METRIC REGULARITY OF THE SUM OF MULTIFUNCTIONS AND APPLICATIONS

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Abstract. In this work, we use the theory of error bounds to study of metric regularity of the sum of two multifunctions, as well as some important properties of variational systems. We use an approach based on the metric regularity of epigraphical multifunctions. Our results subsume some recent results by Durea and Strugariu in [12].

1. Introduction

The study of a wide range of problems from variational analysis, optimization, variational inequalities, and many other areas in mathematics leads to generalized equations of the form

$$0 \in F(x,p), \tag{1}$$

where $F : X \times P \rightrightarrows Y$ is a multifunction, X, Y are metric spaces, and P is a metric space considered as the space of parameters. A typical example of equation (1) is given by a parametrized system of inequalities/equalities. Indeed, let us consider a mapping $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^{k+d}$ with

$$f(x,p) = (f_1(x,p), \cdots, f_k(x,p), f_{k+1}(x,p), \cdots, f_{k+d}(x,p)),$$

and if we set $F(x,p) = f(x,p) - \mathbb{R}^k_- \times \{0\}^d)$, then the system $(\mathcal{S})$ consisting of those points x for which

$$\begin{array}{l} f_i(x,p) \le 0,\ i \in \{1, \cdots, k\}, \\ f_i(x,p) = 0,\ i \in \{k+1, \cdots, d\}, \end{array}$$

can be rewritten in the form (1). Let us notice also that equation (1) subsumes the important subcase of parametrized inclusions of the type:

$$0 \in H(x) + f(x,p), \tag{2}$$

where $H : X \rightrightarrows Y$ is a set-valued mapping and $f : X \times P \to Y$ is a mapping.

For a given parameter $p \in P$, we denote by $S(p)$ the set of solutions of (2) , i.e.,

$$S(p) := \{x \in X : 0 \in F(x) + f(x,p)\}. \tag{3}$$

Let us consider the perturbed optimization problem $(\mathcal{P})$

$$\min_{x \in C} g(x) - \langle p, x \rangle$$

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where $g : \mathbb{R}^n \to \mathbb{R}$ is a convex C^1-function, and $p \in \mathbb{R}^n$ is a given parameter.The first order optimality condition of problem $(\mathcal{P})$ is given by

$$\nabla g(x) - p \in N_C(x),$$

where $N_C(x) = \{p \in \mathbb{R}^n \ : \ \langle p, y - x\rangle \leq 0 \quad \forall y \in C\}$ stands for the normal cone to C at x. Setting $f(x,p) = p - \nabla g(x)$, the first order optimality condition is given by

$$0 \in f(x,p) + N_C(x)$$

and appears as a special case of equation (2), as well as are parametrized variational inequalities, i.e., the problem of finding $x \in C$ such that

$$\langle f(x,p), u - x\rangle \geq 0 \quad \text{for all} \ \ u \in C.$$

The study of variational properties and the stability of the solutions of equation (1) has attracted a lot of interest from an important number of authors and we refer the reader to the monographs by Mordukhovich [23], Rockafellar & Wets [33], Dontchev & Rockafellar [11] and the references therein.

Let us first provide definitions and properties of some essential notions from set-valued analysis that will be used throughout this paper. In what follows X, Y etc., unless specified otherwise, are metric spaces and we use the same symbol $d(\cdot,\cdot)$ to denote the distance in all of them or between a point x to a subset S of one of them : $d(x,S) := \inf_{u\in S} d(x,u)$. By $B(x,\rho)$ and $\bar{B}(x,\rho)$ we denote the open and closed balls of radius ρ around x, while if X is a Banach space, we use the notations $B_X, \bar{B}_X$ for the open and the closed unit balls, respectively. By a multifunction (set-valued mapping) $S : X \rightrightarrows Y$, we mean a mapping from X into the subsets (possibly empty) of Y. We denote by gph S the graph of S, that is the set $\{(x,y) \in X \times Y \ : \ y \in S(x)\}$ and by $\mathrm{D}(S) := \{x \in X \ : \ S(x) \neq \emptyset\}$ the domain of S. When S has a closed graph, we say that S is a closed multifunction.

Since various types of multifunctions arise in a considerable number of models ranging from mathematical programs, through game theory and to control and design problems, they represent probably the most developed class of objects. A number of useful regularity properties have been introduced and investigated (see [11], [33] and the references therein). Among them, the most popular is that of metric regularity ([6], [9], [11], [18], [19], [20], [21], [23], [28], [29], [33]), the root of which can be traced back to the classical Banach open mapping theorem and the subsequent fundamental results of Lyusternik and Graves.

S is said to be *metrically regular* around $(\bar{x},\bar{y}) \in$ gph S with modulus $\tau > 0$ if there exist neighborhoods $\mathcal{U},\mathcal{V}$ of $\bar{x},\bar{y}$, respectively such that, for every $(x,y) \in \mathcal{U}\times\mathcal{V}$,

$$d(x, S^{-1}(y)) \leq \tau d(y, S(x)).$$

A classical illustration of this concept concerns the case when S is a bounded linear continuous operator. Then, metric regularity of S amounts to saying that S is surjective.

Another well known concept is the *Pseudo-Lipschitz property*, also called *Aubin property* (see [2]): let $S : X \rightrightarrows Y$ be a set-valued mapping and fix $(\bar{x},\bar{y}) \in$ gph S. S is said to have the *Aubin property around* $(\bar{x},\bar{y})$ if there exists a constant $\kappa \geq 0$ and neighborhoods $\mathcal{U} \in \mathcal{N}(\bar{x}) \subset X$ of $\bar{x}$ and $\mathcal{V} \in \mathcal{N}(\bar{y}) \subset Y$ of $\bar{y}$ such that

$$S(x') \cap \mathcal{V} \subset S(x) + \kappa d(x,x')\bar{B}_Y \quad \text{for all } x,\ x' \in \mathcal{U}. \tag{4}$$

The Lipschitz modulus of S for $\bar{y}$ at $\bar{x}$, denoted by $\mathrm{lip}(S;(\bar{x},\bar{y}))$, is defined by

$$\mathrm{lip}(S;(\bar{x},\bar{y})) = \inf\{\kappa \in \mathbb{R}^+ \ : \ \exists\, \mathcal{U} \in \mathcal{N}(\bar{x}),\ \mathcal{V} \in \mathcal{N}(\bar{y}) \text{ such that condition (4) is satisfied}\}. \tag{5}$$

The concept of *openness* or *covering (at a linear rate)* is also widely used: one says that $S : X \rightrightarrows Y$ is *open at linear rate* $\tau > 0$ around $(\bar{x}, \bar{y}) \in \operatorname{gph} S$ if there exist neighborhoods $\mathcal{U}, \mathcal{V}$ of $\bar{x}, \bar{y}$, respectively and a positive number $\varepsilon > 0$ such that, for every $(x, y) \in \operatorname{gph} S \cap (\mathcal{U} \times \mathcal{V})$ and every $\rho \in (0, \varepsilon)$,

$$B(y, \rho\tau) \subset S(B(x, \rho)).$$

We refer to Aubin [2], Dmitruk, Milyutin & Osmolovsky [10], Ioffe [18], Kruger [22], Mordukhovich [23], Penot [29], Rockafellar & Wets [33] and the references therein for different developments of these notions. The following relation is well established:

$$\text{Metric regularity} \iff \text{Covering} \iff \text{Aubin property of the inverse.} \tag{6}$$

In this paper, we are especially interested in metric regularity of the sum of two multifunctions. The starting point of the study is the paper by J. Aragón Artacho, A. L. Dontchev, M. Gaydu, M. H. Geoffroy, and V. M. Veliov [1], in which it is proved that if F is metrically regular and if we perturb F by a mapping $g(\cdot, \cdot)$, Lipschitz with respect to x, uniformly in p, with a suitable Lipschitz constant, then the perturbed mapping $F(\cdot) + g(\cdot, p)$ is metrically regular for every p near $\bar{p}$.

When we perturb a metrically regular multifunction F by a set-valued mapping G which is Lipschitz-like, the perturbed mapping $F + G$ fails in general to be metrically regular. However, if for example the so-called " sum-stable" property (introduced below) holds, then metric regularity as well as the Aubin property of the variational system remains. Recently Durea & Strugariu [12] considered the sum of two set-valued mappings and obtained a result very similar to openness of the sum of two set-valued mappings. They also gave some applications to the generalized variational system.

Motivated by the ideas and results from [12], we attack these problems by using a different approach and with rather different assumptions. Indeed, using an approach based on the theory of error bounds, we study metric regularity of a special multifunction called the epigraphical multifunction associated to F and G. This intermediate result allows us to study metric regularity/ linear openness of the sum of two set-valued mappings, as well as metric regularity of the general variational system avoiding the strong assumption of the closedness of the multifunction $F + G$.

From the point of view of applications to optimization (sensitivity analysis, convergence analysis of algorithms, and penalty functions methods), one of the most important regularity properties seems to be that of error bounds providing an estimate for the distance of a point from the solution set. This theory was initiated by the pioneering work by Hoffman [16]. However, it has been pointed out to the authors by J.-B. Hiriart-Urruty recently that traces of the error bounds property can be found in a work by P.C. Rosenbloom [32], published in 1951. Applications of the theory of error bounds to the investigation of metric regularity of multifunctions have been recently studied and developed by many authors, including for instance [24], [14], [4], [5], [28], [26].

The paper is structured as follows. In section 2, we recall some recent results on error bounds of parametrized systems and give, sometimes with some modifications, characterizations of metric regularity of multifunctions given in Huynh & Théra [28], and Huynh, Nguyen & Théra [24]. In the next section, in the context of Asplund spaces, we estimate the strong slope of the function $\varphi_{\mathcal{E}}((x, k), y)$, (see page 5 below for the definition) and give sufficient conditions as well as a point-based condition for metric regularity of the epigraphical multifunction under a coderivative condition. In the last section, we study Robinson metric regularity and Aubin property of a generalized variational system.

2. Metric regularity of epigraphical multifunctions via error bound

Let us remind some basic notions used in the paper. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a given extended-real-valued function. As usual, $\mathrm{Dom}\, f := \{x \in X : f(x) < +\infty\}$ denotes the domain of f. We recall the concept of error bounds that is one of the most important regularity properties. We set

$$S := \{x \in X : \quad f(x) \le 0\}. \tag{7}$$

We use the symbol $[f(x)]_+$ to denote $\max\{f(x), 0\}$. We shall say that the system (7) admits *an error bound* if there exists a real $c > 0$ such that

$$d(x, S) \le c\big[f(x)\big]_+ \quad \text{for all} \quad x \in X. \tag{8}$$

For $x_0 \in S$, we shall say that system (7) has an error bound at x_0, when there exist reals $c > 0$ and $\varepsilon > 0$ such that relation (8) is satisfied for all x around x_0, i.e., in an open ball $B(x_0, \varepsilon)$.

Here and in what follows the convention $0 \cdot (+\infty) = 0$ is used.

We now consider a parametrized inequality system, that is, the problem of finding $x \in X$ such that

$$f(x, p) \le 0, \tag{9}$$

where $f : X \times P \to \mathbb{R} \cup \{+\infty\}$ is an extended-real-valued function, X is a complete metric space and P is a metric space. We denote by $S(p)$ the set of solutions of system (9):

$$S(p) := \{x \in X : \quad f(x, p) \le 0\}.$$

Recall from De Giorgi, Marino & Tosques [8], that the strong slope $|\nabla f|(x)$ of a lower semicontinuous function f at $x \in \mathrm{Dom}\, f$ is the quantity defined by $|\nabla f|(x) = 0$ if x is a local minimum of f, and

$$|\nabla f|(x) = \limsup_{y \to x, y \neq x} \frac{f(x) - f(y)}{d(x, y)},$$

otherwise. For $x \notin \mathrm{Dom}\, f$, we set $|\nabla f|(x) = +\infty$.

In what follows, given a multifunction $F : X \rightrightarrows Y$, we make use of the lower semicontinuous envelope $(x, y) \mapsto \varphi_F(x, y)$ of the function $(x, y) \mapsto d(y, F(x))$, i.e., for $(x, y) \in X \times Y$,

$$\varphi_F(x, y) := \liminf_{(u,v) \to (x,y)} d(v, F(u)) = \liminf_{u \to x} d(y, F(u)).$$

In the sequel, we use the notation F_p for $F(\cdot, p)$ and φ_p for φ_{F_p} and the metric defined on the cartesian product $X \times Y$ by

$$d((x, y), (u, v)) = \max\{d(x, u), d(y, v)\}, \quad (x, y), (u, v) \in X \times Y.$$

Let us now recall a result given in Huynh, Nguyen and Théra [24]. This result which is valid for all y in a neighborhood of $\bar{y}$, instead of $\bar{y}$ only, gives a necessary and sufficient condition for metric regularity of implicit multifunctions. For given $\bar{y} \in Y$, we denote

$$S(\bar{y}, p) := \{x \in X : \bar{y} \in F(x, p)\}.$$

Theorem 1. *Let X be a complete metric space and Y be a metric space. Let P be a topological space and suppose that the set-valued mapping $F : X \times P \rightrightarrows Y$ satisfies the following conditions for some $(\bar{x}, \bar{y}, \bar{p}) \in X \times Y \times P$:*
(a) $\bar{x} \in S(\bar{y}, \bar{p})$;
(b) the multifunction $p \rightrightarrows F(\bar{x}, p)$ is lower semicontinuous at $\bar{p}$;

(c) for any p near $\bar{p}$, the set-valued mapping $x \rightrightarrows F(x,p)$ is a closed multifunction.

Let $\tau \in (0,+\infty)$, be fixed. Then, the following three statements (i), (ii), (iii) are equivalent. Moreover, (iv) $\Rightarrow$ (iii), and (iv) $\Leftrightarrow$ (iii) provided Y is a normed space.

