LINEAR COMPLEMENTARITY PROBLEMS OVER SYMMETRIC CONES: CHARACTERIZATION OF $\mathbf{Q}_b$-TRANSFORMATIONS AND EXISTENCE RESULTS

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Abstract. This paper is devoted to the study of the symmetric cone linear complementarity problem (SCLCP). In this context, our aim is to characterize the class $\mathbf{Q}_b$ in terms of larger classes, such as $\mathbf{Q}$ and $\mathbf{R}_0$. For this, we introduce the class $\mathbf{F}$ and García's transformations. We studied them for concrete particular instances (such as second-order and semidefinite linear complementarity problems) and for specific examples (Lyapunov, Stein functions, among others). This naturally permits to establish noncoercive existence results for SCLCPs.

Key words: Euclidean Jordan algebra, linear complementarity problem, symmetric cone, $\mathbf{Q_b}$-transformation, $\mathbf{Q}$-transformation, García's transformation.

1. Introduction

Consider a Euclidean Jordan algebra $(\mathbb{V}, \circ, \langle\cdot,\cdot\rangle)$, where $\circ$ denotes the Jordan product and $\mathbb{V}$ a finite-dimensional vector space over the real field $\mathbb{R}$ equipped with the inner product $\langle\cdot,\cdot\rangle$. Let $L : \mathbb{V} \to \mathbb{V}$ be a linear transformation (for short $L \in \mathcal{L}(\mathbb{V})$) and $q \in \mathbb{V}$. This paper is devoted to the study of the *symmetric cone linear complementarity problem* (SCLCP), which consists of finding an element $\bar{x}$ such that:

$$\bar{x} \in \mathcal{K}, \quad \bar{y} = L(\bar{x}) + q \in \mathcal{K} \ \text{ and } \ \langle \bar{y}, \bar{x}\rangle = 0. \tag{1}$$

Here, $\mathcal{K} := \{y = x \circ x \,:\, x \in \mathbb{V}\}$, denotes the set of squares elements in $\mathbb{V}$. In what follows, this problem will be denoted by $\mathrm{LCP}(L, \mathcal{K}, q)$, and its solution will be denoted by $\mathrm{SOL}(L, \mathcal{K}, q)$. Also, its feasible set is defined to be $\mathrm{FEAS}(L, \mathcal{K}, q) := \{x \in \mathcal{K} : L(x) + q \in \mathcal{K}\}$. This problem is a particular case of a variational inequality problem (e.g. [1]), and it provides a simple unified framework for various existing complementarity problems such as the linear complementarity problem over the nonnegative orthant (LCP) (e.g. [2]), the second-order cone linear complementarity problem (SOCLCP) (e.g. [3, 4]) and the semidefinite linear complementarity problem (SDLCP) (e.g. [5, 6]), and hence has extensive applications in engineering, economics, game theory, management science, and other fields; see [1, 7, 8, 9] and references therein.

In the last years, SCLCP has been studied by divers authors, with special emphasis in its particular cases: SOCLP and SDLCP. For instance, Gowda et al. [10] extended divers

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types of **P**-transformations and the **GUS**-transformation for linear transformation from LCP to the SCLCP setting. These notions are further exploited in the papers [11, 12, 13].

A key issue in linear complementarity problems -of the form SCLCP- consists in finding necessary and sufficient conditions on the linear transformation L that ensures the nonemptyness and bounded of the solution set $\mathrm{SOL}(L, \mathcal{K}, q)$ for all q. The class of linear functions L satisfying this condition is called $\mathbf{Q}_b$-transformation.

The aim of this paper is to characterize class $\mathbf{Q}_b$ in the context of SCLCP. More precisely, we are interested in to find large classes of linear functions L for which $\mathbf{Q}_b$ behaves similar than larger classes, such as $\mathbf{Q}$ and $\mathbf{R}_0$. For this, on the one hand, we extend from LCPs [14] and SDLCPs [15] particular class called $\mathbf{F}$. Within this class we prove that classes $\mathbf{Q}_b$ and $\mathbf{Q}$ coincide. Then, we consider subclasses of $\mathbf{F}$ (called $\mathbf{F}_1$ and $\mathbf{F}_2$) and study their connections as well as different examples of linear functions L belonging to them. We also extended a particular class, called $\mathbf{T}$, which was originally defined in [16] in the LCP framework, and compare it with class $\mathbf{F}$. Actually, we prove that $\mathbf{T} \subseteq \mathbf{F}_2$. We also specialize all these classes to particular SCLCP such as LCP, SOCLCP and SDLCP. On the other hand, we define the class of García's transformations for SCLCPs. The latter is an extension from LCPs to this setting (cf. [17]). See [6] to see its extension to SDLCP. Within this class, we are able to prove that classes $\mathbf{Q}$ and $\mathbf{R}_0$ coincide. This allows to state some existence result for SCLCPs.

The existing literature on SCLCPs includes only some few works about the class $\mathbf{Q}_b$. For instance, in one of this articles, Gowda and Tao [18] show that, within the class $\mathbf{Z}$, classes $\mathbf{Q}$ and $\mathbf{S}$ behave similarly (see the definitions of $\mathbf{S}$- and $\mathbf{Z}$-transformations in Section 2.2).

This paper is organized as follows. Section 2 is devoted to the preliminaries. It is split into two subsections; first one recalls basic results on Euclidean Jordan algebras, while second one summarizes some classes of linear transformations in $\mathcal{L}(\mathbb{V})$ with their respective connections. In Sections 3 and 4, we established our main results described above. Indeed, Section 3 is dedicated to the study of linear functions for which classes $\mathbf{Q}$ and $\mathbf{Q_b}$ coincide, while Section 4 is devoted to existence results for SCLCPs associated with García's transformations, for which we prove that $\mathbf{R}_0$ and $\mathbf{Q_b}$.

2. Preliminaries

2.1. **Euclidean Jordan algebras review.** In this subsection, we briefly describe some concepts, properties, and results from Euclidean Jordan algebras that are needed in this paper and that have become important in the study of conic optimization; see, e.g., Schmieta and Alizadeh [19]. Most of this material can be found in Faraut and Koyányi [20].

