!B_\eta(\bar{x},\bar{y}))$ is continuous at $(\bar{x},\bar{y})$, it follows from the ε-subdifferential sum rule (2.4) that the latter inclusion reduces to

$$\begin{aligned} \partial_\varepsilon \varphi_2(\bar{x},\bar{y}) \quad &\subset \bigcup_{\substack{\varepsilon_1+\varepsilon_2=\varepsilon \\ \varepsilon_1,\varepsilon_2 \ge 0}} \Big[\partial_{\varepsilon_1}(K\varphi_1)(\bar{x},\bar{y}) + \partial_{\varepsilon_2}\delta(\cdot; I\!B_\eta(\bar{x},\bar{y}))(\bar{x},\bar{y})\Big] \\ &= \bigcup_{\substack{\varepsilon_1+\varepsilon_2=\varepsilon \\ \varepsilon_1,\varepsilon_2 \ge 0}} \Big[\partial_{\varepsilon_1}(K\varphi_1)(\bar{x},\bar{y}) + \frac{\varepsilon_2}{\eta} I\!B_{X^*\times Y^*}\Big] \end{aligned} \tag{3.7}$$

due to the fact that $\partial_\varepsilon \delta(\cdot; I\!B_r(x))(x) = \frac{\varepsilon}{r} I\!B_{X^*}$ for all $\varepsilon \ge 0$ and $r > 0$.

Next we compute the ε-subdifferentials of the functions $K\varphi_1$ and φ_2 at $(\bar{x},\bar{y})$ by using the conjugate functions. It follows from (2.3) that $(x^*, y^*) \in \partial_{\varepsilon_1}(K\varphi_1)(\bar{x},\bar{y})$ if and only if

$$(K\varphi_1)^*(x^*, y^*) \le \langle x^*, \bar{x}\rangle + \langle y^*, \bar{y}\rangle + \varepsilon_1. \tag{3.8}$$

Furthermore, elementary transformations ensure the equalities:

$$\begin{aligned} (K\varphi_1)^*(x^*, y^*) \quad &= \sup_{x,y}\Big(\langle x^*, x\rangle + \langle y^*, y\rangle - K d(y; F(x))\Big) \\ &= \sup_{x,y}\Big(\langle x^*, x\rangle + \langle y^*, y\rangle - \inf_u (K\|y-u\| + \delta(u; F(x)))\Big) \\ &= \sup_{u,x,y}\Big(\langle x^*, x\rangle + \langle y^*, y-u\rangle + \langle y^*, u\rangle - K\|y-u\| - \delta(u; F(x))\Big) \\ &= \sup_{u,x,y}\Big(\langle x^*, x\rangle + \langle y^*, u\rangle - \delta(u; F(x)) + \langle y^*, y\rangle - K\|y\|\Big) \\ &= \sigma_{\operatorname{gph} F}(x^*, y^*) + \delta(y^*; K I\!B_{Y^*}). \end{aligned}$$

By using (2.6) and (3.8), the latter yields that

$$\partial_{\varepsilon_1}(K\varphi_1)(\bar{x},\bar{y}) = N_{\varepsilon_1}\big((\bar{x},\bar{y});\operatorname{gph} F\big) \cap \big(X^* \times K\mathbb{B}_{Y^*}\big). \tag{3.9}$$

Similarly, by taking into account the form of φ_2 in (3.4), we arrive at the representation

$$\partial_\varepsilon\varphi_2(\bar{x},\bar{y}) = N_\varepsilon\big((\bar{x},\bar{y});\operatorname{gph} F\big) \cap \big(\mathbb{B}_{X^*} \times Y^*\big). \tag{3.10}$$

Thus the inclusion in (3.7) reduces to

$$\begin{aligned} N_\varepsilon\big((\bar{x},\bar{y});\operatorname{gph} F\big) \cap \big(\mathbb{B}_{X^*} \times Y^*\big) \quad \subset & \bigcup_{\substack{\varepsilon_1+\varepsilon_2=\varepsilon\\ \varepsilon_1,\varepsilon_2\ge 0}} N_{\varepsilon_1}\big((\bar{x},\bar{y});\operatorname{gph} F\big) \cap \big(X^* \times K\mathbb{B}_{Y^*}\big) \\ & +\frac{\varepsilon_2}{\eta}\mathbb{B}_{X^*\times Y^*}. \end{aligned} \tag{3.11}$$

To justify the equality in (3.1), fix $\varepsilon > 0$ and pick any $(x^*,y^*) \in \mathbb{B}_{X^*} \times Y^*$ satisfying $y^* \in D^*_\varepsilon F^{-1}(\bar{y},\bar{x})(x^*)$, which means that $(-x^*,y^*) \in N_\varepsilon((\bar{x},\bar{y});\operatorname{gph} F)$. It follows from (3.11) that there are $\varepsilon_1 \in [0,\varepsilon]$ and $(u^*,v^*) \in N_{\varepsilon_1}((\bar{x},\bar{y});\operatorname{gph} F)$ such that $\|v^*\| \le K$ and $\|y^* - v^*\| \le (\varepsilon - \varepsilon_1)\eta^{-1}$. Hence we get the estimate

$$\|y^*\| \le \|v^*\| + (\varepsilon-\varepsilon_1)\eta^{-1} \le K + \varepsilon\eta^{-1}.$$

Observe from the definition in (2.8) that the function $\|D^*_\varepsilon F^{-1}(\bar{y},\bar{x})\|$ is nondecreasing with respect to $\varepsilon \ge 0$, which implies therefore that

$$\lim_{\varepsilon\downarrow 0}\|D^*_\varepsilon F^{-1}(\bar{y},\bar{x})\| = \inf_{\varepsilon>0}\|D^*_\varepsilon F^{-1}(\bar{y},\bar{x})\| \le \inf_{\varepsilon>0}\Big(K+\varepsilon\eta^{-1}\Big) = K.$$

Letting $K \downarrow \operatorname{reg} F(\bar{x};\bar{y})$ above gives us the estimate

$$\lim_{\varepsilon\downarrow 0}\|D^*_\varepsilon F^{-1}(\bar{y},\bar{x})\| \le \operatorname{reg} F(\bar{x},\bar{y}). \tag{3.12}$$

It follows from (3.12) that the equality in (3.1) is trivial if $\operatorname{reg} F(\bar{x},\bar{y}) = 0$. Considering further the case of $\operatorname{reg} F(\bar{x},\bar{y}) > 0$, we conclude from the definition of the exact regularity bound that $(\bar{x},\bar{y})$ is not a local minimizer of the DC problem (3.5) when $0 < K < \operatorname{reg} F(\bar{x},\bar{y})$. Then Lemma 2.2 allows us to find sequences $\varepsilon_n \downarrow 0$ and $(x^*_n,y^*_n) \in \partial_{\varepsilon_n}\varphi_2(\bar{x},\bar{y})$ such that $(x^*_n,y^*_n) \notin \partial_{\varepsilon_n}(K\varphi_1)(\bar{x},\bar{y})$. Combining this with (3.9) and (3.10) implies that

$$\|x^*_n\| \le 1 \text{ and } \|y^*_n\| > K. \tag{3.13}$$

Since $\mathbb{B}_{2r}(\bar{y}) \subset F(\bar{x}+\mathbb{B}_X)$ as mentioned above, we get from (3.10) and (3.13) that

$$\begin{aligned} \varepsilon_n \quad &\ge \sup_{(x,y)\in\operatorname{gph} F}\Big(\langle x^*_n, x-\bar{x}\rangle + \langle y^*_n, y-\bar{y}\rangle\Big) \ge \sup_{y\in\mathbb{B}_{2r}(\bar{y})}\big(\langle y^*_n, y-\bar{y}\rangle\big) - \|x^*_n\| \\ &\ge 2r\|y^*_n\| - \|x^*_n\| \ge 2rK - \|x^*_n\|. \end{aligned} \tag{3.14}$$

Suppose without loss of generality that $\|x^*_n\| \ge rK$ for all $n \in \mathbb{N}$ and define

$$\widetilde{y}^*_n := y^*_n\|x^*_n\|^{-1},\ \widetilde{x}^*_n := -x^*_n\|x^*_n\|^{-1},\ \text{and}\ \widetilde{\varepsilon}_n := \varepsilon_n\|x^*_n\|^{-1}.$$

We have $\|\widetilde{x}^*_n\| = 1$, $\widetilde{\varepsilon}_n \downarrow 0$, and $\widetilde{y}^*_n \in D^*_{\widetilde{\varepsilon}_n}F^{-1}(\bar{y},\bar{x})(\widetilde{x}^*_n)$. It follows from (3.13) that

$$\sup\big\{\|y^*\| \bigm| y^* \in D^*_{\varepsilon_n}F^{-1}(\bar{y},\bar{x})(y^*),\ x^* \in S_{X^*}\big\} \ge \|\widetilde{y}^*_n\| = \|y^*_n\|\|x^*_n\|^{-1} > K.$$

Letting $n \to \infty$ and $K \uparrow \operatorname{reg} F(\bar{x}, \bar{y})$ gives us

$$\lim_{\varepsilon \downarrow 0} \sup \left\{ \|y^*\| \,\Big|\, y^* \in D^*_\varepsilon F^{-1}(\bar{y}, \bar{x})(x^*),\ x^* \in S_{X^*} \right\} \geq \operatorname{reg} F(\bar{x}; \bar{y}),$$

which implies the equality in (3.1) by taking in to account the estimate in (3.12).

