CLOSED MEANS CONTINUOUS IFF POLYHEDRAL: A CONVERSE OF THE GKR THEOREM

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ABSTRACT. Given x_0, a point of a convex subset C of an Euclidean space, the two following statements are proven to be equivalent: (i) any convex function $f : C \to \mathbb{R}$ is upper semi-continuous at x_0, and (ii) C is polyhedral at x_0. In the particular setting of closed convex mappings and F_σ domains, we prove that any closed convex function $f : C \to \mathbb{R}$ is continuous at x_0 if and only if C is polyhedral at x_0. This provides a converse to the celebrated Gale-Klee-Rockafellar theorem.

1. INTRODUCTION

One basic fact about real-valued convex mappings on Euclidean spaces, is that they are continuous at any point of their domain's relative interior (see for instance [13, Theorem 10.1]).

On the other hand, it is not difficult to define a convex function which is discontinuous at each and every point of the relative boundary of its domain. As stated by Carter in his treatise "Foundations of mathematical economics" [3, page 334], " *this is not a mere curiosity. Economic life often takes place at the boundaries of convex sets, where the possibility of discontinuities must be taken into account.*"

The celebrated Gale-Klee-Rokafellar theorem ([5, Theorem 2]; see also [13, Theorem 10.2]) is a major step toward an accurate understanding of continuity properties for convex mappings at points belonging to the relative boundary of their domain. This result is particularly meaningful when applied to the class of closed convex functions, as defined in the seminal work of W. Fenchel ([4]).

GKR theorem: *Any convex function is upper semi-continuous at any point at which its domain is polyhedral. Accordingly, any closed convex mapping is continuous at any such point.*

Besides its intrinsic interest, this theorem has proved itself a fertile source of applications. Taking one example out of many, let us remark that, since a polyhedra is polyhedral at any of its points, the GKR theorem proves the ubiquitous mathematical economics and game theory lemma ([1, Theorem 4.2]) which says that any concave function defined on P^n_+, the cone of the vectors from $\mathbb{R}^n$ with positive coordinates, is lower semi-continuous.

The GKR theorem also provides powerful tools in establishing continuity of special convex functions issued from particular optimization problems, as the M-convex and L-convex functions of Murota and Shioura ([12]).

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The example of the closed convex function

$$f : C \to \mathbb{R}, \quad f(x,y) = \frac{x^2}{y}$$

defined on the disk

$$C = \{(x,y) \in \mathbb{R}^2 : \; x^2 + (1-y)^2 \leq 1\},$$

yet discontinuous at the point $(0,0) \in C$ is well-known ([13, page 83]).

Let us remark that the point at which the previously-defined mapping is discontinuous may (inter alia) be characterized as being the limit of a non-constant sequence made of extreme points of the disk. Lemma at page 870 in the article of Gale, Klee and Rockafellar proves that this is a very general feature.

Converse GKR theorem: *Let C be a closed and convex subset of X, and $x_0 \in C$ be the limit of a non-constant sequence of extreme points of C (such a point exists if and only if C is not polyhedral at each and every of its points). Then there exists at least one closed convex function $f : C \to \mathbb{R}$ which is not continuous at x_0.*

A standard observation proves that, if C is conical at some point $x_0 \in C$, then none of the non-constant sequences of extreme points of C can converge to x_0. In this respect, the following result by Howe ([7, Proposition 2]), provides an extension of the reciprocal GKR theorem.

Howe's theorem: *Let C be a closed and convex subset of X and $x_0 \in C$ be a point at which C is not conical. Then there exists at least one closed convex function $f : C \to \mathbb{R}$ which is not continuous at x_0.*

An obvious limitation of the previous theorem is that Howe's result is bound to the setting of closed domains, and no conclusion can be drawn for the larger class of convex domains over which closed convex functions may be defined (that is F_σ convex sets).

Moreover, this result leaves unanswered the decidedly non-trivial question of the continuity of a closed convex function at points at which the domain is conical without being polyhedral (typically the apex of a circular cone). Indeed, the hypothesis that a closed convex function is automatically continuous at such type of points seems very natural, and this claim have been made (in an implicit form) at least once ([2, Proposition 5, p. 183]). However, this conjecture have been proved false when Goossens ([6, p. 609]) provided a (very elaborate) example of a closed convex function defined on a circular cone and discontinuous at its apex.