A *Euclidean Jordan algebra* is a triple $(\mathbb{V}, \circ, \langle \cdot, \cdot \rangle)$, where $(\mathbb{V}, \langle \cdot, \cdot \rangle)$ is a finite-dimensional vector space over $\mathbb{R}$ equipped with an inner product $\langle \cdot, \cdot \rangle$, the jordan product $(x, y) \mapsto x \circ y : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ is a bilinear mapping satisfying the following three conditions:

- **(i)** $x \circ y = y \circ x$ for all $x, y \in \mathbb{V}$,
- **(ii)** $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in \mathbb{V}$ where $x^2 = x \circ x$, and
- **(ii)** $\langle x \circ y, z \rangle = \langle y, x \circ z \rangle$ for all $x, y, z \in \mathbb{V}$,

and there exists a unitary element $e \in \mathbb{V}$ satisfying that $x \circ e = x$ for all $x \in \mathbb{V}$. Henceforth, we simply say that $\mathbb{V}$ is a Euclidean Jordan algebra and $x \circ y$ is called the *Jordan product* of x and y. A Euclidean Jordan algebra is said to be *simple* if it is not a direct sum of two Euclidean Jordan algebras.

In an Euclidean Jordan algebra $\mathbb{V}$, it is known that the set of squares $\mathcal{K} = \{x^2 : x \in \mathbb{V}\}$ is a *symmetric cone* (see [20, Theorem III.2.1]). This means that $\mathcal{K}$ is a self-dual closed convex cone with nonempty interior $\text{int}(\mathcal{K})$ and for any two elements $x, y \in \text{int}(\mathcal{K})$, there exists an invertible linear transformation $\Gamma : \mathbb{V} \to \mathbb{V}$ such that $\Gamma(\mathcal{K}) = \mathcal{K}$ and $\Gamma(x) = y$.

The *rank* of $(\mathbb{V}, \circ, \langle \cdot, \cdot \rangle)$ is defined as

$$r = \max\{\deg(x) : x \in \mathbb{V}\},$$

where $\deg(x)$ is the degree of $x \in \mathbb{V}$ given by

$$\deg(x) = \min\{k > 0 : \{e, x, x^2, \dots, x^k\} \text{ is linearly dependent}\}.$$

Example 2.1. Typical examples of Euclidean Jordan algebras are:

(i) *Euclidean Jordan algebra of n-dimensional vectors:*

$$\mathbb{V} = \mathbb{R}^n, \quad \mathcal{K} = \mathbb{R}^n_+, \quad r = n, \quad \langle x, y \rangle = \sum_{i=1}^{n} x_i y_i, \quad x \circ y = x * y,$$

where $x * y$ denotes the componentwise product of vectors x and y. Here, the unitary element is $e = (1, \dots, 1) \in \mathbb{R}^n$.

(ii) *Euclidean Jordan algebra of quadratic forms:*

$$\mathbb{V} = \mathbb{R}^n, \quad \mathcal{K} = \mathcal{L}^n_+ = \{x = (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1} : \ \|\bar{x}\| \leq x_1\},, \quad r = 2,$$

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i, \quad x \circ y = (x_1, \bar{x}) \circ (y_1, \bar{y}) = (\langle x, y \rangle, x_1 \bar{y} + y_1 \bar{x}).$$

In this algebra, the cone of squares is called the Lorentz cone (or the second-order cone). Moreover, the unitary element is $e = (1, 0, \dots, 0) \in \mathbb{R}^n$.

(iii) *Euclidean Jordan algebra of n-dimensional symmetric matrices:* Let $\mathcal{S}^n$ be the set of all $n \times n$ real symmetric matrices and $\mathcal{S}^n_+$ be the cone of $n \times n$ symmetric positive semidefinite matrices.

$$\mathbb{V} = \mathcal{S}^n, \quad \mathcal{K} = \mathcal{S}^n_+, \quad r = n, \quad \langle X, Y \rangle = \text{tr}(XY), \quad X \circ Y = \frac{1}{2}(XY + YX).$$

Here tr denotes the trace of a matrix $X = (X_{ij}) \in \mathcal{S}^n$. In this setting, the identity matrix $I \in \mathbb{R}^{n \times n}$ is the unit element e.

Other examples are the set of $n \times n$ hermitian positive semidefinite matrices made of complex numbers, the set of $n \times n$ positive semidefinite matrices with quaternion entries, the set of 3×3 positive semidefinite matrices with octonion entries, the exceptional 27-dimensional Albert octonion cone (see [20, 21]).

An element $c \in \mathbb{V}$ is an *idempotent* if $c^2 = c$; it is a *primitive idempotent* if it is nonzero and cannot be written as a sum of two nonzero idempotents. We say that a finite set $\{e_1, \dots, e_r\}$ of primitive idempotents in $\mathbb{V}$ is a *Jordan frame* if

$$e_i \circ e_j = 0 \ \text{ for all } \ i \neq j, \ \text{ and } \ \sum_{i=1}^{r} e_i = e.$$

Note that $\langle e_i, e_j \rangle = \langle e_i \circ e_j, e \rangle = 0$ whenever $i \neq j$. The following theorem gives us a spectral decomposition for the elements in an Euclidean Jordan algebra (see Theorem III.1.2 of [20]).

Theorem 2.1 (Spectral decomposition theorem)**.** *Suppose that $(\mathbb{V}, \circ, \langle \cdot, \cdot \rangle)$ is a Euclidean Jordan algebra with rank r. Then, for every $x \in \mathbb{V}$, there exists a Jordan frame $\{e_1, \dots, e_r\}$ and real numbers $\lambda_1(x), \dots, \lambda_r(x)$ such that $x = \lambda_1(x)e_1 + \ldots + \lambda_r(x)e_r$. The numbers $\lambda_i(x)$'s, called the eigenvalues of x, are uniquely determined.*

It is easy to show that $x \in \mathcal{K}$ (resp. $\mathrm{int}(\mathcal{K})$) if and only if every eigenvalue $\lambda_i(x)$ of x is nonnegative (resp. positive). Due to the uniqueness of the eigenvalues $\lambda_i(x)$ we can define the trace of a element x as $\mathrm{tr}(x) = \sum_{i=1}^r \lambda_i(x)$. Notice that the latter also implies that $\mathrm{tr}(c) = 1$ for every primitive idempotent c in $\mathbb{V}$.