(i) There exists a neighborhhood $\mathcal{U}\times\mathcal{V}\times\mathcal{W} \subseteq X\times P\times Y$ of $(\bar{x},\bar{p},\bar{y})$ such that $\mathcal{U}\cap S(y,p)\neq\emptyset$ for any $(p,y)\in\mathcal{V}\times\mathcal{W}$ and

$$d(x,S(y,p)) \le \tau d(y,F(x,p)) \quad \text{for all} \quad (x,p,y)\in\mathcal{U}\times\mathcal{V}\times\mathcal{W};$$

(ii) There exists a neighborhhood $\mathcal{U}\times\mathcal{V}\times\mathcal{W} \subseteq X\times P\times Y$ of $(\bar{x},\bar{p},\bar{y})$ such that $\mathcal{U}\cap S(y,p)\neq\emptyset$ for any $(p,y)\in\mathcal{V}\times\mathcal{W}$ and

$$d(x,S(y,p)) \le \tau \varphi_p(x,y) \quad \text{for all} \quad (x,p,y)\in\mathcal{U}\times\mathcal{V}\times\mathcal{W};$$

(iii) There exist a neighborhood $\mathcal{U}\times\mathcal{V}\times\mathcal{W} \subseteq X\times P\times Y$ of $(\bar{x},\bar{p},\bar{y})$ and a real $\gamma\in(0,+\infty)$ such that for any $(x,p,y)\in\mathcal{U}\times\mathcal{V}\times\mathcal{W}$ with $y\notin F(x,p)$ and $\varphi_p(x,y)<\gamma$, any $\varepsilon>0$, and any sequence $\{x_n\}_{n\in\mathbb{N}}\subseteq X$ converging to x with

$$\lim_{n\to\infty} d(y,F(x_n,p)) = \liminf_{u\to x} d(y,F(u,p)),$$

there exists a sequence $\{u_n\}_{n\in\mathbb{N}}\subseteq X$ with $\liminf_{n\to\infty} d(u_n,x)>0$ such that

$$\limsup_{n\to\infty} \frac{d(y,F(x_n,p)) - d(y,F(u_n,p))}{d(x_n,u_n)} > \frac{1}{\tau+\varepsilon}; \tag{10}$$

(iv) There exist a neighborhood $\mathcal{U}\times\mathcal{V}\times\mathcal{W} \subseteq X\times P\times Y$ of $(\bar{x},\bar{p},\bar{y})$ and a real $\gamma>0$ such that

$$|\nabla\varphi_p(\cdot,y)|(x) \ge \frac{1}{\tau} \quad \text{for all } (x,p,y)\in\mathcal{U}\times\mathcal{W}\times\mathcal{V} \quad \text{with} \quad \varphi_p(x,y)\in(0,\gamma). \tag{11}$$

Given two multifunctions $F,G : X \rightrightarrows Y$, ($Y$ is a normed linear space) we define the *epigraphical multifunction* associated with F and G as the multifunction $\mathcal{E}_{(F,G)} : X\times Y \rightrightarrows Y$ defined by

$$\mathcal{E}_{(F,G)}(x,k) = \begin{cases} F(x)+k, & \text{if } k\in G(x), \\ \emptyset, & \text{otherwise.} \end{cases}$$

For given $y\in Y$, we set

$$\mathbb{S}_{\mathcal{E}_{(F,G)}}(y) := \{(x,k)\in X\times Y : \quad y\in\mathcal{E}_{(F,G)}(x,k)\}. \tag{12}$$

The lower semicontinuous envelope $((x,k),y)\mapsto\varphi_{\mathcal{E}}((x,k),y)$ of the distance function $d(y,\mathcal{E}_{(F,G)}(x,k))$ is defined for $(x,k,y)\in X\times Y\times Y$ by

$$\begin{aligned}\varphi_{\mathcal{E}}((x,k),y) &:= \liminf_{(u,v,w)\to(x,k,y)} d(w,\mathcal{E}_{(F,G)}(u,v)) \\ &= \begin{cases} \liminf_{u\to x} d(y,F(u)+k) & \text{if } k\in G(x) \\ +\infty & \text{otherwise.} \end{cases}\end{aligned}$$

The next lemma is useful.

Lemma 2. *Assume that $F : X \rightrightarrows Y$ and $G : X \rightrightarrows Y$ are closed multifunctions. Then, the epigraphical multifunction $\mathcal{E}_{(F,G)}$ has a closed graph and for each $y\in Y$,*

(13)

$$\mathbb{S}_{\mathcal{E}_{(F,G)}}(y) = \{(x,k)\in X\times Y : \quad \varphi_{\mathcal{E}}((x,k),y)=0\} = \{(x,k)\in X\times Y : \quad k\in G(x), y\in F(x)+k\}.$$

Proof. Observe that if $F : X \rightrightarrows Y$ and $G : X \rightrightarrows Y$ are closed multifunctions, then so is the epigraphical multifunction $\mathcal{E}_{(F,G)}$.

Let us prove (13). Obviously, for each $y \in Y$ if $(x,k) \in \mathbb{S}_{\mathcal{E}_{(F,G)}}(y)$ then $\varphi_{\mathcal{E}}((x,k),y) = 0$. Conversely, suppose that $\varphi_{\mathcal{E}}((x,k),y) = 0$. Then, $k \in G(x)$ and there exists a sequence $\{x_n\} \to x$ such that $d(y-k, F(x_n)) \to 0$. By the closedness of the graph of F, one has that $y - k \in F(x)$, i.e., $y \in F(x) + k$. Hence, $(x,k) \in \mathbb{S}_{\mathcal{E}_{(F,G)}}(y)$ establishing the proof. □

By virtue of to Lemma 2, we adapt Theorem 1 to the multifunction $\mathcal{E}_{(F,G)}$.

Lemma 3. *Let X be a complete metric space, let Y be a Banach space and let $F, G : X \rightrightarrows Y$ be closed multifunctions. Suppose that $(\bar{x}, \bar{k}, \bar{y}) \in X \times Y \times Y$ such that $\bar{y} \in F(\bar{x}) + \bar{k}, \bar{k} \in G(\bar{x})$.*

Let $\tau \in (0, +\infty)$, be fixed. Then, the following statements are equivalent:
(i) There exists a neighborhood $\mathcal{U}\times\mathcal{V}\times\mathcal{W} \subseteq X\times Y\times Y$ of $(\bar{x}, \bar{k}, \bar{y})$ such that $(\mathcal{U}\times\mathcal{V})\cap\mathbb{S}_{\mathcal{E}_{(F,G)}}(y) \neq \emptyset$ for any $y \in \mathcal{W}$ and

$$d((x,k), \mathbb{S}_{\mathcal{E}_{(F,G)}}(y)) \leq \tau\varphi_{\mathcal{E}}((x,k),y) \quad \textit{for all} \quad (x,k,y) \in \mathcal{U}\times\mathcal{V}\times\mathcal{W};$$

(ii) There exist a neighborhood $\mathcal{U}\times\mathcal{V}\times\mathcal{W} \subseteq X\times Y\times Y$ of $(\bar{x}, \bar{k}, \bar{y})$ and a real $\gamma \in (0, +\infty)$ such that for any $(x,k,y) \in \mathcal{U}\times\mathcal{V}\times\mathcal{W}$ with $y \notin F(x)+k, k \in G(x)$ and $\varphi_{\mathcal{E}}((x,k),y) < \gamma$, any $\varepsilon > 0$, and any sequences $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ converging to x, $\{k_n\}_{n\in\mathbb{N}} \subseteq Y$ converging to k with

$$\lim_{n\to\infty} d(y-k_n, F(x_n)) = \liminf_{u\to x} d(y-k, F(u)),$$

there exist sequences $\{u_n\}_{n\in\mathbb{N}} \subseteq X$, $\{z_n\}_{n\in\mathbb{N}} \subseteq Y$ with $\liminf_{n\to\infty} d((u_n,z_n),(x,k)) > 0$ such that

$$\limsup_{n\to\infty} \frac{d(y-k_n, F(x_n)) - d(y-z_n, F(u_n))}{d((x_n,u_n),(k_n,z_n))} > \frac{1}{\tau+\varepsilon}; \tag{14}$$

(iii) there exist a neighborhood $\mathcal{U}\times\mathcal{V}\times\mathcal{W}$ of $(\bar{x}, \bar{k}, \bar{y})$ and a real $\gamma > 0$ such that

$$|\nabla\varphi_{\mathcal{E}}((\cdot,\cdot),y)|(x,k) \geq \frac{1}{\tau} \text{ for all } (x,k,y) \in \mathcal{U}\times\mathcal{V}\times\mathcal{W} \text{ with } \varphi_{\mathcal{E}}((x,k),y) \in (0,\gamma).$$

Proposition 4. *Let X be a complete metric space, Y be a Banach space and let $F, G : X \rightrightarrows Y$ be closed multifunctions. Suppose that $(\bar{x}, \bar{k}, \bar{y}) \in X \times Y \times Y$ is such that $\bar{y} \in F(\bar{x})+\bar{k}, \bar{k} \in G(\bar{x})$. Consider the following statements:*
(i) there exist a neighborhood $\mathcal{U}\times\mathcal{V}\times\mathcal{W}$ of $(\bar{x}, \bar{k}, \bar{y})$ and $\tau > 0$ such that

$$d((x,k), \mathbb{S}_{\mathcal{E}_{(F,G)}}(y)) \leq \tau\varphi_{\mathcal{E}}((x,k),y) \quad \textit{for all} \quad (x,k,y) \in \mathcal{U}\times\mathcal{V}\times\mathcal{W};$$

(ii) there exist a neighborhood $\mathcal{U}\times\mathcal{V}\times\mathcal{W}$ of $(\bar{x}, \bar{k}, \bar{y})$ and $\tau > 0$ such that

$$d(x, (F+G)^{-1}(y)) \leq \tau d(y, F(x) + G(x) \cap \mathcal{V}) \quad \textit{for all} \quad (x,y) \in \mathcal{U}\times\mathcal{W}; \tag{15}$$

(iii) there exist a neighborhood $\mathcal{U}\times\mathcal{V}\times\mathcal{W}$ of $(\bar{x}, \bar{k}, \bar{y}-\bar{k})$ and $\varepsilon, \tau > 0$ such that for every $(x,k,z) \in \mathcal{U}\times\mathcal{V}\times\mathcal{W}, k \in G(x), z \in F(x)$, and $\rho \in (0,\varepsilon)$,

$$B(k+z, \rho\tau^{-1}) \subset (F+G)(B(x,\rho)).$$

Then one has the following implications: (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii).

Proof. For $(i) \Rightarrow (ii)$. By (i), there exist $\delta_1, \delta_2, \delta_3 > 0$ such that for every $\varepsilon > 0$ and for every $(x,k,y) \in B(\bar{x},\delta_1)\times[B(\bar{k},\delta_2)\cap G(x)]\times B(\bar{y},\delta_3)$, there is $(u,z) \in X\times Y$ with $y \in F(u)+z, z \in G(u)$ such that

$$d((x,k),(u,z)) < (1+\varepsilon)\tau\varphi_{\mathcal{E}}((x,k),y).$$

Consequently,

$$d(x,u) \leq \max\{d(x,u), \|k-z\|\} < (1+\varepsilon)\tau d(y, F(x)+k).$$

Noticing that $y \in F(u)+G(u)$, i.e., $u \in (F+G)^{-1}(y)$; it follows that

$$d(x,(F+G)^{-1}(y)) < (1+\varepsilon)\tau d(y, F(x)+k).$$

In conclusion, we have that

$$d(x,(F+G)^{-1}(y)) < (1+\varepsilon)\tau d(y, F(x)+G(x)\cap B(\bar{k},\delta_2)) \quad \text{for all} \quad (x,y) \in B(\bar{x},\delta_1)\times B(\bar{y},\delta_3).$$

Hence, taking the limit as $\varepsilon > 0$ goes to 0 yields the desired conclusion.

For $(ii) \Rightarrow (iii)$. Suppose that (ii) holds for the neighborhood $B(\bar{x},\delta_1)\times B(\bar{k},\delta_2)\times B(\bar{y},\delta_3)$ with $\delta_1,\delta_2,\delta_3 > 0$ and $\tau > 0$. Choose $\rho_1 = \delta_1, \rho_2 = 1/4\min\{\delta_2,\delta_3\}, \rho_3 = 1/4\delta_3, \varepsilon < \tau\delta_3/2$.
Then, for $(x,k,z) \in B(\bar{x},\rho_1)\times B(\bar{k},\rho_2)\times B(\bar{y}-\bar{k},\rho_3), k \in G(x), z \in F(x)$, we take $y \in B(k+z,\rho\tau^{-1})$.
Consequently,

$$\|y-k-z\| < \rho\tau^{-1},$$

and

$$\begin{aligned}\|y-\bar{y}\| &\leq \|y-k-z\| + \|k-\bar{k}\| + \|\bar{k}-\bar{y}+z\|,\\ &< \rho\tau^{-1} + \rho_2 + \rho_3,\\ &< \varepsilon\tau^{-1} + \delta_3/4 + \delta_3/4,\\ &< \delta_3/2 + \delta_3/2 = \delta_3.\end{aligned}$$

Therefore, we have that

$$d(y, F(x)+G(x)\cap B(\bar{k},\delta_2)) \leq \|y-k-z\| < \rho\tau^{-1}.$$

Hence,

$$d(x,(F+G)^{-1}(y)) < \tau\rho\tau^{-1} = \rho.$$

Let $\gamma > 0$ with $d(x,(F+G)^{-1}(y)) + \gamma < \rho$. Find $u \in (F+G)^{-1}(y)$, i.e., $y \in (F+G)(u)$ such that

$$d(x,u) < d(x,(F+G)^{-1}(y)) + \gamma.$$

Thus, $d(x,u) < \rho$. It follows that

$$y \in (F+G)(B(x,\rho)).$$

For $(iii) \Rightarrow (ii)$. Suppose that (iii) holds for the neighborhood $B(\bar{x},\rho_1)\times B(\bar{k},\rho_2)\times B(\bar{y},\rho_3)$ with $\rho_1,\rho_2,\rho_3 > 0$ and $\tau > 0, \varepsilon > 0$.
Take ρ_1,ρ_3 smaller if neccesary and consider a positive real η sufficiently small so that the quantity $\rho := \tau d(y, F(x)+G(x)\cap B(\bar{k},\rho_2)) + \eta$ satisfies the conclusion of (iii) together with $y \in B(k+z,\rho\tau^{-1})$. Then, there is a $u \in B(x,\rho)$ such that $y \in (F+G)(u)$, that is, $u \in (F+G)^{-1}(y)$.
Thus,

$$d(x,(F+G)^{-1}(y)) \leq d(x,u) < \rho = \tau d(y, F(x)+G(x)\cap B(\bar{k},\rho_2)) + \eta.$$

Since $\eta > 0$ is arbitrary, the proof is complete. □

In the next result, we give conditions for the sum of two metrically regular mappings F, G to remain metrically regular. Before stating this result, we need to recall the so-called "locally sum-stable" property introduced by Durea & Strugariu [12].