This note attempts to fill in the gap between the direct GKR theorem and Howe's result, by proving (Theorem 2.4, Section 2) the following statement.

Second converse GKR theorem: *Given C, a convex subset of the Euclidean space X , and x_0, a point at which C is not polyhedral, then there is a convex mapping $f : C \to \mathbb{R}$ which is not upper semi-continuous at x_0.*

When, in addition, C is a F_σ set, then there is $f : C \to \mathbb{R}$, a closed convex mapping which is discontinuous at x_0.

1.1. **Definitions and notations.** Let us consider X, an Euclidean space endowed with the usual topology, and let us set $x \cdot y$ for the scalar product between the vectors x and y of X, and $\|\cdot\|$ for the associated norm.

Given A a subset of X, let X_A be its affine span (that is the intersection of all the hyperplanes of X containing A). The relative boundary of A is defined by the formula

$$r\partial(A) = \overline{A} \,\cap\, \overline{X_A \setminus A},$$

where a superposed bar denotes the closure of a set, while relation

$$ri(A) = A \setminus r\partial(A)$$

defines the relative interior of the set A. Let us recall ([13, Theorem 6.2, p. 45]) that the relative interior of a non empty convex set is non empty.

As customary, a subset A of X is said to be a F_σ set, if it is the countable union of a family of closed subsets of X:

$$A = \bigcup_{i=1}^{\infty} A_i \qquad A_i = \overline{A_i}\ \forall i \in \mathbb{N};$$

a function $f : A \to \mathbb{R}$ is called closed if its epigraph

$$\operatorname{epi} f = \{(x, r) \in A \times \mathbb{R} :\ f(x) \le r\}$$

is a closed subset of $X \times \mathbb{R}$. Let us notice that the domain of a closed function is necessarily a F_σ set. The mapping $f : A \to \mathbb{R}$ is upper semi-continuous at x_0 if

$$f(x_0) \ge \limsup_{x\in A,\, x\to x_0} f(x).$$

In this article, by polyhedron we mean any set obtained as the intersection of a finite family of closed half-spaces of X; accordingly, polyhedra are closed convex sets, not always bounded. Following Klee ([9, p. 86]), we call the set A polyhedral at $x_0 \in A$ if there are U, a neighborhood of x_0, and B, a polyhedron, such that

$$A \cap U = A \cap B.$$

Similarly, we call a set A conical at $x_0 \in A$ if there are U, a neighborhood of x_0, and K, a closed convex cone, such that

$$A \cap U = A \cap K;$$

in other words (Howe, [7, p. 1198]), "near x_0, the set A looks like a [...] cone". Obviously, a convex set is polyhedral at any of the points of its relative interior. Moreover, if a set is polyhedral at some point, it is also conical at the same point, but the converse does not generally holds.

2. Continuity of convex mappings at points of the relative boundary of their domain

A key step in proving our main result is provided by Theorem 2.2. This result features a geometrical property of points belonging to the relative boundary of a convex set, which, at our best knowledge, has never been addressed.

Following Klee ([8, p. 448]), we call a point $x \in X$ linearly accessible from the subset A of X if there is a point a such that the half-open segment $[a; x[$ is contained in A. Of course, any linearly accessible point belongs to the closure of A, but the converse does not generally holds.

For convex sets, however, any point in the closure is linearly accessible (an obvious application of the fact that their relative interior is always non-empty). Theorem 2.2 addresses the question of the linear accessibility of the boundary points for sets which can be expressed as the difference between two convex sets.

Let us first establish to what extent studying this topic helps to demonstrate the converse GKR theorem.

Proposition 2.1. *Let C be a subset of X, x_0 one of its points, and assume that there is a closed convex set D containing x_0 such that $x_0 \in \overline{C \setminus D}$, yet x_0 is not linearly accessible form $C \setminus D$.*

i) If C is convex, then there is a convex function $f : C \to \mathbb{R}$ which is not upper semi-continuous at x_0.

ii) If C is a F_σ convex set, then it is possible to find a closed convex mapping $f : C \to \mathbb{R}$ which is not continuous at x_0.