Remark 2.2. *We recall that in any simple Euclidean Jordan algebra $\mathbb{V}$, there exists a $\theta > 0$ such that $\langle x, y\rangle = \theta \cdot \mathrm{tr}(x \circ y)$ (see* [20, Proposition III.4.1]*). Hence, $\theta = \langle c, e\rangle = \|c\|^2$ for every primitive idempotent c in $\mathbb{V}$. In particular, we have $\|e_i\|^2 = \theta$ for every element e_i of a Jordan frame $\{e_1, \dots, e_r\}$..*

For any $a \in \mathbb{V}$, the *Lyapunov transformation* $L_a : \mathbb{V} \to \mathbb{V}$ and the *quadratic representation* $P_a : \mathbb{V} \to \mathbb{V}$ are defined as

$$L_a(x) := a \circ x, \quad P_a(x) := (2L_a^2 - L_{a^2})(x) = 2a \circ (a \circ x) - a^2 \circ x, \quad \text{for all } x \in \mathbb{V}. \tag{2}$$

These transformations are linear and self-adjoint on $\mathbb{V}$ (see [20]). In the following example, we describe these transformations in the Euclidean Jordan algebras defined in Example 2.1.

Example 2.2. (*i*) *For the Euclidean Jordan algebra of n-dimensional vectors, the above transformations are given by*

$$L_a(x) = \mathrm{Diag}(a)x, \quad P_a(x) = \mathrm{Diag}(a^2)x,$$

where $\mathrm{Diag}(q)$ *denotes a diagonal matrix of size n whose diagonal entries are given by the entries of q.*

(*ii*) *For Euclidean Jordan algebra of quadratic forms, the above transformations are*

$$L_a(x) = \begin{pmatrix} a_1 & \bar{a}^\top \\ \bar{a} & a_1 I \end{pmatrix} \begin{pmatrix} x_1 \\ \bar{x} \end{pmatrix}, \quad P_a(x) = \begin{pmatrix} \|a\|^2 & 2a_1\bar{a}^\top \\ 2a_1\bar{a} & (a_1^2 - \|\bar{a}\|^2)I + 2\bar{a}\bar{a}^\top \end{pmatrix} \begin{pmatrix} x_1 \\ \bar{x} \end{pmatrix}.$$

(*iii*) *For the Euclidean Jordan algebra of n-dimensional symmetric matrices, the above transformations are given by*

$$L_A(X) = A \circ X = \frac{1}{2}(AX + XA), \quad P_A(X) = AXA,$$

A useful tool in the theory of Euclidean Jordan algebras is the *Peirce decomposition* theorem which is stated as follows (see Theorem IV.2.1 of [20]).

Theorem 2.3. *Let $(\mathbb{V}, \circ, \langle\cdot,\cdot\rangle)$ be a Euclidean Jordan algebra with rank r and let $\{e_1, \dots, e_r\}$ be a Jordan frame in $\mathbb{V}$. For $i, j \in \{1, 2, \dots, r\}$, define the eigenspaces*

$$\mathbb{V}_{ii} := \{x \in \mathbb{V} : x \circ e_i = x\} = \mathbb{R}e_i, \quad \mathbb{V}_{ij} := \{x \in \mathbb{V} : x \circ e_i = \tfrac{1}{2}x = x \circ e_j\},\ i \neq j.$$

Then, the space $\mathbb{V}$ is the orthogonal direct sum of subspaces $\mathbb{V}_{ij}$ $(i \leq j)$. Furthermore,

(*a*) $\mathbb{V}_{ij} \circ \mathbb{V}_{ij} \subseteq \mathbb{V}_{ii} + \mathbb{V}_{jj}$;
(*b*) $\mathbb{V}_{ij} \circ \mathbb{V}_{jk} \subseteq \mathbb{V}_{ik}$ *if* $i \neq k$;
(*c*) $\mathbb{V}_{ij} \circ \mathbb{V}_{kl} = \{0\}$ *if* $\{i, j\} \cap \{k, l\} = \emptyset$.

Thus, given any Jordan frame $\{e_1, \dots, e_r\}$, we can write any element $x \in \mathbb{V}$ as

$$x = \sum_{1 \leq i \leq j \leq r} x_{ij} = \sum_{i=1}^{r} x_i e_i + \sum_{1 \leq i < j \leq r} x_{ij}, \tag{3}$$

where $x_i \in \mathbb{R}$ and $x_{ij} \in \mathbb{V}_{ij}$. Equation (3) corresponds to the Peirce decomposition of x associated with $\{e_1, \dots, e_r\}$.

Example 2.3. (*i*) Let $\mathbb{V} = \mathbb{R}^n$ and $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbb{R}^n$, that is, e_i is a vector with 1 in the i-th entry and 0's elsewhere. It is easily seen that $\{e_1, \ldots, e_n\}$ is a Jordan frame in $\mathbb{R}^n$, called *canonical Jordan frame*, and also that

$$\mathbb{V}_{ii} = \{\kappa e_i : \kappa \in \mathbb{R}\}, \text{ for } i = 1, \ldots, n, \quad \mathbb{V}_{ij} = \{0\}, \ i \neq j.$$

Hence, any element $x \in \mathbb{R}^n$ can be written as $x = \sum_{i=1}^n \kappa_i e_i$, which denotes its Pierce decomposition associated with $\{e_1, \ldots, e_n\}$.

(*ii*) Let $\mathbb{V} = \mathbb{R}^n$ and $\{e_1, e_2\}$ defined by $e_1 = (\frac{1}{2}, \frac{1}{2}, \mathbf{0}_{n-2})$, $e_2 = (\frac{1}{2}, -\frac{1}{2}, \mathbf{0}_{n-2})$, where $\mathbf{0}_{n-2}$ is a vector of zeros in $\mathbb{R}^{n-2}$. Clearly, this set is a Jordan frame in $\mathbb{R}^n$, called *canonical Jordan frame*. It is easy to verify that

$$\mathbb{V}_{ii} = \{\kappa e_i : \kappa \in \mathbb{R}\}, \text{ for } i = 1, 2, \quad \mathbb{V}_{12} = \{x \in \mathbb{R}^n : x_1 = x_2 = 0\}.$$

Thus, given an $x \in \mathbb{V}$ we can write

$$x = (x_1 + x_2)e_1 + (x_1 - x_2)e_2 + (0, 0, x_3 \ldots, x_n),$$

which denotes its Pierce decomposition associated with $\{e_1, e_2\}$.

