

EXISTENCE AND STABILITY RESULTS BASED ON ASYMPTOTIC ANALYSIS FOR SEMIDEFINITE LINEAR COMPLEMENTARITY PROBLEMS

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ABSTRACT. This work is devoted to the study of existence and stability results of semidefinite linear complementarity problems (SDLCP). Our approach consists of approximating the variational inequality formulation of the SDLCP by a sequence of suitable chosen variational inequalities. This provides particular estimates for the asymptotic cone of the solution set of the SDLCP. We thus obtain new coercive and noncoercive existence results, as well as new properties related to the continuity of the solution sets of the SDLCP (such as outer/upper semicontinuity, Lipschitz-type continuity, among others). Moreover, this asymptotic approach leads to a natural extension of the class of García linear transformations, formerly defined in the context of linear complementarity problems, to this SDLCP setting.

Key words: Semidefinite linear complementarity problems, asymptotic analysis, existence stability results.

1. INTRODUCTION

The *semidefinite linear complementarity problem* consists, for a given linear transformation $L : S^n \rightarrow S^n$ (in short $L \in \mathcal{L}(S^n)$) and a symmetric matrix $Q \in S^n$, in finding a matrix $\bar{X} \in S^n$ such that

$$\bar{X} \in S_+^n, \quad \bar{Y} = L(\bar{X}) + Q \in S_+^n \quad \text{and} \quad \langle \bar{Y}, \bar{X} \rangle = 0. \quad (\text{SDLCP})$$

As usual, the space of all real symmetric $n \times n$ matrices has been denoted by S^n , which is equipped with the inner product $\langle X, Y \rangle := \sum_{i,j=1}^n X_{ij}Y_{ij}$, and the cone of symmetric positive semidefinite matrices by S_+^n . Throughout this paper the dependence on the data will be explicitly stressed by (SDLCP) as $\text{SDLCP}(L, S_+^n, Q)$. So, its solution set will be denoted by $\mathcal{S}(L, S_+^n, Q)$ and its feasibility set will be denoted by $\text{Feas}(L, S_+^n, Q) := \{X \in S_+^n : L(X) + Q \in S_+^n\}$. Also, its strict feasibility set is defined as $\text{Feas}_s(L, S_+^n, Q) := \{X \in S_+^n : L(X) + Q \in S_{++}^n\}$.

The SDLCP formulation is able to cover a wide range of applications, including primal-dual semidefinite linear programs, control theory, linear and bilinear matrix inequalities, among others. This explains the important attention that SDLCPs have received in optimization theory during the last 20 years. For more details, we address the reader to [11] and the references therein.

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The SDLCP was introduced by Kojima-Shindoh-Hara in [19] but in a slightly different form, which is now called the geometric-SDLCP. It is well known that the linear complementarity problem (LCP) and the geometric-SDLCP can be reformulated as a SDLCP (see Example 5.3 below and [28], respectively). However, it is also well recognized that properties of LCP cannot be trivially generalized to the SDLCP context. This is mainly explained by two known facts: the cone S_+^n is nonpolyhedral and the matrix multiplication is not commutative.

On the other hand, (SDLCP) can be directly cast as a variational inequality (VI) of the form:

$$\text{find } \bar{X} \in S_+^n \text{ such that } \langle L(\bar{X}) + Q, X - \bar{X} \rangle \geq 0, \text{ for all } X \in S_+^n. \quad (\text{SDVIP})$$

Once again, the latter is denoted by $\text{SDVIP}(L, S_+^n, Q)$ in order to stress the dependence on the data. Thanks to this reformulation, the existence of solutions of the problem (SDLCP) could be analyzed via very well-known existence results for VIs. Nevertheless, most of these existence results are *coercive*, in the sense that they are based on the compactness of the feasible set of the studied VI. This fact is illustrate with the following seminal result:

Theorem 1.1 (Hartman-Stampacchia [17]). *Let $C \subseteq \mathbb{R}^N$ be a compact convex set and let $F : C \rightarrow \mathbb{R}^N$ be a continuous function. Then there exists $\bar{x} \in C$ such that*

$$\langle F(\bar{x}), x - \bar{x} \rangle \geq 0 \quad \text{for all } x \in C.$$

This compactness assumption is not satisfied by SDLCPs, where the set $C = S_+^n$ is clearly unbounded. This problematic is at the core of our work. To overcome it we adapt an asymptotic method developed in [7, 22] for studying LCPs, to the SDLCP context. This approach naturally gives the opportunity to introduce new classes in $\mathcal{L}(S^n)$, namely the classes of García's and #-linear transformations. Moreover, the aforementioned asymptotic approach also provides new stability results, in terms of the data, for the problem (SDLCP). These results are noncoercive in the sense they deal with new estimations for the asymptotic cone of the solution set of (SDLCP), when the latter is not necessary compact. Thus, the existence and stability aspects of SDLCP are systematically studied via the application of the asymptotic analysis of ad-hoc approximations of (SDVIP).

As far as we are aware, this paper is the first systematic study of existence and stability results for SDLCPs based on an asymptotic analysis. We thus extend to SDLCPs the coercive/noncoercive existence results given, within LCPs, by Gowda and Pang in [10] and by Flores and López in [7] for García's and q -pseudomonotone matrices, respectively. Then, our approach allows the recovery of some existence results, previously obtained via Degree Theory, for q -quasimonotone transformations; see Zhang [29]. Moreover, we extend to the SDLCP framework some stability results formerly proven in [4, 7, 21, 22] for linear, nonlinear, and polyhedral complementarity problems.

This paper is organized as follows. Section 2 is devoted to the preliminaries. It is split into two subsections: the first one recalls some basic results on matrix analysis, while the second summarizes some notions about set convergence and set-valued mappings. Section 3 provides a detailed exposition of the asymptotic method. Here we develop two schemes to approximate (SDLCP), one is oriented to establish existence results, while the other is oriented to obtain stability results. In Section 4, we introduce and study the class of García's and #-linear transformations. In this section we also deal with the class of Q -pseudomonotone

linear transformations. Regarding these classes, we perform the asymptotic analysis in order to obtain new bounds/formulas for the asymptotic cone of the solution set of (SDLCP). Finally, in Section 5, our main existence and stability results are stated and proved.

2. PRELIMINARIES

In this section, we describe our preliminary results. They are presented in two subsections: the first one contains notations and some well-known matrix results needed in the sequel, while the second recalls notions of set convergence and set-valued mappings.

2.1. Notation and basic results on matrix analysis. In this paper we shall use the following notation: vectors $x \in \mathbb{R}^n$ in the text are expressed as rows while in the matrix computations they are understood as columns; $I := \{1, \dots, n\}$ is an index set, $J \subseteq I$ is an index subset of I ; $\bar{J} := I \setminus J$ is its complementary set; $\text{supp}\{x\} := \{i \in I : x_i \neq 0\}$ is the support of x ; $x|_J$ is the vector whose indices are in $J \subseteq I$. Let us denote by $x * y$ the componentwise (Hadamard) product of vectors x and y , that is, $x * y = (x_i y_i) \in \mathbb{R}^n$ for all $x = (x_i), y = (y_i) \in \mathbb{R}^n$. Let A be a set in \mathbb{R}^n , $\text{co } A$ is its convex hull; $\text{int } A$ is its topological interior; $\text{ri } A$ is its relative interior, A^+ (resp. A^{s+}) is its positive (resp. strictly positive) polar cone; and A^∞ is its asymptotic cone.

The trace and the diagonal of a square matrix $X = (X_{ij}) \in \mathbb{R}^{n \times n}$ are defined by $\text{tr}(X) := \sum_{i=1}^n X_{ii}$ and $\text{diag}(X) := (X_{11}, X_{22}, \dots, X_{nn})^\top$, respectively. It is well-known that the set S^n of real $n \times n$ symmetric matrices is a finite dimensional real Hilbert space when it is equipped with the inner product $\langle X, Y \rangle = \text{tr}(XY)$. As usual, this product defines a (Frobenius) norm $\|X\|_F := \sqrt{\langle X, X \rangle} = \sqrt{\sum_{i=1}^n \lambda_i(X)^2}$, where $\lambda_i(X)$ stands for the i -th eigenvalue (arranged in nonincreasing order) of X . Thus, $\|X\|_F = \|\lambda(X)\|$ for all $X \in S^n$, where $\|\cdot\|$ denotes the Euclidian norm in \mathbb{R}^n and we have set $\lambda(X) := (\lambda_1(X), \dots, \lambda_n(X))^\top$. Regarding a mapping $L \in \mathcal{L}(S^n)$, $L^\top : S^n \rightarrow S^n$ denotes its transpose, which is defined by $\langle L^\top(X), Y \rangle = \langle X, L(Y) \rangle$ for all $X, Y \in S^n$. By $\lambda_{\max}(X)$ (resp. $\lambda_{\min}(X)$) we denote the largest (resp. smallest) eigenvalue of a squared matrix X . Also, 0_n and I_n denote the zero and the identity matrices respectively, of size n , but index n will be omitted if the size is clear from the context. Typically, throughout the paper D stands for a matrix in S_{++}^n .

Finally, we denote by $\mathbb{B}(0_n, 1)$ the closed unit ball with center 0_n within S^n and, for a vector $q \in \mathbb{R}^n$, we define $\text{Diag}(q)$ as the diagonal matrix of size n whose diagonal entries are given by the entries of q .

We end this subsection by recalling some matrix properties that we shall employ throughout this paper. Their proofs and more details can be found for instance in [15, 16, 20].

Proposition 2.1. *The following results hold:*

- (a): (Von Neumann-Theobald's inequality) For any $X, Y \in S_+^n$, it holds that $\langle X, Y \rangle \leq \lambda(X)^\top \lambda(Y)$, with equality if and only if X and Y are simultaneously diagonalizable (that is, there exists an orthogonal matrix U such that $X = U \text{Diag}(\lambda(X)) U^\top$ and $Y = U \text{Diag}(\lambda(Y)) U^\top$);
- (b): (Fejer's theorem) For any $X \in S^n$, it holds that $\langle X, Y \rangle \geq 0$ for all $Y \in S_+^n$ if and only if $X \in S_+^n$. Moreover, $\langle X, Y \rangle > 0$ for all $Y \in S_+^n \setminus \{0\}$ if and only if $X \in S_{++}^n$, where S_{++}^n denotes the cone of real $n \times n$ symmetric positive definite matrices;
- (c): For any $X, Y \in S_+^n$, $P \in \mathbb{R}^{n \times n}$ being a nonsingular matrix and $\alpha \geq 0$, it holds that $X + Y$, αX , and PXP^\top belong to S_+^n .

- (d): Let $X, Y \in S_+^n$. If $\langle X, Y \rangle = 0$, then X and Y commute (that is $XY = YX$);
- (e): (Simultaneous diagonalization) Let $X, Y \in S_+^n$. If X and Y commute, then X and Y are simultaneously diagonalizable;
- (f): As a direct corollary of (d) and (e), it follows that for any $X, Y \in S_+^n$ such that $\langle X, Y \rangle = 0$, X and Y are simultaneously diagonalizable;
- (g): (Farkas Lemma) Let $L \in \mathcal{L}(S^n)$ and $Q \in S^n$. The following assertions are equivalent:
- (i): The system $\{Q - L(X) \in S_+^n : X \in S_+^n\}$ has solutions.
- (ii): $[L^\top(U) \in S_+^n \text{ and } U \in S_+^n \text{ implies } \langle Q, U \rangle \geq 0]$ and $[L^\top(U) + I_n \in S_+^n \text{ and } U \in S_+^n \text{ implies } \langle Q, U \rangle \geq -\delta_0 \text{ for some } \delta_0 > 0]$.

