CONTINUOUS CONVEX SETS AND ZERO DUALITY GAP FOR CONVEX PROGRAMS

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ABSTRACT. This article uses classical notions of convex analysis over euclidean spaces, like Gale & Klee's boundary rays and asymptotes of a convex set, or the inner aperture directions defined by Larman and Brøndsted for the same class of sets, to provide a new zero duality gap criterion for ordinary convex programs.

On this ground, we are able to characterize objective functions and respectively feasible sets for which the duality gap is always zero, regardless of the value of the constraints and respectively of the objective function.

1. INTRODUCTION

The aim of this paper is to refine the well-known convex optimization Theorem of primal attainement (see for instance [2, Proposition 5.3.3], and also the Appendix of this article) which proves that the duality gap of a convex program defined over an euclidean space amounts to zero provided that it does not possess any direction of recession (that is a direction of recession which is common to both the objective function and the feasible set).

Indeed, an observant analysis points out that, beside the class of convex programs free of directions of recession, another remarkable family of zero duality gap convex programs may also be addressed.

More precisely, our main result identifies among the directions of recession of a closed and convex function or set, a special sub-class of vectors with the following property: a convex program has no duality gap provided that at least one of its directions of recession is special in the above-mentioned way for its objective function or for its feasible set.

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We are thus able to characterize all the directions of recessions of a convex program which may entail a positive duality gap. On this ground, we achieve a complete characterization of the objective functions and respectively feasible sets, such that the duality gap is zero regardless of the constraints and respectively of the objective function of the convex program.

1.1. **Basic notions and definitions for convex programs.** Throughout this article, X stands for an euclidean space, while $\langle \cdot, \cdot \rangle$, and respectively $\| \cdot \|$ denote the dot-product over X, and respectively the associated norm; the class of convex and lower semi-continuous functions $f : X \to \mathbb{R} \cup \{+\infty\}$ which are proper (meaning that their effective domain

$$\mathrm{dom}\,(f) = \{x \in X : \; f(x) < +\infty\}$$

is non-empty) is denoted by $\Gamma_0(X)$. Another standard notation used in the sequel is $[f \leq r]$ for the sub-level set $\{x \in X : \; f(x) \leq r\}$ of the set f.

An important use will be made of notions of recession analysis (for a detailed approach of this topic, the reader is referred to [13, §8]). Thus, the recession cone C^∞ of a closed and convex set C is defined as being the maximal closed convex cone whose translate at every point of C lies in C; if f is a function from $\Gamma_0(X)$, the recession function f^∞ of f is the function whose epigraph is the recession cone of the epigraph of f. A non-null vector v from X is a direction of recession of the closed and convex C if it belongs to C^∞, and of the function f from $\Gamma_0(X)$ if $f^\infty(v) \leq 0$.

Equivalently, it is possible to define the directions of recession of a function f from $\Gamma_0(X)$ as the non-null vectors v such that f is non-increasing over any half-line of direction v, and the recession directions of a closed and convex set C as being the directions of recession of the ι_C, the indicator function of C,

$$\iota_C(x) = \begin{cases} 0 & x \in C \\ +\infty & x \notin C \end{cases} .$$

Let us consider the ordinary convex program

$$P(f, g_i): \qquad \text{Find} \quad \inf\{f(x) : \; g_i(x) \leq 0, \;\; \forall\, 1 \leq i \leq n\},$$

where the objective function f and the n constraint mappings $g_1, \ldots, g_n$ belong to $\Gamma_0(X)$. We assume that the program is consistent, meaning that the feasible set A, that is the set over which all the constraint mappings are non-positive, is non-void:

$$A = \bigcap_{i=1}^{n} [g_i \leq 0] \neq \emptyset.$$

Remark 1. *Our results do not assume any qualification condition on the objective function and feasible set. In particular, we do not suppose that* $\operatorname{dom} f \cap A \neq \emptyset$*; accordingly, the infimum of f over A may take one of the values $-\infty$ or $+\infty$.*

The most obvious way to address this problem, is to adapt to the constrained case one of the algorithms used to minimize the objective function f alone. To this respect, it is customary to address a sequential minimizing algorithm, by modifying the input needed in computing the next term of the minimizing sequence $(x_n)_{n\in\mathbb{N}}$: instead of imputing into the algorithm the previous term x_{n-1}, we use a convex combination of form $[\alpha_{n-1}\, x_{n-1} + (1-\alpha_{n-1})\, y_{n-1}]$, between x_{n-1} and y_{n-1}, its projection on the feasible set A. Cases $\alpha_n = 0$ and $\alpha_n = 1/2$ are among the most occurrent choices for the relaxation parameter α_n.

However, as stated by Bauschke and Borwein in their highly influential article [4, p. 368], it is possible to numerically implement this approach only if "*[the feasible set] is "simple" in the sense that the projection [...] onto [it] can be calculated explicitly*".

Indeed, if it is an easy job to compute the projection operator onto A when an affine or a quadratic mapping is the only constraint of the problem, this is no longer the case for most of the real word applications. For instance, computing the projection onto the feasible set A may be a very difficult task even when all the constraint mappings g_i are affine. In this case, a projection algorithm onto A which is polynomial-time with respect to M has recently been provided - see [12] - but it works only for the particular case when A is a cone, and the dot product of any pair of vectors of A is non-negative; at our best knowledge, the general problem is still open.

A different method, less sensitive to the shape of the effective set, consists in solving the dual problem of $P(f, g_i)$:

$$D(f, g_i) : \sup \left\{ \inf \left\{ f(x) + \sum_{i=1}^{n} r_i\, g_i(x) : x \in X \right\} : r_1, \dots, r_n \geq 0 \right\};$$

we make the convention asking that $0 \cdot (+\infty) = +\infty$.