(*iii*) Let $\mathbb{V} = \mathcal{S}^n$ and consider the set $\{E_1, \ldots, E_n\}$, where E_i is the diagonal matrix with 1 in the (i,i)-entry and 0' elsewhere. It is easily seen that this set is a Jordan frame in $\mathcal{S}^n$, called *canonical Jordan frame*. Also, associated with this Jordan frame, it is easy to verify that

$$\mathbb{V}_{ii} = \{\kappa E_i : \kappa \in \mathbb{R}\}, \text{ for } i = 1, \ldots, n, \quad \mathbb{V}_{ij} = \{\theta E_{ij} : \theta \in \mathbb{R}\}, \ i \neq j,$$

where E_{ij} is a matrix with 1 in the (i,j) and (j,i)-entries and 0' elsewhere. Thus, any $X \in \mathcal{S}^n$ can be written as

$$X = \sum_{i=1}^n x_{ii} E_i + \sum_{1 \le i < j \le n} x_{ij} E_{ij}.$$

This expression denotes the Pierce decomposition of X associated with $\{E_1, \ldots, E_n\}$.

Orthogonal projection. In $\mathbb{V}$, fix a Jordan frame $\{e_1, \ldots, e_r\}$ and define

$$\mathbb{V}^{(\alpha)} := \{x \in \mathbb{V} : x \circ (e_1 + \ldots + e_l) = x\}$$

for $\alpha = \{1, \ldots, l\}$ with $1 \le l \le r$. This set is a subalgebra of $\mathbb{V}$ with rank l (see [20, Proposition IV.1.1]). The symmetric cone in this subalgebra is defined by $\mathcal{K}^{(\alpha)} := \{y \circ y : y \in \mathbb{V}^{(\alpha)}\} = \mathbb{V}^{(\alpha)} \cap \mathcal{K}$ (see [12, Theorem 3.1]). Corresponding to $\mathbb{V}^{(\alpha)}$, we consider the (orthogonal) projection $P^{(\alpha)} : \mathbb{V} \to \mathbb{V}^{(\alpha)}$. Let $x \in \mathbb{V}$ be written as $x = u + v$, where $u \in \mathbb{V}^{(\alpha)}$ and $v \in (\mathbb{V}^{(\alpha)})^{\perp}$. Then, $P^{(\alpha)}(x) = u$. Now, we consider that x has the following Peirce decomposition corresponding to $\{e_1, \ldots, e_r\}$:

$$x = \sum_{i=1}^r x_i e_i + \sum_{1 \le i < j \le r} x_{ij},$$

then (see [10, Lemma 20])

$$P^{(\alpha)}(x) = \sum_{i=1}^l x_i e_i + \sum_{1 \le i < j \le l} x_{ij}.$$

Note that for a given Jordan frame $\{e_1, \ldots, e_r\}$, we can permute the objects and select the first l objects (for any $1 \le l \le r$). Thus there are $2^r - 1$ projections $P^{(\alpha)}$ corresponding to a Jordan frame.

In a similar way one can define the subalgebra $\mathbb{V}^{(\bar{\alpha})}$ by using the set $\{e_{l+1},\ldots,e_r\}$ and also the projection $P^{(\bar{\alpha})}$ on $\mathbb{V}^{(\bar{\alpha})}$, where $\bar{\alpha}=\{1,\ldots,r\}\backslash\alpha$.

Example 2.4. ([10, Example 1.2]) For $\mathbb{V}=\mathcal{S}^n$, consider the Jordan frame $\{E_1,E_2,\ldots,E_n\}$ (defined in Example 2.3(iii)). Let $\alpha:=\{1,\ldots,l\}$ with $1\leq l\leq n$. Then $X\in\mathbb{V}^{(\alpha)}$ has the form

$$X=\left(\begin{array}{cc} X_{\alpha\alpha} & 0\\ 0 & 0\end{array}\right),$$

where $X_{\alpha\alpha}$ is the principal submatrix of X corresponding to the index set α. Thus, we may view $\mathbb{V}^{(\alpha)}$ as $\mathcal{S}^{|\alpha|}$. Hence, the projection $P^{(\alpha)}:\mathcal{S}^n\to\mathbb{V}^{(\alpha)}$ is given by

$$P^{(\alpha)}(Y)=\left(\begin{array}{cc} Y_{\alpha\alpha} & 0\\ 0 & 0\end{array}\right).$$

The following result characterizes all Euclidean Jordan algebras (See [20, Propositions III.4.4 and III.4.5, Theorem V.3.7]).

Theorem 2.4. *Any Euclidean Jordan algebra is, in a unique way, a direct sum of simple Euclidean Jordan algebras. Moreover, the symmetric cone in a given Euclidean Jordan algebra is, in a unique way, a direct sum of symmetric cones in the constituent simple Euclidean Jordan algebras.*

We note that the 'direct sum' in the theorem refers to the orthogonal as well as the Jordan product direct sum. Thus given a Euclidean Jordan algebra $\mathbb{V}$ and the corresponding symmetric cone $\mathcal{K}$, we may write

$$\mathbb{V}=\mathbb{V}_1\times\mathbb{V}_2\times\cdots\times\mathbb{V}_{\bar{j}}\quad\text{and}\quad\mathcal{K}=\mathcal{K}_1\times\mathcal{K}_2\times\cdots\times\mathcal{K}_{\bar{j}},$$

where each $\mathbb{V}_j$ is a simple Jordan Algebra with the corresponding symmetric cone $\mathcal{K}_j$. Moreover, for $x=(x^{(1)},x^{(2)},\ldots,x^{(\bar{j})})$ and $y=(y^{(1)},y^{(2)},\ldots,y^{(\bar{j})})$ in $\mathbb{V}$ with $x^{(j)},y^{(j)}\in\mathbb{V}_j$, we have

$$x\circ y=(x^{(1)}\circ y^{(1)},\ldots,x^{(\bar{j})}\circ y^{(\bar{j})}),\quad\langle x,y\rangle=\sum_{j=1}^{\bar{j}}\langle x^{(j)},y^{(j)}\rangle,\quad\|x\|^2=\sum_{j=1}^{\bar{j}}\|x^{(j)}\|^2.$$

Remark 2.5. *When the Jordan Algebra $\mathbb{V}$ is not simple (that is, when $\bar{j}>1$ in the previous setting), it can be verified that every primitive idempotent element c of $\mathbb{V}$ has necessarily the form $c=(0,0,\ldots,c^{(j)},0,\ldots,0)$ for some primitive idempotent element $c^{(j)}$ in $\mathbb{V}_j$.*

In any Euclidean Jordan algebra $\mathbb{V}$, one can define automorphism groups in the following way (Faraut and Korányi [20]).