Proof of Proposition 2.1: Let us consider the cone of D at x_0,

$$C(x_0, D) = \{x \in X :\ x_0 + \lambda\,(x - x_0) \in D \ \text{ for some } \ \lambda > 0\},$$

and $\mu_{(x_0,D)} : C(x_0, D) \to \mathbb{R}$, the Minkowski gauge of D at x_0,

$$\mu_{(x_0,D)}(x) = \inf \left\{ \gamma > 0 :\ x_0 + \frac{1}{\gamma}\,(x - x_0) \in D \right\}.$$

It is clear that $C(x_0, D)$ is a convex cone of apex x_0, and $\mu_{(x_0,D)}(x_0) = 0$. Moreover, it is well-known (see for instance [13, Corollary 9.7.1, p. 79]), that $\mu_{(x_0,D)}$ is a closed convex function.

We claim that $C \subset C(x_0, D)$, and that the restriction

$$f : C \to \mathbb{R} \qquad f(x) = \mu_{(x_0,D)}(x)$$

of $\mu_{(x_0,D)}$ to C fulfills point $i)$ in Proposition 2.1.

Indeed, let $x \in C$; as x_0 is not linearly accessible from $C \setminus D$, it follows in particular that the segment $[x; x_0[$ is not entirely contained in $C \setminus D$, and since $[x; x_0[\subset C$, it results that

$$\lambda\, x_0 + (1 - \lambda)\, x \in D \ \text{ for some } \ 0 < \lambda < 1. \tag{2.1}$$

But $\lambda\, x_0 + (1 - \lambda)\, x = x_0 + (1 - \lambda)(x - x_0)$, so from relation (2.1) it yields that

$$x_0 + (1 - \lambda)(x - x_0) \in D, \quad (1 - \lambda) > 0,$$

that is $x \in C(x_0, D)$.

We have thus proved that $C \subset C(x_0, D)$; to the end of analyzing the upper semi-continuity of the function f at x_0, let us recall that the point x_0 belongs to the closure of the set $C \setminus D$. One can thus find a sequence, say $(x_n)_n$, of elements from $C \setminus D$ converging to x_0. Pick any of the vectors x_n; as it does not belong to D, the definition of the Minkowski gauge implies that $f(x_n) \geq 1$ for any $n \in \mathbb{N}$. The lack of upper semi-continuity of f at x_0 is therefore established.

In order to address the point $ii)$ of Proposition 2.1, let us state the standard convex analysis result saying that, given C a convex F_σ set, there exists at least one closed convex mapping $g : C \to \mathbb{R}$ (the proof of Theorem 4.1 from the article ([9]) may easily be adapted to provide a demonstration of this fact).

If the mapping g is discontinuous at x_0, then it fulfills point *ii*). Assume now that the mapping g is continuous at x_0; the application

$$f : C \to \mathbb{R} \qquad f(x) = g(x) + \mu_{(x_0,D)}(x)$$

is closed and convex as the sum of to closed and convex functions. Moreover, f is the sum between a mapping which is continuous at x_0, and a mapping which is discontinuous at the same point. Thus f is a closed convex application discontinuous at x_0, and Proposition 2.1 is completely proved. □

With the conclusions of Proposition 2.1 in mind, let us address Theorem 2.2, the most technical part of our paper.

Theorem 2.2. *Let C be a convex subset of X, and x_0 be one of its points. The two following statements are equivalent.*
i) C is not polyhedral at x_0
ii) there is a closed convex set D containing x_0 such that $x_0 \in \overline{C \setminus D}$, yet x_0 is not linearly accessible form $C \setminus D$.

Proof of Theorem 2.2: *i*) ⇒ *ii*) Let x_0 be a point of C at which C is not polyhedral. By virtue of Corollary 3.3 ([9, p. 88]), it results that the convex cone $C(x_0, C)$ is not polyhedral. Let us first prove a general result on non-polyhedral cones.

Lemma 2.3. *Let E be a non-polyhedral convex cone, and x_0 its appex. Then there is a sequence $(y_n)_{n\in\mathbb{N}} \subset X$ such that:*
i) for any $x \in E$ and n large enough, the sequence $((x - x_0) \cdot y_n)_n$ takes only non-positive values,
ii) for each and every $n \in \mathbb{N}$, there is $x_n \in E$ such that $(x_n - x_0) \cdot y_n > 0$.

Proof of Lemma 2.3: A far-reaching characterization of polyhedrality for cones was achieved by Klee ([9, Theorem 4.11, p. 92]; the particular case of closed convex cones have had previously been provided by Mirkil [11, Theorem, p. 1]), which says that a convex cone is polyhedral if and only if its projection on every two-dimensional affine manifold (in other words, on any plane) of X is a closed set.