Definition 5. *Let F, G be two multifunctions and $(\bar{x}, \bar{y}, \bar{z}) \in X \times Y \times Y$ such that $\bar{y} \in F(\bar{x}), \bar{z} \in G(\bar{x})$. We say that the pair (F, G) is locally sum-stable around $(\bar{x}, \bar{y}, \bar{z})$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $x \in B(\bar{x}, \delta)$ and every $w \in (F+G)(x) \cap B(\bar{y}+\bar{z}, \delta)$, there are $y \in F(x) \cap B(\bar{y}, \varepsilon)$ and $z \in G(x) \cap B(\bar{z}, \varepsilon)$ such that $w = y + z$.*

A simple case which ensures the local sum-stability of (F, G) is as follows.

Proposition 6. *Let $F : X \rightrightarrows Y, G : X \rightrightarrows Y$ be two multifunctions and $(\bar{x}, \bar{y}, \bar{z}) \in X \times Y \times Y$ such that $\bar{y} \in F(\bar{x}), \bar{z} \in G(\bar{x})$. If $G(\bar{x}) = \{\bar{z}\}$ and G is upper semicontinuous at $\bar{x}$, then the pair (F, G) is locally sum-stable around $(\bar{x}, \bar{y}, \bar{z})$.*

Proof. Since G is upper semicontinuous at $\bar{x}$, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$G(x) \subset G(\bar{x}) + B(0, \varepsilon/2) = \bar{z} + B(0, \varepsilon/2) = B(\bar{z}, \varepsilon/2), \quad \text{for all } x \in B(\bar{x}, \delta).$$

Set

$$\eta := \min\{\delta, \varepsilon/2\}$$

and take $x \in B(\bar{x}, \eta)$ and $w \in (F+G)(x) \cap B(\bar{y}+\bar{z}, \eta)$. Then, there are $y \in F(x), z \in G(x)$ such that

$$w = y + z \text{ and } w \in B(\bar{y}+\bar{z}, \eta).$$

Clearly, $z \in B(\bar{z}, \varepsilon/2) \subset B(\bar{z}, \varepsilon)$.
Moreover,

$$\|y - \bar{y}\| = \|w - z - \bar{y}\| \leq \|w - \bar{y} - \bar{z}\| + \|z - \bar{z}\| < \eta + \varepsilon/2 \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Consequently,

$$w = y + z, y \in F(x) \cap B(\bar{y}, \varepsilon), z \in G(x) \cap B(\bar{z}, \varepsilon).$$

Hence we have established that (F, G) is locally sum-stable around $(\bar{x}, \bar{y}, \bar{z})$.

Proposition 7. *Let X be a complete metric space, Y be a Banach space and let $F, G : X \rightrightarrows Y$ be closed multifunctions. Suppose that $(\bar{x}, \bar{k}, \bar{y}) \in X \times Y \times Y$ is such that $\bar{y} \in F(\bar{x}) + \bar{k}, \bar{k} \in G(\bar{x})$. If the pair (F, G) is locally sum-stable around $(\bar{x}, \bar{y} - \bar{k}, \bar{k})$ and there exist a neighborhood $\mathcal{U} \times \mathcal{V}$ of $(\bar{x}, \bar{y})$ and $\tau, \theta > 0$ such that*

$$d(x, (F+G)^{-1}(y)) \leq \tau d(y, F(x) + G(x) \cap B(\bar{k}, \theta)) \quad \text{for all} \quad (x, y) \in \mathcal{U} \times \mathcal{V}, \tag{16}$$

then $F + G$ is metrically regular around $(\bar{x}, \bar{y})$ with modulus τ.

As a result, if G is upper semicontinuous at $\bar{x}$ and $G(\bar{x}) = \{\bar{k}\}$, then $F + G$ is metrically regular around $(\bar{x}, \bar{y})$ with modulus τ.

Proof. Suppose that (16) holds for every $(x, y) \in B(\bar{x}, \delta_1) \times B(\bar{y}, \delta_2)$ for some $\delta_1, \delta_2 > 0$. Since (F, G) is locally sum-stable around $(\bar{x}, \bar{y} - \bar{k}, \bar{k})$, there exists $\delta > 0$ such that for every $x \in B(\bar{x}, \delta)$ and every $w \in (F+G)(x) \cap B(\bar{y}, \delta)$, there are $y \in F(x) \cap B(\bar{y} - \bar{k}, \theta)$ and $z \in G(x) \cap B(\bar{k}, \theta)$ such that $w = y + z$.
Taking δ smaller if necessary, we can assume that $\delta < \delta_1$. Fix $(x, y) \in B(\bar{x}, \delta/2) \times B(\bar{y}, \delta/2)$. We consider two following cases:

Case 1. $d(y, F(x) + G(x)) < \delta/2$.

Fix $\gamma > 0$, small enough in order to have

$$d(y, F(x) + G(x)) + \gamma < \delta/2,$$

and take $t \in F(x)+G(x)$ such that $\|y-t\| < d(y, F(x)+G(x))+\gamma$. Hence we have $\|y-t\| < \delta/2$, and since we also have $\|y-\bar{y}\| < \delta/2$, this yields

$$\|t-\bar{y}\| \leq \|y-t\| + \|y-\bar{y}\| < \delta/2 + \delta/2 = \delta.$$

It follows that

$$t \in [F(x)+G(x)] \cap B(\bar{y}, \delta).$$

Since (F, G) is locally sum-stable around $(\bar{x}, \bar{y}-\bar{k}, \bar{k})$, there are $y \in F(x) \cap B(\bar{y}-\bar{k}, \theta)$ and $z \in G(x) \cap B(\bar{k}, \theta)$ such that

$$t = y + z.$$

Consequently,

$$t \in F(x) \cap B(\bar{y}-\bar{k}, \theta) + G(x) \cap B(\bar{k}, \theta) \subset F(x) + G(x) \cap B(\bar{k}, \theta).$$

Therefore,

$$d(y, F(x) + G(x) \cap B(\bar{k}, \theta)) \leq \|y-t\|,$$

from which we derive

$$d(y, F(x) + G(x) \cap B(\bar{k}, \theta)) \leq d(y, F(x) + G(x)) + \gamma,$$

and therefore, as γ is arbitrarily small, we obtain that

$$d(y, F(x) + G(x) \cap B(\bar{k}, \theta)) \leq d(y, F(x) + G(x)).$$

By (16), one gets that

$$d(x, (F+G)^{-1}(y)) \leq \tau d(y, F(x) + G(x)).$$

Since (x, y) is arbitrary in $B(\bar{x}, \delta/2) \times B(\bar{y}, \delta/2)$, this yields

$$d(x, (F+G)^{-1}(y)) \leq \tau d(y, F(x) + G(x)),$$

for all $(x, y) \in B(\bar{x}, \delta/2) \times B(\bar{y}, \delta/2)$.

Case 2. If $d(y, F(x) + G(x)) \geq \delta/2$.

Choose δ sufficiently small so that $\tau\delta/4 < \delta_1$. For every $(x, y) \in B(\bar{x}, \tau\delta/4) \times B(\bar{y}, \delta/4)$ and any $\varepsilon > 0$, by (16), there exists $u \in (F+G)^{-1}(y)$ such that

$$d(\bar{x}, u) < (1+\varepsilon)\tau d(y, F(\bar{x})+G(\bar{x})) \leq (1+\varepsilon)\tau\|y-\bar{y}\| < (1+\varepsilon)\tau\delta/2 \leq (1+\varepsilon)\tau/2d(y, F(x)+G(x)).$$

So,

$$\begin{aligned} d(x, u) &\leq d(x, \bar{x}) + d(\bar{x}, u) \\ &< \tau\delta/4 + (1+\varepsilon)\tau/2d(y, F(x) + G(x)) \\ &< \tau/2d(y, F(x) + G(x)) + (1+\varepsilon)\tau/2d(y, F(x) + G(x)). \end{aligned}$$

Taking the limit as $\varepsilon > 0$ goes to 0, it follows that

$$d(x, (F+G)^{-1}(y)) \leq \tau d(y, F(x) + G(x)).$$

So,

$$d(x, (F+G)^{-1}(y)) \leq \tau d(y, F(x) + G(x)),$$

for all $(x, y) \in B(\bar{x}, \tau\delta/4) \times B(\bar{y}, \delta/4)$. The proof is complete. □

The following theorem establishes metric regularity of the multifunction $\mathcal{E}_{(F,G)}$ as well as metric regularity of a metrically regular set-valued mapping perturbed by a Lipschitz-like one.

Theorem 8. *Let X be a complete metric space, let Y be a Banach space and let $F, G : X \rightrightarrows Y$ be closed multifunctions. Suppose that $(\bar{x}, \bar{k}, \bar{y}) \in X \times Y \times Y$ is such that $\bar{y} \in F(\bar{x}) + \bar{k}, \bar{k} \in G(\bar{x})$, F is metrically regular around $(\bar{x}, \bar{y} - \bar{k})$ with modulus $\tau > 0$ and G is Lipschitz-like around $(\bar{x}, \bar{k})$ with modulus $\lambda > 0$ with $\tau\lambda < 1$. Suppose that the product space $X \times Y$ is endowed with the metric defined by*

$$d((x, k), (u, z)) = \max\{d(x, u), \|z - k\|/\lambda\}.$$

Then $\mathcal{E}_{(F,G)}$ is metrically regular around $(\bar{x}, \bar{k}, \bar{y})$ with modulus $(\tau^{-1} - \lambda)^{-1}$.

If in addition we suppose that the pair (F, G) is locally sum-stable around $(\bar{x}, \bar{y} - \bar{k}, \bar{k})$, then $F + G$ is metrically regular around $(\bar{x}, \bar{y})$ with modulus $(\tau^{-1} - \lambda)^{-1}$.

Proof. Since by assumption G is Lipschitz-like around $(\bar{x}, \bar{k})$ with modulus $\lambda > 0$, there exist $\delta_1, \delta_2 > 0$ such that

$$G(x_1) \cap B(\bar{k}, \delta_1) \subset G(x_2) + \lambda\|x_1 - x_2\|\bar{B}_Y, \quad \text{for all } x_1, x_2 \in B(\bar{x}, \delta_2). \tag{17}$$

Furthermore, since F is metrically regular around $(\bar{x}, \bar{y} - \bar{k})$ with modulus $\tau > 0$, there exist $\delta_3, \delta_4 > 0$ and a real $\gamma > 0$ such that

$$|\nabla\varphi_F(\cdot, y)|(x) \geq \frac{1}{\tau} \quad \text{for all } (x, y) \in B(\bar{x}, \delta_3) \times B(\bar{y} - \bar{k}, \delta_4) \quad \text{with } \varphi_F(x, y) \in (0, \gamma). \tag{18}$$

So, for any $\varepsilon > 0$, there exists $u \in B(x, \delta_3), u \neq x$ such that

$$\frac{\varphi_F(x, y) - \varphi_F(u, y)}{d(x, u)} > \frac{1}{\tau + \varepsilon/2}.$$

Taking δ_1, δ_3 smaller if neccesary, we can assume that $\delta_1 < \delta_4$, and $\delta_3 < \delta_2$. Then, for every $(x, k, y) \in B(\bar{x}, \min\{\delta_2, \delta_3\}/2) \times B(\bar{k}, \delta_1) \times B(\bar{y}, \delta_4 - \delta_1)$ with $y - k \notin F(x), k \in G(x)$, any $\varepsilon > 0$ and any sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ converging to x, $\{k_n\}_{n\in\mathbb{N}} \subseteq X$ converging to k with $k_n \in G(x_n)$, and

$$\lim_{n\to\infty} d(y - k_n, F(x_n)) = \liminf_{u\to x} d(y - k, F(u)),$$

we deduce that

$$\frac{\varphi_F(x, y - k) - \varphi_F(u, y - k)}{d(x, u)} > \frac{1}{\tau + \varepsilon/2}, \text{(since } y - k \in B(\bar{y} - \bar{k}, \delta_4)), \tag{19}$$

and

$$\lim_{n\to\infty} d(y - k, F(x_n)) = \lim_{n\to\infty} d(y - k_n, F(x_n)) = \liminf_{u\to x} d(y - k, F(u)) = \varphi_F(x, y - k).$$

On the other hand, by definition of the function $\varphi_{\mathcal{E}}$, there is a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ converging to u such that

$$\lim_{n\to\infty} d(y - k, F(u_n)) = \varphi_F(u, y - k).$$

Because $u \in B(x, \delta_3), x \in B(\bar{x}, \min\{\delta_2, \delta_3\}/2), \{u_n\}_{n\in\mathbb{N}} \to u$, for n large enough, one has that $u_n \in B(\bar{x}, \delta_2)$. Similarly, since $k \in B(\bar{k}, \delta_1)$ and $\{k_n\}_{n\in\mathbb{N}} \subseteq X$ converges to k, for n large enough, one has that $k_n \in B(\bar{k}, \delta_1)$.