2.2. Notions of set convergence and set-valued mappings. In order to approximate S_+^n by sequences of sets $\{\Omega_k\}$. For this purpose we recall some notions of set convergence from [27]. Let $\{C_k\}$ be a sequence of sets from \mathbb{R}^n , the following limits are defined:

$$\begin{aligned} \limsup_k C_k &:= \{x : \exists x^{k_j} \in C_{k_j}, x^{k_j} \rightarrow x\} \text{ is its outer limit;} \\ \liminf_k C_k &:= \{x : \exists x^k \in C_k, x^k \rightarrow x\} \text{ is its inner limit;} \\ \limsup_k^\infty C_k &:= \{v : \exists x^{k_j} \in C_{k_j}, t_{k_j} \rightarrow +\infty : \frac{x^{k_j}}{t_{k_j}} \rightarrow v\} \text{ is its horizon outer limit and} \\ \liminf_k^\infty C_k &:= \{v : \exists x^k \in C_k, t_k \rightarrow +\infty : \frac{x^k}{t_k} \rightarrow v\} \text{ is its horizon inner limit.} \end{aligned}$$

We say that the sequence $\{C^k\}$ converges to C in the sense of Painlevé-Kuratowski when $\limsup_k C_k \subseteq C \subseteq \liminf_k C_k$. This will be simply denoted by $C_k \rightarrow C$.

For obtaining stability properties for SDLCPs we study continuity properties of the solution set mappings. To this end we recall the following notions of continuity for set-valued mappings from [27]. Let X and Y be metric spaces, the set-valued mapping or multifunction $\Phi : X \rightrightarrows Y$ is said to be: outer semicontinuous (osc) at x if for any sequence $\{x^k\}$ such that $x^k \rightarrow x$ it holds that $\limsup_k \Phi(x^k) \subseteq \Phi(x)$; inner semicontinuous (isc) at x if for any sequence $\{x^k\}$ such that $x^k \rightarrow x$ it holds that $\Phi(x) \subseteq \liminf_k \Phi(x^k)$; upper semicontinuous (usc) at x if for any open set $W \subseteq Y$ with $\Phi(x) \subseteq W$ there exists a neighborhood U of x such that $\Phi(x') \subseteq W$ for all $x' \in U$; lower semicontinuous (lsc) at x if for any open set $W \subseteq Y$ with $\Phi(x) \cap W \neq \emptyset$, there exists a neighborhood U of x such that $\Phi(x') \cap W \neq \emptyset$ for all $x' \in U$; continuous at x if it is osc and isc at x ; K-continuous at x if it is usc and lsc at x ; locally bounded at x if for some neighborhood V of x the set $\Phi(V) = \cup_{z \in V} \Phi(z)$ is bounded. It is known that inner and lower semicontinuity properties coincide within this framework.

3. ASYMPTOTIC ANALYSIS

In this section we approximate the equivalent problem $\text{SDVIP}(L, S_+^n, Q)$ by a sequence of problems $\text{SDVIP}(L^k, \Omega_k, Q^k)$ such that the sequence $\{(L^k, \Omega_k, Q^k)\}$ converges to (L, S_+^n, Q) in some sense to be specified later on. This approach was employed in [7] for studying linear complementarity problems and it was further developed in [22] for studying polyhedral complementarity problems that deal with mappings such that its graph is the union of a finite collection of polyhedral sets.

We employ two schemes of approximation. The first one, which is termed general, consists of approximating all the data. This scheme allows us to obtain stability results. The second, which is termed particular, consists of approximating only the feasible set S_+^n by a particular

sequence of compact convex sets $\{\Omega_k\}$. This scheme allows us to obtain coercive and noncoercive existence results.

3.1. General scheme of approximation. The general scheme of approximation consists of approximating the problem $\text{SDVIP}(L, S_+^n, Q)$ by the following sequence of VIs:

$$\text{find } X^k \in \Omega_k \text{ such that } \langle L^k(X^k) + Q^k, X - X^k \rangle \geq 0, \text{ for all } X \in \Omega_k. \quad (\text{SDVIP}_k)$$

where $L^k : S^n \rightarrow S^n$ are linear transformations such that $\|L^k - L\| \rightarrow 0$ with respect to the subordinate norm to the Frobenius-norm $\|\cdot\|_F$, $Q^k \in S^n$ are matrices such that $\|Q^k - Q\|_F \rightarrow 0$, and Ω_k are subsets of S^n such that $\Omega_k \rightarrow S_+^n$ in the sense of Painlevé-Kuratowski. As before, (SDVIP_k) will be denoted by $\text{SDVIP}(L^k, \Omega_k, Q^k)$ in order to stress the dependence on the data of the problem.

The next result asserts that any accumulation point of a sequence $\{X^k\}$, composed by solutions of $\text{SDVIP}(L^k, \Omega_k, Q^k)$, solves the variational inequality $\text{SDVIP}(L, S_+^n, Q)$, and consequently, solves the complementarity problem $\text{SDLCP}(L, S_+^n, Q)$.

Theorem 3.1. $\limsup_k \mathcal{S}(L^k, \Omega_k, Q^k) \subseteq \mathcal{S}(L, S_+^n, Q)$.

Proof. If X is on the left-hand side, then there exists a subsequence $\{X^{k_j}\}$ such that $X^{k_j} \in \mathcal{S}(L^{k_j}, \Omega_{k_j}, Q^{k_j})$ for all j and $X^{k_j} \rightarrow X$. Let $Z \in S_+^n$ be fixed. As $\Omega_k \rightarrow S_+^n$, by definition there exists $\{Z^k\}$ such that $Z^k \in \Omega_k$ for all k and $Z^k \rightarrow Z$. Therefore, $\langle L^{k_j}(X^{k_j}) + Q^{k_j}, Z^{k_j} - X^{k_j} \rangle \geq 0$ for all j . After taking limit we get $\langle L(X) + Q, Z - X \rangle \geq 0$ and since Z was arbitrary we conclude that X is in the right-hand set. \square

In this paper we consider two types of sequences $\{\Omega_k\}$:

Type I:: $\Omega_k = \{X \in S_+^n : \langle D, X \rangle \leq \sigma_k\}$ where $D \in S_{++}^n$ and $\sigma_k \rightarrow +\infty$. These sets are compact and convex (see [5, Lemma 3.2].)

Type II:: $\Omega_k = S_+^n$ for all k .

Obviously, both types of sequences converge to S_+^n . Type I will be used to establish the existence results, while Type II will allow an estimate the asymptotic cone of the solution set, which will provide the stability results.

Notice that when the sequence $\{X^k\}$, composed by solutions to (SDVIP_k) , is bounded, it is easy to see that each of its accumulation points \bar{X} is a solution of problem $\mathcal{S}(L, S_+^n, Q)$. For this reason, it is important to study the behavior of unbounded sequences $\{X^k\}$. We do this in the next lemma by employing the notion of horizon outer limit.

Lemma 3.2. $\limsup_k^\infty \mathcal{S}(L^k, \Omega_k, Q^k) \subseteq \begin{cases} \bigcup_{\tau \geq 0} \mathcal{S}(L, S_+^n, \tau D), & \text{if } \{\Omega_k\} \text{ is of Type I;} \\ \mathcal{S}(L, S_+^n, 0_n), & \text{if } \{\Omega_k\} \text{ is of Type II.} \end{cases}$

Proof. We first prove the inclusion for $\{\Omega_k\}$ being of Type I. Let V be in the left-hand side set. If $V = 0_n$ then the inclusion is trivial. So, it is enough to consider $V \neq 0_n$. By definition there exists $\{X^{k_j}\}$ and $\{t_{k_j}\}$ such that $X^{k_j} \in \mathcal{S}(L^{k_j}, \Omega_{k_j}, Q^{k_j})$ for all j , $t_{k_j} \rightarrow +\infty$, and $\frac{X^{k_j}}{t_{k_j}} \rightarrow V$ as $j \rightarrow +\infty$. Clearly, $V \in (S_+^n)^\infty = S_+^n$ and since $V \neq 0_n$ we have that $\langle D, X^{k_j} \rangle \rightarrow +\infty$.

Let $X \in S_+^n$ be fixed. There exists $j_X \in \mathbb{N}$ such that $X \in \Omega_{k_j}$ for all $j \geq j_X$. Therefore, $\langle L^{k_j}(X^{k_j}) + Q^{k_j}, X - X^{k_j} \rangle \geq 0$ for all j . After dividing both sides by $t_{k_j}^2$ and taking limit $j \rightarrow +\infty$ we obtain $\langle L(V), V \rangle \leq 0$.

If $Z^{k_j} = \frac{\langle D, X^{k_j} \rangle}{\langle D, Z \rangle} Z$, with $Z \in S_+^n \setminus \{0_n\}$, then $Z^{k_j} \in \Omega_{k_j}$ for all j . Therefore, we have $\langle L^{k_j}(X^{k_j}) + Q^{k_j}, Z^{k_j} - X^{k_j} \rangle \geq 0$ for all j . As $\frac{X^{k_j}}{\langle D, X^{k_j} \rangle} \rightarrow \frac{V}{\langle D, V \rangle}$, by dividing the last inequality by $t_{k_j} \langle D, X^{k_j} \rangle$ and taking limit $j \rightarrow +\infty$ we get $\langle L(V), \frac{Z}{\langle D, Z \rangle} - \frac{V}{\langle D, V \rangle} \rangle \geq 0$. Since $\langle D, Z \rangle, \langle D, V \rangle \geq 0$ (see Section 2), we can write this inequality in the following form $\langle \langle D, V \rangle L(V) - \langle L(V), V \rangle D, Z \rangle \geq 0$. As $Z \in S_+^n$ was arbitrary, by Fejer's Theorem we conclude that $W = L(V) - \frac{\langle L(V), V \rangle}{\langle D, V \rangle} D \in S_+^n$. Moreover, it is easy to check that $\langle W, V \rangle = 0$. From this, we conclude that $V \in \mathcal{S}(L, S_+^n, \tau_V D)$ with $\tau_V = -\frac{\langle L(V), V \rangle}{\langle D, V \rangle} \geq 0$ which proves the first inclusion.

If $\{\Omega_k\}$ is of Type II, then $X^{k_j} \in S_+^n, Y^{k_j} = L^{k_j}(X^{k_j}) + Q^{k_j} \in S_+^n$ and $\langle Y^{k_j}, X^{k_j} \rangle = 0$ for all j . After dividing by t_{k_j} each of these expressions and taking limit $j \rightarrow +\infty$ we get $V \in S_+^n, L(V) \in S_+^n$, and $\langle L(V), V \rangle = 0$; thus, $V \in \mathcal{S}(L, S_+^n, 0_n)$. \square

The set $\mathcal{S}(L, S_+^n, 0_n)$, termed kernel of the SDLCP, plays an important role in the SDLCP theory. It is a closed cone and satisfies $\text{int}[\mathcal{S}(L, S_+^n, 0_n)^+] = \mathcal{S}(L, S_+^n, 0_n)^{++}$ (see [27, Exercise 6.22]).

As a consequence of Lemma 3.2 we obtain bounds for the asymptotic cone of the solution set to our problem in Corollary 3.3 and Proposition 4.11 below. The next result is a particular case of [6, Proposition 2.5.6] given for LCPs with a feasible set being a closed convex cone (not necessarily self-dual). In order to illustrate our approach, we give here a different proof based on Lemma 3.2 and Theorem 3.1.

Corollary 3.3. (a): $\bigcup_{Q \in S^n} \mathcal{S}(L, S_+^n, Q)^\infty = \mathcal{S}(L, S_+^n, 0_n)$;

(b): If $\mathcal{S}(L, S_+^n, 0_n) = \{0_n\}$, then $\mathcal{S}(L, S_+^n, Q)$ is bounded (possibly empty) for all $Q \in S^n$;

(c): $\mathcal{S}(L, S_+^n, 0_n) = \{0_n\}$ if and only if there exists a constant $c > 0$ such that

$$\|X\|_F \leq c\|Q\|_F, \quad \forall Q \in S^n, X \in \mathcal{S}(L, S_+^n, Q).$$

Proof. (a): The inclusion $\mathcal{S}(L, S_+^n, Q)^\infty \subseteq \mathcal{S}(L, S_+^n, 0_n)$ follows by setting $(L^k, \Omega_k, Q^k) = (L, S_+^n, Q)$ for all k in Lemma 3.2 and since $\limsup_k^\infty \mathcal{S}(L, S_+^n, Q) = \mathcal{S}(L, S_+^n, Q)^\infty$ by [27, Exercise 4.21]. The reverse inclusion follows from the inclusion $\mathcal{S}(L, S_+^n, 0_n) \subseteq \mathcal{S}(L, S_+^n, 0_n)^\infty$ which holds since $\mathcal{S}(L, S_+^n, 0_n)$ is a cone.

(b): This part follows from the fact that a set is bounded if and only if its asymptotic cone is equal to $\{0_n\}$ (see [1, Proposition 2.1.2].)