We are interested in the apparently more complicated dual problem simply because it demands to minimize convex functions over the whole underlying space; from a numerical point of view, it is much easier to compute its solution, sup $D(f, g_i)$ than the solution inf $P(f, g_i)$ of the initial convex program. Characterizing the convex programs for which the duality gap $\delta(f, g_i) = \inf\ P(f, g_i) - \sup\ D(f, g_i)$ amounts to zero (by convention, $(+\infty) - (+\infty) = 0$ and $(-\infty) - (-\infty) = 0$), is thus of an important concern in constrained convex optimization.

Remark 2. *Even when the duality gap is non-null, it may still be possible to reduce the convex program to an unconstrained setting, by using instead of the combination $\sum_{i=1}^{n} r_i\, g_i(x)$ a different convex mapping. One of the most prevailing such examples is provided by* [10, Theorem 1.2.3, Chapter VII]*, a result which proves that*

$$\inf_A f = \sup \{\inf \{f(x) + r\, \mathrm{dist}(x, A) :\ x \in X\} :\ r \geq 0\}\,.$$

However, the practical interest of this result is meager, as it is impossible to actually compute the distance to A when the feasible set is not simple in the sense of Bauschke and Borwein. In other words, using the distance to the feasible set as a penalty function does not work for precisely the same type of problems for which the projection-based methods fail.

Arguably one of the most versatile and useful zero duality gap criterion is provided by the following result.

Theorem of the primal attainement [2, Proposition 5.3.3]: *The convex program $P(f, g_i)$ has no duality gap if:*
(Q1) there is no direction of recession common to all the functions f, g_i.
Moreover, the primal value inf $P(f, g_i)$ *is attained when finite.*

Remark 3. *In their textbook* [2]*, Auslender and Teboulle prove Proposition 5.3.3 under the blanket assumption that*

$$\text{(1)}\quad \mathrm{dom}\, f \subset \mathrm{dom}\, g_i,\quad \mathrm{ri}(\mathrm{dom}\, f) \subset \mathrm{ri}(\mathrm{dom}\ g_i),\qquad \forall i \in \{1, \ldots, n\},$$

(here ri C stands for the relative interior of the convex set C), but implicitly claim that there is no need of such conditions for the Theorem of the primal attainement to hold true. Indeed, at page 158 of their treatise, Auslender and Teboulle write that "the assumption on the domains can be always enforced for any optimization problem by appropriately redefining the objective function when necessary", although no indication about how this goal may be achieved is given.

In order to fill this gap in the mathematical literature, we provide in the Appendix of this article, a complete proof of the Theorem of the primal attainement, under no blanket or qualification condition.

Remark 4. *A different proof of the Theorem of the primal attainement is provided in Borwein & Lewis' book* [5, Theorem 4.3.8]*, for the particular case when condition $(Q1)$ is replaced by:*
(Q2) the mapping $(r_0\, f + \Sigma_{i=1}^n\, r_i\, g_i)$ has compact sub-level sets for some non-negative coefficients r_0, r_i.

Let us also recall that, given $C_1,\ldots,C_n$, a collection of closed convex sets in X whose intersection is not empty, then the recession cone of the intersection of the sets C_i is the intersection of the recession cones of the sets C_i (see [13, Corollary 8.3.3]).

Combining the theorem of the primal attainement and the above result, it yields that the duality gap of $P(f, g_i)$ is zero provided that:
(Q3) the convex program does not have any direction of recession (that is directions of recession common to both the objective function and the feasible set):

$$[f^\infty \leq 0] \cap\, A^\infty = \{0\}. \tag{2}$$

1.2. **Statement of the problem and plan of the paper.** Our article addresses the characterization of zero duality gap convex programs that do have directions of recession (and thus lie outside of the scope of the Theorem of primal attainement).

The main concern of our paper is to define and study, both for closed and convex subsets of X, and for functions belonging to $\Gamma_0(X)$, the notion of a special direction of recession.

Definition 1. *Let v be a direction of recession of the function f from $\Gamma_0(X)$. The vector v is called a* special direction of recession of f *if for any convex program $P(f, g_i)$ admitting v as a direction of recession, it holds that* $\delta(f, g_i) = 0$.

Definition 2. *Let v be a direction of recession of the closed and convex set A. The vector v is called a* special direction of recession of A *if, for any convex program $P(f, g_i)$ such that A is the feasible set of $P(f, g_i)$, and v one of its directions of recession, it holds that $\delta(f, g_i) = 0$.*

Section 2 deals with the study of the special directions of recession for functions and sets. Our main results (Theorems 1 and 2) characterize the special directions by using the notion of an ia-direction of recession, which generalizes to the closed and convex functions the inner aperture directions defined for sets by Larman ([11]) and Brøndsted ([6]).

Our new duality gap criterion, as well as three applications of this result in which an important role is played by the continuous convex sets as defined by Gale $\&$ Klee ([9]), are addressed in the last section of our study.

2. SPECIAL DIRECTION OF RECESSION FOR CLOSED AND CONVEX SETS AND FUNCTIONS

The following well-known result (see [13, Theorem 8.6]) restates the notion of direction of recession for a closed and convex function f. Indeed, instead of imposing monotony properties of f over all the half-lines of a given direction, Proposition 1 focuses on the limit of f along one half-line.

Proposition 1. *Let f be a function from $\Gamma_0(X)$ and v a non-null vector from X. The two following sentences are equivalent:*

$i)$ *The mapping f is non-increasing over any half-line of direction v (in other words, the vector v is a direction of recession of f)*

$ii)$ *There is a half-line of direction v along which the limit of f does not amount to $+\infty$.*

In the same spirit, let us refine the notion of direction of recession by a more detailed investigation of the limits of a function along various half-lines with the same direction.