Definition 2.6. *A linear transformation $\Lambda:\mathbb{V}\to\mathbb{V}$ is said to be an* automorphism *of $\mathbb{V}$ if Λ is invertible and*

$$\Lambda(x\circ y)=\Lambda(x)\circ\Lambda(y)\quad\textit{for all }\ x,y\in\mathbb{V}.\tag{4}$$

The set of all automorphisms of $\mathbb{V}$ is denoted by $\mathrm{Aut}(\mathbb{V})$*.*

Definition 2.7. *A linear transformation $\Lambda:\mathbb{V}\to\mathbb{V}$ is said to be an* automorphism *of $\mathcal{K}$ if $\Lambda(\mathcal{K})=\mathcal{K}$. Note that this transformation constrained to $\mathcal{K}$ is necessarily invertible. We denote the set of all automorphisms of $\mathcal{K}$ by* $\mathrm{Aut}(\mathcal{K})$*, and each element of it by Γ.*

It directly follows from (4) that $\mathrm{Aut}(\mathbb{V})\subseteq\mathrm{Aut}(\mathcal{K})$. Moreover, if $\Gamma\in\mathrm{Aut}(\mathcal{K})$, then Γ^{-1} and $\Gamma^\top\in\mathrm{Aut}(\mathcal{K})$ ([12, Proposition 4.1]).

Example 2.5. (*i*) For $\mathbb{V}=\mathbb{R}^n$, it is easily seen that $\mathrm{Aut}(\mathbb{R}^n)$ consists of permutation matrices, and any element in $\mathrm{Aut}(\mathbb{R}^n_+)$ is a product of a permutation matrix and a diagonal matrix with positive diagonal entries.

(*ii*) For $\mathbb{V}=\mathbb{R}^n$, it is known [10, example 2.1] that any automorphism Λ in $\mathrm{Aut}(\mathbb{R}^n)$ can be written as $\Lambda=\begin{pmatrix}1 & 0\\ 0 & D\end{pmatrix}$, where D is an $(n-1)\times(n-1)$ orthogonal matrix. Also, an $n\times n$ matrix $\Gamma\in\mathrm{Aut}(\mathcal{L}^n_+)$ if and only if there exists $\mu>0$ such that

$$\Gamma^\top J\Gamma=\mu J,$$

where $J=\mathrm{Diag}(1,-1,\ldots,-1)\in\mathbb{R}^{n\times n}$. .

(*iii*) For $\mathbb{V}=\mathcal{S}^n$, it is known (see [22, Theorem 2]) that corresponding to any $\Lambda\in\mathrm{Aut}(\mathcal{S}^n)$, there exists an orthogonal matrix U such that

$$\Lambda(X)=UXU^\top \quad (\forall X\in\mathcal{S}^n).$$

Also, for $\Gamma\in\mathrm{Aut}(\mathcal{S}^n_+)$, there exists an invertible matrix $Q\in\mathbb{R}^{n\times n}$ such that

$$\Gamma(X)=QXQ^\top \quad (\forall X\in\mathcal{S}^n).$$

From now on, $(\mathbb{V},\circ,\langle\cdot,\cdot\rangle)$ will be an Euclidean Jordan algebra of rank r and $\{e_1,\ldots,e_r\}$ will be a Jordan frame in $\mathbb{V}$.

We end this subsection by recalling properties that we shall employ throughout this paper. Their proofs and more details can be found in [10, 12, 19, 20, 23].

Proposition 2.8. *The following results hold:*

(*a*) *$x\in\mathcal{K}$ if and only if $\langle x,y\rangle\geq 0$ holds for all $y\in\mathcal{K}$. Moreover, $x\in\mathrm{int}(\mathcal{K})$ if and only if $\langle x,y\rangle>0$ for all $y\in\mathcal{K}\setminus\{0\}$.*

(*b*) *For $x,y\in\mathbb{V}$ the following conditions are equivalent:*
  (*i*) *$x,y\in\mathcal{K}$, and $\langle x,y\rangle=0$.*
  (*ii*) *$x,y\in\mathcal{K}$, and $x\circ y=0$.*
  In each case, the elements x and y operator commute, that is, $L_xL_y=L_yL_x$.

(*c*) *The elements x and y operator commute if and only if x and y have their spectral decompositions with respect to a common Jordan frame.*

(*d*) *If $x\in\mathcal{K}$, then $P^{(\alpha)}(x)\in\mathcal{K}^{(\alpha)}$. Moreover, if $x\in\mathrm{int}(\mathcal{K})$, then $P^{(\alpha)}(x)\in\mathrm{int}(\mathcal{K}^{(\alpha)})$.*

(*e*) *Suppose that $x\in\mathcal{K}$ and let $x=\sum_{i=1}^r x_ie_i+\sum_{1\leq i<j\leq r}x_{ij}$ be its Peirce decomposition. If $x_k=0$ for some index k, then $\sum_{1\leq k<j\leq r}x_{kj}+\sum_{1\leq i<k\leq r}x_{ik}=0$.*

(*f*) *For any $x,y\in\mathbb{V}$, we have $\mathrm{tr}(x\circ y)\leq\sum_{i=1}^r\lambda_i(x)\lambda_i(y)$. The latter holds with equality if and only if x and y operator commute.*

(*g*) *Let $x,y\in\mathcal{K}$. Then $\mathrm{tr}(x\circ y)\geq 0$. Moreover, $\mathrm{tr}(x\circ y)=0$ if and only if $x\circ y=0$.*