Accordingly, the convex cone $\Pi(E)$ is not closed, where $\Pi : X \to X_1$ is the operator of projection onto some plane X_1 of X. Set v_0 for the projection of x_0, and let v be a vector belonging to the closure of $\Pi(E)$ but not to $\Pi(E)$ itself (of course, $v \neq v_0$).

As the relative interiors of any convex set and of its closure coincide, the fact that the vector v belongs to $\overline{\Pi(E)} \setminus \Pi(E)$ implies that v lies within the relative boundary of $\overline{\Pi(E)}$. A standard support hyperplane argument reads now that there exists an element $w \in X_1$ such that the mapping $x \to x \cdot w$ achieves its maximum over $\overline{\Pi(E)}$ at v; in particular, it holds that

$$\Pi(x) \cdot w \leq v \cdot w \qquad \forall x \in E. \tag{2.2}$$

Since both the vectors v_0 and $v_0 + 2(v - v_0)$ belong to $\overline{\Pi(E)}$, we infer that

$$v_0 \cdot w \leq v \cdot w, \qquad (v_0 + 2(v - v_0)) \cdot w \leq v \cdot w;$$

thus

$$(v - v_0) \cdot w = 0. \tag{2.3}$$

For every $n \in \mathbb{N}$, let us set $y_n = w + \frac{v - v_0}{n}$. As $y_n \in X_1$, it results that

$$(x - x_0) \cdot y_n = (\Pi(x) - v_0) \cdot \left(w + \frac{v - v_0}{n} \right) \quad \forall x \in E,\ n \in \mathbb{N}. \tag{2.4}$$

We claim that the sequence $(y_n)_n$ fulfills relation $i)$. Let us pick $x \in E$; in view of relation (2.2), there are two possible cases: $a)$ $\Pi(x) \cdot w < v \cdot w$, and $b)$ $\Pi(x) \cdot w \le v \cdot w$.

In case $a)$, from relation (2.3) we infer that

$$(\Pi(x) - v_0) \cdot w < 0. \tag{2.5}$$

As obviously

$$\lim_{n \to \infty} (\Pi(x) - v_0) \cdot \frac{v - v_0}{n} = 0, \tag{2.6}$$

statement $i)$ yields from relations (2.4), (2.5) and (2.6).

In case $b)$, since X_1 is a to-dimensional manifold, it results that $\Pi(x)$ lies on the line $v_0 + \mathbb{R}\,(v - v_0)$. But the half-line $v_0 + \mathbb{R}_+^* (v - v_0)$ is disjoint from $\Pi(E)$ (recall that $v_0 \in \Pi(E)$, $v \notin \Pi(E)$ and that $\Pi(E)$ is a cone of appex v_0), so we may affirm that

$$\Pi(x) = v_0 - \lambda(v - v_0) \quad \text{for some } \lambda \ge 0. \tag{2.7}$$

By combining relations (2.3), (2.4) and (2.7), we conclude that

$$(x - x_0) \cdot y_n = -\lambda \, \frac{\|v - v_0\|^2}{n} \le 0 \quad \forall\, n \in \mathbb{N}.$$

Statement $i)$ is therefore fulfilled in both situations $a)$ and $b)$.

Let us now address relation $ii)$. As $v \in \overline{\Pi(E)}$, there is a sequence $(z_n)_{n \in \mathbb{N}} \subset E$ such that the sequence $(\Pi(z_n))_n$ converges to v. Pick $k \in \mathbb{N}$, and apply relation (2.4) for $x = z_n$ and y_k:

$$(z_n - x_0) \cdot y_k = (\Pi(z_n) - v_0) \cdot \left(w + \frac{v - v_0}{k} \right) \quad \forall n \in \mathbb{N}.$$

Accordingly,

$$\lim_{n \to \infty} ((z_n - x_0) \cdot y_k) = (v - v_0) \cdot \left(w + \frac{v - v_0}{k} \right);$$

by virtue of relation (2.3), we obtain that

$$\lim_{n \to \infty} ((z_n - x_0) \cdot y_k) = \frac{\|v - v_0\|^2}{k} > 0.$$

The set $L_k = \{n \in \mathbb{N} : \ (z_n - x_0) \cdot y_k > 0\}$ is therefore non-empty. Set $u(k) = \min L_k$; the sequences $(x_n)_{n \in \mathbb{N}} \subset E$, where $x_n = z_{u(n)}$, and $(y_n)_n$, obviously fulfills relation $ii)$. □