Therefore, by (17), and (19), there exists $z_n \in G(u_n)$ such that

$$\|z_n - k_n\| \leq \lambda d(x_n, u_n). \tag{20}$$

and

$$\lim_{n\to\infty} \frac{d(y - k, F(x_n)) - d(y - k, F(u_n))}{d(x, u)} > \frac{1}{\tau + \varepsilon}.$$

Thus, noticing that $u \neq x$, one has that

$$
\begin{aligned}
&\limsup_{n\to\infty} \frac{d(y-k,F(x_n))-d(y-k,F(u_n))}{d(x_n,u_n)} \\
&= \lim_{n\to\infty} \frac{d(y-k,F(x_n))-d(y-k,F(u_n))}{d(x_n,u_n)} \\
&= \lim_{n\to\infty} \frac{d(y-k,F(x_n))-d(y-k,F(u_n))}{d(x,u)} \frac{d(x,u)}{d(x_n,u_n)} \\
&= \lim_{n\to\infty} \frac{d(y-k,F(x_n))-d(y-k,F(u_n))}{d(x,u)} \lim_{n\to\infty} \frac{d(x,u)}{d(x_n,u_n)} \\
&\geq \frac{1}{\tau+\varepsilon}.
\end{aligned}
$$

On the other hand,

$$
d(y-z_n,F(u_n)) \leq d(y-k_nF(u_n)) + \|k_n-z_n\|. \tag{21}
$$

From relations (20), (21), we deduce that for any $(x,k,y) \in B(\bar{x},\min\{\delta_2,\delta_3\}/2) \times B(\bar{k},\delta_1) \times B(\bar{y},\delta_4-\delta_1)$ with $y-k \notin F(x), k \in G(x)$, and any $\varepsilon > 0$, any sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ converging to x, $\{k_n\}_{n\in\mathbb{N}} \subseteq X$ converging to k, there exists $\{(u_n,z_n)\}_{n\in\mathbb{N}}$ with

$$
\liminf_{n\to\infty} d((u_n,z_n),(x,k)) = \liminf_{n\to\infty} \max\{d(u_n,x), \|z_n-k\|/\lambda\} \geq \liminf_{n\to\infty} d(u_n,x) > 0,
$$

(since $0 < d(x,u) \leq d(u_n,x) + d(u_n,u)$ and $u_n \to u$)
such that

$$
\begin{aligned}
&\limsup_{n\to\infty} \frac{d(y-k_n,F(x_n))-d(y-z_n,F(u_n))}{d((x_n,k_n),(u_n,z_n))} \\
&\geq \limsup_{n\to\infty} \frac{d(y-k_n,F(x_n))-d(y-k_n,F(u_n))-\|k_n-z_n\|}{d((x_n,k_n),(u_n,z_n))} \\
&= \limsup_{n\to\infty} \frac{d(y-k_n,F(x_n))-d(y-k_n,F(u_n))-\|k_n-z_n\|}{\max\{d(x_n,u_n),\|k_n-z_n\|/\lambda\}} \\
&\geq \limsup_{n\to\infty} \frac{d(y-k_n,F(x_n))-d(y-k_n,F(u_n))}{\max\{d(x_n,u_n),\|k_n-z_n\|/\lambda\}} - \lambda \\
&= \limsup_{n\to\infty} \frac{d(y-k_n,F(x_n))-d(y-k_n,F(u_n))}{d(x_n,u_n)} - \lambda > \frac{1}{\tau+\varepsilon} - \lambda,
\end{aligned}
$$

(since $\|z_n-k_n\|/\lambda \leq d(x_n,u_n)$).

By Lemma 3 ($(i) \Leftrightarrow (ii)$), one concludes that $\mathcal{E}_{(F,G)}$ is metrically regular around $(\bar{x},\bar{k},\bar{y})$ with modulus $(\tau^{-1}-\lambda)^{-1}$.

If the pair (F,G) is locally sum-stable around $(\bar{x},\bar{y}-\bar{k},\bar{k})$, then by combining the hypothesis with Proposition 7 and Proposition 4, we complete the proof. □

Combining Proposition 4 and Theorem 8, we obtain the following corollary which is equivalent to the main result (Theorem 3.3) in [12], which is stated for the difference of an open mapping and a Lipschitz-like one.

Corollary 9. *Let X be a complete metric space, let Y be a Banach space and let $F,G : X \rightrightarrows Y$ be closed multifunctions. Suppose that $(\bar{x},\bar{k},\bar{y}) \in X \times Y \times Y$ is such that $\bar{y} \in F(\bar{x}) + \bar{k}, \bar{k} \in G(\bar{x})$*

and F is metrically regular around $(\bar{x}, \bar{y}-\bar{k})$ with modulus $\tau > 0$ and G is Lipschitz-like around $(\bar{x}, \bar{k})$ with modulus $\lambda > 0$ with $\tau\lambda < 1$. Then, there exist a neighborhood $\mathcal{U}\times\mathcal{V}\times\mathcal{W}$ of $(\bar{x}, \bar{k}, \bar{y}-\bar{k})$ and $\varepsilon, \tau > 0$ such that for every $(x,k,z) \in \mathcal{U}\times\mathcal{V}\times\mathcal{W}, k \in G(x), z \in F(x)$, and $\rho \in (0, \varepsilon)$,

$$B(k+z, \rho\tau^{-1}) \subset (F+G)(B(x,\rho)).$$

3. Metric regularity of the epigraphical multifunction under coderivative conditions

In this section, X, Y are assumed to be Asplund spaces, i.e., Banach spaces for which each separable subspace has a separable dual (in particular, any reflexive space is Asplund; see, e.g., [6] for more details). We recall some notations, terminology and definitions basically standard and conventional in the area of variational analysis and generalized differentials. As usual, $\|\cdot\|$ stands for the norm on X or Y indifferently and $\langle\cdot,\cdot\rangle$ signifies for the canonical pairing between X and its topological dual $X^\star$ with the symbol $\xrightarrow{w^\star}$ indicating the convergence in the weak* topology of $X^\star$ and the symbol cl* standing for the weak* topological closure of a set. Given a set-valued mapping $F\colon X \rightrightarrows X^\star$ between X and $X^\star$, recall that the symbol

$$\underset{x\to\bar{x}}{\mathrm{Lim\,sup}}\, F(x) := \left\{x^\star \in X^\star \,\middle|\, \exists\, x_n \to \bar{x},\ \exists\, x_n^\star \xrightarrow{w^\star} x^\star \ \text{ with } \ x_n^\star \in F(x_n), \quad n \in \mathbb{N}\right\} \tag{22}$$

stands for the *sequential Painlevé-Kuratowski outer/upper limit* of F as $x \to \bar{x}$ with respect to the norm topology of X and the weak* topology of $X^\star$. Let us give an extended-real-valued lower semicontinuous function $f : X \to \mathbb{R}\cup\{+\infty\}$ and $\bar{x} \in X$. The notation $x \xrightarrow{f} \bar{x}$ means that with $x \to \bar{x}$ with $f(x) \to f(\bar{x})$. For $\varepsilon \geq 0$, the ε-Fréchet subdifferential of f at $\bar{x} \in \mathrm{Dom}\, f$ is the set

$$\hat{\partial}_\varepsilon f(\bar{x}) := \{x^\star \in X^\star : \liminf_{x\to\bar{x},\, x\neq\bar{x}} \frac{f(x)-f(\bar{x})-\langle x^\star, x-\bar{x}\rangle}{\|x-\bar{x}\|} \geq -\varepsilon\}, \tag{23}$$

and if $\bar{x} \notin \mathrm{Dom}\, f$, we set $\hat{\partial}_\varepsilon f(\bar{x}) = \emptyset$.
When $\varepsilon = 0$, formula (23) defines the Fréchet subdifferential $\hat{\partial} f(\bar{x})$ of f at $\bar{x}$:

$$\hat{\partial} f(\bar{x}) = \{x^\star \in X^\star : \liminf_{x\to\bar{x},\, x\neq\bar{x}} \frac{f(x)-f(\bar{x})-\langle x^\star, x-\bar{x}\rangle}{\|x-\bar{x}\|} \geq 0\},$$

and $\hat{\partial} f(\bar{x}) = \emptyset$ if $\bar{x} \notin \mathrm{Dom}\, f$.

The notation $\partial f(\bar{x})$ is used to denote the limiting subdifferential of f at $\bar{x} \in \mathrm{Dom}\, f$. It is defined by

$$\partial f(\bar{x}) := \underset{x\xrightarrow{f}\bar{x},\, \varepsilon\downarrow 0}{\mathrm{Lim\,sup}}\, \hat{\partial}_\varepsilon f(x),$$

i.e.,

$$\partial f(\bar{x}) = \{x^\star \in X^\star : \exists \varepsilon_n \downarrow 0, x_n \xrightarrow{f} \bar{x}, x_n^\star \xrightarrow{w^\star} x^\star \text{ with } x_n^\star \in \hat{\partial}_{\varepsilon_n} f(x_n) \text{ for all } n \in \mathbb{N}\}.$$

The following formula holds:

$$\partial f(\bar{x}) := \underset{x\xrightarrow{f}\bar{x}}{\mathrm{Lim\,sup}}\, \hat{\partial} f(x).$$

For a closed set $C \subset X$ and $\bar{x} \in C$, the Fréchet normal cone to C at $\bar{x}$ is denoted $\hat{N}(\bar{x}; C)$ and is defined as the Fréchet subdifferential of indicator function δ_C of C at $\bar{x}$, i.e.,

$$\hat{N}(\bar{x}; C) := \hat{\partial}\delta_C(\bar{x}),$$

where $\delta_C(x) = 0$ if $x \in C$, and $\delta_C(x) = +\infty$ if $x \notin C$.
The limiting normal cone of C at $\bar{x}$ is defined and denoted by

$$N(\bar{x}; C) = \partial\delta_C(\bar{x}).$$

Let us consider a closed multifunction $F : X \rightrightarrows Y$ and $\bar{y} \in F(\bar{x})$. The Fréchet coderivative of F at $(\bar{x}, \bar{y})$ is the mapping $\hat{D}^\star F(\bar{x}, \bar{y}) : Y^\star \rightrightarrows X^\star$ defined by

$$x^\star \in \hat{D}^\star F(\bar{x}, \bar{y})(y^\star) \Leftrightarrow (x^\star, -y^\star) \in \hat{N}((\bar{x}, \bar{y}); \text{gph } F),$$

while the Mordukhovich (limiting) coderivative of F at $(\bar{x}, \bar{y})$ is the mapping $D^\star F(\bar{x}, \bar{y}) : Y^\star \rightrightarrows X^\star$ defined by

$$x^\star \in D^\star F(\bar{x}, \bar{y})(y^\star) \Leftrightarrow (x^\star, -y^\star) \in N((\bar{x}, \bar{y}); \text{gph } F).$$

Here, $\hat{N}((\bar{x}, \bar{y}); \text{gph } F)$ and $N((\bar{x}, \bar{y}); \text{gph } F)$ are the Fréchet and the limiting normal cone to gph F at $(\bar{x}, \bar{y})$, respectively.

To obtain a point-based condition for metric regularity of multifunctions in infinite dimensional spaces, one often uses the so-called *partial sequential normal compactness* (PSNC) property. A multifunction $F : X \rightrightarrows Y$ is *partially sequentially normally compact* at $(\bar{x}, \bar{y}) \in \text{gph } F$, if for any sequences $\{(x_k, y_k, x_k^\star, y_k^\star)\} \in \text{gph } F \times X^\star \times Y^\star$ satisfying

$$(x_k, y_k) \to (\bar{x}, \bar{y}), x_k^\star \in \hat{D}^\star(x_k, y_k)(y_k^\star), x_k^\star \xrightarrow{w^\star} 0, \|y_k^\star\| \to 0$$

one has $\|x_k^\star\| \to 0$ as $k \to \infty$.

Remark 10. *Condition (PSNC) at $(\bar{x}, \bar{y}) \in \text{gph } F$ is satisfied if X, or Y is finite dimensional space, or F is Lipschitz-like around that point.*

In the following, we need a result on *the metric inequality* (see, e.g., Ioffe [18], Huynh &Théra [25]). Let us recall that the sets $\{\Omega_1, \Omega_2\}$ satisfy the metric inequality at $\bar{x}$ if there are $\tau > 0$ and $r > 0$ such that

$$d(x, \Omega_1 \cap \Omega_2) \le \tau[d(x, \Omega_1) + d(x, \Omega_2)] \text{ for all } x \in B(\bar{x}, r).$$

Proposition 11. *Let $\{\Omega_1, \Omega_2\}$ be two closed subsets of X and fix $\bar{x} \in \Omega_1 \cap \Omega_2$. Suppose that the following hypothesis $(\mathcal{H})$ is satisfied at $\bar{x}$:*
$(\mathcal{H})$ for any sequences $\{x_{ik}\}_{k\in\mathbb{N}} \subset \Omega_i$, $\{x_{ik}^\star\}_{k\in\mathbb{N}} \subset \hat{N}(x_{ik}; \Omega_i)$ such that $\{x_{ik}\}_{k\in\mathbb{N}} \to \bar{x}$, i=1, 2 and

$$\|x_{1k}^\star + x_{2k}^\star\|_{k\in\mathbb{N}} \to 0 \textit{ implies } x_{1k}^\star \to 0, x_{2k}^\star \to 0,$$

then, the sets $\{\Omega_1, \Omega_2\}$ satisfy the metric inequality at $\bar{x}$. Under this assumption, there is $r > 0$ such that for every $\varepsilon > 0$, and $x \in B(\bar{x}, r)$, there exist $x_1, x_2 \in B(x, \varepsilon)$ such that

$$\hat{N}(x; \Omega_1 \cap \Omega_2) \subset \hat{N}(x_1; \Omega_1) + \hat{N}(x_2; \Omega_2) + \varepsilon B_{X^\star}. \tag{24}$$

Let us consider two multifunctions $F, G : X \rightrightarrows Y$. To these multifunctions, we associate the two sets

$$C_1 := \{(x, y, z) \in X \times Y \times Y : y \in G(x)\} \text{ and } C_2 := \{(x, y, z) \in X \times Y \times Y : z \in F(x)\}.$$

Remark 12. *Hypothesis $(\mathcal{H})$ can be restated for the sets $\{C_1, C_2\}$ at $(\bar{x}, \bar{y}, \bar{z}) \in C_1 \cap C_2$ as follows:*
(i) $(\mathcal{H})$: for any sequences

$$\{(x_k, y_k)\}_{k\in\mathbb{N}} \subset \text{gph } G, \{(v_k, z_k)\}_{k\in\mathbb{N}} \subset \text{gph } F,$$

$$x_k^\star \in \hat{D}^\star G(x_k, y_k)(y_k^\star), u_k^\star \in \hat{D}^\star F(v_k, z_k)(z_k^\star),$$

such that if

$$(x_k, y_k) \to (\bar{x}, \bar{y}),$$
$$(v_k, z_k) \to (\bar{x}, \bar{z}),$$
$$\|x_k^\star + u_k^\star\| \to 0,$$
$$y_k^\star \to 0, z_k^\star \to 0,$$

then

$$x_k^\star \to 0, u_k^\star \to 0, \textit{as } k \to 0.$$

It holds whenever one of following conditions is fulfilled:
(ii) F^{-1} *or* G^{-1} *is Lipschitz-like around* $(\bar{z}, \bar{x})$ *and* $(\bar{y}, \bar{x})$*, respectively;*
(iii) either F *is PSNC at* $(\bar{x}, \bar{z})$ *or* G *is PSNC at* $(\bar{x}, \bar{y})$*, and*

$$D^\star F(\bar{x}, \bar{z})(0) \cap -D^\star G(\bar{x}, \bar{y})(0) = \{0\}.$$

Proof. Observe that if F^{-1} or G^{-1} is Lipschitz-like around $(\bar{z}, \bar{x})$ and $(\bar{y}, \bar{x})$, respectively, then assumption $(\mathcal{H})$ always holds (see for instance Mordukhovich [23]).
We now assume that (iii) holds. Take

$$\{(x_k, y_k)\}_{k\in\mathbb{N}} \subset \text{gph } G, \{(v_k, z_k)\}_{k\in\mathbb{N}} \subset \text{gph } F,$$
$$x_k^\star \in \hat{D}^\star G(x_k, y_k)(y_k^\star), u_k^\star \in \hat{D}^\star F(v_k, z_k)(z_k^\star),$$

such that

$$(x_k, y_k) \to (\bar{x}, \bar{y}),$$
$$(v_k, z_k) \to (\bar{x}, \bar{z}),$$
$$\|x_k^\star + u_k^\star\| \to 0,$$
$$y_k^\star \to 0, z_k^\star \to 0.$$