(*h*) *The smallest and the largest eigenvalue of $x\in\mathbb{V}$ are given by*

$$\lambda_{\min}(x)=\min_{u\neq 0}\frac{\mathrm{tr}(x\circ u^2)}{\mathrm{tr}(u^2)},\quad \lambda_{\max}(x)=\max_{u\neq 0}\frac{\mathrm{tr}(x\circ u^2)}{\mathrm{tr}(u^2)}.$$

In particular, when $\mathbb{V}$ is simple, these eigenvalues can be equivalently written as

$$\lambda_{\min}(x)=\min_{u\neq 0}\frac{\langle x,u^2\rangle}{\|u\|^2},\quad \lambda_{\max}(x)=\max_{u\neq 0}\frac{\langle x,u^2\rangle}{\|u\|^2}.$$

2.2. **Linear transformations review.** The literature on symmetric cone LCP (see [10, 11, 13, 18]) has already be extended, from the LCP theory. Most of the well-known classes of matrices used in that context have been extended to symmetric cone LCP. We list these classes here below to be employed in the sequel. Given a linear transformation $L \in \mathcal{L}(\mathbb{V})$, we say that:

- L has the $\mathbf{Q}$-property if $\mathrm{SOL}(L, \mathcal{K}, q) \neq \emptyset$, for all $q \in \mathbb{V}$.
- L has the $\mathbf{Q_b}$-property if $\mathrm{SOL}(L, \mathcal{K}, q) \neq \emptyset$ and bounded, for all $q \in \mathbb{V}$.
- L is an $\mathbf{R_0}$-transformation if $\mathrm{SOL}(L, \mathcal{K}, 0) = \{0\}$.
- L is copositive (resp. strictly copositive) if $\langle L(x), x\rangle \geq 0$ (resp. > 0) for all $x \in \mathcal{K}$ (resp. for all $x \in \mathcal{K}$, $x \neq 0$).
- L is monotone (resp. strongly monotone) if $\langle L(x), x\rangle \geq 0$ (resp. > 0) for all $x \in \mathbb{V}$ (resp. for all $x \in \mathbb{V}$, $x \neq 0$).
- L has the $\mathbf{P}$-property if [x and $L(x)$ operator commute and $x \circ L(x) \in -\mathcal{K} \Rightarrow x = 0$].
- L has the $\mathbf{Q_0}$-property if [$\mathrm{FEAS}(L, \mathcal{K}, q) \neq \emptyset \Rightarrow \mathrm{SOL}(L, \mathcal{K}, q) \neq \emptyset$].
- L has the $\mathbf{S}$-property if there is a $x \in \mathrm{int}(\mathcal{K})$ such that $L(x) \in \mathrm{int}(\mathcal{K})$.
- L is normal if L commutes with $L^\top$. Here, $L^\top : \mathbb{V} \to \mathbb{V}$ denotes the transpose of L, which is defined by $\langle L(x), y\rangle = \langle x, L^\top(y)\rangle$ for all $x, y \in \mathbb{V}$.
- L is a star-transformation if [$v \in \mathrm{SOL}(L, \mathcal{K}, 0) \Rightarrow L^\top(v) \in -\mathcal{K}$].
- L has the $\mathbf{Z}$-property if [$x, y \in \mathcal{K},\ \langle x, y\rangle = 0 \Rightarrow \langle L(x), y\rangle \leq 0$].
- L is a Lyapunov-like transformation if [$x, y \in \mathcal{K},\ \langle x, y\rangle = 0 \Rightarrow \langle L(x), y\rangle = 0$].

It is easy to check that monotone (resp. strongly monotone) transformations are copositive (resp. strictly copositive) and that Lyapunov-like transformations has $\mathbf{Z}$-property.

The next proposition establishes some links between the classes mentioned above.

Proposition 2.9. *Let $L \in \mathcal{L}(\mathbb{V})$ and $q \in \mathbb{V}$ be given. The following relations hold:*

(a): *L is strongly monotone $\Longrightarrow L \in \mathbf{P} \Longrightarrow L \in \mathbf{R_0}$;*

(b): *$L \in \mathbf{S} \Longleftrightarrow \mathrm{FEAS}(L, \mathcal{K}, q) \neq \emptyset$ for all $q \in \mathbb{V}$;*

(c): *$\mathbf{Q} = \mathbf{Q_0} \cap \mathbf{S}$;*

(d): *If L is a Lyapunov-like transformation or $L \in \mathbf{R_0}$ or skew-symmetric (that is, $L^\top = -L$) or monotone $\Longrightarrow L$ is a star-transformation.*

Proof. Statement (a) is proven in [10]. The equality in (c) follows from (b).
(b): ($\Leftarrow$) Let $d \in \mathrm{int}(\mathcal{K})$. By hypothesis $\mathrm{FEAS}(L, \mathcal{K}, -d) \neq \emptyset$, that is, there exists $x \in \mathcal{K}$ such that $y = L(x) - d \in \mathcal{K}$. From this we get $L(x) = y + d \in \mathrm{int}(\mathcal{K})$, since $\mathcal{K} + \mathrm{int}(\mathcal{K}) = \mathrm{int}(\mathcal{K})$. Hence $L \in \mathbf{S}$.
($\Rightarrow$) As $\mathcal{K}$ is self-dual closed convex cone with $\mathrm{int}(\mathcal{K}) \neq \emptyset$, then by [24, Theorem 2.2.13] we conclude that $\mathcal{K}$ has a closed bounded base, that is, $\mathcal{K} = \mathrm{cone}(B)$, where B is a compact set such that $0 \notin B$. By hypothesis there is $x \in \mathcal{K}$ such that $L(x) \in \mathrm{int}(\mathcal{K})$. Fix $q \in \mathbb{V}$. As B is compact, there exists e_1, e_1 such that $\min_{e \in B}\langle L(x), e\rangle = \langle L(x), e_1\rangle > 0$ and $\min_{e \in B}\langle q, e\rangle = \langle q, e_2\rangle$. Clearly, there is some $t > 0$ such that $t\langle L(x), e_1\rangle + \langle q, e_2\rangle > 0$. For each $y \in \mathcal{K}$, there exist $\gamma \geq 0$ and $e \in B$ such that $y = \gamma e$. Therefore,

$$\langle tL(x) + q, y\rangle = \gamma(t\langle L(x), e\rangle + \langle q, e\rangle) \geq \gamma(t\langle L(x), e_1\rangle + \langle q, e_2\rangle) > 0.$$

Then, as $y \in \mathcal{K}$ was arbitrary, by Proposition 2.8, Part (a) we obtain that $tL(x) + q = L(tx) + q \in \mathcal{K}$. Thus, $tx \in \mathrm{FEAS}(L, \mathcal{K}, q)$. The desired equivalence follows.