It remains to prove formula (3.2). By arguments similar to those following (3.14) we get that

$$D^*_\varepsilon F(\bar{x}, \bar{y})(y^*) \cap r\mathbb{B}_{X^*} = \emptyset \quad \text{for all } \ 0 < \varepsilon < r \ \text{ and } \ y^* \in S_{Y^*}. \tag{3.15}$$

Pick any $(x^*, y^*) \in X^* \times S_{Y^*}$ such that $x^* \in D^*_\varepsilon F(\bar{x}, \bar{y})(y^*)$ for some $0 < \varepsilon < r$. Define further $\widehat{x}^* := -x^* \|x^*\|^{-1}$, $\widehat{y}^* := -y^* \|x^*\|^{-1}$, and $\widehat{\varepsilon} := \varepsilon \|x^*\|^{-1}$. This ensures that $\widehat{x}^* \in S_{X^*}$, $\|\widehat{y}^*\| = \|x^*\|^{-1}$, and $\widehat{y}^* \in D^*_{\widehat{\varepsilon}} F^{-1}(\bar{y}, \bar{x})(\widehat{x}^*)$. Observe from (3.15) that $\widehat{\varepsilon} \leq \varepsilon r^{-1}$, and thus we have

$$\|x^*\|^{-1} = \|\widehat{y}^*\| \leq \|D^*_{\widehat{\varepsilon}} F^{-1}(\bar{y}, \bar{x})\| \leq \|D^*_{\varepsilon r^{-1}} F^{-1}(\bar{y}, \bar{x})\|.$$

This together with (3.1) yields the inequality "$\geq$" in (3.2) by letting $\varepsilon \downarrow 0$.

To justify the converse inequality in (3.2), note first that it is obvious when $\operatorname{reg} F(\bar{x}, \bar{y}) = 0$. If $\operatorname{reg} F(\bar{x}, \bar{y}) > 0$, we get from the equality in (3.1) that there exists a sufficiently small number $0 < s < \operatorname{reg} F(\bar{x}, \bar{y})$ ensuring the condition

$$D^*_\varepsilon F^{-1}(\bar{y}, \bar{x})(x^*) \cap s\mathbb{B}_{X^*} = \emptyset \quad \text{for all } \ 0 < \varepsilon < s \ \text{ and } \ x^* \in S_{X^*}.$$

By arguments similar to the above proof we also derive that

$$\|D^*_\varepsilon F^{-1}(\bar{y}, \bar{x})\| \leq \sup \left\{ \|x^*\|^{-1} \,\Big|\, x^* \in D^*_{\varepsilon s^{-1}} F(\bar{x}, \bar{y})(y^*),\ y^* \in S_{Y^*} \right\}.$$

This together with (3.1) justifies the inequality "$\leq$" in (3.2) and completes the proof. $\triangle$

The following consequence of Theorem 3.1 and the classical Brøndsted-Rockafellar theorem establishes a precise formula for the exact regularity bound of a closed and convex multifunction F between Banach spaces by using the coderivative of F^{-1} instead of its ε-counterparts while involving points close to the reference one.

Corollary 3.2 (calculating the exact regularity bound via the coderivative norm at nearby points). *In the setting of Theorem* 3.1 *we have*

$$\operatorname{reg} F(\bar{x}, \bar{y}) = \lim_{\varepsilon \downarrow 0} \left[\sup \left\{ \|D^* F^{-1}(y, x)\| \,\Big|\, (x, y) \in \operatorname{gph} F \cap \mathbb{B}_\varepsilon(\bar{x}, \bar{y}) \right\} \right]. \tag{3.16}$$

Proof. To verify the inequality "$\geq$" in (3.16), observe from (3.3) that for any $K > \operatorname{reg} F(\bar{x}, \bar{y})$ and any sufficiently small $\varepsilon > 0$ we get

$$d\big(x; F^{-1}(y)\big) \leq K d\big(y; F(x)\big) \quad \text{for all } \ (x, y) \in \mathbb{B}_\varepsilon(\widetilde{x}, \widetilde{y})$$

whenever $(\widetilde{x}, \widetilde{y}) \in \mathbb{B}_\varepsilon(\bar{x}, \bar{y})$. It follows from (3.1) that

$$K \geq \lim_{\eta \downarrow 0} \|D^*_\eta F^{-1}(\widetilde{y}, \widetilde{x})\| \geq \|D^* F^{-1}(\widetilde{y}, \widetilde{x})\| \quad \text{for all } \ (\widetilde{x}, \widetilde{y}) \in \mathbb{B}_\varepsilon(\bar{x}, \bar{y}).$$

This clearly implies the estimate

$$K \geq \lim_{\varepsilon \downarrow 0} \left[\sup \left\{ \|D^* F^{-1}(y, x)\| \,\Big|\, (x, y) \in \operatorname{gph} F \cap \mathbb{B}_\varepsilon(\bar{x}, \bar{y}) \right\} \right].$$

Letting there $K \downarrow \operatorname{reg} F(\bar{x}, \bar{y})$, we arrive at the inequality "$\geq$" in (3.16).

To prove the converse inequality in (3.16), take an arbitrary $\varepsilon > 0$ and observe from Theorem 3.1 that $\mathrm{reg}\, F(\bar{x}, \bar{y}) \le \|D^*_{\varepsilon^2} F^{-1}(\bar{y}, \bar{x})\|$. This allows us to find $(x^*, y^*) \in X^* \times Y^*$ satisfying the condition $y^* \in D^*_{\varepsilon^2} F^{-1}(\bar{y}, \bar{x})(x^*)$, i.e., $(-x^*, y^*) \in N_{\varepsilon^2}((\bar{x}, \bar{y}); \mathrm{gph}\, F)$. We have furthermore that

$$\|x^*\| \le 1 \quad \text{and} \quad \|y^*\| + \varepsilon \ge \mathrm{reg}\, F(\bar{x}, \bar{y}). \tag{3.17}$$

Due to the Brøndsted-Rockafellar theorem (see, e.g., [2]) there are $(x_\varepsilon, y_\varepsilon) \in \mathrm{gph}\, F \cap I\!B_\varepsilon(\bar{x}, \bar{y})$ and $(-x^*_\varepsilon, y^*_\varepsilon) \in N((x_\varepsilon, y_\varepsilon); \mathrm{gph}\, F)$ satisfying $\|x^*_\varepsilon - x^*\| \le \varepsilon$ and $\|y^*_\varepsilon - y^*\| \le \varepsilon$. Thus we get $\|x^*_\varepsilon\| \le \|x^*\| + \varepsilon \le 1 + \varepsilon$ and $\|y^*\| \le \|y^*_\varepsilon\| + \varepsilon$ arriving in the way at

$$\|y^*\| \le (1 + \varepsilon)\|D^* F^{-1}(y_\varepsilon, x_\varepsilon)\| + \varepsilon.$$

Combining this with (3.17) implies the estimate

$$\mathrm{reg}\, F(\bar{x}, \bar{y}) \le (1 + \varepsilon) \sup\Big\{\|D^* F^{-1}(y, x)\| \;\Big|\; (x, y) \in \mathrm{gph}\, F \cap I\!B_\varepsilon(\bar{x}, \bar{y})\Big\} + 2\varepsilon,$$

which ensures the inequality "$\le$" in (3.16) by passing to the limit as $\varepsilon \downarrow 0$. $\triangle$

It is worth mentioning that the exact bound formulas similar to (3.16) are established in [3, Theorem 4.5] and [12, Theorem 5.6] for general closed-graph multifunctions between Asplund spaces via the so-called Fréchet/regular coderivative. Though Corollary 3.2 concerns convex multifunctions, both spaces X and Y in it are arbitrary Banach.

The next consequence of Theorem 3.1 gives the main result of [8] proved in a different way.