If the sequences $\{x_k^\star\}$, $\{u_k^\star\}$ are unbounded, we can assume that

$$\|x_k^\star\| \to \infty, \|u_k^\star\| \to \infty,$$

and

$$\frac{x_k^\star}{\|x_k^\star\|} \xrightarrow{w^\star} x^\star, \frac{u_k^\star}{\|u_k^\star\|} \xrightarrow{w^\star} u^\star.$$

Then,

$$y_k^\star/\|x_k^\star\| \to 0 \text{ and } z_k^\star/\|u_k^\star\| \to 0.$$

Consequently,

$$x^\star \in D^\star G(\bar{x}, \bar{y})(0), u^\star \in D^\star F(\bar{x}, \bar{z})(0).$$

On the other hand,

$$u^\star + x^\star = 0, \text{ (since } \|x_k^\star + u_k^\star\| \to 0).$$

It follows that

$$u^\star \in D^\star F(\bar{x}, \bar{z})(0) \cap -D^\star G(\bar{x}, \bar{y})(0).$$

Therefore, by assumption, this yields $x^\star = u^\star = 0$.
Hence,

$$\frac{x_k^\star}{\|x_k^\star\|} \to 0, \quad \text{or} \quad \frac{u_k^\star}{\|u_k^\star\|} \to 0, \text{ (by PSNC property of } F \text{ or } G).$$

This contradicts the fact $\frac{x_k^\star}{\|x_k^\star\|}$, and $\frac{u_k^\star}{\|u_k^\star\|}$ are in the unit sphere $S_{Y^\star}$ of $Y^\star$. So, the sequences $\{x_k^\star\}$, $\{u_k^\star\}$ are bounded. Without any loss of generality, we can assume that

$$x_k^\star \xrightarrow{w^\star} x^\star, u_k^\star \xrightarrow{w^\star} u^\star.$$

It follows that

$$x^\star \in D^\star G(\bar{x},\bar{y})(0), u^\star \in D^\star F(\bar{x},\bar{z})(0).$$

Moreover,

$$x^\star + u^\star = 0.$$

Hence,

$$u^\star \in D^\star F(\bar{x},\bar{z})(0) \cap -D^\star G(\bar{x},\bar{y})(0).$$

Therefore, by assumption, we obtain $x^\star = u^\star = 0$, and $x_k^\star \to 0$, or, $u_k^\star \to 0$, (by PSNC property of F or G). The proof is complete. □

The following lemma gives an estimation for the strong slope of the function $\varphi_{\mathcal{E}}((x,k),y)$.

Lemma 13. *Let $(\bar{x},\bar{y}-\bar{k},\bar{k}) \in X \times F(\bar{x}) \times G(\bar{x})$ be given. Assume that the sets $\{C_1, C_2\}$ defined as above satisfy hypothesis $(\mathcal{H})$ at $(\bar{x},\bar{k},\bar{y}-\bar{k})$. Then there exists $\rho > 0$ such that for all $(x,k,y) \in B((\bar{x},\bar{k},\bar{y}),\rho)$ with $y \notin F(x)+k, k \in G(x)$ as well as $d(y,F(x)+k) < \rho$, one has*

$$|\nabla\varphi_{\mathcal{E}}((\cdot,\cdot),y)|(x,k) \geq \lim_{\delta\downarrow 0}\left\{\inf\left\{\|x^\star\| : \begin{array}{l} (u,w) \in \text{gph } F, (v,z) \in \text{gph } G, u,v \in B(x,\delta) \\ u^\star \in \hat{D}^\star G(v,z)(y^\star), \|y^\star\| = 1, z \in B(k,\delta) \\ x^\star \in \hat{D}^\star F(u,w)(y^\star + z^\star) + u^\star, z^\star \in \delta B_{Y^\star} \\ d(y,F(u)+k) \leq \varphi_{\mathcal{E}}((x,k),y) + \delta \\ \|w+k-y\| \leq d(y,F(u)+k) + \delta \\ |\langle y^\star + z^\star, w+k-y\rangle - d(y,F(u)+k)| < \delta \end{array}\right\}\right\}.$$

Proof. Obviously, if $(\mathcal{H})$ is satisfied at $(\bar{x},\bar{k},\bar{y}-\bar{k})$ then it is‘also satisfied at all points $(u,v,w) \in X \times G(u) \times F(u)$ near $(\bar{x},\bar{k},\bar{y}-\bar{k})$, say $(u,v,w) \in X \times G(u) \times F(v) \cap B_{X\times Y\times Y}((\bar{x},\bar{k},\bar{y}-\bar{k}),3\rho)$. Let $(x,k,y) \in B_{X\times Y\times Y}((\bar{x},\bar{k},\bar{y}-\bar{k}),\rho)$ be such that $y \notin F(x)+k, k \in G(x)$ and $d(y,F(x)+k) < \rho$. Set $|\nabla\varphi_{\mathcal{E}}((\cdot,\cdot),y)|(x,k) := m$. By the lower semicontinuity of $\varphi_{\mathcal{E}}$ as well as the definition of the strong slope, for each $\varepsilon \in (0,\varphi_{\mathcal{E}}((x,k),y))$, there is $\eta \in (0,\varepsilon)$ with $4\eta+\varepsilon < \varphi_{\mathcal{E}}((x,k),y)$ and $1-(m+\varepsilon+3)\eta > 0$ such that $d(y,F(u)+k) \geq \varphi_{\mathcal{E}}((x,k),y) - \varepsilon$, for all $u \in B(x,4\eta)$ and

$$m+\varepsilon \geq \frac{\varphi_{\mathcal{E}}((x,k),y) - \varphi_{\mathcal{E}}((z,k'),y)}{\max\{\|x-z\|,\|k-k'\|\}} \quad \text{for all } z \in \bar{B}(x,\eta), k' \in \bar{B}(k,\eta) \cap G(x).$$

Consequently,

$$\varphi_{\mathcal{E}}((x,k),y) \leq \varphi_{\mathcal{E}}((z,k'),y) + (m+\varepsilon)\|z-x\| + (m+\varepsilon)\|k-k'\| \quad \text{for all } z \in \bar{B}(x,\eta), k' \in \bar{B}(k,\eta) \cap G(x).$$

Take $u \in B(x,\eta^2/4)$, $v \in F(u)$ such that $\|y-k-v\| \leq \varphi_{\mathcal{E}}((x,k),y) + \eta^2/4$.
Taking into account that $\varphi_{\mathcal{E}}((z,k'),y) \leq d(y,F(z)+k')$ with $k' \in G(z)$, then $\varphi_{\mathcal{E}}((z,k'),y) \leq \|y-k'-w\|$ with $w \in F(z)$ and $k' \in G(z)$. It follows that

$$\varphi_{\mathcal{E}}((z,k'),y) \leq \|y-k'-w\| + \delta_{C_2}(z,k',w) + \delta_{C_1}(z,k',w).$$

From the inequality,

$$\|y-k-v\| \leq \varphi_{\mathcal{E}}((z,k'),y) + (m+\varepsilon)\|z-x\| + \eta^2/4,$$

we obtain that

$$\|y-k-v\| \leq \|y-k'-w\| + \delta_{C_2}(z,k',w) + \delta_{C_1}(z,k',w) + (m+\varepsilon)\|z-u\| + (m+\varepsilon)\eta + \eta^2/4,$$

for all $(z,w) \in \bar{B}(x,\eta) \times Y, k' \in \bar{B}(k,\eta)$. Ekeland variational principle to the function

$$(z,k',w) \mapsto \|y-k'-w\| + \delta_{C_2}(z,k',w) + \delta_{C_1}(z,k',w) + (m+\varepsilon)\|z-u\|$$

on $\bar{B}(x,\eta)\times\bar{B}(k,\eta)\times Y$, we can select $(u_1,k_1,w_1)\in(u,k,v)+\frac{\eta}{4}B_{X\times Y\times Y}$ with $(u_1,k_1,w_1)\in C_2\cap C_1$ such that

$$\|y-k_1-w_1\|\le\|y-k-v\|(\le\varphi_{\mathcal{E}}((x,k),y)+\eta^2/4); \tag{25}$$

and the function

$$(z,k',w)\mapsto\|y-k'-w\|+\delta_{C_2}(z,k',w)+\delta_{C_1}(z,k',w)+(m+\varepsilon)\|z-u\|+(m+\varepsilon+1)\eta\|(z,k',w)-(u_1,v_1,w_1)\|$$

attains a minimum on $\bar{B}(x,\eta)\times\bar{B}(k,\eta)\times Y$ at (u_1,k_1,w_1). Hence, using the sum rule for Fréchet subdifferentials, we can find

$$(u_2,k_2,w_2),(u_4,k_4,w_4)\in B_{X\times Y\times Y}((u_1,k_1,w_1),\eta);\ (u_3,k_3,w_3)\in B_{X\times Y\times Y}((u_1,k_1,w_1),\eta)\cap C_2\cap C_1;$$

such that

$$\begin{aligned}(0,k_2^\star,w_2^\star)&\in\hat{\partial}\|y-\cdot-\cdot\|(u_2,k_2,w_2),\\(u_3^\star,k_3^\star,w_3^\star)&\in\hat{\partial}(\delta_{C_2}(\cdot,\cdot,\cdot)+\delta_{C_1}(\cdot,\cdot,\cdot))(u_3,k_3,w_3),\\(u_4^\star,0,0)&\in\hat{\partial}((m+\varepsilon)\|\cdot-u\|)(u_4,k_4,w_4)\end{aligned}$$

and

$$(0,0,0)\in(0,k_2^\star,w_2^\star)+(u_3^\star,k_3^\star,w_3^\star)+(u_4^\star,0,0)+(m+\varepsilon+2)\eta[\bar{B}_{X^\star}\times\bar{B}_{Y^\star}\times\bar{B}_{Y^\star}]. \tag{26}$$

Notice that

$$\begin{aligned}\|y-k_2-w_2\|&\ge\|y-v-k\|-\|w_2-v\|-\|k_2-k\|\\&\ge\varphi_{\mathcal{E}}((x,k),y)-\varepsilon-(\|w_2-w_1\|+\|w_1-v\|)-(\|k_2-k_1\|+\|k-k_1\|)\\&>\varphi_{\mathcal{E}}((x,k),y)-\varepsilon-2\eta-2\eta=\varphi_{\mathcal{E}}((x,k),y)-\varepsilon-4\eta>0.\end{aligned} \tag{27}$$

Then, by [[34]Theorem 2.8.3](see, also, [13][proof of Theorem 3.6]), we know that

$$\hat{\partial}\|y-\cdot-\cdot\|(u_2,k_2,w_2)=\{(0,y^\star,y^\star):y^\star\in S_{Y^\star},\langle y^\star,w_2+k_2-y\rangle=\|y-w_2-k_2\|\}.$$

Hence,

$$w_2^\star=k_2^\star\in S_{Y^\star}\text{ and }\langle w_2^\star,w_2+k_2-y\rangle=\|y-w_2-k_2\|.$$

Now, in order to have $(u_3,k_3,w_3)\in B_{X\times Y\times Y}((\bar{x},\bar{k},\bar{y}-\bar{k}),3\rho)$, we take η smaller if necessary, and by virtue of Proposition 11 one has

$$(u_3^\star,k_3^\star,w_3^\star)\in\hat{N}((u_5,k_5,w_5);C_2)+\hat{N}((u_6,k_6,w_6);C_1)+\eta[\bar{B}_{X^\star}\times\bar{B}_{Y^\star}\times\bar{B}_{Y^\star}], \tag{28}$$

where $(u_5,k_5,w_5)\in C_2\cap B_{X\times Y\times Y}((u_3,k_3,w_3),\eta),(u_6,k_6,w_6)\in C_1\cap B_{X\times Y\times Y}((u_3,k_3,w_3),\eta)$. From (26) and (28), one deduces that

$$\begin{gathered}(0,0,0)\in(0,k_2^\star,w_2^\star)+\hat{N}((u_5,k_5,w_5);C_2)+\\\hat{N}((u_6,k_6,w_6);C_1)+(u_4^\star,0,0)+(m+\varepsilon+3)\eta[\bar{B}_{X^\star}\times\bar{B}_{Y^\star}\times\bar{B}_{Y^\star}].\end{gathered}$$

Therefore, there exist $(u_5^\star,k_5^\star,w_5^\star)\in[\bar{B}_{X^\star}\times\bar{B}_{Y^\star}\times\bar{B}_{Y^\star}],(u_6^\star,k_6^\star,0)\in\hat{N}((u_6,k_6,w_6);C_1)$, i.e., $u_6^\star\in\hat{D}^\star G(u_6,k_6)(-k_6^\star)$ such that

$$(-u_4^\star-(m+\varepsilon+3)\eta u_5^\star-u_6^\star,-k_2^\star-(m+\varepsilon+3)\eta k_5^\star-k_6^\star,-w_2^\star-(m+\varepsilon+3)\eta w_5^\star)\in\hat{N}((u_5,k_5,w_5);C_2).$$

It follows that

$$-k_2^\star-(m+\varepsilon+3)\eta k_5^\star-k_6^\star=0,$$

and

$$(-u_4^\star-(m+\varepsilon+3)\eta u_5^\star-u_6^\star,-w_2^\star-(m+\varepsilon+3)\eta w_5^\star)\in\hat{N}((u_5,w_5);\text{gph }F).$$

Consequently,

$-k_6^\star = k_2^\star + (m+\varepsilon+3)\eta k_5^\star$ and $(-u_4^\star - (m+\varepsilon+3)\eta u_5^\star - u_6^\star) \in \hat{D}^\star F(u_5, w_5)(w_2^\star + (m+\varepsilon+3)\eta w_5^\star)$.