(c): (\Rightarrow) Suppose that $\mathcal{S}(L, S_+^n, 0_n) = \{0_n\}$ but such a constant c does not exist. Then, there exist sequences $\{X^k\}$ and $\{Q^k\}$ with $Q^k \in S^n$ and $X^k \in \mathcal{S}(L, S_+^n, Q^k)$ such that $\|X^k\|_F > k\|Q^k\|_F$ for all k . If $Q^k = 0_n$ for some k , then $0_n \neq X^k \in \mathcal{S}(L, S_+^n, 0_n)$, a contradiction. Therefore, $X^k \neq 0_n, Q^k \neq 0_n$ for all k , and $\frac{\|X^k\|_F}{\|Q^k\|_F} \rightarrow +\infty$. By Weierstrass theorem there exists $X \in S^n$ such that up to subsequences $\frac{X^k}{\|X^k\|_F} \rightarrow X$. It is easy to check that $X \in \limsup_k \mathcal{S}(L, S_+^n, \frac{Q^k}{\|X^k\|_F})$. As $\frac{Q^k}{\|X^k\|_F} \rightarrow 0_n$, by Theorem 3.1 with $\{\Omega_k\}$ being of Type II we conclude that $0_n \neq X \in \mathcal{S}(L, S_+^n, 0_n)$, a contradiction.

(\Leftarrow) If the constant $c > 0$ with the desired property exists, then by setting $Q = 0_n$ we find that $\mathcal{S}(L, S_+^n, 0_n) = \{0_n\}$. \square

3.2. Particular scheme of approximation. In order to obtain existence results for SDLCPs, we approximate SDVIP(L, S_+^n, Q) by the following sequence of problems SDVIP(L, Ω_k, Q) where $\{\Omega_k\}$ is of Type I; i.e.,:

$$\text{find } X^k \in \Omega_k \text{ such that } \langle L(X^k) + Q, X - X^k \rangle \geq 0, \text{ for all } X \in \Omega_k. \quad (\text{SDLCP}_k)$$

As Ω_k is compact convex and nonempty, by Theorem 1.1 such a solution X^k exists for all k . It is clear that X^k is a solution of this problem if and only if $X^k \in \Omega_k$ is an optimal solution of the linear program

$$\inf_X \left[\langle L(X^k) + Q, X \rangle : X \in S_+^n, \langle D, X \rangle \leq \sigma_k \right].$$

Applying optimality conditions (see [18]), we find that X^k is a solution of this problem if and only if there exists $Y^k \in S^n$ such that (X^k, θ_k) is a solution of the following problem, called the augmented semidefinite linear complementarity problem:

$$\begin{aligned} & \text{find } X^k \in S_+^n \text{ and } \theta_k \geq 0, \text{ such that} \\ & Y^k = L(X^k) + Q + \theta_k D \in S_+^n, \langle D, X^k \rangle \leq \sigma_k, \\ & \langle Y^k, X^k \rangle = 0, \text{ and } \theta_k(\sigma_k - \langle D, X^k \rangle) = 0. \end{aligned} \quad (\text{ASDLCP}_k)$$

From this we can observe that

$$(3.1) \quad \langle D, X^k \rangle < \sigma_k \implies [\theta_k = 0] \implies X^k \in \mathcal{S}(L, S_+^n, Q).$$

Moreover, we have from (ASDLCP_k) that

$$(3.2) \quad \theta_k = - \left\langle L(X^k) + Q, \frac{X^k}{\sigma_k} \right\rangle.$$

Implications (3.1) show that only the case when $\langle D, X^k \rangle = \sigma_k$, for all k , deserves to be analyzed. This analysis is carried out below by extending the arguments from [7] to the semidefinite framework via the simultaneous diagonalization theorems from Section 2. Indeed, since $\langle D, X^k \rangle = \sigma_k$ for all k , we are interested in obtaining asymptotic properties of sequences $\{\frac{X^k}{\sigma_k}\}$ such that

$$(3.3) \quad \frac{X^k}{\sigma_k} \in \left\{ Z \in S_+^n : \langle D, Z \rangle = 1 \right\}.$$

Moreover, since $X^k, Y^k \in S_+^n$ and $\langle Y^k, X^k \rangle = 0$, by part (f) of Proposition 2.1, there exists an orthogonal matrix U^k such that

$$(3.4) \quad \frac{X^k}{\sigma_k} = U^k \text{Diag} \left(\lambda \left(\frac{X^k}{\sigma_k} \right) \right) (U^k)^\top \quad \text{and} \quad Y^k = U^k \text{Diag} \left(\lambda(Y^k) \right) (U^k)^\top.$$

This implies that

$$\left\langle D, \frac{X^k}{\sigma_k} \right\rangle = \left\langle D, \sum_{i=1}^n \lambda_i \left(\frac{X^k}{\sigma_k} \right) u_i \left(\frac{X^k}{\sigma_k} \right) u_i \left(\frac{X^k}{\sigma_k} \right)^\top \right\rangle = \sum_{i=1}^n \lambda_i \left(\frac{X^k}{\sigma_k} \right) d_i^k = 1,$$

where $u_i \left(\frac{X^k}{\sigma_k} \right)$ is the i -th column vector of U^k and $d_i^k := u_i \left(\frac{X^k}{\sigma_k} \right)^\top D u_i \left(\frac{X^k}{\sigma_k} \right) \geq \lambda_{\min}(D) > 0$. The latter is a direct consequence of the Rayleigh-Ritz ratio (see [15, Theorem 4.2.2]). Denote $d^k := (d_1^k, \dots, d_n^k)$, it is thus clear that

$$\lambda \left(\frac{X^k}{\sigma_k} \right) * d^k \in \Delta := \left\{ \gamma \in \mathbb{R}_+^n : \sum_{i=1}^n \gamma_i = 1 \right\}.$$

As stated in [26, Theorem 18.2], the simplex Δ can be decomposed as the disjoint union of the relative interior of its extreme faces $\Delta_{J_i} = \text{co}\{e^k : k \in J_i\}$, with J_i being a nonempty subindex set of I , for each $i = 1, \dots, 2^n - 1$. This is stated here below:

$$(3.5) \quad \Delta = \bigsqcup_{i=1}^{2^n-1} \text{ri}(\Delta_{J_i}).$$

The next result describes the asymptotic behavior of the sequence $\{\frac{X^k}{\sigma_k}\}$. This result is an extension to SDLCPs of [7, Lemma 2.1] given for LCPs.

Lemma 3.4. *Let $D \in S_{++}^n$. Let $\{X^k\}$ be a sequence of solutions to (SDLCP_k) such that $\langle D, X^k \rangle = \sigma_k$ for all k and $\frac{X^k}{\sigma_k} \rightarrow V$, as $k \rightarrow +\infty$, for some $V \in S^n$. Then*

(a): $V \in \mathcal{S}(L, S_+^n, \tau_V D)$ with $\tau_V = -\langle L(V), V \rangle \geq 0$.

Moreover, there exists a nonempty subindex set $J_V \subseteq I$ and subsequences $\{\frac{X^{k_m}}{\sigma_{k_m}}\}$, $\{Y^{k_m}\}$ and $\{d^{k_m}\}$ such that for all m one has

(b): $\lambda\left(\frac{X^{k_m}}{\sigma_{k_m}}\right) * d^{k_m} \in \text{ri}(\Delta_{J_V})$. Thus, $\text{supp}\{\lambda\left(\frac{X^{k_m}}{\sigma_{k_m}}\right)\} = J_V$ and $\lambda(Y^{k_m})|_{J_V} = 0$ (hence $\text{supp}\{\lambda(V)\} \subseteq J_V$);

(c): $\langle Y^{k_m}, Z \rangle = 0$ for all $Z \in S_+^n$ such that $\text{supp}\{\lambda(Z)\} \subseteq J_V$;

(d): $\left\langle L(X^{k_m}) + Q, \frac{Z}{\langle D, Z \rangle} \right\rangle = \langle L(X^{k_m}) + Q, V \rangle$ for all $Z \in S_+^n \setminus \{0_n\}$ and $\text{supp}\{\lambda(Z)\} \subseteq J_V$;

(e): $\langle L(V), \frac{Z}{\langle D, Z \rangle} \rangle = \langle L(V), V \rangle$ for all $Z \in S_+^n$ such that $Z \neq 0_n$ and $\text{supp}\{\lambda(Z)\} \subseteq J_V$.

Proof. (a): Since $\sigma_k = \langle D, X^k \rangle \rightarrow +\infty$ (because of $\{\Omega_k\}$ is of Type I) and $\frac{X^k}{\sigma_k} \rightarrow V$, by definition we have $V \in \limsup_k^\infty \mathcal{S}(L, \Omega_k, Q)$. This assertion follows from the proof of Lemma 3.2 since $\langle D, V \rangle = 1$.

(b): Since $\lambda\left(\frac{X^k}{\sigma_k}\right) * d^k \in \Delta$ for all k , from decomposition (3.5) it is easy to check that there exists a subsequence $\{\frac{X^{k_m}}{\sigma_{k_m}}\}$ and a nonempty subindex set $J_V \subseteq I$ such that $\lambda\left(\frac{X^{k_m}}{\sigma_{k_m}}\right) * d^{k_m} \in \text{ri}(\Delta_{J_V})$ for all m . Hence, $\text{supp}\{\lambda\left(\frac{X^{k_m}}{\sigma_{k_m}}\right)\} = J_V$ for such m (see [27, Exercise 2.28(e)]). So, (3.4) implies that

$$0 = \left\langle \frac{X^{k_m}}{\sigma_{k_m}}, Y^{k_m} \right\rangle = \left\langle \lambda\left(\frac{X^{k_m}}{\sigma_{k_m}}\right), \lambda(Y^{k_m}) \right\rangle = \left\langle \lambda\left(\frac{X^{k_m}}{\sigma_{k_m}}\right)|_{J_V}, \lambda(Y^{k_m})|_{J_V} \right\rangle$$

and thus $\lambda(Y^{k_m})|_{J_V} = 0$. Finally, on the one hand, the continuity of eigenvalues implies that $\lambda(X^{k_m}/\sigma_{k_m}) \rightarrow \lambda(V)$ and, on the other hand, matrices U^k can be chosen such that $u_i(X^{k_m}/\sigma_{k_m}) \rightarrow u_i(V)$, for all $i = 1, \dots, n$. This yields to

$$d_i^{k_m} \rightarrow d_i = u_i(V)^\top D u_i(V) \quad \text{and} \quad \lambda\left(\frac{X^{k_m}}{\sigma_{k_m}}\right) * d^{k_m} \rightarrow \lambda(V) * d, \quad \text{as } m \rightarrow +\infty$$

Consequently, $d_i \geq \lambda_{\min}(D) > 0$, $\lambda(V) * d \in \Delta_{J_V}$, and hence $\text{supp}\{\lambda(V)\} \subseteq J_V$.

(c): Let $Z \in S_+^n$ be such that $\text{supp}\{\lambda(Z)\} \subseteq J_V$. By employing part (a) of Proposition 2.1 and item (b) above, we obtain:

$$0 \leq \langle Y^{k_m}, Z \rangle \leq \langle \lambda(Y^{k_m}), \lambda(Z) \rangle = \langle \lambda(Y^{k_m})|_{J_V}, \lambda(Z)|_{J_V} \rangle = 0, \quad \text{for all } m.$$

Therefore, $\langle Y^{k_m}, Z \rangle = 0$ for all m .

(d): Let $Z \in S_+^n \setminus \{0_n\}$ such that $\text{supp}\{\lambda(Z)\} \subseteq J_V$. Equation (3.2) and item (c) above yield to

$$\langle L(X^{k_m}) + Q, \frac{X^{k_m}}{\sigma_{k_m}} \rangle = \langle L(X^{k_m}) + Q, \frac{Z}{\langle D, Z \rangle} \rangle.$$

Since above, Z can be replaced by V , we obtain item (d).