Corollary 3.3 (calculating the exact covering bound). *Given a point $(\bar{x}, \bar{y}) \in \mathrm{gph}\, F$ with $\bar{y} \in \mathrm{int}\,(\mathrm{rge}\, F)$, the exact covering bound of F at $(\bar{x}, \bar{y})$ is calculated by*

$$\mathrm{cov}\, F(\bar{x}, \bar{y}) = \lim_{\varepsilon \downarrow 0} \Big[\inf_{x^* \in X^*} \inf_{y^* \in S_{Y^*}} \Big(\|x^*\| + \frac{\sigma_{\mathrm{gph}\, F - (\bar{x}, \bar{y})}(x^*, y^*)}{\varepsilon} \Big) \Big].$$

Proof. Define $\Omega := \mathrm{gph}\, F - (\bar{x}, \bar{y})$. Since the number $\mathrm{cov}\, F(\bar{x}, \bar{y})$ is the reciprocal of $\mathrm{reg}\, F(\bar{x}, \bar{y})$, it suffices to show that

$$\mathrm{reg}\, F(\bar{x}, \bar{y}) = \lim_{\varepsilon \downarrow 0} \Big[\sup_{x^* \in X^*} \sup_{y^* \in S_{Y^*}} \Big(\|x^*\| + \frac{\sigma_\Omega(x^*, y^*)}{\varepsilon} \Big)^{-1} \Big] =: \alpha. \tag{3.18}$$

By (3.2) we find sequences $\varepsilon_n \downarrow 0$ and $(x^*_n, y^*_n) \in X^* \times S_{Y^*}$ such that $x^*_n \in D^*_{\varepsilon_n} F(\bar{x}, \bar{y})(y^*_n)$, which means $\sigma_\Omega(x^*_n, -y^*_n) \le \varepsilon_n$ due to (2.6), and that $\|x^*_n\|^{-1} \to \mathrm{reg}\, F(\bar{x}, \bar{y})$ as $n \to \infty$. Hence

$$\sup_{x^* \in X^*} \sup_{y^* \in S_{Y^*}} \Big(\|x^*\| + \frac{\sigma_\Omega(x^*, y^*)}{\sqrt{\varepsilon_n}} \Big)^{-1} \ge \Big(\|x^*_n\| + \frac{\sigma_\Omega(x^*_n, -y^*_n)}{\sqrt{\varepsilon_n}} \Big)^{-1} \ge \Big(\|x^*_n\| + \sqrt{\varepsilon_n} \Big)^{-1},$$

which implies the inequality "$\le$" in (3.18) by passing to the limit as $n \to \infty$.

Conversely, if the right-hand side of (3.18) equals to 0, the equality in (3.18) is obvious. Otherwise, we find sequences $\widetilde{\varepsilon}_n \downarrow 0$ and $(\widetilde{x}^*_n, \widetilde{y}^*_n) \in X^* \times S_{Y^*}$ such that

$$\beta < \Big(\|\widetilde{x}^*_n\| + \frac{\sigma_\Omega(\widetilde{x}^*_n, \widetilde{y}^*_n)}{\widetilde{\varepsilon}_n} \Big)^{-1} \to \alpha \;\text{ as }\; n \to \infty \tag{3.19}$$

for some $\beta > 0$. It follows that $\sigma_\Omega(\widetilde{x}^*_n, \widetilde{y}^*_n) \le \widetilde{\varepsilon}_n \beta^{-1}$ for all $n \in I\!N$, which yields $\widetilde{x}^*_n \in D^*_{\widehat{\varepsilon}_n} F(\bar{x}, \bar{y})(-\widetilde{y}^*_n)$ with $\widehat{\varepsilon}_n := \widetilde{\varepsilon}_n \beta^{-1} \to 0$ by (2.6). Hence we have

$$\Big(\|\widetilde{x}^*_n\| + \frac{\sigma_\Omega(\widetilde{x}^*_n, \widetilde{y}^*_n)}{\widetilde{\varepsilon}_n} \Big)^{-1} \le \|\widetilde{x}^*_n\|^{-1} \le \sup\Big\{ \|x^*\|^{-1} \Big|\; x^* \in D^*_{\widehat{\varepsilon}_n} F(\bar{x}, \bar{y})(y^*),\; y^* \in S_{Y^*} \Big\}. \tag{3.20}$$

Employing now the regularity formula (3.2) in (3.20) and taking (3.19) into account, we get the estimate $\alpha \le \operatorname{reg} F(\bar{x},\bar{y})$ and thus complete the proof of the corollary. $\triangle$

Finally in this section, we introduce a condition improving the one in (1.5), which helps us to remove $\varepsilon > 0$ in the exact bound formula (3.1) and get the result in form (1.2) for convex multifunctions between general Banach spaces.

Theorem 3.4 (pointwise calculating the exact regularity bound via the coderivative norm). *In the setting of Theorem* 3.1*, assume in addition that*

$$\Lambda(S_{Y^*}) \subset S_{Y^*}, \tag{3.21}$$

where the set $\Lambda(S_{Y^*})$ *is defined sequentially by*

$$\Lambda(S_{Y^*}) := \Big\{ y^* \in Y^* \Big|\ \exists\, \varepsilon_n \downarrow 0,\ y^*_n \in S_{Y^*}\ \text{s.t.}\ D^*_{\varepsilon_n} F(\bar{x},\bar{y})(y^*_n) \ne \emptyset\ \text{and}\ y^*\ \text{is a weak}^*\ \text{cluster point of}\ y^*_n \Big\}.$$

Then we have the equality

$$\operatorname{reg} F(\bar{x},\bar{y}) = \|D^* F^{-1}(\bar{y},\bar{x})\|. \tag{3.22}$$

If furthermore $\operatorname{reg} F(\bar{x},\bar{y}) > 0$*, we get the improved formula*

$$\operatorname{reg} F(\bar{x},\bar{y}) = \max\Big\{ \|x^*\|^{-1}\ \Big|\ x^* \in D^* F(\bar{x},\bar{y})(y^*),\ y^* \in S_{Y^*} \Big\}. \tag{3.23}$$

Proof. Note that the equality in (3.22) is trivial when $\operatorname{reg} F(\bar{x},\bar{y}) = 0$. Otherwise, it follows from (3.2) that there are sequences $\varepsilon_n \downarrow 0$ and $x^*_n \in D^*_{\varepsilon_n} F(\bar{x},\bar{y})(y^*_n)$ such that $\|x^*_n\| > 0$, $\|y^*_n\| = 1$, and

$$\operatorname{reg} F(\bar{x},\bar{y}) = \lim_{n\to\infty} \|x^*_n\|^{-1}. \tag{3.24}$$

Since the sequence $\{x^*_n\}$ is bounded by (3.24), we get from (3.21) and the classical Alaoglu-Bourbaki theorem that there is a subnet $(x^*_\alpha, y^*_\alpha, \varepsilon_\alpha)$ of $(x^*_n, y^*_n, \varepsilon_n)$ weak* converging to some $(\bar{x}^*, \bar{y}^*, 0) \in X^* \times S_{Y^*} \times I\!R$. Note further that

$$\langle \bar{x}^*, x - \bar{x}\rangle - \langle \bar{y}^*, y - \bar{y}\rangle = \lim_\alpha \langle x^*_\alpha, x - \bar{x}\rangle - \langle y^*_\alpha, y - \bar{y}\rangle \le \limsup_\alpha \varepsilon_\alpha = 0$$

for all $(x,y) \in \operatorname{gph} F$, which yields $\bar{x}^* \in D^* F(\bar{x},\bar{y})(\bar{y}^*)$. Moreover, the classical uniform boundedness principle tells us that $\|\bar{x}^*\| \le \liminf_\alpha \|x^*_\alpha\|$. This together with (3.24) ensures that

$$\operatorname{reg} F(\bar{x},\bar{y}) \le \|\bar{x}^*\|^{-1} \le \sup\Big\{ \|x^*\|^{-1}\ \Big|\ x^* \in D^* F(\bar{x},\bar{y})(y^*),\ \|y^*\| = 1 \Big\}.$$

Combining the latter with (3.2) yields (3.22), which easily implies (3.23) under the additional assumption made. This completes the proof of the theorem. $\triangle$

It is obvious that assumption (3.21) automatically holds when the space Y is finite-dimensional. In this case Theorem 3.4 agrees with [3, Theorem 4.21] obtained under the Asplund property of X while for nonconvex set-valued mappings. Furthermore, the equality in (3.22) is equivalent to the *perfect regularity* property of F introduced and studied in [8] under assumption (1.5). Observe that the latter assumption with $\Omega = \operatorname{gph} F - (\bar{x},\bar{y})$ is stronger than (3.21) due to the proper inclusion

$$\Lambda(S_{Y^*}) \subset \operatorname{cl}^*\Big[\bigcup_{\varepsilon \ge 0} \Big\{ y^* \in S_{Y^*} \Big|\ D^*_\varepsilon F(\bar{x},\bar{y})(y^*) \ne \emptyset \Big\} \Big] = \operatorname{cl}^*\big\{ y^* \in S_{Y^*} \big|\ \sigma_\Omega(x^*, y^*) < \infty,\ x^* \in X^* \big\}.$$

4 Applications to Infinite Convex Systems

In this section we develop applications of the results obtained in Section 3 to the special class of set-valued mappings $F : X \rightrightarrows Y := Z \times l_\infty(T)$ given by

$$F(x) := \begin{cases} \big\{(z,p) \in Z \times l_\infty(T) \big|\ Ax = z,\ f_t(x) \le p_t,\ t \in T\big\} & \text{if } x \in C \\ \emptyset \quad \text{otherwise} \end{cases} \tag{4.1}$$

and describing, in particular, sets of feasible solutions in parameterized semi-infinite/infinite programs with inequality, equality constraints, and geometric constraints.