Definition 3. *Let f be a function from $\Gamma_0(X)$. A non-null vector $v \in X$ is a ba-direction of recession for f if there are two half-lines of direction v along of which f has different limits.*

A non-null vector $v \in X$ is an ia-direction of recession for f if the limit of f along any half-line of direction v amounts to $\inf_X f$.

Definition 4. *Given C a non-empty closed and convex subset of X, we call ba and respectively ia-directions of recession of C the vectors which are ba and respectively ia-directions of recession of the indicator function of C.*

Let us first state a standard remark.

Proposition 2. *Let f be a function from $\Gamma_0(X)$ and v a non-null vector. The two following sentences are equivalent.*
$i)$ v is a direction of recession for f
$ii)$ v is either a ba, or an ia-direction of recession of f.

Proof of Proposition 2. $i) \Rightarrow ii)$. Let us consider v, a direction of recession of f which is not a ba-direction of recession, and x a point in X. All what we have to prove is that the limit of f along the half-line $x + \mathbb{R}_+ \, v$ amounts to $\inf_X \, f$.

Let us assume, to the end of achieving a contradiction, that this fact does not holds true, that is that $\lim_{s \to \infty} \, f(x + s\, v) > \inf_X \, f$. Accordingly, there exists $y \in X$ such that

$$f(y) < \lim_{s \to \infty} \, f(x + s\, v). \tag{3}$$

Since v is a direction of recession of f, it yields that f is non-increasing over $y + \mathbb{R}_+ \, v$; thus

$$\lim_{s \to \infty} \, f(y + s\, v) \leq f(y). \tag{4}$$

Relations (3) and (4) prove that the limits of f along the two half-lines $x + \mathbb{R}_+ \, v$ and $y + \mathbb{R}_+ \, v$, both of direction v, are different; consequently, v is a ba-direction of recession of f. This contradiction shows that our initial assumption was false.

In other words, we have proved that any direction of recession of f which is not a ba-direction of recession is necessarily an ia-direction of recession.

$ii) \Rightarrow i)$. Let v be a ba-direction of recession of f. Accordingly, there are two half-lines of direction v along of which the function f has different limits. Along at least one of those two half-lines, the limit of f must be different from $+\infty$, so (see Proposition 1), it follows that v is a direction of recession for f.

Let us now consider w, a ia-direction of recession of f. As f is proper, its infimum over the underlying space X does not amount to $+\infty$; thus, the same holds for the limit of f along any half-line of direction w. Hence, by using once more Proposition 1, it results that w is one of the directions of recession of the mapping f.

Accordingly, we have proved that any ba or ia-direction of recession of f is a direction of recession of f. □

In order to justify the notations from Definition 3, let us recall, following Gale and Klee ([9]), that a boundary ray of a set is a half-line which is contained in the boundary of the set. An asymptote of a set C is a half-line d disjoint from C, such that the gap between d and C,

$$\mathrm{gap}(d, C) = \inf\{\|x - y\| : \ x \in d, \ y \in C\},$$

amounts to zero. Another key notion for our study is borrowed from Larman ([11]) and Brøndsted ([6]): a direction of inner aperture of a closed and convex set C is a non-null vector v such that any half-line of direction v intersects C along a half-line.

For a given closed and convex set C, Larman proved (see [11, Theorem 4, page 225]) that the set of all its inner aperture directions is an evenly convex cone (that is a cone which is the intersection of a family of open half-spaces) containing the interior of the recession cone of C, but not, in general, its relative interior. It is thus possible to find an unbounded closed and convex set with no inner aperture.

The connection between these those important notions was achieved by Bair ([3, Theorem 1, p. 237]), who proved that any recession direction of a closed and convex set C is either an inner aperture direction of C, or the direction of some boundary ray or some asymptote of C

We are now in a position to give the following characterization of ba and ia-directions of recession of a mapping in terms of the boundary rays, asymptotes and inner aperture directions of its sub-level sets.

Proposition 3. *Let f be a function from $\Gamma_0(X)$, and v a non-null vector from X. Then v is a ba-direction of recession for f if and only if any sub-level set $[f \leq r]$ with $r > \inf_X f$ possesses a boundary ray or an asymptote of direction v.*

In particular, the ba-directions of recession of a closed and convex set C are simply the directions of its boundary rays and asymptotes.

Proof of Proposition 3. The if part: let us suppose that the half-line $d = (x_0 + \mathbb{R}_+ v)$ is a boundary ray or an asymptote of the sub-level set $[f \leq r]$, where $r > \inf_X f$.

The following result is well-known (see [8, Lemma 1], and also Lemma 1.2 in [9]).

Lemma 1. *Let C be a closed and convex subset of X and $v \in X$ a non-null vector. Then v is the direction of some boundary ray or asymptote of C if and only if there are two half-lines of direction v, one completely contained within the set C, the other disjoint from that set.*

Thus, there is $x + \mathbb{R}_+ v$, a half-line of direction v which lies entirely without $[f \leq r]$; in particular,

$$\lim_{s\to\infty} f(x + s\,v) \geq r. \tag{5}$$

On the other hand, there exists a half-line of direction v which is contained in $[f \leq r]$; hence, the limit of f along this half-line is not $+\infty$, and (see Proposition 1), we may infer that v is a direction of recession of f.

Finally, let us pick a point $y \in X$ such that $f(y) < r$; since v is a direction of recession of f, it follows that f is non-increasing over the half-line $y + \mathbb{R}_+ v$. Thus

$$\lim_{s\to\infty} f(y + s\,v) \leq f(y) < r. \tag{6}$$

Relations (5) and (6) prove that v fulfills the definition of a ba-direction of recession of f.