(e): After dividing the equality in item (d) by σ_{k_m} and taking the limit $m \rightarrow +\infty$ we obtain the desired result. \square

4. GARCÍA, #, AND Q -PSEUDOMONOTONE LINEAR TRANSFORMATIONS

As declared previously, our existence results are established via an approximation scheme where problem (SDLCP) is approximated by a sequence of problems (SDLCP $_k$), with $\{\Omega_k\}$ being of Type I. By Theorem 1.1 each problem (SDLCP $_k$) has solutions and by Lemma 3.2 we have the following inclusion:

$$\limsup_k^\infty \mathcal{S}(L^k, \Omega_k, Q^k) \subseteq \bigcup_{\tau \geq 0} \mathcal{S}(L, S_+^n, \tau D).$$

So, if the following condition holds:

$$(4.1) \quad \mathcal{S}(L, S_+^n, \tau D) = \{0_n\} \quad \text{for all } \tau \geq 0,$$

then $\limsup_k^\infty \mathcal{S}(L, \Omega_k, Q) = \{0_n\}$. Due to [27, Exercise 4.22] this implies that the sequence $\{\mathcal{S}(L, \Omega_k, Q)\}$ is eventually locally bounded, that is, there exists $N_0 \in \mathbb{N}$ such that the set $\bigcup_{k \geq N_0} \mathcal{S}(L, \Omega_k, Q)$ is bounded. Therefore, any sequence $\{X^k\}$, composed by $X^k \in \mathcal{S}(L, \Omega_k, Q)$, is bounded and, consequently, it has a cluster point, denoted by \bar{X} . By Theorem 3.1 we conclude that \bar{X} is a solution of (SDLCP). Thus, $\mathcal{S}(L, S_+^n, Q) \neq \emptyset$. Moreover, as $\mathcal{S}(L, S_+^n, 0_n) = \{0_n\}$, we conclude from Corollary 3.3(b) that $\mathcal{S}(L, S_+^n, Q)$ is a bounded set. We thus conclude the following result.

Proposition 4.1 (cf. [11]). *Let $L \in \mathcal{L}(S^n)$ such that condition (4.1) holds. Then, $\mathcal{S}(L, S_+^n, Q)$ is nonempty and compact for every $Q \in S^n$.*

One of the motivations of the paper is to show that the less restrictive condition:

$$(4.2) \quad \mathcal{S}(L, S_+^n, \tau D) = \{0_n\} \quad \text{for all } \tau > 0,$$

also yields existence of solutions under some additional hypotheses. The class of linear transformations satisfying this condition is a generalization to SDLCPs of a class of matrices introduced by García (see [8, 7, 10]) within the LCP context. The formal extension to SDLCPs is stated here below.

Definition 4.2. *A linear transformation $L : S^n \rightarrow S^n$ is said to be of García's type or $L \in \mathbf{G}$ if there exists $D \in S_{++}^n$ such that condition (4.2) holds. In this case we say that L is a \mathbf{G} -linear transformation with respect to D and we write $L \in \mathbf{G}(D)$.*

Remark 4.3. *By linearity one has $L \in \mathbf{G}(D)$ if and only if $\mathcal{S}(L, S_+^n, D) = \{0_n\}$.*

Let us recall the classes of linear transformations that will be used in the sequel. They play an important role in the SDLCP theory (see, for instance, [11, 12, 13, 23]).

A linear transformation $L : S^n \rightarrow S^n$ is said to be:

- regular or $L \in \mathbf{R}$ if there exists $D \in S_{++}^n$ such that condition (4.1) holds. We say that L is an \mathbf{R} -transformation with respect to D and we write $L \in \mathbf{R}(D)$;
- an \mathbf{R}_0 -transformation if $\mathcal{S}(L, S_+^n, 0_n) = \{0_n\}$;
- positive or $L \geq 0$ if $L(X) \in S_+^n$ for all $X \in S_+^n$;
- copositive if $\langle L(X), X \rangle \geq 0$ for all $X \in S_+^n$;
- monotone (resp. strictly monotone) if $\langle L(X), X \rangle \geq 0$ (resp. > 0) for all $X \in S^n$ (resp. for all $X \in S^n, X \neq 0_n$);
- strictly semimonotone SSM or $L \in \mathbf{E}$ if $[X \in S_+^n, XL(X) = L(X)X \in -S_+^n \Rightarrow X = 0_n]$;

- semimonotone or $L \in \mathbf{E}_0$ if $L(\cdot) + \varepsilon I_n \in \mathbf{E}$ for all $\varepsilon > 0$;
- a \mathbf{P} -transformation if $[XL(X) = L(X)X \in -S_+^n \Rightarrow X = 0_n]$;
- a \mathbf{P}_0 -transformation if $L(\cdot) + \varepsilon I_n \in \mathbf{P}$, $\forall \varepsilon > 0$;
- a star-transformation if $[V \in \mathcal{S}(L, S_+^n, 0) \Rightarrow L^\top(V) \in -S_+^n]$;
- a \mathbf{Q} -transformation if $\mathcal{S}(L, S_+^n, Q) \neq \emptyset$ for all $Q \in S^n$;
- a \mathbf{Q}_b -transformation if $\mathcal{S}(L, S_+^n, Q) \neq \emptyset$ and bounded for all $Q \in S^n$;
- a Q -pseudomonotone if $X, Y \in S_+^n$ $\langle L(X) + Q, Y - X \rangle \geq 0 \Rightarrow \langle L(Y) + Q, Y - X \rangle \geq 0$;
- a Q -quasimonotone if $X, Y \in S_+^n$ $\langle L(X) + Q, Y - X \rangle > 0 \Rightarrow \langle L(Y) + Q, Y - X \rangle \geq 0$;
- a \mathbf{Z} -transformation if $[X, Y \in S_+^n, \langle X, Y \rangle = 0 \Rightarrow \langle L(X), Y \rangle \leq 0]$.

Remark 4.4. Notice that Proposition 4.1 can be rewritten as follows: $L \in \mathbf{R}$ implies $L \in \mathbf{Q}_b$.

We now list some known relationships between the above linear transformations and García's linear transformations.

Proposition 4.5. Let $L \in \mathcal{L}(S^n)$ and $Q \in S^n$ be given.

- (a): $\mathbf{G} = \bigcup_{D \in S_{++}^n} \mathbf{G}(D)$, $\mathbf{R} = \bigcup_{D \in S_{++}^n} \mathbf{R}(D)$ and $\mathbf{R}(D) = \mathbf{G}(D) \cap \mathbf{R}_0$;
- (b): L is monotone or $L \geq 0$ or $L^\top \geq 0 \Rightarrow L$ is copositive $\Rightarrow L \in \mathbf{G}(D)$ for all $D \in S_{++}^n$;
- (c): L is monotone $\Rightarrow L \in \mathbf{P}_0 \Rightarrow L \in \mathbf{E}_0 \Rightarrow L \in \mathbf{G}(I_n)$;
- (d): L is strictly monotone $\Rightarrow L$ is monotone $\Rightarrow L$ is Q -pseudomonotone $\Rightarrow L$ is Q -quasimonotone;
- (e): L is Q -pseudomonotone and $\text{Feas}(L, S_+^n, Q) \neq \emptyset \Rightarrow L$ is copositive;
- (f): L is Q -quasimonotone and $Q \neq 0_n \Rightarrow L$ is Q -pseudomonotone. Moreover, if L is Q -quasimonotone but is not monotone, $L(X) + Q \in S_+^n$ and $L(X) + Q \neq 0_n$ for some $X \in S_+^n \Rightarrow L$ is copositive.

Proof. Parts (a), (d), and the first implication for \mathbf{P}_0 in part (c) are trivial. See [24, Theorem 5] for part (b). See [11, Theorem 4] for the third implication in part (c). See [7, Remark 4.1] for part (e) and [14, Propositions 4.1 and 5.4] for part (f). \square

The next proposition provides two characterizations of the class of García's linear transformations. This is an SDLCP version of Proposition 3.1 from [7] proved for LCPs.

Proposition 4.6. Let $D \in S_{++}^n$. Let $L : S^n \rightarrow S^n$ be a linear transformation. The following assertions are equivalent:

- (a): $L \in \mathbf{G}(D)$;
- (b): $\left[V \in S_+^n, L(V) - \langle L(V), V \rangle D \in S_+^n, \langle D, V \rangle = 1 \right] \Rightarrow \langle L(V), V \rangle \geq 0$;
- (c): $\left[V \in S_+^n, \langle D, V \rangle = 1, \langle L(V), V \rangle < 0 \right] \Rightarrow \exists i \in I : \lambda_i(L(V) - \langle L(V), V \rangle D) < 0$.

Proof. (a) \Rightarrow (b): Suppose that the left-hand side of the implication in item (b) holds. Then, it is easy to check that $V \in \mathcal{S}(L, S_+^n, \tau D) \setminus \{0_n\}$ for $\tau = -\langle L(V), V \rangle$. Then, since $L \in \mathbf{G}(D)$, this necessarily implies that $\tau = -\langle L(V), V \rangle \leq 0$.

(b) \Rightarrow (c): Note that if $L(V) - \langle L(V), V \rangle D \in S_+^n$ and the left-hand side of the implication in item (c) holds, then the desired result follows directly from item (b).

(c) \Rightarrow (a): We argue by contradiction. Suppose that for some $\tau > 0$ there exists $V \in \mathcal{S}(L, S_+^n, \tau D) \setminus \{0_n\}$, that is, $V \in S_+^n \setminus \{0_n\}$, $L(V) + \tau D \in S_+^n$, and $\langle L(V) + \tau D, V \rangle = 0$. By

linearity, we can assume without loss of generality (we can change τ and V if necessary) that $\langle D, V \rangle = 1$. So, it follows that $\langle L(V), V \rangle = -\tau < 0$. Therefore, item (c) implies that the existence of an index $i \in I$ such that $\lambda_i(L(V) - \langle L(V), V \rangle D) < 0$, which contradicts $L(V) + \tau D \in S_+^n$. \square

We now extend to SDLCPs a class of matrices for LCPs introduced by Gowda and Pang in [10].

Definition 4.7. A linear transformation $L : S^n \rightarrow S^n$ is said to be a

- $\#$ -transformation if $[V \in \mathcal{S}(L, S_+^n, 0_n) \implies (L + L^\top)(V) \in S_+^n]$.
- $\mathbf{G}^\#$ -transformation if $L \in \mathbf{G}$ and it is a $\#$ -transformation. Similarly, a $\mathbf{G}(D)^\#$ -transformation is defined for $D \in S_{++}^n$.

We now provide some examples of $\#$ -transformations.

Proposition 4.8. $L \in \#$ if any of the following conditions is satisfied.

- (a): L is self-adjoint (that is, $L^\top = L$), or skew-symmetric (that is, $L^\top = -L$);
- (b): $L \in \mathbf{R}_0$;
- (c): L is copositive;
- (d): $-L$ is a star-transformation;
- (e): L is a star-transformation and $-L^\top \in \mathbf{Z}$.

Proof. (a)-(b), (d): They follow directly from the definitions.

(c): Let $V \in \mathcal{S}(L, S_+^n, 0_n)$. Since L is copositive, it holds that $\langle L(X + tV), X + tV \rangle \geq 0$ for all $t > 0$ and $X \in S_+^n$. After some basic computations the latter implies that $\langle (L + L^\top)(V), X \rangle \geq 0$, for all $X \in S_+^n$. Hence, by Fejer's theorem (Proposition 2.1(b)), we conclude $(L + L^\top)(V) \in S_+^n$.

(e): Let $V \in \mathcal{S}(L, S_+^n, 0_n)$. Clearly, $\langle V, -L^\top(V) \rangle = 0$. Since L is a star-transformation, we have $-L^\top(V) \in S_+^n$. And since $-L^\top \in \mathbf{Z}$, we find that $\langle -L^\top(V), -L^\top(V) \rangle \leq 0$, that is, $L^\top(V) = 0_n$. Therefore, $(L + L^\top)(V) = L(V) \in S_+^n$. \square

Remark 4.9. (1) The class of \mathbf{R}_0 -transformations contains the following classes of linear transformations: strictly monotone, isomorphisms, \mathbf{P} , \mathbf{E} , and nondegenerate (that is, $XL(X) = 0_n$ implies $X = 0_n$).

- (2) It is not difficult to see that if $-L$ is 0-pseudomonotone (in particular, if $-L$ is monotone), then $-L$ is a star-transformation.

The next result provides description of the asymptotic behavior of the sequence $\{\frac{X^k}{\sigma_k}\}$, which is sharper than the one stated in Lemma 3.4. To this end, we restrict our analysis to \mathbf{G} , copositive and Q -pseudomonotone linear transformations. This result extends [7, Lemmas 4.1 and 4.2] proved for LCPs.