The data of (4.1) are as follows: $A : X \to Z$ is a bounded linear operator between two Banach spaces; the functions $f_t : X \to \overline{I\!R}$ are proper, lower semicontinuous (l.s.c.), and convex for all t from the arbitrary index set T; C is a closed and convex subset of X with nonempty interior. These assumptions clearly imply that F in (4.1) is closed and convex multifunction, and so we can implement the results on metric regularity at $(x,(z,p)) \in \operatorname{gph} F$ obtained above to the infinite constraint system (4.1) provided the validity of the underlying condition

$$(z,p) \in \operatorname{int}(\operatorname{rge} F). \tag{4.2}$$

Note that this condition clearly implies that $z \in \operatorname{int}(AX)$, which ensures that A is an open mapping, and hence it must be surjective.

Throughout this section we denote $f := \sup_{t \in T} f_t$ and suppose that the space $Z \times l_\infty(T)$ is equipped with the maximum product norm

$$\|(z,p)\| = \max\big\{\|z\|, \|p\|\big\} \quad \text{for all } z \in Z,\ p \in l_\infty(T).$$

As mentioned in Section 1, F is metrically regular around $(x,(z,p)) \in \operatorname{gph} F$ *if and only if* condition (4.2) holds (Robinson-Ursescu). This motivates us to introduce a qualification condition via the initial data of (4.1), which ensures (4.2) and extend the well-known strong Slater constraint qualification for infinite convex inequality systems to the more general constraint case of (4.1).

Definition 4.1 (strong Slater condition). *We say the infinite system* (4.1) *satisfies the strong Slater condition* (*SSC*) *at* $(z,p) \in Z \times l_\infty(T)$ *if there is* $\widehat{x} \in \operatorname{int} C$ *such that the function* f *is bounded from above around* $\widehat{x}$*, that* $A\widehat{x} = z$*, and that*

$$\sup_{t \in T}[f_t(\widehat{x}) - p_t] < 0. \tag{4.3}$$

The following proposition shows that the SSC is a sufficient condition for the validity of (4.2), being in fact "almost" necessary for this, up to the upper boundedness of the supremum function f.

Proposition 4.2 (strong Slater condition and metric regularity). *Let* $(z,p) \in \operatorname{rge} F$ *for the infinite system* (4.1)*. Then the SSC for* F *at* (z,p) *implies the validity of* (4.2)*. Conversely, if* (4.2) *holds, then there is* $\widehat{x} \in \operatorname{int} C$ *such that* $A\widehat{x} = z$ *and that* (4.3) *is satisfied.*

Proof. To justify the first part, assume that F satisfies the SSC at (z,p). Then there are $\widehat{x} \in \operatorname{int} C$ and $\varepsilon > 0$ such that the supremum function f is bounded from above around $\widehat{x}$, $A(\widehat{x}) = z$, and that $f(\widehat{x}) < -\varepsilon$ for some $\varepsilon > 0$. Note that f is also a proper, l.s.c., and convex. It follows from [2, Theorem 2.2.9] that it is actually continuous at $\widehat{x}$. Since A is surjective and $\widehat{x} \in \operatorname{int} C$, the Open

Mapping Theorem allows us to find $0 < s \le \frac{\varepsilon}{2}$ such that $I\!B_s(z) \subset A(I\!B_r(\widehat{x}) \cap C)$ for $r > 0$. Picking any $(z', p') \in I\!B_s(z, p)$, there is $x \in I\!B_r(\widehat{x}) \cap C$ such that $Ax = z'$, and for each $t \in T$ we have

$$f_t(x) - p_t \le f(x) + s \le f(x) - f(\widehat{x}) + s + f(\widehat{x}) \le f(x) - f(\widehat{x}) + s - \varepsilon \le f(x) - f(\widehat{x}) - \frac{\varepsilon}{2} \le 0$$

when r is sufficiently small. This yields $(z', p') \in \operatorname{rge} F$, which implies that $I\!B_s(z, p) \subset \operatorname{rge} F$.

To verify the converse part, observe that $(z, (p_t - \varepsilon)_{t \in T}) \in \operatorname{rge} F$ for some $\varepsilon > 0$ provided that $(z, p) \in \operatorname{int}(\operatorname{rge} F)$. Hence there is $\widehat{x} \in X$ such that $A\widehat{x} = z$ and $f_t(\widehat{x}) - p_t \le -\varepsilon$ for all $t \in T$, which completes the proof of the proposition. △

Now we proceed with calculating the exact regularity bound for the constraint system (4.1) at $(\bar{x}, (\bar{z}, 0)) \in \operatorname{gph} F$ based on the results of Section 3. It follows from Theorem 3.1 that $\operatorname{reg} F(\bar{x}, (\bar{z}, 0))$ can be calculated via the norms of ε-coderivatives. The next result, which is certainly of its own interest, accomplishes an important step in this direction.

Theorem 4.3 (explicit form of ε-coderivatives for infinite convex systems). *Let F be the infinite constraint system* (4.1), *and let $(\bar{x}, (\bar{z}, 0)) \in \operatorname{gph} F$. Then we have the following ε-coderivative representation on the unit sphere:*

$$D^*_\varepsilon F\big(\bar{x}, (\bar{z}, 0)\big)\big(S_{(Z \times l_\infty(T))^*}\big) = \big\{x^* \in X^* \big|\ (x^*, \langle x^*, \bar{x}\rangle + \varepsilon) \in M\big\} \quad \text{for each } \ \varepsilon \ge 0 \tag{4.4}$$

with $M := \displaystyle\bigcup_{z^* \in I\!B_{Z^*}} \operatorname{cl}^*\Big[(1 - \|z^*\|) \operatorname{co}\Big(\bigcup_{t \in T} \operatorname{epi} f_t^*\Big) + \operatorname{epi} \delta^*(\cdot; C_0)\Big] + (A^* z^*, \langle z^*, \bar{z}\rangle)$ *and* $C_0 := C \cap \operatorname{dom} f$.

Proof. To prove the inclusion "$\subset$" in (4.4), pick $(z^*, p^*) \in S_{(Z \times l_\infty(T))^*}$ and $x^* \in D^*_\varepsilon F(\bar{x}, (\bar{z}, 0))(z^*, p^*)$. Then we have $\|z^*\| + \|p^*\| = 1$ and

$$\langle x^*, x - \bar{x}\rangle - \langle z^*, z - \bar{z}\rangle - \langle p^*, p\rangle \le \varepsilon \quad \text{for all } \ (x, z, p) \in \operatorname{gph} F,$$

which can be equivalently written as

$$\langle x^* - A^* z^*, x - \bar{x}\rangle - \langle p^*, p\rangle \le \varepsilon \ \text{ whenever } \ (x, p) \in C_0 \times l_\infty(T),\ f_t(x) - \langle \delta_t, p\rangle \le 0,\ t \in T, \tag{4.5}$$

where $\delta_t \in l^*_\infty(T)$ stands for the classical *Dirac function* at each $t \in T$ satisfying

$$\langle \delta_t, p\rangle = p_t \quad \text{as} \quad t \in T \quad \text{for} \quad p = (p_t)_{t \in T} \in l_\infty(T).$$

It follows from the extended Farkas lemma in [13, Theorem 4.1] that (4.5) is equivalent to

$$\big(p^*, x^* - A^* z^*, \langle x^* - A^* z^*, \bar{x}\rangle + \varepsilon\big) \in \operatorname{cl}^*\Big[\operatorname{cone}\Big\{\bigcup_{t \in T} \{\delta_t\} \times \operatorname{epi} f_t^*\Big\} + \{0\} \times \operatorname{epi} \delta^*(\cdot; C_0)\Big]. \tag{4.6}$$

Hence there exist nets $\{\lambda_\nu\}_{\nu \in \mathcal{N}} \subset \widetilde{I\!R}^T_+$, $\{(w^*_\nu, s_\nu)\}_{\nu \in \mathcal{N}} \subset \operatorname{epi} \delta^*(\cdot; C_0)$, and $\{(u^*_{t\nu}, r_{t\nu})\}_{\nu \in \mathcal{N}} \subset \operatorname{epi} f_t^*$ for each $t \in T$ such that

$$\big(p^*, x^* - A^* z^*, \langle x^* - A^* z^*, \bar{x}\rangle + \varepsilon\big) = w^* - \lim_\nu \sum_{t \in T} \lambda_{t\nu}(\delta_t, u^*_{t\nu}, r_{t\nu}) + (0, w^*_\nu, s_\nu). \tag{4.7}$$