The $only\ if$ part: let us suppose that v is a ba-direction of recession of f. Accordingly, there are two half-lines of direction v along of which the function f has different limits.

On one hand, we deduce that the limit of f along one of this two half-lines (say $x + \mathbb{R}_+ v$) is greater then $\inf_X f$. Pick now a real number t such that

$$\inf_X f < t < \lim_{s\to\infty} f(x + s\,v), \tag{7}$$

and a point $y \in X$ such that $f(y) \leq t$.

On the other hand, we know (see Proposition 2), that v is a direction of recession for f. In particular, the mapping f is non-increasing over the half-line $y + \mathbb{R}_+ v$, fact which allows us to write that

$$\lim_{s\to\infty} f(y + s\,v) \leq f(y) \leq t. \tag{8}$$

From relation (7) it results that the half-line $x + \mathbb{R}_+ v$ lies without the sub-level set $[f \leq t]$, while relation (8) proves that the half-line $y + \mathbb{R}_+ v$ lies within this set. Let us invoke once again Lemma 1 to deduce that v is the direction of some boundary line or asymptote of the sub-level set $[f \leq t]$. Point $d)$ from [8, Proposition 3] proves that the directions of the boundary lines or asymptotes are common to all the sub-level sets of f, perhaps excepting argmin$_X f$; we may therefore deduce that any sub-level set $[f \leq r]$ with $r > \inf_X f$, possesses a boundary ray or an asymptote of direction v. □

Let us complete this sub-section with the following corollary of Propositions 2 and 3, Theorem 1 from [3], and point $d)$ from [8, Proposition 3].

Proposition 4. *Let f be a function from $\Gamma_0(X)$, and v a non-null vector from X. Then v is a ia-direction of recession for f if and only if v is an inner aperture direction for any sub-level set $[f \le r]$ with $r > \inf_X f$.*

In particular, the ia-directions of recession of a closed and convex set C are its inner aperture directions.

2.1. **Special directions of recession for a closed and convex function.**

Theorem 1. *Let f be a function from $\Gamma_0(X)$, and v one of its directions of recession. The two following statements are equivalent.*

$i)$ v is a ia-direction of recession of f

$ii)$ $\delta(f, g_i) = 0$ provided that v is a direction of recession for the convex program $P(f, g_i)$.

Proof of Theorem 1. $i) \Rightarrow ii)$ Let us consider f an element of $\Gamma_0(X)$, v one of its ia-directions of recession, and $P(f, g_i)$, a convex program such that v is a direction of recession for its feasible set, A.

Pick x_0, a point belonging to the set A; as v is a direction of recession of A, the entire half-line $x_0 + \mathbb{R}_+ v$ lies within A, and hence

$$\inf_A f \le \inf_{x_0+\mathbb{R}_+ v} f \le \lim_{s\to\infty} f(x_0 + s\,v). \tag{9}$$

Let us also pick a real number $r > \inf_X f$; Proposition 4 implies that v is an inner aperture direction for the set $[f \le r]$. Accordingly, the half-line $x_0 + \mathbb{R}_+ v$ meets $[f \le r]$ over a half-line; it follows that

$$\lim_{s\to\infty} f(x_0 + s\,v) \le r \qquad \forall r > \inf_X f. \tag{10}$$

From relations (9) and (10) it yields that $\inf_A f = \inf_X f$. Since for any convex program it obviously holds that

$$\inf_X f \le \sup D(f, g_i) \le \inf P(f, g_i) = \inf_A f, \tag{11}$$

we may infer that the duality gap of the convex program $P(f, g_i)$ amounts to zero.

$ii) \Rightarrow i)$ Let us consider f a mapping belonging to $\Gamma_0(X)$, and v one of its ba-directions of recession; without restraining the generality, we may assume that $\|v\| = 1$.

Theorem 1 is completely proved provided that we define a convex program $P(f, g_i)$ with a positive duality gap such that v is a recession direction for the feasible set A.

As v is a ba-direction of recession, there are two half-lines of direction v along which the limits of f are different. One of those two limits must be greater then $\inf_X\ f$; hence, there are a real number $r > \inf_X\ f$, and a point $x_0 \in X$ such that

$$\lim_{s\to\infty} f(x_0 + s\,v) \geq r. \tag{12}$$

Let us define the mapping $g : X \to \mathbb{R}$ by the formula

$$g(x) = \|x - x_0\| - \langle x - x_0, v\rangle\,.$$

Obviously, g belongs to $\Gamma_0(X)$, and $[g \leq 0] = x_0 + \mathbb{R}_+ v$. As v is a ba-direction of recession of f, let us use Proposition 2 and deduce that the objective function f in non-increasing over any half-line of direction v; in particular, it holds that

$$\inf\ P(f, g) = \inf_{x_0+\mathbb{R}_+ v} f = \lim_{s\to\infty} f(x_0 + s\,v). \tag{13}$$

To the purpose of estimating $\sup\ D(f, g)$, let us consider a point $y \in X$ such that $f(y) < r$. As v is a direction of recession of f, it results that

$$\lim_{s\to\infty} f(y + s\,v) \leq f(y) < r. \tag{14}$$

On the other hand,

$$\begin{aligned} g(y + s\,v) &= \|y - x_0 + s\,v\| - \langle y - x_0 + s\,v, v\rangle \\ &= \sqrt{\|y - x_0\|^2 + 2\,s\,\langle y - x_0, v\rangle + s^2} - (\langle y - x_0, z\rangle + s)\,. \end{aligned}$$