Lemma 4.10. Let $D \in S_{++}^n$. Let $\{X^k\}$ be a sequence of solutions to (SDLCP_k) such that $\langle D, X^k \rangle = \sigma_k$ for all k and $\frac{X^k}{\sigma_k} \rightarrow V$, as $k \rightarrow +\infty$, for some $V \in S^n$. Then, in addition to the properties established in Lemma 3.4, for that nonempty subindex set $J_V \subseteq I$ one has

- (a): If $L \in \mathbf{G}(D)$, then $V \in \mathcal{S}(L, S_+^n, 0_n)$. Moreover, $\langle L(V), Z \rangle = 0$ for all $Z \in S_+^n$ such that $\text{supp}\{\lambda(Z)\} \subseteq J_V$. In particular, if L is copositive, then we also have $\langle Q, V \rangle \leq 0$;
- (b): If L is Q -pseudomonotone, then $L^\top(V) \in -S_+^n$ and $\langle Q, V \rangle \leq 0$.

Proof. (a): By Lemma 3.4(a) and Proposition 4.6(b) we have $\langle L(V), V \rangle \geq 0$; thus, $\tau_V = 0$. Hence $V \in \mathcal{S}(L, S_+^n, 0_n)$. Taking into account this and Lemma 3.4(e), we conclude the second part.

Now, assume that L is copositive. As $0_n \in \Omega_k$ for all k , we have $\langle L(X^k) + Q, 0_n - X^k \rangle \geq 0$ for all k . By assumption we obtain $\langle Q, X^k \rangle \leq 0$. After dividing by σ_k and taking limit $k \rightarrow +\infty$ we deduce the desired inequality.

(b): Let $Y \in S_+^n$. As $Y \in \Omega_k$ for k large enough, we have $\langle L(X^k) + Q, Y - X^k \rangle \geq 0$ for all k . By assumption we obtain $\langle L(Y) + Q, Y - X^k \rangle \geq 0$ for all k . After dividing by σ_k and taking limit $k \rightarrow +\infty$ we get $\langle L(Y) + Q, V \rangle \leq 0$. On one hand, setting $Y = 0_n$ we get $\langle Q, V \rangle \leq 0$. On the other hand, as $\langle L(\frac{1}{t}Y) + Q, V \rangle \leq 0$ for all $Y \in S_+^n$ and $t > 0$, after multiplying by t and taking limit $t \searrow 0$ we obtain $\langle Y, L^\top(V) \rangle \leq 0$ for all $Y \in S_+^n$, which in turn by Fejer's theorem (Proposition 2.1(b)) we get $-L^\top(V) \in S_+^n$. \square

We now improve the bound for the asymptotic cone of the solution set to SDLCP. We do this within the classes of Garcia's and Q -pseudomonotone linear transformations.

Proposition 4.11. (a): If $L \in \mathbf{G}^\#$, then $\mathcal{S}(L, S_+^n, Q)^\infty \subseteq \mathcal{S}(L, S_+^n, 0_n) \cap \{-Q\}^+$;
 (b): If L is Q -pseudomonotone and $\mathcal{S}(L, S_+^n, Q) \neq \emptyset$, then $\mathcal{S}(L, S_+^n, Q)^\infty = \{V \in S_+^n : L^\top(V) \in -S_+^n, \langle Q, V \rangle \leq 0\}$.

Proof. (a): Let $V \in \mathcal{S}(L, S_+^n, Q)^\infty$. If $V = 0_n$ then the inclusion is trivial. So, in the rest of the proof we only consider the case when $V \neq 0_n$. We can assume, without loss of generality, that $\langle D, V \rangle = 1$. By definition there exists $\{X^k\}$ and $\{t_k\}$ such that $X^k \in \mathcal{S}(L, S_+^n, Q)$ for all k , $t_k \rightarrow +\infty$ and $\frac{X^k}{t_k} \rightarrow V$ as $k \rightarrow +\infty$. If we define $\sigma_k := \langle D, X^k \rangle$ for all k , then it is not difficult to check that $\sigma_k \rightarrow +\infty$ and $\frac{X^k}{\sigma_k} \rightarrow V$. By Lemma 4.10(a) we have $V \in \mathcal{S}(L, S_+^n, 0_n)$. Taking into account this and Lemma 3.4(e) we deduce that $\langle L(V), \frac{X^{k_m}}{\sigma_{k_m}} \rangle = 0$. Then,

$$0 = \left\langle L(X^{k_m}) + Q, \frac{X^{k_m}}{\sigma_{k_m}} \right\rangle = \langle L(X^{k_m}) + Q, V \rangle = \langle X^{k_m}, (L + L^\top)(V) \rangle + \langle Q, V \rangle,$$

where we have used Lemma 3.4(d) and the fact that X^{k_m} is a solution to problem (SDLCP) for all m . By using the fact that $L \in \#$, we conclude that $\langle X^{k_m}, (L + L^\top)(V) \rangle \geq 0$. Hence, from the above equality we get $\langle Q, V \rangle \leq 0$.

(b): For L being Q -pseudomonotone by [5] we have

$$\mathcal{S}(L, S_+^n, Q) = \bigcap_{Y \in S_+^n} \{X \in S_+^n : \langle L(Y) + Q, X - Y \rangle \leq 0\}.$$

From this, by properties of asymptotic cones (see [1, Proposition 2.1.9]) we get

$$\mathcal{S}(L, S_+^n, Q)^\infty = \bigcap_{Y \in S_+^n} \{X \in S_+^n : \langle L(Y) + Q, X - Y \rangle \leq 0\}^\infty.$$

Moreover, from [1, Proposition 2.5.3] we have

$$\{X \in S_+^n : \langle L(Y) + Q, X - Y \rangle \leq 0\}^\infty = \{V \in S_+^n : \langle L(Y) + Q, V \rangle \leq 0\}.$$

Hence, the same arguments given in the proof of Lemma 4.10(b) provide the desired equality. \square

5. MAIN RESULTS

In this section we present the main results of this paper. As far as we are aware, we present the first systematic study, based on asymptotic analysis, on existence and stability properties for SDLCPs.

5.1. Existence results. By using our approach we now obtain existence results for different classes of linear transformations. The next result extends [10, Theorems 9 and 11] to SDLCPs.

Theorem 5.1. *Let $Q \in S^n$ and $L \in \mathbf{G}^\#(D)$.*

- (a): *If $Q \in \mathcal{S}(L, S_+^n, 0_n)^+$, then $\mathcal{S}(L, S_+^n, Q)$ is nonempty (possibly unbounded);*
- (b): *If $Q \in \text{int}[\mathcal{S}(L, S_+^n, 0_n)^+]$, then $\mathcal{S}(L, S_+^n, Q)$ is nonempty and compact.*

Proof. (a): We approximate the problem (SDLCP) by the sequence of problems (SDLCP_k). Let $\{X^k\}$ be a sequence of solutions to such problems. If there exists k such that $\langle D, X^k \rangle < \sigma_k$, then by implication (3.1) we conclude that X^k is a solution to the problem (SDLCP). On the contrary, if $\langle D, X^k \rangle = \sigma_k$ for all k , then up to subsequences $\frac{X^k}{\sigma_k} \rightarrow V$ for some V . By Lemmas 3.4(b) and 4.10(a) there exists a nonempty subindex set $J_V \subseteq I$ and a subsequence $\{\frac{X^{k_m}}{\sigma_{k_m}}\}$ such that $\text{supp}\{\lambda(\frac{X^{k_m}}{\sigma_{k_m}})\} = J_V$, $V \in \mathcal{S}(L, S_+^n, 0_n)$, and $\langle L(V), X^{k_m} \rangle = 0$ for all m . By using this equality, (3.2), and Lemma 3.4(d) we obtain

$$0 \leq \theta_{k_m} = -\left\langle L(X^{k_m}) + Q, \frac{X^{k_m}}{\sigma_{k_m}} \right\rangle = -\langle L(X^{k_m}) + Q, V \rangle = -\langle X^{k_m}, (L + L^\top)(V) \rangle - \langle Q, V \rangle.$$

As $V \in \mathcal{S}(L, S_+^n, 0_n)$, we have $\langle Q, V \rangle \geq 0$ by hypothesis, and $(L + L^\top)(V) \in S_+^n$ since $L \in \#$; thus, $\langle X^{k_m}, (L + L^\top)(V) \rangle \geq 0$. Consequently, $\theta_{k_m} = 0$ and by implication (3.1) we conclude that X^{k_m} is a solution to the problem (SDLCP).

(b): The first part, is a direct consequence of item (a). For the second one, we will prove that $\mathcal{S}(L, S_+^n, Q)^\infty = \{0_n\}$ by verifying that $\mathcal{S}(L, S_+^n, 0_n) \cap \{-Q\}^+ = \{0_n\}$ (cf. Proposition 4.11(a)). We argue by contradiction. Suppose that there exists $W \neq 0_n$ such that $W \in \mathcal{S}(L, S_+^n, 0_n)$ and $\langle Q, W \rangle \leq 0$. Since $Q \in \text{int}[\mathcal{S}(L, S_+^n, 0_n)^+] = \mathcal{S}(L, S_+^n, 0_n)^{s+}$ (see [27, Exercise 6.22]), we deduce that $\langle Q, W \rangle > 0$. Both inequalities yield a contradiction. Hence, the solution set is bounded. \square

Remark 5.2. (1) *As a consequence of this theorem we conclude that if $L \in \mathbf{G}(D)$, then $L \in \mathbf{R}_0$ if and only if $L \in \mathbf{Q}_b$. Another characterization of \mathbf{Q}_b -transformations can be seen in the recent paper [23].*

(2) *The hypothesis on Q of Theorem 5.1(a) implies the next necessary condition:*

$$Q \in \mathcal{S}(L, S_+^n, 0_n)^+ \implies \lambda_{\max}(Q) \geq 0.$$

Indeed, if $Q \in \mathcal{S}(L, S_+^n, 0_n)^+$, then $\langle Q, X \rangle \geq 0$ for all $X \in \mathcal{S}(L, S_+^n, 0_n)$. So, Von Neumann-Theobald inequality (Proposition 2.1(a)) implies that $\sum_{i=1}^n \lambda_i(Q) \lambda_i(X) \geq 0$. But, since $\lambda_i(X) \geq 0$ for every $i = 1, \dots, n$ (because $X \in S_+^n$), it necessarily follows that $\lambda_{\max}(Q) \geq 0$.

Remark 5.3. *For $M \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$, the linear complementarity problem, denoted by LCP(M, q), reads as follows:*

$$\text{find } \bar{x} \in \mathbb{R}_+^n \text{ such that } \bar{y} = M\bar{x} + q \in \mathbb{R}_+^n \text{ and } \langle \bar{y}, \bar{x} \rangle = 0.$$

Its solution set is denoted by $\mathcal{S}(M, q)$ and it is related to the problem $\text{SDLCP}(\mathcal{M}, S_+^n, \text{Diag}(q))$ with $\mathcal{M} : S^n \rightarrow S^n$ defined by $\mathcal{M}(X) := \text{Diag}(M \text{diag}(X))$ in the following way (see [28]):

$$\begin{aligned} \bar{X} \in \mathcal{S}(\mathcal{M}, S_+^n, \text{Diag}(q)) &\implies \bar{x} := \text{diag}(\bar{X}) \in \mathcal{S}(M, q) \\ \bar{x} \in \mathcal{S}(M, q) &\implies \bar{X} := \text{Diag}(\bar{x}) \in \mathcal{S}(\mathcal{M}, S_+^n, \text{Diag}(q)) \end{aligned}$$

From these implications, we deduce that for $d \in \mathbb{R}_{++}^n$ one has¹

$$\begin{aligned} M \in \mathbf{G}(d) &\iff \mathcal{M} \in \mathbf{G}(\text{Diag}(d)) \\ M \in \# &\iff \mathcal{M} \in \# \\ q \in \mathcal{S}(M, 0)^+ &\iff \text{Diag}(q) \in \mathcal{S}(\mathcal{M}, S_+^n, 0_n)^+ \end{aligned}$$

Indeed, the first and third equivalences follow directly from the definitions. To prove the remaining equivalence we employ that $\mathcal{M}^\top(V) = \text{Diag}(M^\top \text{diag}(V))$.