Remark that $\|k_6^\star\| = \| - k_2^\star - (m+\varepsilon+3)\eta k_5^\star\| \geq 1 - (m+\varepsilon+3)\eta > 0$.
Hence, setting

$$\begin{aligned} y^\star &:= (k_2^\star + (m+\varepsilon+3)\eta k_5^\star)/\|k_2^\star + (m+\varepsilon+3)\eta k_5^\star\|, \\ z^\star &:= (w_5^\star - k_5^\star)(m+\varepsilon+3)\eta/\|k_2^\star + (m+\varepsilon+3)\eta k_5^\star\|, \\ x_1^\star &:= u_6^\star/\|k_2^\star + (m+\varepsilon+3)\eta k_5^\star\|, \\ x_2^\star &:= (-u_4^\star - (m+\varepsilon+3)\eta u_5^\star)/\|k_2^\star + (m+\varepsilon+3)\eta k_5^\star\|, \end{aligned}$$

one obtains that

(29) $$x_1^\star \in \hat{D}^\star G(u_6, k_6)(y^\star) \text{ and } (x_2^\star - x_1^\star) \in \hat{D}^\star F(u_5, w_5)(y^\star + z^\star),$$

where,

(30) $$\|y^\star\| = 1, \|z^\star\| \leq \frac{2(m+\varepsilon+3)\eta}{1-(m+\varepsilon+3)\eta} := \delta, \|x_2^\star\| \leq \frac{m+\varepsilon+(m+\varepsilon+3)\eta}{1-(m+\varepsilon+3)\eta}.$$

On the other hand, according to relation (25) one has that

(31)

$$\begin{aligned} \varphi_{\mathcal{E}}((x,k),y) - \varepsilon \leq d(y, F(u_5)+k) \quad &\leq \|y-k-w_5\| \leq \|y-k_1-w_1\| + \|w_5-w_1\| + \|k_1-k\| \\ &\leq \|y-k-v\| + \eta + 2\eta \\ &\leq \varphi_{\mathcal{E}}((x,k),y) + \eta^2/4 + 3\eta. \end{aligned}$$

Consequently,

(32) $$\langle y^\star + z^\star, k + w_5 - y\rangle \leq (1+\delta)\|y-k-w_5\| \leq (1+\delta)d(y, F(u_5)+k) + (1+\delta)(\eta^2/4+3\eta+\varepsilon),$$

whence

(33) $$\langle y^\star + z^\star, y - k - w_5\rangle - d(y, F(u_5)+k) \leq \delta d(y, F(u_5)+k) + (1+\delta)(\eta^2/4+3\eta+\varepsilon).$$

Furthermore,

$$\begin{aligned} &\langle y^\star + z^\star, k + w_5 - y\rangle \\ &= \frac{\langle w_2^\star + (m+\varepsilon+3)\eta w_5^\star, k + w_5 - y\rangle}{\|k_2^\star + (m+\varepsilon+3)\eta k_5^\star\|} \\ &= \frac{\langle w_2^\star, w_2 + k_2 - y\rangle + \langle w_2^\star, w_5 - w_2\rangle + \langle w_2^\star, k - k_2\rangle + (m+\varepsilon+3)\eta\langle w_5^\star, w_5 + k - y\rangle}{\|k_2^\star + (m+\varepsilon+3)\eta k_5^\star\|} \\ &\geq \frac{\|w_2 + k_2 - y\| - 3\eta - 2\eta - (m+\varepsilon+3)\eta\|w_5 + k - y\|}{(1+(m+\varepsilon+3)\eta)}, \end{aligned}$$

and by (27) and (31) we have that
$\|y - k_2 - w_2\| \geq \varphi_{\mathcal{E}}((x,k),y) - \varepsilon - 4\eta \geq d(y, F(u_5)+k) - \eta^2/4 - 8\eta - \varepsilon$. Along with (33), one

deduces that

$$\begin{aligned}&\langle y^\star, w_5+k-y\rangle\\ &\geq \frac{d(y,F(u_5)+k)-\eta^2/4-8\eta-\varepsilon-4\eta-(m+\varepsilon+3)\eta(d(y,F(u_5)+k)+\eta^2/4+\eta+\varepsilon}{1+(m+\varepsilon+3)\eta}\\ &= \frac{d(y,F(u_5)+k)(1-(m+\varepsilon+3)\eta)-(\eta^2/4+\eta+\varepsilon)(1+(m+\varepsilon+3)\eta)-11\eta}{1+(m+\varepsilon+3)\eta}.\end{aligned}$$

So,

$$\langle y^\star, w_5+k-y\rangle \geq \frac{d(y,F(u_5)+k)(1-(m+\varepsilon+3)\eta)-11\eta}{1+(m+\varepsilon+3)\eta}-(\eta^2/4+\eta+\varepsilon). \tag{34}$$

As $\varepsilon,\eta,\delta>0$ are arbitrary small, by combining relations (29)-(34), we complete the proof. □

Theorem 14. *Let X,Y be Asplund spaces, and let $F,G: X\rightrightarrows Y$ be closed multifunctions. Suppose that $(\bar{x},\bar{k},\bar{y})\in X\times Y\times Y$ is such that $\bar{y}\in F(\bar{x})+\bar{k},\bar{k}\in G(\bar{x})$ and the sets $\{C_1,C_2\}$ satisfy the hypothesis $(\mathcal{H})$ at $(\bar{x},\bar{k},\bar{y}-\bar{k})$. Let $m>0$. If there exist a neighborhood $\mathcal{U}\times\mathcal{V}\times\mathcal{W}$ of $(\bar{x},\bar{k},\bar{y})$ and $\gamma>0$ such that for each $(x,y,k)\in\mathcal{U}\times\mathcal{V}\times\mathcal{W}$ with $y\notin F(x)+k, k\in G(x)$,*

$$m\leq \lim_{\delta\downarrow 0}\left\{\inf\left\{\|x^\star\|:\begin{array}{l}(u,w)\in \text{gph } F,(v,z)\in\text{gph } G, u,v\in B(x,\delta)\\ u^\star\in\hat{D}^\star G(v,z)(y^\star),\|y^\star\|=1, z\in B(k,\delta)\\ x^\star\in\hat{D}^\star F(u,w)(y^\star+z^\star)+u^\star, z^\star\in\delta B_{Y^\star}\\ d(y,F(u)+k)\leq\gamma+\delta\\ \|w+k-y\|\leq d(y,F(u)+k)+\delta\\ |\langle y^\star+z^\star,w+k-y\rangle-d(y,F(u)+k)|<\delta\end{array}\right\}\right\}$$

then there exists a neighborhood $\mathcal{U}_1\times\mathcal{V}_1\times\mathcal{W}_1$ of $(\bar{x},\bar{k},\bar{y})$ such that

$$md((x,k),\mathbb{S}_{\mathcal{E}_{(F,G)}}(y))\leq\varphi_{\mathcal{E}}((x,k),y)\quad\textit{for all}\quad(x,k,y)\in\mathcal{U}_1\times\mathcal{V}_1\times\mathcal{W}_1.$$

This theorem implies the following result:

Theorem 15. *Let X,Y be Asplund spaces, and let $F,G: X\rightrightarrows Y$ be closed multifunctions, and let $(\bar{x},\bar{k},\bar{y})\in X\times Y\times Y$ be such that $\bar{y}\in F(\bar{x})+\bar{k},\bar{k}\in G(\bar{x})$. Let $m>0$. If the sets $\{C_1,C_2\}$ satisfy the hypothesis $(\mathcal{H})$ at $(\bar{x},\bar{k},\bar{y}-\bar{k})$ and*

$$m<\liminf_{(x_1,w)\xrightarrow{F}(\bar{x},\bar{y}-\bar{k}),(x_2,z)\xrightarrow{G}(\bar{x},\bar{k}),\delta\downarrow 0}\left\{\|x^\star\|:\begin{array}{l}x^\star\in\hat{D}^\star F(x_1,w)(y^\star+\delta B_{Y^\star})+u^\star\\ u^\star\in\hat{D}^\star G(x_2,z)(y^\star),\|y^\star\|=1,\end{array}\right\} \tag{35}$$

where the notations $(x_1,w)\xrightarrow{F}(\bar{x},\bar{y}-\bar{k})$, $(x_2,z)\xrightarrow{G}(\bar{x},\bar{k})$ mean that

$$(x_1,w)\to(\bar{x},\bar{y}-\bar{k}),(x_2,z)\to(\bar{x},\bar{k})\text{ and }(x_1,w)\in\text{gph } F,(x_2,z)\in\text{gph } G,$$

then there exists a neighborhood $\mathcal{U}_1\times\mathcal{V}_1\times\mathcal{W}_1$ of $(\bar{x},\bar{k},\bar{y})$ such that

$$md((x,k),\mathbb{S}_{\mathcal{E}_{(F,G)}}(y))\leq\varphi_{\mathcal{E}}((x,k),y)\quad\textit{for all}\quad(x,k,y)\in U_1\times V_1\times W_1.$$

The next result gives a point-based condition for metric regularity of the epigraphical multifunction.

Theorem 16. *Let X,Y be Asplund spaces, and let $F,G: X\rightrightarrows Y$ be closed multifunctions, and let $(\bar{x},\bar{k},\bar{y})\in X\times Y\times Y$ be such that $\bar{y}\in F(\bar{x})+\bar{k},\bar{k}\in G(\bar{x})$. Suppose that*
(i) F or G is PSNC at $(\bar{x},\bar{y}-\bar{k})$ and $(\bar{x},\bar{k})$, respectively;
(ii) $D^\star F(\bar{x},\bar{y}-\bar{k})(0)\cap -D^\star G(\bar{x},\bar{k})(0)=\{0\}$;

(iii) for any $u_n^\star \in \hat{D}^\star F(x_n, y_n - k_n)(y_n^\star + (1/n)B_{Y^\star}), v_n^\star \in \hat{D}^\star G(x_n, k_n)(y_n^\star)$ *such that*

$$\|u_n^\star + v_n^\star\| \to 0, y_n^\star \xrightarrow{w^\star} 0 \text{ it follows that } y_n^\star \to 0;$$

Under the condition that

$$(\star) \qquad Ker\big(D^\star F(\bar{x}, \bar{y} - \bar{k}) + D^\star G(\bar{x}, \bar{k})\big) = \{0\},$$

the multifunction $\mathcal{E}_{(F,G)}$ *is metrically regular around* $(\bar{x}, \bar{k}, \bar{y})$.

Proof. We prove the result by contradiction. Suppose that $\mathcal{E}_{(F,G)}$ fails to be metrically regular around $(\bar{x}, \bar{k}, \bar{y})$. Then, by Theorem 15, there exist sequences

$$(x_n, y_n - k_n) \xrightarrow{F} (\bar{x}, \bar{y} - \bar{k}), (x_n, k_n) \xrightarrow{G} (\bar{x}, \bar{k}), (x_n^\star, u_n^\star, y_n^\star, z_n^\star) \in X^\star \times X^\star \times Y^\star \times Y^\star,$$

with

$$\begin{aligned} &x_n^\star \in \hat{D}^\star F(x_n, y_n - k_n)(y_n^\star + z_n^\star) + u_n^\star, \\ &u_n^\star \in \hat{D}^\star G(x_n, k_n)(y_n^\star), \\ &y_n^\star \in S_{Y^\star}, z_n^\star \in (1/n)B_{Y^\star}, \end{aligned}$$

and

$$x_n^\star \to 0.$$

Then there is $v_n^\star \in \hat{D}^\star F(x_n, y_n - k_n)(y_n^\star + z_n^\star)$ such that $x_n^\star = u_n^\star + v_n^\star$.
Since Y is an Asplund space, we can assume that $y_n^\star \xrightarrow{w^\star} y^\star \in Y^\star$.

We consider the following cases:

Case 1. The sequences $\{u_n^\star\}$, $\{v_n^\star\}$ are unbounded. We can assume that

$$\|u_n^\star\| \to \infty, \|v_n^\star\| \to \infty,$$

and

$$\frac{u_n^\star}{\|u_n^\star\|} \xrightarrow{w^\star} u^\star, \frac{v_n^\star}{\|v_n^\star\|} \xrightarrow{w^\star} v^\star.$$

Then,

$$y_n^\star/\|u_n^\star\| \to 0 \text{ and } (y_n^\star + z_n^\star)/\|v_n^\star\| \to 0.$$

Consequently,

$$u^\star \in D^\star G(\bar{x}, \bar{k})(0), v^\star \in D^\star F(\bar{x}, \bar{y} - \bar{k})(0).$$

On the other hand,

$$u^\star + v^\star = 0, (\text{since} \|u_n^\star + v_n^\star\| \to 0).$$

It follows that

$$v^\star \in D^\star F(\bar{x}, \bar{y} - \bar{k})(0) \cap -D^\star G(\bar{x}, \bar{k})(0).$$

Therefore, by (ii), we have that $u^\star = v^\star = 0$.
So,

$$\frac{u_n^\star}{\|u_n^\star\|} \to 0, \quad \text{or} \quad \frac{v_n^\star}{\|v_n^\star\|} \to 0, \text{ (by PSNC property of F or G)}.$$

This contradicts the fact $\frac{u_n^\star}{\|u_n^\star\|}$ and $\frac{v_n^\star}{\|v_n^\star\|}$ belong to the unit sphere $S_{Y^\star}$ of $Y^\star$.