Let us remark that the expression $\sqrt{\|y - x_0\|^2 + 2\,s\,\langle y - x_0, v\rangle + s^2}$ may be written under the form

$$\sqrt{\left(\|y - x_0\|^2 - \langle y - x_0, v\rangle^2\right) + \left(\langle y - x_0, v\rangle + s\right)^2};$$

moreover, when s is greater than $|\,\langle y - x_0, z\rangle\,|$, then it holds that

$$\langle y - x_0, z\rangle + s = \sqrt{\left(\langle y - x_0, z\rangle + s\right)^2}.$$

Consequently, when $s > |\,\langle y - x_0, z\rangle\,|$, it follows that

$$g(y + s\,v) = \sqrt{\alpha + \beta^2} - \sqrt{\beta^2},$$

where

$$\alpha = \|y - x_0\|^2 - \langle y - x_0, v\rangle^2 \geq 0, \quad \beta = \langle y - x_0, z\rangle + s > 0.$$

But when $\alpha \geq 0$ and $\beta > 0$ it holds that

$$\sqrt{\alpha + \beta^2} - \sqrt{\beta^2} \quad = \quad \frac{\alpha}{\sqrt{\alpha + \beta^2} + \sqrt{\beta^2}};$$

thus

$$g(y + s\,v) = \frac{\|y - x_0\|^2 - \langle y - x_0, v\rangle^2}{\|y - x_0 + s\,v\| + \langle y - x_0, z\rangle + s} \qquad \forall s > |\,\langle y - x_0, z\rangle\,|.$$

It is obvious that

$$\lim_{s\to\infty} g(y + s\,v) = 0. \tag{15}$$

By combining relations (14) and (15), we infer that

$$\lim_{s\to\infty} \left(f(y + s\,v) + t\,(y + s\,v)\right) < r \qquad \forall t \geq 0. \tag{16}$$

As a direct consequence of relation (16) we deduce that

$$\inf_X\,(f + t\,g) \leq \inf_{y+\mathbb{R}_+\,v} (f + t\,g) < r \qquad \forall t \geq 0,$$

fact which implies that

$$\sup D(f, g) < r. \tag{17}$$

Relations (12), (13) and (17) prove that $\delta(f, g) > 0$. □

2.2. **Special directions of recession for a closed and convex set.**

Theorem 2. *Let A be a non-void closed and convex subset of X, and v one of its directions of recession. The two following statements are equivalent.*

$i)$ v is a ia-direction of recession of A

$ii)$ $\delta(f, g_i) = 0$ provided that v is a direction of recession for the convex program $P(f, g_i)$.

Proof of Theorem 2. $i) \Rightarrow ii)$ Let us consider a convex program such that v, one of its directions of recession, is an ia-direction of recession of the feasible set A.

Pick a real number $r > \inf_X f$, and a point $x \in X$ such that $f(x) \leq r$. As v is a direction of recession for the objective mapping, it follows that f is non-increasing over the half-line $x + \mathbb{R}_+\,v$; accordingly,

$$f(x + s\,v) \leq f(x) \leq r \qquad \forall s \geq 0. \tag{18}$$

On the other hand, v is a ia-direction of recession for A; Proposition 4 proves that the half-line $x + \mathbb{R}_+\,v$ meets the feasible set. Hence

$$\exists s \geq 0 \;\text{ s.t. }\; (x + s\,v) \in\; A. \tag{19}$$

Combining relations (9) and (10), it results that $\inf_A f = \inf_X f$; as already noticed (see relation (11), this fact entails that the duality gap of the convex program $P(f, g_i)$ amounts to zero.

$ii) \Rightarrow i)$ Let us consider A a non-void closed and convex subset of X, and v one of its ba-directions of recession; without restraining the generality, we may assume that $\|v\| = 1$.

Our aim is to define a convex program $P(f, g_i)$ with a positive duality gap such that v is a recession direction for the objective function f. To this respect, we distinguish two cases: $a)$, when A admits a boundary line of direction v, and $b)$, when there is an asymptote of A of direction v.

• Case $a)$. Let $x_0 + \mathbb{R}_+ v$ be a boundary ray of A. The point $x_0 + v$ belongs accordingly to the boundary of the closed and convex subset A of the euclidean space X. Thus (see [13, Theorem 11.6]), there exists an element $y \in X$ such that

$$\langle x - (x_0 + v), y \rangle \le 0 \qquad \forall x \in A. \tag{20}$$

Let us consider the convex program $P(f, g)$, where relations

$$f(x) = \begin{cases} -\sqrt{\lambda\, s}, & x = x_0 + \lambda\, y + s\, v \\ +\infty, & x \notin (x_0 + [0,1]\, y + \mathbb{R}_+ v) \end{cases} \tag{21}$$

denotes the objective function, while the constraint mapping is defined by the formula

$$g(x) = \operatorname{dist}(x, A) = \inf\{\|x - y\| :\ y \in A\}.$$

In order to determine $\inf\, P(f, g)$, let us first study the intersection between the half-strip $(x_0 + [0,1]\, y + \mathbb{R}_+ v)$ and the feasible set A. Set $x = x_0$ and respectively $x = x_0 + 2\, v$ in relation (20) to deduce that $\langle v, y \rangle \le 0$ and respectively $\langle v, y \rangle \ge 0$. Thus,

$$\langle v, y \rangle = 0. \tag{22}$$

Pick two real numbers, $\lambda \in [0, 1]$ and $s \ge 0$; relation (22) implies that

$$\langle x_0 + \lambda\, y + s\, v, y \rangle = \langle x_0, y \rangle + \lambda\, \|y\|^2; \tag{23}$$

by combining relations (20), (22) and (23) it results that :

$$(x_0 + [0,1]\, y + \mathbb{R}_+ v) \cap\, A = (x_0 + \mathbb{R}_+ v)\,. \tag{24}$$