These equivalences allow the extension, in a straightforward way, examples from LCPs to SDLCPs. Thus, from [10, Example 2] and [7, Example 6.1] we obtain instances showing that Theorem 5.1 may be false if either $L \notin \#$ or $Q \notin \mathcal{S}(L, S_+^n, 0_n)^+$ or $L \notin \mathbf{G}(D)$.

We now recall three linear transformations that are intensively studied in the SDLCP literature since they play an important role when studying the stability of continuous and discrete dynamical systems. We list some conditions under which these transformations satisfy the hypotheses of Theorem 5.1.

Example 5.4. Let $A \in \mathbb{R}^{n \times n}$ be a given matrix.

- (1) The Lyapunov transformation $L_A : S^n \rightarrow S^n$ is defined by $L_A(X) = AX + XA^\top$. If A is positive stable (i.e., each of its eigenvalues has a positive real part), then $L_A \in \mathbf{E}_0 \cap \mathbf{R}_0$ by [11, Theorem 5]; thus, $L_A \in \mathbf{G}^\#(I_n)$ by Propositions 4.5(c) and 4.8(b). On the other hand, if A is positive definite, then L_A is strictly monotone by [25, Theorem 5]; thus, $L_A \in \mathbf{G}^\#(D)$ for any $D \in S_{++}^n$ by Proposition 4.5(b) and Remark 4.9.
- (2) The Multiplicative transformation $M_A : S^n \rightarrow S^n$ is defined by $M_A(X) = AXA^\top$. If A is symmetric positive or negative definite, then M_A is strictly monotone by [25, Theorem 6]; thus, $M_A \in \mathbf{G}^\#(D)$ for any $D \in S_{++}^n$ as above.
- (3) The Stein transformation $S_A : S^n \rightarrow S^n$ is defined by $S_A(X) = X - AXA^\top$. If $\|A\|_2 < 1$, then S_A is strictly monotone by [12, Theorem 3]; thus, $S_A \in \mathbf{G}^\#(D)$ for any $D \in S_{++}^n$ as above.

Now, we obtain existence theorems for Q -pseudomonotone and Q -quasimonotone linear transformations. To do this, we first introduce the following classes of linear transformations that are related to [29, Definition 3.12].

Definition 5.5. A linear transformation $L : S^n \rightarrow S^n$ is said to be weakly (resp. strictly weakly) proper on S_+^n if there exists $D \in S_{++}^n$ and $\hat{X} \in S_+^n$ such that for every sequence $\{X^k\} \subseteq S_+^n$ with $\|X^k\|_F \rightarrow +\infty$, there exists some k such that

$$\langle L(\hat{X}), X^k - \hat{X} \rangle \geq 0 \text{ (resp. } > 0) \text{ and } \langle D, X^k \rangle > \langle D, \hat{X} \rangle.$$

First, we provide a noncoercive existence result under Q -pseudomonotonicity.

¹We recall that $M \in \mathbf{G}(d)$ for $d \in \mathbb{R}_{++}^n$ if $\mathcal{S}(M, d) = \{0\}$ and $M \in \#$ if $v \in \mathcal{S}(M, 0)$ implies $(M + M^\top)v \in \mathbb{R}_+^n$ (see [7]).

Theorem 5.6. *Let $Q \in S^n$ and $L \in \mathcal{L}(S^n)$. Consider the following statements:*

- (a): $\text{Feas}(L, S_+^n, Q) \neq \emptyset$;
- (b): $[V \in S_+^n, L^\top(V) \in -S_+^n, \langle Q, V \rangle \leq 0 \implies \langle Q, V \rangle = 0]$ and $[L^\top(V) - I_n \in -S_+^n, V \in S_+^n \implies \langle Q, V \rangle \geq -\delta_0 \text{ for some } \delta_0 > 0]$;
- (c): $\mathcal{S}(L, S_+^n, Q) \neq \emptyset$;
- (d): $L(\cdot) + Q$ is weakly proper on S_+^n .

Then (a) \iff (b) \iff (c) \implies (d). In addition, if L is Q -pseudomonotone, then all the statements are equivalent.

Proof. (a) \iff (b): It is a consequence of Farkas Lemma (Proposition 2.1(g)) applied to $-L$.

(c) \implies (b): Part (c) implies part (a) that is equivalent to part (b) by the first part.

(c) \implies (d): If $\bar{X} \in \mathcal{S}(L, S_+^n, Q)$, then $\langle L(\bar{X}) + Q, X - \bar{X} \rangle \geq 0$ for all $X \in S_+^n$. From this we conclude that $L(\cdot) + Q$ is weakly proper on S_+^n with $\hat{X} = \bar{X}$ since for every sequence $\{X^k\} \subseteq S_+^n$ such that $\|X^k\|_F \rightarrow +\infty$ we have $\langle D, X^k \rangle \rightarrow +\infty$ for every $D \in S_{++}^n$.

From now on we employ the Q -pseudomonotone assumption.

(a) \implies (c): We approximate the problem (SDLCP) by the sequence of problems (SDLCP $_k$) and let $X^k \in \mathcal{S}(L, \Omega_k, Q)$ for all k with $\{\Omega_k\}$ being of Type I for $D \in S_{++}^n$. If there exists k such that $\langle D, X^k \rangle < \sigma_k$, then by implication (3.1) we conclude that X^k is a solution to (SDLCP). On the contrary, if $\langle D, X^k \rangle = \sigma_k$ for all k , then up to subsequences $\frac{X^k}{\sigma_k} \rightarrow V$ for some V . By hypothesis and Proposition 4.5(e) we conclude that L is copositive. From Lemmas 3.4 and 4.10 there exists a nonempty subindex set $J_V \subseteq I$ and subsequences $\{\frac{X^{k_m}}{\sigma_{k_m}}\}$ such that $\text{supp}\{\lambda(\frac{X^{k_m}}{\sigma_{k_m}})\} = J_V$, $V \in \mathcal{S}(L, S_+^n, 0_n)$, $\langle L(V), X^{k_m} \rangle = 0$, $\langle Q, V \rangle \leq 0$, and $L^\top(V) \in -S_+^n$. Moreover, as $L \in \#$ we have $(L + L^\top)(V) \in S_+^n$. From this, and the first implication of item (b) we conclude that $\langle Q, V \rangle = 0$. As $\langle (L + L^\top)(V), X^{k_m} \rangle \geq 0$, we get $\langle L^\top(V), X^{k_m} \rangle \geq 0$. Because of $L^\top(V) \in -S_+^n$ we obtain that $\langle L^\top(V), X^{k_m} \rangle = 0$. Therefore, from equality (3.2) we get

$$\theta_{k_m} = -\left\langle L(X^{k_m}) + Q, \frac{X^{k_m}}{\sigma_{k_m}} \right\rangle = -\langle L(X^{k_m}) + Q, V \rangle = -\langle X^{k_m}, L^\top(V) \rangle - \langle Q, V \rangle = 0,$$

where the second equality follows from Lemma 3.4(d). Hence, $\theta_{k_m} = 0$ and the existence of solutions follows from implication (3.1).

(d) \implies (c): We approximate the problem (SDLCP) by the sequence of problems (SDLCP $_k$) and let $X^k \in \mathcal{S}(L, \Omega_k, Q)$ for all k with $\{\Omega_k\}$ being of Type I for $D \in S_{++}^n$ being that of Definition 5.5. If there exists k such that $\langle D, X^k \rangle < \sigma_k$, then by implication (3.1) we have $X^k \in \mathcal{S}(L, S_+^n, Q)$ and we are done. On the contrary, if $\langle D, X^k \rangle = \sigma_k$ for all k , then $\|X^k\|_F \rightarrow +\infty$. As $L(\cdot) + Q$ is weakly proper on S_+^n , there exists $\hat{X} \in S_+^n$ and k such that $\langle L(\hat{X}) + Q, X^k - \hat{X} \rangle \geq 0$ and $\langle D, \hat{X} \rangle < \sigma_k$. By Q -pseudomonotonicity we have $\langle L(X^k) + Q, X^k - \hat{X} \rangle \geq 0$. We claim that $X^k \in \mathcal{S}(L, S_+^n, Q)$. If this is not the case, there exists $Z \in S_+^n$ such that $\sigma_k < \langle D, Z \rangle$ and $\langle L(X^k) + Q, Z - X^k \rangle < 0$. It is easy to see that there exists $t \in (0, 1)$ such that $X^t = t\hat{X} + (1-t)Z$ satisfies $\langle D, X^t \rangle < \sigma_k$; that is $X^t \in \Omega_k$. As X^k is a solution on Ω_k , we have $\langle L(X^k) + Q, X^t - X^k \rangle \geq 0$. On the other hand, as

$$\langle L(X^k) + Q, X^t - X^k \rangle = t\langle L(X^k) + Q, \hat{X} - X^k \rangle + (1-t)\langle L(X^k) + Q, Z - X^k \rangle < 0,$$

we get a contradiction. \square

Remark 5.7. *The equivalences (a)-(c) are extensions to SDLCPs of [7, Theorems 6.7] given for LCPs. It is important to point out that condition (b) differs from that for LCPs since the Farkas*

lemma in S^n has a different form than in \mathbb{R}^n (see [20]). This is because in general the image set $L(S^n)$ is not closed (see Example 5.12 below). The equivalence (c)-(d) is a particular case of [29, Theorems 3.13] that has been proved via exceptional families for nonlinear transformations based on Degree Theory. We provide a new proof by using our approach.

Second, we provide a coercive existence result under Q -pseudomonotonicity. This result is the SDLCP counterpart of [7, Theorem 6.8].

Theorem 5.8. *Let $Q \in S^n$ and $L \in \mathcal{L}(S^n)$. Consider the following statements:*

- (a): $\text{Feas}_s(L, S_+^n, Q) \neq \emptyset$;
- (b): $[V \in S_+^n, L^\top(V) \in -S_+^n, \langle Q, V \rangle \leq 0 \implies V = 0_n]$;
- (c): $\mathcal{S}(L, S_+^n, Q) \neq \emptyset$ and compact;
- (d): $\text{Feas}(L, S_+^n, Q) \neq \emptyset$ and $[V \in S_+^n, L^\top(V) \in -S_+^n, \langle Q, V \rangle = 0 \implies V = 0_n]$.

Then (a) \implies (b). In addition, if L is Q -pseudomonotone, then all the statements are equivalent.

Proof. (a) \implies (b): We argue by contradiction. Suppose that there exists $V \neq 0_n$ such that $V \in S_+^n$, $L^\top(V) \in -S_+^n$, and $\langle Q, V \rangle \leq 0$. Let $X^0 \in S_+^n$ be such that $L(X^0) + Q \in S_{++}^n$. Clearly, $\langle L(X^0) + Q, V \rangle > 0$. On the other hand, $\langle X^0, L^\top(V) \rangle + \langle Q, V \rangle \leq 0$, obtaining a contradiction.

From now on we employ the Q -pseudomonotone assumption.

(b) \implies (c): To prove the first part, we approximate the problem (SDLCP) by the sequence of problems (SDLCP $_k$) and let $X^k \in \mathcal{S}(L, \Omega_k, Q)$ for all k with $\{\Omega_k\}$ being of Type I for $D \in S_{++}^n$. There are two cases: there exists k such that $\langle D, X^k \rangle < \sigma_k$ and $\langle D, X^k \rangle = \sigma_k$ for all k . In the first case we conclude that X^k is a solution to (SDLCP) by implication (3.1). Whereas in the second, there exists $V \in S_+^n \setminus \{0_n\}$ such that $\frac{X^k}{\sigma_k} \rightarrow V$ up to subsequences. By using Lemma 4.10(c) we conclude that $L^\top(V) \in -S_+^n$ and $\langle Q, V \rangle \leq 0$, which by hypothesis implies that $V = 0_n$, a contradiction. This means that the second case does not hold. Hence, $\mathcal{S}(L, S_+^n, Q)$ is nonempty.

For the second part, we will prove that $\mathcal{S}(L, S_+^n, Q)^\infty = \{0_n\}$ by verifying that $\{V \in S_+^n : L^\top(V) \in -S_+^n, \langle Q, V \rangle \leq 0\} = \{0_n\}$ (cf. Proposition 4.11(b)). Indeed, this holds by hypothesis.