Case 2. The sequences $\{u_n^\star\}$, $\{v_n^\star\}$ are bounded. Assume that $u_n^\star \xrightarrow{w^\star} u^\star$, $v_n^\star \xrightarrow{w^\star} v^\star$.
It follows that

$$u^\star \in D^\star G(\bar{x}, \bar{k})(y^\star), v^\star \in D^\star F(\bar{x}, \bar{y} - \bar{k})(y^\star).$$

Moreover,

$$u^\star + v^\star = 0.$$

So,

$$0 \in [D^\star G(\bar{x},\bar{k})(y^\star) + D^\star F(\bar{x},\bar{y}-\bar{k})(y^\star)] = [D^\star G(\bar{x},\bar{k}) + D^\star F(\bar{x},\bar{y}-\bar{k})](y^\star),$$

which means that

$$y^\star \in Ker[D^\star G(\bar{x},\bar{k}) + D^\star F(\bar{x},\bar{y}-\bar{k})].$$

By $(\star)$, one has that $y^\star = 0$.
Now, by assumption, one gets $y_n^\star \to 0$ which contradicts to $\|y_n^\star\| = 1$. □

Remark 17. *If X, Y are finite dimensional spaces, then conditions (i), (iii) hold true automatically, while condition (ii) holds if F or G is Lipschitz-like at $(\bar{x},\bar{y}-\bar{k})$ or $(\bar{x},\bar{k})$, respectively.*

4. Applications to variational systems

In this section, we use the above results to study some properties of variational systems of the form

$$0 \in F(x) + G(x,p), \tag{36}$$

where X is a complete metric space, Y is a Banach space, P is a topological space considered as a parameter space, $F : X \rightrightarrows Y, G : X \times P \rightrightarrows Y$ are given multifunctions. The solution set of (36) is defined by the notation below should also be defined

$$\mathbf{S}_{(F+G)}(p) := \{x \in X : 0 \in F(x) + G(x,p)\}, \tag{37}$$

and we denote

$$\mathbb{S}_{(F+G)}(y,p) := \{x \in X : y \in F(x) + G(x,p)\}.$$

For every $(y,p) \in Y \times P$,

$$\mathbb{S}_{\mathcal{E}_{(F,G)}}(y,p) = \{(x,k) \in X \times Y : y \in F(x) + k, k \in G(x,p)\},$$

and, for every $p \in P$,

$$\mathbf{S}_{\mathcal{E}_{(F,G)}}(p) = \{(x,k) \in X \times Y : 0 \in F(x) + k, k \in G(x,p)\}.$$

We say that the multifunction $\mathbf{S}_{(F+G)}$ is Robinson metrically regular (see [30], [31]) around $(\bar{x},\bar{p})$ with modulus τ, if there exist neighborhoods $\mathcal{U}, \mathcal{V}$ of $\bar{x}, \bar{p}$, respectively, such that

$$d(x, \mathbf{S}_{(F+G)}(p)) \le \tau d(0, F(x) + G(x,p)), \text{ for all } (x,p) \in \mathcal{U} \times \mathcal{V}.$$

We also recall that the multifunction $G : X \times P \rightrightarrows Y$ is said to be Lipschitz-like around $(\bar{x},\bar{p},\bar{y})$ with $\bar{y} \in G(\bar{x},\bar{p})$ with respect to x, uniformly in p with constant $\kappa > 0$ if there is a neighborhood $\mathcal{U} \times \mathcal{V} \times \mathcal{W}$ of $(\bar{x},\bar{p},\bar{y})$ such that

$$G(x,p) \cap \mathcal{W} \subset G(u,p) + \kappa d(x,u)\bar{B}_Y \text{ for all } x, u \in \mathcal{U}, \text{and for all } p \in \mathcal{V}.$$

The lower semicontinuous envelope $(x,p,k,y) \mapsto \varphi_{p,\mathcal{E}}((x,k),y)$ of the distance function $d(y, \mathcal{E}_{(F,G)}((x,p),k))$ is defined by, for each $(x,p,k,y) \in X \times P \times Y \times Y$

$$\varphi_{p,\mathcal{E}}((x,k),y) := \liminf_{(u,v,w)\to(x,k,y)} d(w, \mathcal{E}_{(F,G)}((u,p),v))$$

$$= \begin{cases} \liminf\limits_{u\to x} d(y, F(u) + k) & \text{if } k \in G(x,p) \\ +\infty & \text{otherwise.} \end{cases}$$

Lemma 18. *Let X be a complete metric space and Y be a Banach space and let P be a topological space. Suppose that the set-valued mappings $F : X \rightrightarrows Y$, $G : X \times P \rightrightarrows Y$ satisfy the following conditions for some $(\bar{x}, \bar{k}, \bar{p}) \in X \times Y \times P$:*
(a) $(\bar{x}, \bar{k}) \in \mathbf{S}_{\mathcal{E}_{(F,G)}}(\bar{p})$;
(b) the set-valued mapping $p \rightrightarrows G(\bar{x}, p)$ is lower semicontinuous at $\bar{p}$;
(c) the set-valued mapping F is a closed multifunction, and for any p near $\bar{p}$, the set-valued mapping $x \rightrightarrows G(x, p)$ is a closed multifunction.
Then
(i) for ever p near $\bar{p}$, the epigraphical multifunction $\mathcal{E}_{(F,G)}$ has closed graph, and, $\mathcal{E}_{(F,G)}((\bar{x}, \cdot), \bar{k})$ is lower semicontinuous at $\bar{p}$;
(ii) the function $p \mapsto \varphi_{p,\mathcal{E}}((\bar{x}, \bar{k}), 0)$ is upper semicontinuous at $\bar{p}$;
(iii) for each $(y, p) \in Y \times P$;

$$\{(x, k) \in X \times Y : \varphi_{p,\mathcal{E}}((x, k), y) = 0\} = \mathbb{S}_{\mathcal{E}_{(F,G)}}(y, p).$$

Proof. We only note that if the multifunction $p \rightrightarrows G(\bar{x}, p)$ is lower semicontinuous at $\bar{p}$, then so is the mapping $\mathcal{E}_{(F,G)}((\bar{x}, \cdot), \bar{k})$.

By using the strong slope of the lower semicontinuous envelope $\varphi_{p,\mathcal{E}}$, one has the following result.

Theorem 19. *Let X be a complete metric space, Y be a Banach space and let P be a topological space. Suppose that the set-valued mappings $F : X \rightrightarrows Y, G : X \times P \rightrightarrows Y$ satisfy conditions $(a), (b), (c)$ from Lemma 18 around $(\bar{x}, \bar{k}, \bar{p}) \in X \times Y \times P$. If there exist a neighborhood $\mathcal{T}_1 \times \mathcal{U}_1 \times \mathcal{V}_1 \times \mathcal{W}_1$ of $(\bar{x}, \bar{p}, \bar{k}, 0)$ and reals $m, \gamma > 0$ such that $|\nabla \varphi_{p,\mathcal{E}}((\cdot, \cdot), y)|(x, k) \geq m$ for all $(x, p, k, y) \in \mathcal{T}_1 \times \mathcal{U}_1 \times \mathcal{V}_1 \times \mathcal{W}_1$ with $\varphi_{p,\mathcal{E}}((x, k), y) \in (0, \gamma)$, then there exists a neighborhood $\mathcal{T} \times \mathcal{U} \times \mathcal{V} \times \mathcal{W}$ of $(\bar{x}, \bar{p}, \bar{k}, 0)$ such that*

$$md((x, k), \mathbb{S}_{\mathcal{E}_{(F,G)}}(y, p)) \leq \varphi_{p,\mathcal{E}}((x, k), y),$$

for all $(x, p, k, y) \in \mathcal{T} \times \mathcal{U} \times \mathcal{V} \times \mathcal{W}$.

Proof. Applying Theorem 1 and Lemma 18 for the mapping $\mathcal{E}_{(F,G)}(\cdot, \cdot)$, one obtains the proof. □

Proposition 20. *Let X be a complete metric space and Y be a Banach space and let P be a topological space. Suppose that the set-valued mappings $F : X \rightrightarrows Y, G : X \times P \rightrightarrows Y$ satisfy conditions $(a), (b), (c)$ from Lemma 18 around $(\bar{x}, \bar{k}, \bar{p}) \in X \times Y \times P$. If there exist a neighborhood $\mathcal{T} \times \mathcal{U} \times \mathcal{V} \times \mathcal{W} \subset X \times P \times Y \times Y$ of $(\bar{x}, \bar{p}, \bar{k}, 0)$ and $m > 0$ such that*

$$md((x, k), \mathbb{S}_{\mathcal{E}_{(F,G)}}(y, p)) \leq \varphi_{p,\mathcal{E}}((x, k), y) \quad \text{for all} \quad (x, p, k, y) \in \mathcal{T} \times \mathcal{U} \times \mathcal{V} \times \mathcal{W}$$

then there exists $\theta > 0$ such that

$$md(x, \mathbb{S}_{(F+G)}(y, p)) \leq d(y, F(x) + G(x, p) \cap B(\bar{k}, \theta)) \quad \text{for all} \quad (x, p, y) \in \mathcal{T} \times \mathcal{U} \times \mathcal{W}.$$

Therefore,

$$md(x, \mathbf{S}_{(F+G)}(p)) \leq d(0, F(x) + G(x, p) \cap B(\bar{k}, \theta)) \quad \text{for all} \quad (x, p) \in \mathcal{T} \times \mathcal{U}.$$

Proof. By the hypothesis, there exist a neighborhood $\mathcal{T} \times \mathcal{U} \times \mathcal{V} \times \mathcal{W} \subset X \times P \times Y \times Y$ of $(\bar{x}, \bar{p}, \bar{k}, 0)$ and $m > 0$ such that for every $(x, p, k, y) \in \mathcal{T} \times \mathcal{U} \times \mathcal{V} \times \mathcal{W}$, it holds

$$md((x, k), \mathbb{S}_{\mathcal{E}_{(F,G)}}(y, p)) \leq \varphi_{p,\mathcal{E}}((x, k), y).$$

Here, we can assume $\mathcal{V} = B(\bar{k}, \theta)$, with certain positive θ. Then, for every small $\varepsilon > 0$ and for every $(x, p, k, y) \in \mathcal{T} \times \mathcal{U} \times [B(\bar{k}, \theta) \cap G(x, p)] \times \mathcal{W}$, there is $(u, z) \in \mathbb{S}_{\mathcal{E}_{(F,G)}}(y, p)$, i.e., $y \in F(u) + z, z \in G(u, p)$ such that

$$md(u, x) \leq m \max\{d(u, x), \|z - k\|\} < (1 + \varepsilon)d(y, F(x) + k).$$

Noticing that $u \in (F + G)^{-1}(y)$, we obtain that

$$md(x, (F + G)^{-1}(y)) < (1 + \varepsilon)d(y, F(x) + k).$$

Thus,

$$md(x, (F + G)^{-1}(y)) \leq (1 + \varepsilon)d(y, F(x) + G(x, p) \cap B(\bar{k}, \theta)),$$

or,

$$md(x, \mathbb{S}_{(F+G)}(y, p)) \leq (1 + \varepsilon)d(y, F(x) + G(x, p) \cap B(\bar{k}, \theta)).$$

Since this inequality does not depend on arbitrarily small $\varepsilon > 0$, we obtain that

$$md(x, \mathbb{S}_{(F+G)}(y, p)) \leq d(y, F(x) + G(x, p) \cap B(\bar{k}, \theta))$$

for all $(x, p, y) \in \mathcal{T} \times \mathcal{U} \times \mathcal{W}$.
Taking $\bar{y} = 0$ and $y = \bar{y}$, we obtain the second conclusion of the Theorem, and the proof is complete. □

In the sequel we use for the parametrized case, the concept of locally sum-stability which was considered in the previous section.

Definition 21. *Let $F : X \rightrightarrows Y, G : X \times P \rightrightarrows Y$ be two multifunctions and $(\bar{x}, \bar{p}, \bar{y}, \bar{z}) \in X \times P \times Y \times Y$ be such that $\bar{y} \in F(\bar{x}), \bar{z} \in G(\bar{x}, \bar{p})$. We say that the pair (F, G) is locally sum-stable around $(\bar{x}, \bar{p}, \bar{y}, \bar{z})$ if for every $\varepsilon > 0$ there exists $\delta > 0$ and a neighborhood W of $\bar{p}$ such that for every $(x, p) \in B(\bar{x}, \delta) \times W$ and every $w \in (F + G)(x) \cap B(\bar{y} + \bar{z}, \delta)$, there are $y \in F(x) \cap B(\bar{y}, \varepsilon)$ and $z \in G(x) \cap B(\bar{z}, \varepsilon)$ such that $w = y + z$.*

A following simple case which ensures the locally sum-stability of the pair (F, G) is analogous to Proposition 6 .

Proposition 22. *Let $F : X \rightrightarrows Y, G : X \times P \rightrightarrows Y$ be two multifunctions and $(\bar{x}, \bar{p}, \bar{y}, \bar{z}) \in X \times P \times Y \times Y$ such that $\bar{y} \in F(\bar{x}), \bar{z} \in G(\bar{x}, \bar{p})$. If $G(\bar{x}, \bar{p}) = \{\bar{z}\}$ and G is upper semicontinuous at $(\bar{x}, \bar{p})$, then the pair (F, G) is locally sum-stable around $(\bar{x}, \bar{p}, \bar{y}, \bar{z})$.*

Proposition 23. *Let X be a complete metric space, Y be a Banach space and let P be a topological pace. Suppose that the set-valued mappings $F : X \rightrightarrows Y, G : X \times P \rightrightarrows Y$ satisfy conditions (a), (b), (c) from Lemma 18 around $(\bar{x}, \bar{k}, \bar{p}) \in X \times Y \times P$. If there exist a neighborhood $\mathcal{T} \times \mathcal{U}$ of $(\bar{x}, \bar{p})$ and $\theta, \tau > 0$ such that*

$$d(x, \mathbf{S}_{(F+G)}(p)) \leq \tau d(0, F(x) + G(x, p) \cap B(\bar{k}, \theta)) \quad \text{for all} \quad (x, p) \in \mathcal{T} \times \mathcal{U}, \tag{38}$$

and (F, G) is locally sum-stable around $(\bar{x}, \bar{p}, -\bar{k}, \bar{k})$, then $\mathbf{S}_{(\mathbf{F}+\mathbf{G})}$ is Robinson metrically regular around $(\bar{x}, \bar{p})$ with modulus τ.

The conclusion remains true if the assumption of local sum stability around $(\bar{x}, \bar{p}, -\bar{k}, \bar{k})$ is replaced by the following one: $G(\bar{x}, \bar{p}) = \{\bar{z}\}$ and G is upper semicontinuous at $(\bar{x}, \bar{p})$.

Proof. The proof of this proposition is very similar to that of Proposition 7. Here, we sketch the proof. Suppose that (38) holds for every $(x,p) \in \mathcal{T} \times \mathcal{U}$. Here, we can assume that $\mathcal{T} = B(\bar{x}, \delta)$, with some positive $\delta > 0$.