From relation (24) it yields that

$$\inf_A\, f = \inf_{x_0 + \mathbb{R}_+ v}\, f, \tag{25}$$

while definition (21) implies that

$$f(x_0 + s\,v) = 0 \qquad \forall s \geq 0. \tag{26}$$

Accordingly,

$$\inf P(f, g) = 0. \tag{27}$$

Let us now compute $\sup\, D(f, g)$. As for any $s \geq 0$ it obviously holds that $x_0 + s\,v \in A$, we deduce that

$$g(y + s\,v) \leq \|(y + s\,v) - (x_0 + s\,v)\| = \|y - x_0\| \qquad \forall s \geq 0; \tag{28}$$

let us combine relations (21) and (28) to obtain that

$$f(y + s\,v) + r\,g(y + s\,v) \leq r\,\|y - x_0\| - \sqrt{s} \qquad \forall s \geq 0.$$

Thus

$$\inf_X\,(f + r\,g) \leq \inf_{y+\mathbb{R}_+\,v}(f + r\,g) = -\infty \qquad \forall r \geq 0,$$

and so

$$\sup D(f, g) = -\infty. \tag{29}$$

Since relations (27) and (29) imply that the duality gap of $P(f, g)$ is equal to infinity, the proof of case $a)$ is complete.

• Case $b)$. Let $x_0 + \mathbb{R}_+\,v$ be an asymptote of the feasible set A.

The convex program addressed in order to prove Theorem 2 under the assumptions of case $b)$ is $P(f, g)$, where the objective mapping f is the indicator function of the half-line $x_0 + \mathbb{R}_+\,v$, and the constraints are expressed by the function $g(x) = \operatorname{dist}(x, A)$.

On one hand, the half-line $x_0 + \mathbb{R}_+\,v$ and the feasible set A are disjoint, and so it results that $\inf\; P(f, g) = +\infty$.

On the other hand, the gap between the set A and the half-line $x_0 + \mathbb{R}_+\,v$ is zero. Accordingly,

$$\inf_{x_0+\mathbb{R}_+\,v}\; g = 0,$$

and hence

$$\inf_X(f + r\,g) \leq \inf_{x_0+\mathbb{R}_+\,v}(f + r\,g) = 0 \qquad \forall r \geq 0.$$

It results that $\sup\; D(f, g) \leq 0$; the vector v is thus a direction of recession for the positive duality gap convex program $P(f, g)$. □

3. A NEW ZERO DUALITY GAP CRITERION AND APPLICATIONS

An obvious consequence of Theorems 1 and 2, combined with the classical Theorem of primal attainement is the following zero duality result.

Theorem 3. *The duality gap of a convex program over an euclidean space is zero provided that either:*
$a)$ *it doesn't admit directions of recession, or*
$b)$ *at least one of its directions of recession is an* ia*-direction of recession of the objective function, or an* ia*-direction of recession of the feasible set.*

An unexpected result of Theorem 3 is the following "folk theorem" (that is a known result which does not figure in the mathematical literature).

Theorem 4. *If* $X = \mathbb{R}$*, then any convex program has a zero duality gap.*

Proof of theorem 4. Let f be a function from $\Gamma_0(\mathbb{R})$ and v one of its directions of recession. Assume that $v > 0$ (the case $v < 0$ is similar). Then f is a non-increasing mapping, so

$$\lim_{x \to +\infty} f(x) = \inf_{\mathbb{R}} f. \tag{30}$$

Remark also that each and every of the half-lines of direction v is necessarily of form $[a, +\infty]$, with $a \in \mathbb{R}$. From relation (30) it follows that the limit of f along any half-line of direction v amounts $\inf_{\mathbb{R}} f$; in other words, v is an ia-direction of recession of f.

We have thus proved that any direction of recession of a function belonging to $\Gamma_0(\mathbb{R})$ is an ia-direction of recession (in dimension one, there are no ba-directions of recession). The desired conclusion stems now from point $ii)$ of Theorem 3. □

The following example states that, unlike for the convex programs fulfilling the condition $Q1$ of the Theorem of primal attainement, the primal value of a convex program satisfying point $ii)$ of Theorem 3 is not necessarily attained when finite.

Example 1. *Let* $X = \mathbb{R}$*,* $f(x) = \exp(x)$ *and* $g(x) = x$*.*

Clearly, Theorem 4 proves that the duality gap of the problem $P(f, g)$ *amounts to zero. Moreover, it is well-known that the infimum of the function* $\exp(x)$ *over the half-line* $]-\infty, 0]$ *is zero, and that the mapping* $\exp(x)$ *does not achieve it.*

We have thus defined a convex program with no duality gap, whose primal value, although finite, is not attained.

To the respect of giving more amenable versions of Theorem 3, let us recall, following Gale & Klee ([9]), that a closed and convex set without any boundary rays or asymptotes is called a continuous convex set.

The class of functions f belonging to $\Gamma_0(X)$, such that all their sub-level sets, excepting perhaps argmin f, are continuous convex sets, have been extensively studied in [7, §4] (see also [1]). Of a particular interest for our study is the statement [7, Corollary 4. 1], which proves that a mapping f belongs to the above-defined class if and only if for each and every of its directions of recession v, its limit along any half-line of direction v amounts to $\inf_X f$.

We are now in a position to state the main consequences of our previous results.