Observe from the latter equality that $p^* = w^* - \lim_\nu \sum_{t \in T} \lambda_{t\nu} \delta_t$. Thus we have

$$\limsup_\nu \sum_{t \in T} \lambda_{t\nu} \ge \sup_{\|p\| \le 1} \lim_\nu \sum_{t \in T} \lambda_{t\nu} p_t = \sup_{\|p\| \le 1} \langle p^*, p\rangle = \|p^*\| \ge \langle p^*, e\rangle = \lim_\nu \sum_{t \in T} \lambda_{t\nu} \tag{4.8}$$

with $e \in l_\infty(T)$ satisfying $e_t = 1$ for all $t \in T$. This yields that

$$1 - \|z^*\| = \|p^*\| = \lim_\nu \sum_{t\in T} \lambda_{t\nu}. \tag{4.9}$$

If $\|z^*\| = 1$, we get from (4.7) and (4.9) the relationships

$$\begin{array}{rcl}\langle x^* - A^*z^*, x - \bar{x}\rangle - \varepsilon & = & \langle x^* - A^*z^*, x\rangle - (\langle x^* - A^*z^*, \bar{x}\rangle + \varepsilon)\\ & = & \lim\limits_\nu \sum\limits_{t\in T} \lambda_{t\nu}\langle u^*_{t\nu}, x\rangle + \langle w^*_\nu, x\rangle - \lim\limits_\nu \sum\limits_{t\in T} \lambda_{t\nu} r_{t\nu} - s_\nu\\ & \le & \limsup\limits_\nu \sum\limits_{t\in T} \lambda_{t\nu}\big(\langle u^*_{t\nu}, x\rangle - f_t(x) - r_{t\nu} + f(x)\big) + \langle w^*_\nu, x\rangle - s_\nu\\ & \le & \limsup\limits_\nu \sum\limits_{t\in T} \lambda_{t\nu}\big([f^*_t(u^*_{t_\nu}) - r_{t\nu}] + f(x)\big) + [\delta^*(\cdot; C_0)(w^*_\nu) - s_\nu]\\ & \le & \limsup\limits_\nu \sum\limits_{t\in T} \lambda_{t\nu} f(x) = 0\end{array}$$

for any $x \in C_0$. It follows from (2.1) that $(x^* - A^*z^*, \langle x^* - A^*z^*, \bar{x}\rangle + \varepsilon) \in \operatorname{epi}\delta^*(\cdot; C_0)$, which yields

$$(x^*, \langle x^*, \bar{x}\rangle + \varepsilon) \in \operatorname{epi}\delta^*(\cdot; C_0) + (A^*z^*, \langle A^*z^*, \bar{x}\rangle) = \operatorname{epi}\delta^*(\cdot; C_0) + (A^*z^*, \langle z^*, \bar{z}\rangle) \subset M.$$

If $\|z^*\| < 1$, by (4.9) we assume without loss of generality that $\sum_{t\in T} \lambda_{t\nu} > 0$ for all $\nu \in \mathcal{N}$ and define $\widetilde{\lambda}_{t\nu} := \dfrac{\lambda_{t\nu}}{\sum_{t'\in T} \lambda_{t'\nu}}$ for each $t \in T$ and $\nu \in \mathcal{N}$. It gives by (4.7) that

$$\begin{aligned}(x^*, \langle x^*, \bar{x}\rangle + \varepsilon) &= \mathrm{w}^* - \lim_\nu \sum_{t\in T} \lambda_{t\nu}(u^*_{t\nu}, r_{t\nu}) + (w^*_\nu, s_\nu) + (A^*z^*, \langle z^*, \bar{z}\rangle)\\ &= \mathrm{w}^* - (1 - \|z^*\|) \lim_\nu \sum_{t\in T} \widetilde{\lambda}_{t\nu}(u^*_{t\nu}, r_{t\nu}) + (w^*_\nu, s_\nu) + (A^*z^*, \langle z^*, \bar{z}\rangle) \subset M.\end{aligned}$$

In both cases above we have $(x^*, \langle x^*, \bar{x}\rangle + \varepsilon) \in M$, which justifies the inclusion "$\subset$" in (4.4).

Conversely, take an arbitrary element $x^* \in X^*$ satisfying $(x^*, \langle x^*, \bar{x}\rangle + \varepsilon) \in M$. We find $z^* \in I\!B_{Z^*}$ and nets $\{\lambda_\nu\}_{\nu\in\mathcal{N}} \subset \widetilde{I\!R}^T_+$, $\{(w^*_\nu, s_\nu)\}_{\nu\in\mathcal{N}} \subset \operatorname{epi}\delta^*(\cdot; C_0)$, and $\{(u^*_{t\nu}, r_{t\nu})\}_{\nu\in\mathcal{N}} \subset \operatorname{epi} f^*_t$ for each $t \in T$ such that $\sum_{t\in T} \lambda_{t\nu} = 1$ and

$$(x^*, \langle x^*, \bar{x}\rangle + \varepsilon) = \mathrm{w}^* - (1 - \|z^*\|) \sum_{t\in T} \lambda_{t\nu}(u^*_{t\nu}, r_{t\nu}) + (w^*_\nu, s_\nu) + (A^*z^*, \langle z^*, \bar{z}\rangle).$$

Defining $p^*_\nu := (1 - \|z^*\|) \sum_{t\in T} \lambda_{t\nu}\delta_t$, note that $\|p^*_\nu\| = 1 - \|z^*\|$ by similar arguments as in the proof of (4.8). It follows from the classical Alaoglu-Bourbaki theorem that there is a subnet of p^*_ν (without relabeling) weak* converging to some $p^* \in I\!B_{l^*_\infty(T)}$. By using the arguments as in (4.8) again, we get $\|p^*\| = 1 - \|z^*\|$ and then obtain (4.6). Due to the equivalence between (4.5) and (4.6), this justifies the inclusion "$\supset$" in (4.4) and thus completes the proof of the theorem. $\triangle$

In the *coderivative* case of Theorem 4.3 (i.e., when $\varepsilon = 0$) we can equivalently modify the representation in (4.4) and provide its further specification.

Proposition 4.4 (explicit forms of the coderivative for infinite convex systems). *Let* $(\bar{x}, (\bar{z}, 0)) \in \operatorname{gph} F$ *for the constraint system* (4.1). *Then we have the coderivative representation*

$$D^*F\big(\bar{x}, (\bar{z}, 0)\big)\big(S_{(Z\times l_\infty(T))^*}\big) = \big\{x^* \in X^* \big|\ (x^*, \langle x^*, \bar{x}\rangle) \in L\big\} \tag{4.10}$$

with $L := \displaystyle\bigcup_{z^*\in I\!B_{Z^*}} \operatorname{cl}^*\Big[(1-\|z^*\|)\operatorname{co}\Big(\bigcup_{t\in T}\operatorname{gph} f^*_t\Big) + \operatorname{gph}\delta^*(\cdot; C_0)\Big] + (A^*z^*, \langle z^*, \bar{z}\rangle)$ *and* $C_0 = C \cap \operatorname{dom} f$. *Furthermore, the term* $\operatorname{gph}\delta^*(\cdot; C_0)$ *can be removed in the expression for* L *if* $\bar{x} \in \operatorname{int} C_0$.