(c) \implies (a): If $\mathcal{S}(L, S_+^n, Q)$ is nonempty and compact, then by Proposition 4.11(b) we get

$$\{V \in S_+^n : L^\top(V) \in -S_+^n, \langle Q, V \rangle \leq 0\} = \{0_n\}.$$

This implies that $\text{Feas}_s(L, S_+^n, Q) \neq \emptyset$. Indeed, if not then we have $[L(S_+^n) + Q] \cap S_{++}^n = \emptyset$ and by a standard separation theorem we prove that there exists a vector $V \in S_+^n \setminus \{0_n\}$ such that $L^\top(V) \in -S_+^n$ and $\langle Q, V \rangle \leq 0$, a contradiction.

(c) \implies (d): The first part is obvious. The second follows from Proposition 4.11(b).

(d) \implies (c): The first part is a direct consequence of Theorem 5.6. For the second part we prove that $\mathcal{S}(L, S_+^n, Q)^\infty = \{0_n\}$. We argue by contradiction. Suppose that there exists $V \in S_+^n \setminus \{0_n\}$ such that $L^\top(V) \in -S_+^n$ and $\langle Q, V \rangle \leq 0$ (cf. Proposition 4.11(b)). First, if we assume that $\langle Q, V \rangle = 0$, then by hypothesis we conclude that $V = 0_n$, obtaining a contradiction. Second, if we now assume that $\langle Q, V \rangle < 0$, then for $\bar{X} \in \text{Feas}(L, S_+^n, Q)$ we have $\langle L(\bar{X}) + Q, V \rangle = \langle \bar{X}, L^\top(V) \rangle + \langle Q, V \rangle < 0$. On the other hand, $\langle L(\bar{X}) + Q, V \rangle \geq 0$, which is a contradiction. \square

Finally, we provide an existence result under Q -quasimonotonicity.

Theorem 5.9. *Let $Q \in S^n$ and L be a Q -quasimonotone linear transformation. If $L(\cdot) + Q$ is strictly weakly proper on S_+^n , then $\mathcal{S}(L, S_+^n, Q) \neq \emptyset$.*

Proof. This follows by the same arguments as in the proof of implication (d) \Rightarrow (c) of Theorem 5.6 by using Q -quasimonotonicity instead of Q -pseudomonotonicity. \square

Remark 5.10. *Theorem 5.9 is a particular case of [29, Theorems 3.14] proved via exceptional families for nonlinear transformations.*

5.2. Stability results. Stability of a SDLCP means that the basic parameters of the problem, the existence conditions, and the solution sets, “do not vary much” if we slightly change the initial data of the problem, i.e., the linear transformation L and the matrix Q .

We first establish some properties of various set-valued mappings related to SDLCPs. The first four items are extensions of various results from [22] given for polyhedral complementarity problems and the last one complements and sheds new light on [24, Theorem 2].

Theorem 5.11. (a): *The mapping $(L, Q) \mapsto \mathcal{S}(L, S_+^n, 0_n) \cap \{-Q\}^+$ is osc at every (L, Q) .*

(b): *The mapping $(L, Q) \mapsto \mathcal{S}(L, S_+^n, Q)$ is osc at every (L, Q) .*

(c): *If $\bar{L} \in \mathbf{R}_0$, then the mapping $(L, Q) \mapsto \mathcal{S}(L, S_+^n, Q)$ is locally bounded and usc at every (\bar{L}, Q) with $Q \in S^n$.*

(d): *If $\mathcal{S}(\bar{L}, S_+^n, \bar{Q})$ is bounded and the mapping $L \mapsto \mathcal{S}(L, S_+^n, \bar{Q})$ is usc at \bar{L} , then $\bar{L} \in \mathbf{R}_0$.*

(e): *If $\text{Ker}(\bar{L}) = \{0_n\}$ and $L^k \rightarrow \bar{L}$, then $L^k(S_+^n) \rightarrow \bar{L}(S_+^n)$ in Painlevé-Kuratowski sense. In particular, the image $\bar{L}(S_+^n)$ is closed.*

Proof. (a): Let $\{(L^k, Q^k)\}$ be a sequence such that $L^k \rightarrow L$ and $Q^k \rightarrow Q$ and let V be in $\limsup_k [\mathcal{S}(L^k, S_+^n, 0_n) \cap \{-Q^k\}^+]$. Then, there exists a subsequence $\{V^{k_j}\}$ such that $V^{k_j} \in S_+^n$, $L(V^{k_j}) \in S_+^n$, $\langle L(V^{k_j}), V^{k_j} \rangle = 0$, $\langle Q^{k_j}, V^{k_j} \rangle \leq 0$ all j , and $V^{k_j} \rightarrow V$ as $j \rightarrow +\infty$. After taking the limit in the above expressions we get $V \in S_+^n$, $L(V) \in S_+^n$, $\langle L(V), V \rangle = 0$, and $\langle Q, V \rangle \leq 0$; thus, $V \in \mathcal{S}(L, S_+^n, 0_n) \cap \{-Q\}^+$.

(b): This result follows from Theorem 3.1 with the sequence $\{\Omega_k\}$ being of Type II.

(c): We prove only the first part since the second one follows from it and [27, Theorem 5.19] which asserts that an osc mapping with bounded values is usc. Let $\bar{L} \in \mathbf{R}_0$. We argue by contradiction. Suppose that the mapping $\mathcal{S}(\cdot, S_+^n, \cdot)$ is not locally bounded at (\bar{L}, Q) ; i.e., there exist sequences $\{L^k\}$, $\{Q^k\}$ and $\{X^k\}$ such that $L^k \rightarrow \bar{L}$, $Q^k \rightarrow Q$, $X^k \in \mathcal{S}(L^k, S_+^n, Q^k)$ for all k , and $\|X^k\|_F \rightarrow +\infty$. Clearly, there exists a subsequence $\{\frac{X^{k_j}}{\|X^{k_j}\|_F}\}$ converging to some V such that $\|V\|_F = 1$. This means that $V \in \limsup_k^\infty \mathcal{S}(L^k, S_+^n, Q^k)$. By Lemma 3.2 with $\{\Omega_k\}$ being of Type II, we conclude that $V \in \mathcal{S}(\bar{L}, S_+^n, 0_n) \setminus \{0_n\}$, a contradiction.

(d): We argue by contradiction. Suppose that there exists $V \in \mathcal{S}(\bar{L}, S_+^n, 0) \setminus \{0_n\}$. Let us define the linear transformation $L^k(X) := \bar{L}(X) - \frac{1}{k} \langle D, X \rangle \bar{Q}$ for some $D \in S_{++}^n$ such that $\langle D, V \rangle = 1$. It is not difficult to check that $kV \in \mathcal{S}(L^k, S_+^n, \bar{Q})$ for all k . As $\mathcal{S}(\bar{L}, S_+^n, \bar{Q})$ is bounded, there exists a scalar $r > 0$ such that $\mathcal{S}(\bar{L}, S_+^n, \bar{Q}) \subseteq \text{rint}(\mathbb{B}(0_n, 1))$. Moreover, as $L^k \rightarrow \bar{L}$ and $\mathcal{S}(\cdot, S_+^n, \bar{Q})$ is usc at \bar{L} , one has $kV \in \text{rint}(\mathbb{B}(0_n, 1))$ for k large enough, a contradiction.

(e): We need to prove the inclusions: $\limsup_k L^k(S_+^n) \subseteq \bar{L}(S_+^n) \subseteq \liminf_k L^k(S_+^n)$. Let us prove the first inclusion. Let $Y \in \limsup_k L^k(S_+^n)$. There exists a subsequence $\{Y^{k_j}\}$ such that $Y^{k_j} \in L^{k_j}(S_+^n)$ for all j and $Y^{k_j} \rightarrow Y$ as $j \rightarrow +\infty$. Clearly, there exists $X^{k_j} \in S_+^n$ such that $Y^{k_j} = L^{k_j}(X^{k_j})$ for all j . If the sequence $\{X^{k_j}\}$ is bounded, then there exists $X \in S_+^n$ such that up to subsequences $X^{k_j} \rightarrow X$ as $j \rightarrow +\infty$. After taking limit we get $Y = \bar{L}(X)$; thus, $Y \in \bar{L}(S_+^n)$ and the inclusion holds. On the contrary, if $\{X^{k_j}\}$ is unbounded, then we can assume that $\|X^{k_j}\|_F \rightarrow +\infty$. Clearly, there exists $V \in S_+^n$ such that up to subsequences $\frac{X^{k_j}}{\|X^{k_j}\|_F} \rightarrow V$

and $\|V\|_F = 1$. Note that $\frac{Y^{k_j}}{\|X^{k_j}\|_F} = L^{k_j} \left(\frac{X^{k_j}}{\|X^{k_j}\|_F} \right)$. Taking limit as $j \rightarrow +\infty$, we deduce that $\bar{L}(V) = 0$, that is, $V \in \text{Ker}(\bar{L})$, which is a contradiction.

Now, we prove the second inclusion. Let $Y \in \bar{L}(S_+^n)$. Then $Y = \bar{L}(X)$ for some $X \in S_+^n$. For $Y^k := L^k(X)$ for all k we have $Y^k \in L^k(S_+^n)$ for all k and $Y^k \rightarrow Y$. Therefore, $Y \in \liminf_k L^k(S_+^n)$.

The last part holds since the limit of a sequence of sets is closed by [27, Proposition 4.4]. \square

Example 5.12. (1) *In general the image of S_+^n under $L \in \mathcal{L}(S^n)$ is not closed as shown in [24, Example 1]. Let*

$$L \left(\begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix} \right) = \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & 0 \end{pmatrix}.$$

We observe that

$$\begin{pmatrix} \frac{1}{k} & -1 \\ -1 & k \end{pmatrix} \in S_+^2 \quad \text{and} \quad L \left(\begin{pmatrix} \frac{1}{k} & -1 \\ -1 & k \end{pmatrix} \right) \rightarrow \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad \text{as } k \rightarrow +\infty.$$

However, there exists no $X \in S_+^2$ such that $L(X) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

(2) *In general the mapping $(L, Q) \mapsto \mathcal{S}(L, S_+^n, Q)$ is not isc. Indeed, let*

$$L(X) = \begin{pmatrix} 2x & y \\ y & 0 \end{pmatrix} \quad \text{and} \quad L^k(X) = \begin{pmatrix} 2x & y \\ y & \frac{1}{k}z \end{pmatrix} \quad \text{where } X = \begin{pmatrix} x & y \\ y & z \end{pmatrix}.$$

We have $L^k \rightarrow L$,

$$\mathcal{S}(L, S_+^2, 0_n) = \left\{ X(t) = \begin{pmatrix} 0 & 0 \\ 0 & t \end{pmatrix} : t \geq 0 \right\} \quad \text{and} \quad \mathcal{S}(L^k, S_+^2, 0_n) = \{0_n\}$$

for all k . Clearly, for $X(1) \in \mathcal{S}(L, S_+^2, 0_n)$ there is no sequence $\{X^k\}$ such that $X^k \in \mathcal{S}(L^k, S_+^2, 0_n)$ and $X^k \rightarrow X(1)$.

As a consequence of Theorem 5.11 we extend [9, Theorem 1] proved for LCPs.

Corollary 5.13. *The following classes of linear transformations are open:*

- (a): *Strictly monotone transformations;*
- (b): $\mathbf{R}(D)$ for $D \in S_{++}^n$ (hence \mathbf{R});
- (c): \mathbf{R}_0 .

Proof. (a): We argue by contradiction. Suppose that there exists a strictly monotone linear transformation L and a sequence $\{L^k\}$ not in such a class that converges to L . By definition there exists $Z^k \in S^n \setminus \{0_n\}$ such that $\langle L^k(Z^k), Z^k \rangle \leq 0$ for all k . By linearity we may assume that $\|Z^k\|_F = 1$ for such k . From this, we conclude that there exists $Z \in S^n \setminus \{0_n\}$ such that, up to subsequences, $Z^k \rightarrow Z$. After taking limit in the last inequality we obtain $\langle L(Z), Z \rangle \leq 0$, a contradiction.