Theorem 5. *The duality gap of the convex program $P(f, g_i)$ over the euclidean space X is zero provided that the function $f + \iota_A$ is proper and all its sub-level sets, excepting perhaps its argmin set, are continuous convex sets.*

Proof of Theorem 5. Let us first remark that a vector v is a direction of recession of the mapping $f + \iota_A$ if and only if it is simultaneously a direction of recession for the objective function f and for the feasible set A. Accordingly, the function $f + \iota_A$ is coercive if and only if the convex program does not admit directions of recession, so the conclusion of the theorem in this case yields from point $a)$, Theorem 3.

In order to address the remaining case, that is when the mapping $f + \iota_A$ admits a direction of recession, say v, let us first recall the obvious fact that any inner aperture direction of some closed and convex set C is also an inner aperture direction for any closed and convex set D which contains C as a subset.

Pick r a real number such that $r > \inf_X (f + \iota_A)$. The set $[(f + \iota_A) \leq r]$ is a continuous convex set and v is one of its directions of recession; hence v is an inner aperture direction for $[(f + \iota_A) \leq r]$.

Both the closed and convex sets $[f \leq r]$ and A contain $[(f + \iota_A) \leq r]$ as a subset. Accordingly, v is an inner aperture direction of $[f \leq r]$, and thus, by Proposition 3, it is an ia-direction of recession of f, as well as an inner aperture direction for A, and hence, by Proposition 4, an ia-direction of recession of A. The duality gap of the convex program is zero by virtue of the point $b)$ of Theorem 3. □

Remark 5. *As any bounded closed and convex subset of X is continuous, Theorem 5 allows us to recapture the well-known result which states that the duality gap of the convex program $P(f, g_i)$ is zero provided that the function $f + \iota_A$ is coercive.*

The last two statements (obvious corollaries of Theorems 1 and 2) characterize the objective functions and the feasible sets for which any convex program has a zero duality gap.

Theorem 6. *Let f be a function from $\Gamma_0(X)$. The two following sentences are equivalent:*

$i)$ *the sub-level sets $[f \leq r]$ with $r > \inf_X f$ are continuous convex sets,*

$ii)$ *the duality gap of any convex program whose objective function is f amounts to zero.*

Theorem 7. *Let A be a non-empty closed and convex subset of X. The two following sentences are equivalent:*

$i)$ *A is a continuous convex set,*

$ii)$ *if A is the feasible set of a convex program, then its duality gap is zero.*

4. Appendix

The main concern of this appendix is to prove that the Theorem of the primal attainement holds true even if the blanket conditions given by relation (1) are not satisfied.

A standard manner to address the duality gap of a convex program, is to study the extended-real-valued mapping $v : \mathbb{R}^n \to \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$, customary called the infimal (or marginal) function, and defined, for every $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$, as the infimum of the objective function f over the closed and convex set $X_{\mathbf{y}}$, where

$$X_{\mathbf{y}} = \{x \in X : \ g_i(x) \leq y_i, \, \forall 1 \leq i \leq n\}.$$

At this point of our study, we need to recall a central notion in convex analysis. Given Y a locally convex space, Y^* its topological dual and $\langle \cdot, \cdot \rangle_Y$ the bilinear form between Y and Y^*, to any extended-real-valued mapping $h : Y \to \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$ we associate its Fenchel-Legendre conjugate, a function $h^* : Y^* \to \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$ defined by the formula

$$h^*(y^*) = \text{Sup } \{\langle y^*, y \rangle_Y - h(y) : \ y \in Y\}.$$

Let us get back to the infimal function of the convex program, and consider $v^{**} : \mathbb{R}^n \to \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$, its Fenchel-Legendre bi-conjugate, as well as $\overline{v} : \mathbb{R}^n \to \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$, its lower semi-continuous envelope,

$$\overline{v}(\mathbf{z}) = \liminf_{\mathbf{y} \to \mathbf{z}} \ v(\mathbf{y});$$

as v is convex, both v^{**} and $\overline{v}$ are convex functions which are lower semi-continuous, and it holds that

$$v^{**} \leq \overline{v} \leq v.$$

A main result in the theory of the convex programs (a very complete account of the problem may be found in [14, Chapters 2.6 and 2.9], or in [5, Chapter 4.3]) states that

$$v(\mathbf{0}) = \inf \ P(f, g), \qquad v^{**}(\mathbf{0}) = \sup \ D(f, g). \tag{31}$$

It has already been remarked that for some classes of convex programs, it is possible to sharpen the second part of relation (31) to

$$\overline{v}(\mathbf{0}) = \sup \ D(f, g). \tag{32}$$

Indeed, relation (32) is obviously satisfied if $\sup \ D(f, g)$ amounts to $+\infty$; a more significant result is provided by the Corollary 4.3.6 from [5], which proves that the same holds true provided that the primal value $\inf \ P(f, g)$ is finite.

The following result is a first step in proving that any convex program with no directions of recession also fulfills relation (32).

Lemma 2. *Let $P(f, g_i)$ a convex program without any direction of recession. Then*

$$-\infty < v(\mathbf{y}) \qquad \forall \ \mathbf{y} \in \mathbb{R}^n. \tag{33}$$

Moreover,

$$[v(\mathbf{y}) \in \mathbb{R}] \Rightarrow [\exists x \in \ X_{\mathbf{y}}, \ \ v(\mathbf{y}) = f(x)]. \tag{34}$$

Prof of Lemma 2. Let $\mathbf{y}$ be a vector from $\mathbb{R}^n$. If the effective domain of f and $X_{\mathbf{y}}$ are two disjoint sets, it obviously results that $v(\mathbf{y}) = +\infty$.