(d): If L is a Lyapunov-like transformation, then $v \in \mathrm{SOL}(L, \mathcal{K}, 0)$ implies $L^\top(v) = 0 \in -\mathcal{K}$. If $L \in \mathbf{R_0}$ or L is skew-symmetric, then the proof is trivial. If L is monotone, then for $v \in \mathrm{SOL}(L, \mathcal{K}, 0)$ from $\langle L(tx - v), tx - v\rangle \geq 0$ for all $t \in \mathbb{R}$ and $x \in \mathcal{K}$ it is not difficult to obtain that $(L + L^\top)(v) \in \mathcal{K} \cap (-\mathcal{K})$. As $\mathrm{int}(\mathcal{K}) \neq \emptyset$, the cone $\mathcal{K}$ is pointed by [25, Exercise 6.22]; that is, $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$; thus, $(L + L^\top)(v) = 0$ and $L^\top(v) = -L(v) \in -\mathcal{K}$. □

3. Characterizations of $\mathbf{Q}$- and $\mathbf{Q_b}$-transformations

A direct consequence of [1, Proposition 2.5.6] is that, within the class $\mathbf{R_0}$, the classes $\mathbf{Q}$ and $\mathbf{Q_b}$ coincide. This result holds even when $\mathcal{K}$ is a general closed convex solid cone in the SCLCP (cf. (1)). Let us recall this result here below.

Lemma 3.1. *Let $L \in \mathbf{R_0}$. Then, $L \in \mathbf{Q_b} \iff L \in \mathbf{Q}$.*

Moreover, since $\mathrm{SOL}(L, \mathcal{K}, 0)$ is always a cone, it also follows that:

Lemma 3.2. *It holds that $\mathbf{Q_b} \subseteq \mathbf{R_0}$. Consequently, $\mathbf{Q_b} = \mathbf{Q} \cap \mathbf{R_0}$.*

Previous lemmas motivate us to study classes of linear transformations in $\mathcal{L}(\mathbb{V})$, larger than $\mathbf{R_0}$, for which the last results are fulfilled.

3.1. The class of F-transformations and its subclasses.

Inspired by [14] and [15, Definition 3.5], we introduce the next new class of linear transformations in $\mathcal{L}(\mathbb{V})$.

Definition 3.3. *We say that $L \in \mathcal{L}(\mathbb{V})$ is an $\mathbf{F}$-transformation or $L \in \mathbf{F}$ if for each $v \in \mathrm{SOL}(L, \mathcal{K}, 0)\backslash\{0\}$ there exists $\chi_v \in \mathbb{V}$ such that*

$$(i)\ \chi_v \in \mathcal{K}, \qquad (ii)\ \langle \chi_v, v\rangle > 0, \qquad (iii)\ L^\top(\chi_v) \in -\mathcal{K}. \tag{5}$$

Indeed, this class was defined in [14] in the LCP context as follows:

Definition 3.4. *A matrix $M \in \mathbb{R}^{n\times n}$ is said to be an F_1-matrix if, for every $v \in \mathrm{SOL}(M, 0) \setminus \{0\}$, there exists a nonnegative diagonal matrix Σ such that $\Sigma v \neq 0$ and $M^\top \Sigma v \in -\mathbb{R}^n_+$. Here $\mathrm{SOL}(M, q)$ denotes, for given $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$, the solution of problem $\mathrm{LCP}(M, q)$.*

Therein, the equivalence of Lemma 3.1 is proven within the class F_1. This class turns to be larger than $\mathbf{R}_0$, which makes the result interesting to be analyzed for more general complementarity problems. So, in [15], this definition was extended to the SDLCP framework as well as the desired equivalence between classes Q and $\mathrm{Q_b}$. As one can expect, both definitions, for the LCP and the SDLCP setting, are particular cases of Definition 3.3 given above. The rest of this section is dedicated to extend the desired equivalence within the class $\mathbf{F}$ and to study its different subclasses.

Given a linear transformation $L : \mathbb{V} \to \mathbb{V}$ and $\Lambda \in \mathrm{Aut}(\mathbb{V})$, we define a linear transformation $\widetilde{L}$ on $\mathbb{V}$ by

$$\widetilde{L} := \Lambda^\top L \Lambda.$$

Example 3.1. Consider in $\mathbb{V} = \mathcal{S}^n$, the automorphism $\Lambda \in \mathrm{Aut}(\mathbb{V})$ defined in Example 2.5(*iii*). Then,

$$\widetilde{L}(X) = U^\top L(UXU^\top)U.$$

The next result shows that this class is invariant under automorphisms.

Lemma 3.5. *Let $L : \mathbb{V} \to \mathbb{V}$ be a linear transformation and $\Lambda \in \mathrm{Aut}(\mathbb{V})$ be orthogonal. Then L has the $\mathbf{F}$-property if and only if $\widetilde{L}$ has the $\mathbf{F}$-property.*

Proof. We first point out that $\Lambda^{-1}(\mathrm{SOL}(L,\mathcal{K},q)) = \mathrm{SOL}(\widetilde{L},\mathcal{K},\Lambda^\top q)$ (see [12, Theorem 5.1]).