(b): We argue by contradiction. Suppose that there exist $L \in \mathbf{R}(D)$ and a sequence $\{L^k\}$ not in such a class that converges to L . By definition there exist scalars $\tau_k \geq 0$ and $X^k \in \mathcal{S}(L^k, S_+^n, \tau_k D) \setminus \{0_n\}$ for all k . By linearity and redefining each τ_k if necessary, we can assume that $\langle D, X^k \rangle = 1$ for all k . Therefore, up to subsequences $X^k \rightarrow X$ for some $X \neq 0_n$. Since $\langle L^k(X^k) + \tau_k D, X^k \rangle = 0$ for all k , it follows that $\tau_k = -\langle L^k(X^k), X^k \rangle$ which implies that

$\tau_k \rightarrow \bar{\tau} = -\langle L(X), X \rangle \geq 0$. As, $X \in \limsup_k \mathcal{S}(L^k, S_+^n, \tau_k D)$ and $\tau_k \rightarrow \bar{\tau}$, by Theorem 5.11(b) we conclude that $X \in \mathcal{S}(L, S_+^n, \bar{\tau} D) \setminus \{0_n\}$, a contradiction

(c): We proceed as in item (b) setting $\tau_k = 0$ for all k . \square

The next theorem is a stability result for SDLCPs under copositivity assumptions. It extends [4, Theorem 7.5.1] given for LCPs.

Theorem 5.14. *Let $\bar{L} \in \mathcal{L}(S^n)$ and $\bar{Q} \in S^n$ be such that*

$$(5.1) \quad \bar{Q} \in \text{int}[\mathcal{S}(\bar{L}, S_+^n, 0_n)^+].$$

Then, there exist scalars $\varepsilon, r > 0$, such that for all $Q \in S^n$ and $L \in \mathcal{L}(S^n)$ satisfying $\|Q - \bar{Q}\|_F + \|L - \bar{L}\| < \varepsilon$ one has

(a): $Q \in \text{int}[\mathcal{S}(L, S_+^n, 0_n)^+]$;

(b): $\mathcal{S}(L, S_+^n, Q) \neq \emptyset$ and $\mathcal{S}(L, S_+^n, Q) \subseteq r\mathbb{B}(0_n, 1)$ for all L being also copositive.

Proof. (a): We argue by contradiction. Suppose that there exist sequences $\{Q^k\}$, $\{L^k\}$ and $\{V^k\}$ satisfying $Q^k \rightarrow \bar{Q}$, $L^k \rightarrow \bar{L}$, $V^k \in \mathcal{S}(L^k, S_+^n, 0_n) \setminus \{0_n\}$, and $\langle Q^k, V^k \rangle \leq 0$ for all k . By linearity we may assume that $\|V^k\|_F = 1$ for such k and therefore, there exists $V \in S_+^n$ such that up to subsequences $V^k \rightarrow V$ and $\|V\|_F = 1$. As $V \in \limsup_k \mathcal{S}(L^k, S_+^n, 0_n)$, by Theorem 3.1 with $\{\Omega_k\}$ being of Type II we obtain $V \in \mathcal{S}(\bar{L}, S_+^n, 0_n) \setminus \{0_n\}$. Moreover, after taking limit we get $\langle \bar{Q}, V \rangle \leq 0$; contradicting condition (5.1).

(b): We have $\mathcal{S}(L, S_+^n, Q) \neq \emptyset$ by item (a) and Theorem 5.1(b). It remains to prove the second part. We argue by contradiction. Suppose that there exist sequences $\{L^k\}$, $\{Q^k\}$, and $\{X^k\}$ such that each L^k is copositive, $L^k \rightarrow \bar{L}$, $Q^k \rightarrow \bar{Q}$, $X^k \in \mathcal{S}(L^k, S_+^n, Q^k)$ for all k , and $\|X^k\|_F \rightarrow +\infty$. There exists some $V \in S_+^n$ such that up to subsequences $\frac{X^k}{\|X^k\|_F} \rightarrow V$ and $\|V\|_F = 1$. From this, we have $V \in \limsup_k^\infty \mathcal{S}(L^k, S_+^n, Q^k)$, which by Lemma 3.2 implies that $V \in \mathcal{S}(L, S_+^n, 0_n) \setminus \{0_n\}$. As each L^k is copositive, we get $\langle L^k(X^k) + Q^k, X^k \rangle \geq \langle Q^k, X^k \rangle$ for all k . Moreover, as $\langle L^k(X^k) + Q^k, X^k \rangle = 0$ we conclude that $\langle Q^k, X^k \rangle \leq 0$ for all k . After dividing by $\|X^k\|_F$ and taking limit we get $\langle \bar{Q}, V \rangle \leq 0$; contradicting condition (5.1). \square

The next theorem is a counterpart to Theorem 5.14 but under pseudomonotonicity assumptions. It extends [7, Theorem 6.9 and Corollary 6.5] given for LCPs.

Theorem 5.15. *Let $\bar{L} \in \mathcal{L}(S^n)$ and $\bar{Q} \in S^n$ be such that*

$$(5.2) \quad V \in S_+^n, \bar{L}^\top(V) \in -S_+^n, \langle \bar{Q}, V \rangle \leq 0 \implies V = 0_n.$$

Then, there exist scalars $\varepsilon, r > 0$, such that for all $Q \in S^n$ and $L \in \mathcal{L}(S^n)$ satisfying $\|Q - \bar{Q}\|_F + \|L - \bar{L}\| < \varepsilon$ one has

(a): $V \in S_+^n, L^\top(V) \in -S_+^n, \langle Q, V \rangle \leq 0 \implies V = 0_n$;

(b): $\mathcal{S}(L, S_+^n, Q) \neq \emptyset$ and $\mathcal{S}(L, S_+^n, Q) \subseteq r\mathbb{B}(0_n, 1)$ for all L being also Q -pseudomonotone.

Proof. (a): We argue by contradiction. Suppose that there exist sequences $\{Q^k\}$, $\{L^k\}$ and $\{V^k\}$ satisfying $Q^k \rightarrow \bar{Q}$, $L^k \rightarrow \bar{L}$, $V^k \in S_+^n \setminus \{0_n\}$, $L^\top(V^k) \in -S_+^n$, and $\langle Q^k, V^k \rangle \leq 0$ for all k . By linearity we may assume that $\|V^k\|_F = 1$ for such k and therefore, there exists some $V \in S_+^n$ such that up to subsequences $V^k \rightarrow V$ and $\|V\|_F = 1$. After taking limit we get $\bar{L}^\top(V) \in -S_+^n$ and $\langle \bar{Q}, V \rangle \leq 0$, contradicting condition (5.2).

(b): We have $\mathcal{S}(L, S_+^n, Q) \neq \emptyset$ by item (a) and Theorem 5.8(a). It remains to prove the second part. We argue by contradiction. Suppose that there exist sequences $\{L^k\}$, $\{Q^k\}$, and $\{X^k\}$ such that each L^k is Q^k -pseudomonotone, $L^k \rightarrow \bar{L}$, $Q^k \rightarrow \bar{Q}$, $X^k \in \mathcal{S}(L^k, S_+^n, Q^k)$ for all k , and $\|X^k\|_F \rightarrow +\infty$. There exists $V \in S_+^n$ such that up to subsequences $\frac{X^k}{\|X^k\|_F} \rightarrow V$ and $\|V\|_F = 1$. As $\langle L^k(X^k) + Q^k, Y - X^k \rangle \geq 0$ for all k and $Y \in S_+^n$, by Q^k -pseudomonotonicity we get $\langle L^k(Y) + Q^k, Y - X^k \rangle \geq 0$ for such k and $Y \in S_+^n$. After dividing by $\|X^k\|_F$ and taking limit we obtain $\langle \bar{L}(Y) + \bar{Q}, V \rangle \leq 0$ for all $Y \in S_+^n$. From this, we obtain that $\bar{L}^\top(V) \in -S_+^n$ and $\langle \bar{Q}, V \rangle \leq 0$, contradicting condition (5.2). \square

Remark 5.16. *In other words, part (b) of Theorem 5.14 (resp. Theorem 5.15) asserts that under condition (5.1) (resp. (5.2)) the mapping $(L, Q) \mapsto \mathcal{S}(L, S_+^n, Q)$ is uniformly locally bounded at (\bar{L}, \bar{Q}) within the class of copositive (resp. pseudomonotone) linear transformations.*

By following some ideas from [4, 21, 22] and by employing our previous results we obtain a locally Lipschitz continuity property of the solution set mapping \mathcal{S} to SDLCPs. To do this we recall the following property of the set-valued map Φ_L defined by

$$Q \in S^n \mapsto \Phi_L(Q) := \mathcal{S}(L, S_+^n, Q).$$

Theorem 5.17. *If L is strictly monotone, then Φ_L satisfies the following properties:*

- (a): $\Phi_L(Q) \neq \emptyset$ for all $Q \in S^n$; i.e., $\text{dom } \Phi_L = S^n$;
- (b): Φ_L is single-valued;
- (c): Φ_L is Lipschitzian; i.e., there exists a scalar $\mu > 0$ such that

$$(5.3) \quad \Phi_L(Q) \subseteq \Phi_L(Q') + \mu \|Q - Q'\|_F \mathbb{B}(0_n, 1), \quad \forall Q, Q' \in S^n.$$

Proof. (a): By Proposition 4.5(d) we have that L is Q -pseudomonotone for every $Q \in S^n$. Moreover, it is easy to check that condition (5.2) holds. From this and Theorem 5.8 we conclude that $\mathcal{S}(L, S_+^n, Q) \neq \emptyset$ for all $Q \in S^n$.

(b): See [6, Theorem 2.3.3].

(c): See [2, Theorem 2.1]. \square

Remark 5.18. *In [2, Example 2.1] it is shown that the monotonicity property is not sufficient to conclude that Φ_L is Lipschitzian.*

We now extend Theorem 5.17 to the case when not only the matrix Q varies but also the linear transformation L .

Theorem 5.19. *If \bar{L} is strictly monotone, then there exist scalars $\eta, r, \varepsilon > 0$ such that for all $Q \in S^n$ and $L \in \mathcal{L}(S^n)$ satisfying $\|Q - \bar{Q}\|_F + \|L - \bar{L}\| < \varepsilon$ one has*

- (a): L is strictly monotone;
- (b): $\mathcal{S}(L, S_+^n, Q)$ is a singleton and $\mathcal{S}(L, S_+^n, Q) \subseteq r\mathbb{B}(0_n, 1)$;
- (c): $\mathcal{S}(L, S_+^n, Q) \subseteq \mathcal{S}(\bar{L}, S_+^n, \bar{Q}) + \eta \left(\|Q - \bar{Q}\|_F + \|L - \bar{L}\| \right) \mathbb{B}(0_n, 1)$.

Proof. (a): By Corollary 5.13 there exists $\varepsilon_1 > 0$ such that every transformation L satisfying $\|L - \bar{L}\| \leq \varepsilon_1$ is strictly monotone.

(b): This follows from Theorem 5.17 and Theorem 5.15(b) for some $r, \varepsilon_2 > 0$.

(c): Let us set $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$ and let L and Q be satisfying $\|Q - \bar{Q}\|_F + \|L - \bar{L}\| < \varepsilon$. If $\mathcal{S}(L, S_+^n, Q) = \{X\}$, then defining the matrix $Q^0 := Q + [L(X) - \bar{L}(X)]$ it is not difficult to check

that $\mathcal{S}(\bar{L}, S_+^n, Q^0) = \{X\}$; thus, $\mathcal{S}(L, S_+^n, Q) = \mathcal{S}(\bar{L}, S_+^n, Q^0)$. By Theorem 5.17 there exists a scalar $\mu > 0$ such that

$$\mathcal{S}(\bar{L}, S_+^n, Q^0) \subseteq \mathcal{S}(\bar{L}, S_+^n, \bar{Q}) + \mu \|Q^0 - \bar{Q}\|_F \mathbb{B}(0_n, 1).$$

By taking into account the definition of Q^0 and item (b) we obtain

$$\|Q^0 - \bar{Q}\|_F \leq \|Q - \bar{Q}\|_F + \|L(X) - \bar{L}(X)\|_F \leq \|Q - \bar{Q}\|_F + r \|L - \bar{L}\|.$$

From these assertions we conclude that

$$\mathcal{S}(L, S_+^n, Q) \subseteq \mathcal{S}(\bar{L}, S_+^n, \bar{Q}) + \mu \left(\|Q - \bar{Q}\|_F + r \|L - \bar{L}\| \right) \mathbb{B}(0_n, 1)$$

and the result follows by setting $\eta := \mu \max\{1, r\}$. \square

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