Since (F,G) is locally sum-stable around $(\bar{x}, \bar{p}, -\bar{k}, \bar{k})$, there exists $\delta > 0$ such that for every $(x,p) \in B(\bar{x},\delta) \times \mathcal{U}$ and every $w \in (F+G)(x) \cap B(0,\delta)$, there are $y \in F(x) \cap B(-\bar{k},\theta)$ and $z \in G(x) \cap B(\bar{k},\theta)$ such that $w = y + z$.

Fix $(x,p) \in B(\bar{x},\delta) \times \mathcal{U}$. We consider two following cases:

Case 1. $d(0, F(x) + G(x,p)) < \delta/2$.

Fix $\gamma > 0$, small enough so that $d(0, F(x) + G(x,p)) + \gamma < \delta/2$, and take $t \in F(x) + G(x,p)$ such that

$$\|t\| < d(0, F(x) + G(x,p)) + \gamma.$$

Hence we have $\|t\| < \delta/2$, i.e., $t \in B(0,\delta/2) \subset B(0,\delta)$. It follows that $t \in [F(x) + G(x,p)] \cap B(0,\delta)$.
Therefore, there are $y \in F(x) \cap B(-\bar{k},\theta)$ and $z \in G(x,p) \cap B(\bar{k},\theta)$ such that $t = y + z$.
Consequently,

$$t \in F(x) \cap B(-\bar{k},\theta) + G(x,p) \cap B(\bar{k},\theta) \subset F(x) + G(x,p) \cap B(\bar{k},\theta).$$

It follows that

$$d(0, F(x) + G(x,p) \cap B(\bar{k},\theta)) \leq \|t\|.$$

This yields

$$d(0, F(x) + G(x,p) \cap B(\bar{k},\theta)) < d(0, F(x) + G(x,p)) + \gamma,$$

and therefore, as $\gamma > 0$ is arbitrarily small, we derive that

$$d(0, F(x) + G(x,p) \cap B(\bar{k},\theta)) \leq d(0, F(x) + G(x,p)).$$

By (38), one derives

$$d(x, \mathbf{S}_{(F+G)}(p)) \leq \tau d(0, F(x) + G(x,p)), \quad \text{for all} \quad (x,p) \in B(\bar{x},\delta) \times \mathcal{U}.$$

Case 2. $d(0, F(x) + G(x,p)) \geq \delta/2$.

According to condition (c), the multifunction $p \rightrightarrows G(\bar{x}, \cdot)$ is lower semicontinuous at $\bar{p}$. It follows that the distance function $d(0, F(\bar{x}) + G(\bar{x}, \cdot))$ is upper semicontinuous at $\bar{p}$, and thus, there exists a neighborhood W of $\bar{p}$ such that

$$d(0, F(\bar{x}) + G(\bar{x},p) \leq \delta/4, \text{ for all } p \in W.$$

Shrinking W smaller if necessary, we can assume that $W \subset \mathcal{U}$. Choosing $0 < \delta_1 < \min\{\delta, \tau\delta/4\}$. For every $(x,p) \in B(\bar{x},\delta_1) \times W$, and for every small $\varepsilon > 0$, there exists $u \in \mathbf{S}_{(F+G)}(p)$ such that

$$d(\bar{x}, u) \leq (1+\varepsilon)\tau d(0, F(\bar{x}) + G(\bar{x},p)).$$

So,

$$\begin{aligned} d(x,u) \leq d(x,\bar{x}) + d(\bar{x},u) & \\ & < \delta_1 + \tau(1+\varepsilon) d(0, F(\bar{x}) + G(\bar{x},p)) \\ & < \tau\delta/4 + \tau(1+\varepsilon)\delta/4 \\ & \leq \tau/2 d(0, F(x) + G(x,p)) \\ & + \tau/2(1+\varepsilon) d(0, F(x) + G(x,p). \end{aligned}$$

Taking the limit as $\varepsilon > 0$ goes to 0, it follows that

$$d(x, \mathbf{S}_{(F+G)}(p)) \leq \tau d(0, F(x) + G(x,p)),$$

establishing the proof.

The following theorem establishes the Lipschitz property for the solution mapping $\mathbb{S}_{\mathcal{E}_{(F,G)}}$.

Theorem 24. *Let X be a complete metric space, Y be a Banach space, P be a topological space. Suppose that $F : X \rightrightarrows Y$ and $G : X \times P \rightrightarrows Y$ are multifunctions satisfying conditions (a), (b), (c) in Lemma 18.*

If F is metrically regular around $(\bar{x}, -\bar{k})$ with modulus $\tau > 0$ and G is Lipschitz-like around $(\bar{x}, \bar{p}, \bar{k})$ with respect to x, uniformly in p with modulus $\lambda > 0$ such that $\tau\lambda < 1$, then $\mathcal{E}_{(F,G)}$ is metrically regular around $(\bar{x}, \bar{p}, \bar{k}, 0)$ with respect to (x, k), uniformly in p, with modulus $(\tau^{-1} - \lambda)^{-1}$.

Moreover, assume in addition that P is a metric space, if G is Lipschitz-like around $(\bar{x}, \bar{p}, \bar{k})$ with respect to p, uniformly in x with modulus $\gamma > 0$ then $\mathbb{S}_{\mathcal{E}_{(F,G)}}$ is Lipschitz-like around $((0, \bar{p}), (\bar{x}, \bar{k}))$ with modulus $L = \gamma + (\gamma + 1)(\tau^{-1} - \lambda)^{-1})$. Especially, $\mathbf{S}_{\mathcal{E}_{(F,G)}}$ is Lipschitz-like around $((0, \bar{p}), (\bar{x}, \bar{k}))$ with modulus $\gamma(1 + (\tau^{-1} - \lambda)^{-1})$.

Proof. The first part is the parametrized version of Theorem 8. Its proof is completely similar to the one of Theorem 8, and is omitted. For the second part, as $\mathcal{E}_{(F,G)}$ is metrically regular around $(\bar{x}, \bar{p}, \bar{k}, 0)$ with respect to (x, k), uniformly in p, with modulus $(\tau^{-1} - \lambda)^{-1}$, there exists $\delta_1 > 0$ such that

$$d((x, k), \mathbb{S}_{\mathcal{E}_{(F,G)}}(y, p)) \leq (\tau^{-1} - \lambda)^{-1} \varphi_{p,\mathcal{E}}((x, k), y), \tag{39}$$

for all $(x, p, k, y) \in B((\bar{x}, \bar{p}, \bar{k}, 0), \delta_1)$.
Now, if G is Lipschitz-like around $(\bar{x}, \bar{p}, \bar{k})$ with respect to p, uniformly in x with modulus $\gamma > 0$ then there is $\delta_2 > 0$ such that

$$G(x, p) \cap B(\bar{k}, \delta_2) \subset G(x, p') + \gamma d(p, p')\bar{B}_Y, \tag{40}$$

for all $p, p' \in B(\bar{p}, \delta_2)$, for all $x \in B(\bar{x}, \delta_2)$.
Set

$$\alpha := \min\{\delta_1/(\gamma + 1), \delta_2\}.$$

Fix $(y, p), (y', p') \in B(0, \alpha) \times B(\bar{p}, \alpha)$. Take $(x, k) \in \mathbb{S}_{\mathcal{E}_{(F,G)}}(y, p)) \cap [B(\bar{x}, \alpha) \times B(\bar{k}, \alpha)]$.
Since $(x, k) \in \mathbb{S}_{\mathcal{E}_{(F,G)}}(y, p)) \cap [B(\bar{x}, \alpha) \times B(\bar{k}, \alpha)]$, then

$$y \in F(x) + k, k \in G(x, p) \text{ and } (x, k) \in B(\bar{x}, \alpha) \times B(\bar{k}, \alpha).$$

Along with (40), we can find that $k' \in G(x, p')$ such that

$$\|k - k'\| \leq \gamma d(p, p') < \gamma\alpha,$$

which follows that $k' \in B(\bar{k}, \delta_1)$. Therefore, by (39), one has

$$\begin{aligned} d((x, k'), \mathbb{S}_{\mathcal{E}_{(F,G)}}(y', p')) &\leq (\tau^{-1} - \lambda)^{-1} \varphi_{p',\mathcal{E}}((x, k'), y'), \\ &\leq (\tau^{-1} - \lambda)^{-1} d(y', F(x) + k')), \end{aligned}$$

Hence, by noting that $y \in F(x) + k$, one deduces that

(41)

$$\begin{aligned} d((x,k),\mathbb{S}_{\mathcal{E}_{(F,G)}}(y',p')) &\leq \|k-k'\| + d((x,k'),\mathbb{S}_{\mathcal{E}_{(F,G)}}(y',p')) \\ &\leq \gamma d(p,p') + (\tau^{-1}-\lambda)^{-1} d(y',F(x)+k')), \\ &\leq \gamma d(p,p') + (\tau^{-1}-\lambda)^{-1}(\|y-y'\|+\|k-k'\|) \\ &\leq \gamma(1+(\tau^{-1}-\lambda)^{-1})d(p,p') + (\tau^{-1}-\lambda)^{-1}\|y-y'\| \end{aligned}$$

and so

$$\begin{aligned} &\mathbb{S}_{\mathcal{E}_{(F,G)}}(y,p)) \cap [B(\bar{x},\alpha)\times B(\bar{k},\alpha)] \\ &\subseteq \mathbb{S}_{\mathcal{E}_{(F,G)}}(y',p') + Ld((y',p'),(y,p))\bar{B}_X \times \bar{B}_Y, \end{aligned}$$

where, $L=\gamma+(\gamma+1)(\tau^{-1}-\lambda)^{-1}$, and by taking $y=y'=0$ in relation (41), one also derives that $\mathbf{S}_{\mathcal{E}_{(F,G)}}$ is Lipschitz-like around $((0,\bar{p}),(\bar{x},\bar{k}))$ with modulus $\gamma(1+(\tau^{-1}-\lambda)^{-1})$. The proof is complete. □

If we add the assumption that (F,G) is locally sum-stable, we obtain the Lipschitz property of $\mathbf{S}_{(F+G)}$.

Theorem 25. *Let X be a complete metric space and Y be a Banach space, P be a metric space. Suppose that $F : X \rightrightarrows Y$ and $G : X \times P \rightrightarrows Y$ satisfy conditions (a), (b), (c) in Lemma 18. Moreover, assume that*
(i) (F,G) is locally sum-stable around $(\bar{x},\bar{p},-\bar{k},\bar{k})$;
(ii) F is metrically regular around $(\bar{x},-\bar{k})$ with modulus $\tau>0$;
(iii) G is Lipschitz-like around $(\bar{x},\bar{p},\bar{k})$ with respect to x, uniformly in p with modulus $\lambda>0$ such that $\tau\lambda<1$;
(iv) G is Lipschitz-like around $(\bar{x},\bar{p},\bar{k})$ with respect to p, uniformly in x with modulus $\gamma>0$. Then $\mathbf{S}_{(F+G)}$ is Robinson metrically regular around $(\bar{x},\bar{p})$ with modulus $(\tau^{-1}-\lambda)^{-1}$. Moreover, $\mathbf{S}_{(F+G)}$ is Lipschitz-like around $(\bar{x},\bar{p})$ with constant $\gamma(\tau^{-1}-\lambda)^{-1}$.

Proof. Applying Proposition 24, Proposition 23 and Proposition 20, respectively, we obtain that $S_{(F+G)}$ is Robinson metrically regular around $(\bar{x},\bar{p})$ with modulus $(\tau^{-1}-\lambda)^{-1}$. Thus, there exists $\delta_1>0$ such that

$$d(x,\mathbf{S}_{(F+G)}(p)) \leq (\tau^{-1}-\lambda)^{-1} d(0,F(x)+G(x,p)), \text{ for all } (x,p)\in B((\bar{x},\bar{p}),\delta_1).$$

On the other hand, since G is Lipschitz-like around $(\bar{x},\bar{p},\bar{k})$ with respect to p, uniformly in x with modulus $\gamma>0$, we can find $\delta_2>0$ such that

$$G(x,p)\cap B(\bar{k},\delta_2) \subset G(x,p') + \gamma d(p,p')\bar{B}_Y,$$

for all $p,p'\in B(\bar{p},\delta_2)$, for all $x\in B(\bar{x},\delta_2)$.
Moreover, since the pair (F,G) is locally sum-stable around $(\bar{x},\bar{p},-\bar{k},\bar{k})$, there is $\delta_3>0$ such that for every $(x,p)\in B(\bar{x},\delta_3)\times B(\bar{p},\delta_3)$ and every $w\in[F(x)+G(x,p)]\cap B(0,\delta_3)$, there are $y\in F(x)\cap B(-\bar{k},\delta_2)$, $z\in G(x,p)\cap B(\bar{k},\delta_2)$ such that $w=y+z$. Set

$$\alpha := \min\{\delta_1,\delta_2,\delta_3\}.$$

Take $p,p'\in B(\bar{p},\alpha)$, and $x\in\mathbf{S}_{(F+G)}(p)\cap B(\bar{x},\alpha)$, i.e., $0\in F(x)+G(x,p)$ and $x\in B(\bar{x},\alpha)$. Moreover, we observe that for every $w\in[F(x)+G(x,p)]\cap B(0,\alpha)$,

$$w\in F(x)\cap B(-\bar{k},\delta_2)) + G(x,p)\cap B(\bar{k},\delta_2) \subseteq F(x)+G(x,p') + \gamma d(p,p')\bar{B}_Y.$$

Thus,

$$[F(x)+G(x,p)]\cap B(0,\alpha) \subseteq F(x)+G(x,p') + \gamma d(p,p')\bar{B}_Y.$$

Since $0 \in F(x) + G(x,p)$, and also $0 \in [F(x) + G(x,p)] \cap B(0,\alpha)$, thus
$$0 \in F(x) + G(x,p') + \gamma d(p,p')\bar{B}_Y.$$
It follows that there is $w \in F(x) + G(x,p')$ such that
$$\|w\| \le \gamma d(p,p').$$
Therefore,
$$d(x, \mathbf{S}_{(F+G)}(p')) \le (\tau^{-1} - \lambda)^{-1} d(0, F(x) + G(x,p')) \le (\tau^{-1} - \lambda)^{-1}\|w\| \le \gamma(\tau^{-1} - \lambda)^{-1} d(p,p').$$
So,
$$\mathbf{S}_{(F+G)}(p) \cap B(\bar{x},\alpha) \subseteq \mathbf{S}_{(F+G)}(p') + \gamma(\tau^{-1} - \lambda)^{-1} d(p,p')\bar{B}_X,$$
establishing the proof. □

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