Proof. To verify the inclusion "$\subset$", for any $x^*\in D^*F(\bar x,(\bar z,0))(z^*,p^*)$ with $\|z^*\|+\|p^*\|=1$ we get from the proof of Theorem 4.3 the validity of inclusion (4.5) with $\varepsilon=0$. This allows us to find nets $\{\lambda_\nu\}_{\nu\in\mathcal N}\subset\widetilde{I\!R}^T_+$, $\{\rho_\nu\}_{\nu\in\mathcal N}\subset I\!R_+$, $\{(w^*_\nu,s_\nu)\}_{\nu\in\mathcal N}\subset\operatorname{gph}\delta^*(\cdot;C_0)$, and $\{(u^*_{t\nu},r_{t\nu})\}_{\nu\in\mathcal N}\subset\operatorname{gph}f^*_t$ for each $t\in T$ providing the limiting representation

$$\big(p^*,x^*-A^*z^*,\langle x^*-A^*z^*,\bar x\rangle\big)=w^*-\lim_\nu\sum_{t\in T}\lambda_{t\nu}(\delta_t,u^*_{t\nu},r_{t\nu})+(0,w^*_\nu,s_\nu)+(0,0,\rho_\nu). \tag{4.11}$$

Similarly to the proof of Theorem 4.3, suppose with no loss of generality that $\sum_{t\in T}\lambda_{t\nu}=1-\|z^*\|$ for all $\nu\in\mathcal N$ and then get

$$r_{t\nu}=f^*_t(u^*_{t\nu})\ge\langle u^*_{t\nu},\bar x\rangle-f_t(\bar x)\ge\langle u^*_{t\nu},\bar x\rangle\quad\text{and}\quad s_\nu=\delta^*(\cdot;C_0)(w^*_\nu)\ge\langle w^*_\nu,\bar x\rangle.$$

This implies together with (0.2) the relationships

$$\begin{aligned}\langle x^*-A^*z^*,\bar x\rangle\ &=\lim_\nu\sum_{t\in T}\lambda_{t\nu}r_{t\nu}+s_\nu+\rho_\nu\ge\limsup_\nu\sum_{t\in T}\lambda_{t\nu}\langle u^*_{t\nu},\bar x\rangle+\langle w^*_\nu,\bar x\rangle+\rho_\nu\\ &\ge\langle x^*-A^*z^*,\bar x\rangle+\limsup_\nu\rho_\nu,\end{aligned}$$

which ensure that $\limsup_\nu\rho_\nu=0$. Then it follows from (4.11) that

$$(x^*,\langle x^*,\bar x\rangle)=w^*-\lim_\nu\sum_{t\in T}\lambda_{t\nu}(u^*_{t\nu},r_{t\nu})+(w^*_\nu,s_\nu)+(A^*z^*,\langle z^*,\bar z\rangle)\in L,$$

and thus we arrive at the inclusion "$\subset$" in (4.10). The verification of the opposite inclusion in (4.10) follows the lines on the proof of Theorem 4.3.

Finally, let $\bar x\in\operatorname{int}C_0$ and $x^*\in D^*F(\bar x,(\bar z,0))(z^*,p^*)$ with $(z^*,p^*)\in S_{(Z\times l_\infty(T))^*}$. Using the same notation as in the proof of (4.10) above, we have

$$\begin{aligned}0\ &=\ \langle x^*-A^*z^*,\bar x\rangle-\langle x^*-A^*z^*,\bar x\rangle=\lim_\nu\sum_{t\in T}\lambda_{t\nu}(\langle u^*_{t\nu},\bar x\rangle-r_{t\nu})+\langle w^*_\nu,\bar x\rangle-s_{t\nu}\\ &\le\ -\limsup_\nu\sup_{x\in C_0}\langle w^*_\nu,x\rangle-\langle w^*_\nu,\bar x\rangle\le-\limsup_\nu\eta\|w^*_\nu\|,\end{aligned}$$

where $\eta>0$ is such that $I\!B_\eta(\bar x)\subset C_0$. This implies that $\limsup_\nu\|w^*_\nu\|=0$, and thus we can remove $\operatorname{gph}\delta^*(\cdot;C_0)$ in the representation of L in (4.10) and complete the proof of the proposition. $\triangle$

The next major result provides a precise calculation of the exact regularity bound of the infinite constraint system (4.1) entirely via its initial data.

Theorem 4.5 (exact regularity bound of infinite constraint systems). *Let $(\bar x,(\bar z,0))\in\operatorname{gph}F$ for the infinite system in* (4.1)*, which is assumed to satisfy the SSC from Definition* 4.1 *at* $(\bar z,0)$*. Then the exact regularity bound of F at $(\bar x,(\bar z,0))$ is calculated by*

$$\operatorname{reg}F(\bar x,(\bar z,0))=\lim_{\varepsilon\downarrow0}\big[\sup\big\{\|x^*\|^{-1}\big|\ (x^*,\langle x^*,\bar x\rangle+\varepsilon)\in M\big\}\big], \tag{4.12}$$

where the set M is defined in Theorem 4.3*. If in addition* $0<\dim Z<\infty$*, then*

$$\operatorname{reg}F(\bar x,(\bar z,0))=\|D^*F^{-1}((\bar z,0),\bar x)\|=\max\Big\{\|x^*\|^{-1}\Big|\ (x^*,\langle x^*,\bar x\rangle)\in L\Big\}, \tag{4.13}$$

where the set L is defined in Proposition 4.4.

Proof. It follows from Proposition 4.2 that $(\bar{z}, 0) \in \text{int}\,(\text{rge}\, F)$, i.e., the mapping F is metrically regular around $(\bar{x}, (\bar{z}, 0))$. Substituting the ε-coderivative expression from Theorem 4.3 into the exact bound formula (3.2) of Theorem 3.1, we arrive at the limiting representation (4.12).

Let us next justify the equalities in (4.13) under the additional assumption made. Employing Theorem 3.4 and Proposition 4.4, we need to check that condition (3.21) holds and that $\text{reg}\, F(\bar{x}, (\bar{z}, 0)) > 0$. To proceed, take any $\varepsilon > 0$ and $(z^*, p^*) \in S_{(Z \times l_\infty(T))^*}$ satisfying $D_\varepsilon F(\bar{x}, (\bar{z}, 0))(z^*, p^*) \neq \emptyset$. By the same arguments as in the proofs of relationships (4.6) and (4.9) we get the inclusion

$$p^* \in \big(1 - \|z^*\|\big)\, \text{cl}^*\text{co}\, \big\{\delta_t \big|\ t \in T\big\}.$$

Thus the set $\text{cl}^*\{(z^*, p^*) \in S_{(Z \times l_\infty(T))^*} |\ D^*_\varepsilon F(\bar{x}, (\bar{z}, 0))(z^*, p^*) \neq \emptyset\}$ is contained in that of

$$\text{cl}^* \bigcup_{z^* \in I\!B_{Z^*}} \Big[\big\{z^*\big\} \times \big(1 - \|z^*\|\big)\, \text{cl}^*\text{co}\, \big\{\delta_t \big|\ t \in T\big\}\Big]. \tag{4.14}$$

Further, it follows from the proof in (4.9) that $\text{cl}^*\text{co}\, \{\delta_t |\ t \in T\} \subset S_{l_\infty(T)^*}$. Since $\dim Z < \infty$, the latter implies that the set in (4.14) is a subset of $S_{(Z \times l_\infty(T))^*}$, which ensures the validity of (3.21).

It remains to verify that $\text{reg}\, F(\bar{x}, (\bar{z}, 0)) > 0$. Indeed, it is easy to see that

$$D^* F^{-1}\big((\bar{z}, 0), \bar{x}\big)(x^*) \supset \big\{(z^*, 0) \in Z^* \times l_\infty(T)^* \big|\ A^* z^* = x^*\big\}.$$

Since the operator A is surjective, it follows from [3, Lemma 1.18] that $\|(A^*)^{-1}\| > 0$. Therefore we conclude that $\|D^* F^{-1}((\bar{z}, 0), \bar{x})\| > 0$, which yields $\text{reg}\, F(\bar{x}, (\bar{z}, 0)) > 0$ by Theorem 3.1 and thus completes the proof of this theorem. △

Observe from Theorem 3.4 that the exact bound formula

$$\text{reg}\, F\big(\bar{x}, (\bar{z}, 0)\big) = \big\|D^* F^{-1}\big((\bar{z}, 0), \bar{x}\big)\big\| \tag{4.15}$$

holds also in the case of $\dim Z = 0$. For the linear functions $f_t(x) = \langle a^*_t, x\rangle - b_t$ with $(a^*_t, b_t) \in X^* \times I\!R$ as $t \in T$ this equality was obtained in [5] when $C = X$ under the additional assumption that the coefficient set $\{a^*_t |\ t \in T\}$ is bounded in X^*. It is easy to check that the latter assumption implies that the supremum function f is bounded from above around the Slater point $\widehat{x}$ and that $\bar{x} \in \text{int}\, C_0 = \text{int}\,(\text{dom}\, f)$. The following example demonstrates that the converse implication is not true even in a simple one-dimensional setting of semi-infinite programming.

Example 4.6 (bounded from above constraint linear functions with unbounded coefficients). Let $X = I\!R$, $Z = \{0\}$, $T = (0, 1)$, and $f_t(x) = -\frac{1}{t}x + t$ in (4.1). Note that

$$f_t(x) = -\frac{1}{t}x + t = -\frac{1}{t}x - t + 2t \leq -2\sqrt{x} + 2t \quad \text{for all}\ \ x > 0\ \ \text{and}\ \ t \in T.$$

Taking $\widehat{x} = 4$ and $\bar{x} = 1$, we observe that $f_t(\widehat{x}) < -2$, $f_t(\bar{x}) \leq 0$, and the supremum function f is bounded from above around $\widehat{x}$. However, the coefficient set $\big\{-\frac{1}{t}\big|\ t \in T\big\}$ is obviously unbounded.