Let us now get back to the proof of the implication $i) \Rightarrow ii)$ from Theorem 2.2, and apply the conclusions of Lemma 2.3 to the non-polyhedral cone $C(x_0, C)$. Accordingly, there are two sequences, $(x_n)_{n \in \mathbb{N}} \subset C(x_0, C)$ and $(y_n)_{n \in \mathbb{N}} \subset X$ such that

$$(x - x_0) \cdot y_n \le 0 \ \text{ for } \ n \ \text{ large enough}, \tag{2.8}$$

and

$$(x_n - x_0) \cdot y_n > 0 \qquad \forall n \in \mathbb{N}. \tag{2.9}$$

By replacing, if necessary, the vectors x_n with vectors of form $\lambda_n x_0 + (1-\lambda_n) x_n$, we may assume that $x_n \in C$. Set

$$\gamma_n = x_0 \cdot y_n + \frac{(x_n - x_0) \cdot y_n}{2n\|x_n - x_0\|},$$

and define the set

$$D = \{x \in X : \ x \cdot y_n \leq \gamma_n \ \ \forall n \in \mathbb{N}\}.$$

Obviously, D is a closed convex set which contains the point x_0. Let us prove that $x_0 \in \overline{C \setminus D}$. Indeed, each end every of the points

$$t_n = x_0 + \frac{x_n - x_0}{n\|x_n - x_0\|}$$

belong to C (they are convex combinations of vectors x_0 and x_n, both lying within C); from relation (2.9) it yields that

$$t_n \cdot y_n - \gamma_n = -\frac{(x_n - x_0) \cdot y_n}{2n\|x_n - x_0\|} < 0,$$

so neither of the points t_n belong to D. Finally, remark that the sequence $(t_n)_n$ converges to x_0, to conclude that $x_0 \in \overline{C \setminus D}$.

To the purpose of proving that x_0 is not linearly accessible from $C \setminus D$, let us pick $x \in C$. Since the sequence $((x - x_0) \cdot y_n)_n$ takes only a finite number of positive values, then for any positive sequence $(u_n)_n$, there is a positive real number a such that

$$\lambda \ ((x - x_0) \cdot y_n) \leq \ u_n \qquad \forall 0 \leq \lambda \leq a, \ \forall n \in \mathbb{N}. \tag{2.10}$$

Inequality (2.9) allows us to apply relation (2.10) for $u_n = \dfrac{(x_n - x_0) \cdot y_n}{2n\|x_n - x_0\|}$, and deduce that there exists a positive real value a such that

$$\lambda \ ((x - x_0) \cdot y_n) \leq \frac{(x_n - x_0) \cdot y_n}{2n\|x_n - x_0\|} \qquad \forall 0 \leq \lambda \leq a, \ \forall n \in \mathbb{N}. \tag{2.11}$$

But

$$\lambda \ ((x - x_0) \cdot y_n) - \frac{(x_n - x_0) \cdot y_n}{2n\|x_n - x_0\|} = ((1 - \lambda) \, x_0 + \lambda \, x) \cdot \ y_n - \gamma_n,$$

so relation (2.11) proves in fact that

$$((1 - \lambda) \, x_0 + \lambda \, x) \in D \qquad \forall 0 \leq \lambda \leq a.$$

Accordingly, there is no point $x \in C$ such that the segment $[x, x_0[$ be entirely contained within $C \setminus D$; in other words, the point x_0 is not linearly accessible from $C \setminus D$.

ii) $\Rightarrow$ *i)* This implication easily follows by combining the classical GKR theorem and Proposition 2.1. □

The main result of this note stems now by combining Proposition 2.1 and Theorem 2.2.

Theorem 2.4. *Given C, a convex subset of the Euclidean space X , and $x_0 \in C$, then any convex mapping $f : C \to \mathbb{R}$ is upper semi-continuous at x_0 if and only if C is polyhedral at x_0.*

When, in addition, C is a F_σ set, then any closed convex mapping $f : C \to \mathbb{R}$ is continuous at x_0 if and only if C is polyhedral at x_0.

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