In order to address the case when the objective function f does take a finite value at some point of the closed and convex set $X_{\mathbf{y}}$, let us invoke the well-known result (see [13, Theorem 8.7]) which states that any two non-void sub-level sets of a function from $\Gamma_0(X)$ have the same recession cone.

Hence, for any $1 \leq i \leq n$, the recession cones of the sets $[g_i \leq y_i]$ and $[g_i \leq 0]$ coincide. The set $X_{\mathbf{y}}$ is the intersection of all the sets $[g_i \leq y_i]$, while the intersection of all the sets $[g_i \leq 0]$ is the efficient set A; the standard result ([13, Corollary 8.3.3]) which states that, given a family of closed and convex subsets of X with a non-empty intersection, then the recession cone of the intersection amounts to the intersection of the recession cones of all the sets in the family, proves now that the sets $X_{\mathbf{y}}$ and A have the same recession cone.

Accordingly, the mapping f from $\Gamma_0(X)$ and the closed and convex set $X_{\mathbf{y}}$ have no common directions of recession; moreover, f takes a finite value at some point of $X_{\mathbf{y}}$. We are thus in a position to apply Corollary 27.3.3 from [13], which reads that there is (at least) a point x lying both in $\operatorname{dom} f$ and $X_{\mathbf{y}}$, such that $f(x) = v(\mathbf{y})$. □

Let us recall the classical result (see for instance [2, Theorem 1.2.5]), which proves that

$$[\forall\, \mathbf{y} \in \mathbb{R}^n,\ -\infty < v(\mathbf{y})] \Rightarrow [\forall\, \mathbf{y} \in \mathbb{R}^n,\ \ -\infty < v^{**}(\mathbf{y}) = \overline{v}(\mathbf{y})]; \tag{35}$$

putting together relation (35) and relation (33) from Lemma 2, we may state the following result.

Proposition 5. *The duality gap of a convex program without any direction of recession amounts to zero if and only if the infimal function v is lower semi-continuous at* $\mathbf{y} = \mathbf{0}$.

Let us conclude this appendix by proving the Theorem of the primal attainement in the absence of any blanket or qualification conditions.

Theorem 8. *The duality gap of a convex program amounts to zero provided that it does not posses any direction of recession.*

Proof of Theorem 8. In view of Proposition 5, all what we have to prove is that the infimal function v is lower semi-continuous. To this end, let us introduce the closed convex set

$$C = \bigcap_{i=0}^{n} C_i,$$

where

$$C_0 = \{(x, \mathbf{y}, t) \in\ X \times \mathbb{R}^n \times \mathbb{R} :\ f(x) \leq t\},$$

and

$$C_i = \{(x, \mathbf{y}, t) \in\ X \times \mathbb{R}^n \times \mathbb{R} :\ g_i(x) \leq y_i\}.$$

Obviously,

$$(C_0)^\infty = \{(x,\mathbf{y},t) :\ (x,t) \in (\text{epi} f)^\infty\} = \{(x,\mathbf{y},t) :\ f^\infty(x) \le t\}, \tag{36}$$

while, for any $1 \le i \le n$, it holds that

$$(C_i)^\infty = \{(x,\mathbf{y},t) : (x,y_i) \in (\text{epi}\, g_i)^\infty\} = \{(x,\mathbf{y},t) :\ g_i^\infty(x) \le y_i\}. \tag{37}$$

Denoting by $\text{epi}_s v = \{(\mathbf{y},t) \in \mathbb{R}^n \times \mathbb{R} :\ v(\mathbf{y}) < t\}$ the strict epigraph of the infimal function v, and by L the projection of $, X \times \mathbb{R}^n \times \mathbb{R}$ onto $\mathbb{R}^n \times \mathbb{R}$; one clearly has

$$\text{epi}_s\, v \subset L(C) \subset \text{epi}\ v. \tag{38}$$

We claim that the epi v is a closed subset of $\mathbb{R}^n \times \mathbb{R}$.

If $C = \emptyset$, thus epi $v = \emptyset$, that is a closed set.

If $C \ne \emptyset$, the family of closed and convex sets C_i, $0 \le i \le n$, has a non-empty interior. We may thus apply Corollary 8.3.3, [13], and infer that $C^\infty = \bigcap_{i=0}^n (C_i)^\infty$. By virtue of relation (37), this fact implies that

$$C^\infty = \{(x,\mathbf{y},t) :\ f^\infty(x) \le t\} \cap \left(\bigcap_{i=1}^n \{(x,\mathbf{y},t) :\ g_i^\infty(x) \le y_i\} \right). \tag{39}$$

Let us also remark that

$$L^{-1}(\mathbf{0},0) = (X,\mathbf{0},0); \tag{40}$$

combining relations (39) and (40), we obtain that

$$L^{-1}(\mathbf{0},0) \cap C^\infty = \bigcap_{i=1}^n \{(x,\mathbf{y},t) :\ f^\infty(x) \le 0,\ g_i^\infty(x) \le 0\} = \left([f^\infty \le 0] \cap A^\infty, \mathbf{0}, 0\right). \tag{41}$$

But the convex program $P(f, g_i)$ does not have any direction of recession. Consequently, $[f^\infty \le 0] \cap A^\infty = \{0\}$, so relation (41) yields that

$$L^{-1}(\mathbf{0},0) \cap C^\infty = \{(0,\mathbf{0},0)\}.$$

Let us now apply Theorem 9.1 from [13], which states that $L(C)$ is closed; relation (38) implies that

$$\text{cl}(\text{epi}\, v) = \text{cl}(\text{epi}_s\, v) \subset L(C) \subset \text{epi}\, v \subset \text{cl}(\text{epi}\, v).$$

Consequently, epi v is a closed set, and hence v is a lower semi-continuous mapping. □

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