A natural question arising from Theorem 4.5 is whether formula (4.15) holds true for infinite-dimensional spaces Z. The following counterexample is constructed for the case of the classical Asplund space $Z = c_0$, which is the space of sequences of real numbers converging to zero and endowed with the supremum norm.

Example 4.7 (failure of the coderivative exact bound formula for countable inequality and infinite-dimensional equality constraints.) Let $X = Z = c_0$, and let $T = I\!N$. Define a linear operator $A : X \to Z$ by $Ax := (x_2, x_3, \ldots)$ for all $x = (x_1, x_2, \ldots) \in X$. It is easy to see that A is bounded and surjective. We form a set-valued mapping $F : c_0 \rightrightarrows c_0 \times l_\infty$ of type (4.1) by

$$F(x) := \big\{(z,p) \in Z \times l_\infty \big|\ Ax = z, x_1 + x_n + 1 \le p_n,\ n \in I\!N\big\} \quad \text{for any} \ \ x \in X. \tag{4.16}$$

Take $\bar{x} := \big(-\frac{1}{n}\big)_{n\in I\!N}$, $\bar{z} := A\bar{x}$, and $\widehat{x} := (-2, -\frac{1}{2}, -\frac{1}{3}, \ldots) \in X$. Observe that the strong Slater condition of Definition 4.1 is satisfied at $\widehat{x}$ for (4.16) and that $\bar{x} \in F^{-1}(\bar{z}, 0)$. Defining further

$$x^k := \Big(-1, -\frac{1}{2}, \ldots, -\frac{1}{k-1}, \frac{1}{k}, -\frac{1}{k+1}, -\frac{1}{k+2}, \ldots\Big),\ z^k := \Big(-\frac{1}{2}, \ldots, -\frac{1}{k-1}, \frac{2}{k}, -\frac{1}{k+1}, -\frac{1}{k+2}, \ldots\Big),$$

shows that $x^k \to \bar{x}$ and $z^k \to \bar{z}$ in c_0. Moreover, we have the equalities

$$d\big((z^k, 0); F(x^k)\big) = \max\Big\{\sup_n (x_1^k + x_n^k + 1)_+,\ \sup_n |\ x_{n+1}^k - z_n^k|\Big\} = \max\Big\{\frac{1}{k}, \frac{1}{k}\Big\} = \frac{1}{k} \tag{4.17}$$

where $(\alpha)_+$ when $\alpha \in I\!R$ stands for $\max\{0, \alpha\}$ as usual. It is easy to calculate the inverse mapping value $F^{-1}(z^k, 0) = \{(a, z_1^k, z_2^k, \ldots) \in c_0 |\ a \le -\frac{2}{k} - 1\}$, which gives us

$$d\big(x^k; F^{-1}(z^k, 0)\big) = \max\Big\{\Big(x_1^k + \frac{2}{k} + 1\Big)_+,\ \sup_n |x_{n+1}^k - z_n^k|\Big\} = \max\Big\{\frac{2}{k}, \frac{1}{k}\Big\} = \frac{2}{k}. \tag{4.18}$$

It follows from (4.17) and (4.18) that $\operatorname{reg} F(\bar{x}, (0, \bar{z})) \ge 2$. Thus formula (4.15) fails if we show that

$$\|x^*\| \ge 1 \ \text{ for all } \ x^* \in D^*F\big(\bar{x}, (0, \bar{z})\big)\big(S_{(Z\times l_\infty)^*}\big). \tag{4.19}$$

To proceed, we employ Proposition 4.4 that gives us some $z^* \in I\!B_{Z^*}$ with

$$(x^*, \langle x^*, \bar{x}\rangle) \in \operatorname{cl}^*\Big[(1 - \|z^*\|) \operatorname{co}\big\{(\delta_1 + \delta_n, -1)\big|\ n \in I\!N\big\}\Big] + (A^*z^*, \langle z^*, \bar{z}\rangle),$$

where $\delta_n \in c_0^*$ and $\langle \delta_n, x\rangle = x_n$ for all $x \in c_0$ and $n \in I\!N$. Hence there is a net $(\lambda_\nu)_{\nu\in\mathcal{N}} \subset \widetilde{I\!R}^{I\!N}$ such that $\sum_{n\in I\!N} \lambda_{n_\nu} = 1 - \|z^*\|$ for all $\nu \in \mathcal{N}$ and that

$$(x^*, \langle x^*, \bar{x}\rangle) = w^* - \lim_\nu \sum_{n\in I\!N} \lambda_{n_\nu}(\delta_1 + \delta_n, -1) + (A^*z^*, \langle z^*, \bar{z}\rangle),$$

which readily implies the limiting relationships

$$0 = \lim_\nu \sum_{n\in I\!N} \lambda_{n_\nu}(-\langle \delta_1 + \delta_n, \bar{x}\rangle - 1) = \lim_\nu \sum_{n\in I\!N} \lambda_{n_\nu}(-\bar{x}_1 - \bar{x}_n - 1) = \lim_\nu \sum_{n\in I\!N} \frac{\lambda_{n_\nu}}{n}. \tag{4.20}$$

Since $c_0^* = l_1$, we write z^* in the form $(z_1^*, z_2^*, \ldots) \in l_1$ and observe that $A^*z^* = (0, z_1^*, z_2^*, \ldots) \in l_1$. Thus for any $\varepsilon > 0$ there is $k \in I\!N$ sufficiently large and such that $\sum_{n=k+1}^\infty |z_n^*| \le \varepsilon$, which ensures that $\|A^*z^* - \widehat{z}_k^*\| \le \varepsilon$ with $\widehat{z}_k^* := (0, z_1^*, \ldots, z_k^*, 0, 0, \ldots) \in l_1$. Define further $\widehat{x}_k^*$ by

$$\widehat{x}_k^* := w^* - \lim_\nu \sum_{n\in I\!N} \lambda_{n_\nu}(\delta_1 + \delta_n) + \widehat{z}_k^*$$

and take $e^k := (1, \operatorname{sign}(z_1^*), \ldots, \operatorname{sign}(z_k^*), 0, 0, \ldots) \in c_0$. We observe that $\|e^k\| = 1$ and that

$$\begin{aligned} \|\widehat{x}_k^*\| \quad &\ge \langle \widehat{x}_k^*, e^k\rangle = \lim_\nu \sum_{n\in I\!N} \lambda_{n_\nu}(e_1^k + e_n^k) + \sum_{n=1}^k z_n^* e_{n+1}^k \\ &\ge \lim_\nu \sum_{n\in I\!N} \lambda_{n_\nu} + \liminf_\nu \sum_{n=1}^k |z_n^*| - \sum_{n=1}^{k+1} \lambda_{n_\nu}. \end{aligned} \tag{4.21}$$

It follows from the equations (4.20) that

$$0 \le \limsup_{\nu} \sum_{n=1}^{k+1} \lambda_{n_\nu} \le (k+1) \limsup_{\nu} \sum_{n \in I\!N} \frac{\lambda_{n_\nu}}{n} = 0.$$

Combining this with (4.21) gives us the estimates

$$\|\widehat{x}_k^*\| \ge 1 - \|z^*\| + \sum_{n=1}^{k} |z_n^*| = 1 - \|z^*\| + \sum_{n=1}^{k} |z_n^*| \ge 1 - \|z^*\| + \|z^*\| - \varepsilon = 1 - \varepsilon.$$

It is clear furthermore that $\|x^* - \widehat{x}_k^*\| = \|A^* z^* - \widehat{z}_k^*\| \le \varepsilon$. Thus we arrive at the inequalities

$$\|x^*\| \ge \|\widehat{x}_k^*\| - \|x^* - \widehat{x}_k^*\| \ge 1 - \varepsilon - \varepsilon = 1 - 2\varepsilon \ \text{ for all } \ \varepsilon > 0,$$

which show that $\|x^*\| \ge 1$, i.e., we get (4.19). This verifies the failure of (4.15).

The next example, which is a modification of Example 4.7 shows that the coderivative formula (4.15) for calculating the exact regularity bound fails when $\dim Z = \infty$ even for constraint systems (4.1) with a *single* convex inequality constraint while both Asplund spaces X and Z.

Example 4.8 ((failure of the coderivative exact bound formula for single inequality and infinite-dimensional equality constraints). Let $X = Z = c_0$, and let $T = \{1\}$. Define the linear operator $A : X \to Z$ as in Example 4.7 and consider $F : X \rightrightarrows Z \times I\!R$ given by

$$F(x) := \big\{(z,p) \in Z \times I\!R \big|\ Ax = z,\ f(x) \le p\big\} \quad \text{for any} \quad \in X,$$

where $f(x) := \sup\{x_1 + x_n + 1 |\ n \in I\!N\}$. With the same notation as in Example 4.7 we get

$$d\big((z^k, 0); F(x^k)\big) = k^{-1} \quad \text{and} \quad d\big(x^k; F^{-1}(z^k, 0)\big) = 2k^{-1},$$

which shows that $\operatorname{reg} F(\bar{x}, (\bar{z}, 0)) \ge 2$. Note further that $\operatorname{dom} f = X$ and that

$$\operatorname{epi} f^* = \operatorname{cl}^* \operatorname{co} \big\{(\delta_1 + \delta_n, -1) \big|\ n \in I\!N\big\} + \{0\} \times I\!R_+;$$

the latter follows from the well-known formula for general supremum functions:

$$\operatorname{epi} f^* = \operatorname{cl}^* \operatorname{co} \bigcup_{t \in T} \Big(\operatorname{epi} f_t^*\Big).$$

Taking now any $x^* \in D^* F^{-1}((\bar{z}, 0), \bar{x})(S_{(Z \times I\!R)^*})$ and using Theorem 4.3 with the above representation of $\operatorname{epi} f^*$ to, we arrive at

$$(x^*, \langle x^*, \bar{x}\rangle) \in \operatorname{cl}^* \Big[(1 - \|z^*\|) \operatorname{co} \big\{(\delta_1 + \delta_n, -1) \big|\ n \in I\!N\big\}\Big] + (A^* z^*, \langle z^*, \bar{z}\rangle).$$

Then repeating the arguments of Example 4.7 gives us $\|x^*\| \ge 1$, and thus (4.15) fails.

The next result provides efficient conditions, which ensure the validity of the major regularity formula (4.15) in the case of infinite-dimensional spaces Z. Note that its proof is different from that of (4.13) in Theorem 4.5 when Z in finite-dimensional. In particular, it does not rely on condition (3.21), or its predecessor (1.5), that may not hold. Indeed, even in the simplest setting of $T = \emptyset$ the left-hand side of (3.21) is $\operatorname{cl}^* S_{Z^*}$, which is obviously not a subset of S_{Z^*} when $\dim Z = \infty$.

Theorem 4.9 (validity of the coderivative exact bound formula for finite inequality and infinite-dimensional equality constraints). *In the case of arbitrary Banach spaces X and Z in* (4.1)*, assume that the index set T is finite, that*

$$f_t(x)=\langle a_t^*,x\rangle-b_t \quad \text{for all } \ x\in X,\ t\in T \quad \text{with } \ (a_t^*,b_t)\in X^*\times I\!R,$$

and, given $(\bar x,(\bar z,0))\in\operatorname{gph}F$, the constraint mapping F satisfies the SSC condition at $(\bar z,0)$ with $\bar x\in\operatorname{int}C$. Then we have the exact bound formula (4.15).

Proof. $T=\{1,\ldots,k\}$ and observe that $\operatorname{dom}f=X$ and $C_0=C$ in our case. Since

$$\operatorname{epi}f_n^*=(a_n^*,b_n)+\{0\}\times I\!R_+ \ \text{ and } \ \{0\}\times I\!R_++\operatorname{epi}\delta^*(\cdot;C)\subset\operatorname{epi}\delta^*(\cdot;C),$$

for any $z^*\in I\!B_{Z^*}$ we get the equality

$$\big(1-\|z^*\|\big)\operatorname{co}\big\{\operatorname{epi}f_t^*\big|\ t\in T\big\}+\operatorname{epi}\delta^*(\cdot;C_0)=\big(1-\|z^*\|\big)\operatorname{co}\big\{(a_n^*,b_n)\big|\ 1\le n\le k\big\}+\operatorname{epi}\delta^*(\cdot;C).$$

It is easy to see that the above set is weak* closed in $X^*\times I\!R$. Thus the set M in Theorem 4.3 is represented as follows:

$$M=\bigcup_{z^*\in I\!B_{Z^*}}\Big\{(1-\|z^*\|)\operatorname{co}\big\{(a_n^*,b_n)\big|\ 1\le n\le k\big\}+\operatorname{epi}\delta^*(\cdot;C)+(A^*z^*,\langle z^*,\bar z\rangle)\Big\}$$

Invoking now the result in the first part of Theorem 4.5, we find sequences of $x_m^*\in X^*$, $\lambda^m\in I\!R_+^k$, $(w_m^*,s_m)\in\operatorname{epi}\delta^*(\cdot;C)$, and $z_m^*\in I\!B_{Z^*}$ such that $\sum_{n=1}^k\lambda_n^m=1-\|z_m^*\|$ and

$$\Big(x_m^*,\langle x_m^*,\bar x\rangle+m^{-1}\Big)=\sum_{n=1}^k\lambda_n^m(a_n^*,b_n)+(w_m^*,s_m)+(A^*z_m^*,\langle z_m^*,\bar z\rangle) \tag{4.22}$$

with the upper estimate of the regularity bound

$$\operatorname{reg}F\big(\bar x,(\bar z,0)\big)\le\|x_m^*\|^{-1}+O(m)=\Big\|\sum_{n=1}^k\lambda_n^m a_n^*+w_m^*+A^*z_m^*\Big\|^{-1}+O(m), \tag{4.23}$$

where $O(m)\to0$ as $m\to\infty$. Since $\|\lambda^m\|\le1$, we suppose that $\lambda^m\to\lambda\in I\!R_+^k$ as $m\to\infty$. Furthermore, observe from (4.22) that

$$\begin{aligned} m^{-1} &= \langle x_m^*,\bar x\rangle+\frac{1}{m}-\langle x_m^*,\bar x\rangle=\sum_{n=1}^k\lambda_n^m b_n+s_m-\sum_{n=1}^k\lambda_n^m\langle a_n^*,\bar x\rangle-\langle w_m^*,\bar x\rangle\\ &\ge \sum_{n=1}^k\lambda_n^m(b_n-\langle a_n^*,\bar x\rangle)+s_m-\langle w_m^*,\bar x\rangle\ge\sum_{n=1}^k\lambda_n^m(b_n-\langle a_n^*,\bar x\rangle)\ge0,\end{aligned}$$

which implies that $\sum_{n=1}^k\lambda_n(b_n-\langle a_n^*,\bar x\rangle)=0$ by passing to the limit as $m\to\infty$. Define

$$\varepsilon_m:=\sum_{n=1}^k|\lambda_n^m-\lambda_n|,\ \eta_m:=\sum_{n=1}^k\lambda_n+\|z_m^*\|,\ \text{and}\ \ \widehat{x}_m^*:=\sum_{n=1}^k\lambda_n a_n^*+A^*z_m^*$$

and note that $\varepsilon_m=O(m)$ and $\eta_m=1-O(m)$. Applying Proposition 4.4, we have that

$$\eta_m^{-1}\widehat{x}_m^*\in D^*F\big(\bar x,(\bar z,0)\big)\big(S_{(Z\times I\!R^n)^*}\big),$$

Moreover, the same arguments as in the proof of the second part of Proposition 4.4 ensure that $\|w_m^*\| \to 0$. It follows therefore that

$$\|x_m^* - \widehat{x}_m^*\| = \Big\| \sum_{n=1}^{k} (\lambda_n^m - \lambda_n) a_n^* + w_m^* \Big\| \le \varepsilon_m \sup_{1 \le n \le k} \|a_n^*\| + \|w_m^*\| = O(m),$$

which implies together with (4.23) the estimates

$$\begin{aligned} \operatorname{reg} F\big(\bar{x}, (\bar{z}, 0)\big) &\le \big(\|\widehat{x}_m^*\| + O(m)\big)^{-1} + O(m) \le \big(\eta_m \|\eta_m^{-1} \widehat{x}_m^*\| + O(m)\big)^{-1} + O(m) \\ &\le \Big[(1 - O(m)) \inf\big\{\|x^*\| \big|\, (x^*, \langle x^*, \bar{x}\rangle) \in L\big\} + O(m)\Big]^{-1} + O(m). \end{aligned}$$

Letting $m \to \infty$ therein, we arrive at

$$\operatorname{reg} F\big(\bar{x}, (\bar{z}, 0)\big) \le \sup\big\{\|x^*\|^{-1} \big|\, (x^*, \langle x^*, \bar{x}\rangle) \in L\big\},$$

which implies (4.15) and thus completes the proof of the theorem. △

Concluding Remarks. The reader can observe from the proofs of the results in this section that the imposed Slater condition can be actually replaced by the weaker condition (4.2) at $(\bar{z}, 0)$, which is necessary and sufficient for the metric regularity $\operatorname{reg} F(\bar{x}, (\bar{z}, 0)) < \infty$ of F in (4.1). When $\dim Z = 0$ therein, an equivalent equality to (4.15) is established in [6, Theorem 15] under the additional assumptions that either the set $\cup_{t \in T} \operatorname{dom} f_t^*$ is bounded in X^*, or the space X is reflexive. A result similar to (4.15) as $\dim Z < \infty$ is presented in [7, Theorem 16] in a different form with a different proof (not completely clear to us) under a certain uniform boundedness condition on the functions f_t.

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