($\Rightarrow$): Let w be a nonzero solution of $\mathrm{LCP}(\widetilde{L},\mathcal{K},0)$. Then, by above equality, we get that for any $\Lambda \in \mathrm{Aut}(\mathbb{V})$ orthogonal, $v = \Lambda(w)$ is nonzero solution of $\mathrm{LCP}(L,\mathcal{K},0)$. So, by hypothesis there exist a $\chi_v \in \mathbb{V}$ such that (5) holds. Clearly, $\chi_w = \Lambda^\top(\chi_v) \in \mathcal{K}$. Also, $\langle \chi_w, w\rangle = \langle \chi_v, v\rangle > 0$. Finally, since $L^\top(\chi_v) \in -\mathcal{K}$, Λ is orthogonal and $\Lambda^\top(\mathcal{K}) = \mathcal{K}$, it follows that $\widetilde{L}^\top(\chi_w) \in -\mathcal{K}$ and hence $\widetilde{L} \in \mathbf{F}$.
($\Leftarrow$): Let v be a nonzero solution of $\mathrm{LCP}(L,\mathcal{K},0)$. Then, by the above equality, we get that for any $\Lambda \in \mathrm{Aut}(\mathbb{V})$ being orthogonal, $w = \Lambda^{-1}(v)$ is nonzero solution of $\mathrm{LCP}(\widetilde{L},\mathcal{K},0)$. Thus, by hypothesis there exist a $\chi_w \in \mathbb{V}$ such that (5) holds. Clearly, $\chi_v = \Lambda(\chi_w) \in \mathcal{K}$ and $\langle \chi_v, v\rangle = \langle \chi_w, \Lambda^\top(v)\rangle > 0$, since $\Lambda(\mathcal{K}) = \mathcal{K}$ and Λ is orthogonal. Finally, since $\widetilde{L}^\top(\chi_w) \in -\mathcal{K}$, $\widetilde{L}^\top(\chi_w) = \Lambda^\top L^\top(\chi_v)$ and $\Lambda^\top(\mathcal{K}) = \mathcal{K}$, it follows that $L^\top(\chi_v) \in -\mathcal{K}$ and hence $L \in \mathbf{F}$. $\square$

We now establish the main properties of the class $\mathbf{F}$. In particular, assertion (b) below extends Lemma 3.1 to this larger class.

Theorem 3.6. *Let $L \in \mathcal{L}(\mathbb{V})$ be given.*

- **(a):** *If $L \in \mathbf{F} \cap \mathbf{S}$, then $L \in \mathbf{R_0}$;*
- **(b):** *Let $L \in \mathbf{F}$. Then, $L \in \mathbf{Q_b} \iff L \in \mathbf{Q}$.*

Proof. (a): Let $L \in \mathbf{F} \cap \mathbf{S}$. We argue by contradiction. Suppose that $L \notin \mathbf{R_0}$, that is, there exist $v \in \mathrm{SOL}(L,\mathcal{K},0)\backslash\{0\}$. Since $L \in \mathbf{F}$, there exists a χ_v satisfying (*i*)-(*iii*) in Definition 3.3. This together with Proposition 2.8, Part (a) implies that $\langle L(x) - v, \chi_v\rangle < 0$ for all $x \in \mathcal{K}$. Consequently, $L(x) - v \notin \mathcal{K}$ for all $x \in \mathcal{K}$. Therefore, by Proposition 2.9, Part (b), it follows that $L \notin \mathbf{S}$, obtaining a contradiction.

(b): Obviously $L \in \mathbf{Q_b}$ implies $L \in \mathbf{Q}$. If $L \in \mathbf{Q}$, then $L \in \mathbf{S}$ (cf. Proposition 2.9, Part (c)). Thus, $L \in \mathbf{F} \cap \mathbf{S}$. By item (a) above we conclude that $L \in \mathbf{R_0}$, and consequently $L \in \mathbf{Q} \cap \mathbf{R_0}$. We thus conclude that $L \in \mathbf{Q_b}$ thanks to equality $\mathbf{Q_b} = \mathbf{Q} \cap \mathbf{R_0}$ established in Lemma 3.2. $\square$

To check whenever a linear transformation L belongs to $\mathbf{F}$ can be a difficult task. This is mainly because there is no a clear guide about how to chose, for a given $v \in \mathrm{SOL}(L,\mathcal{K},0)\backslash\{0\}$, a χ_v satisfying conditions (**i**)–(**iii**) in Definition 3.3. For this, we focus now on the subclass of $\mathbf{F}$ for which χ_v is chosen via Schur product (see [26, Section 5]) of two elements.

Definition 3.7. *For any $A = (a_{ij}) \in \mathcal{S}^r$ and $x \in \mathbb{V}$, with Peirce decomposition $x = \sum_{i\le j} x_{ij}$, we define the Schur product of A and x by*

$$A \bullet x := \sum_{1\le i\le j\le r} a_{ij}x_{ij}. \tag{6}$$

Definition 3.8. *Let $L : \mathbb{V} \to \mathbb{V}$ be a linear transformation. We say that L is an $\mathbf{F}_1$-transformation or $L \in \mathbf{F}_1$, if for each $v \in \mathrm{SOL}(L,\mathcal{K},0)\backslash\{0\}$, with Peirce decomposition $v = \sum_{i\le j} v_{ij}$, there exists a matrix $\Xi = (\xi_{ij}) \in \mathcal{S}^r_+$ such that*

$$(i)\ \Xi \bullet v \in \mathcal{K} \qquad (ii)\ \langle \Xi \bullet v, v\rangle > 0 \qquad (iii)\ L^\top(\Xi \bullet v) \in -\mathcal{K}.$$

Remark 3.9. *Notice that, if Ξ is positive semidefinite and $x \in \mathcal{K}$, then $\Xi \bullet x \in \mathcal{K}$ (cf. [26, Theorem 4]). Hence, condition (i) above becomes superfluous.*

We illustrate this concept in the Euclidean Jordan algebras defined in Example 2.1.

Example 3.2. (i) For $\mathbb{V} = \mathbb{R}^n$ and $\mathcal{K} = \mathbb{R}^n_+$, we consider the Jordan frame $\{e_1, \ldots, e_n\}$ (defined in Example 2.3(i)). Then, for $\Xi = (\xi_{ij}) \in \mathcal{S}^n_+$, the Schur product of Ξ and $v \in \mathrm{SOL}(L, \mathbb{R}^n_+, 0)\backslash\{0\}$ reduces to

$$\Xi \bullet v = \sum_{i=1}^{n} \xi_{ii} v_i e_i = \mathrm{Diag}(\xi_{11}, \ldots, \xi_{nn})v,$$

where $v = \sum_{i=1}^n v_i e_i$ is its Peirce decomposition associated with $\{e_1, \ldots, e_n\}$. Taking $\Sigma = \mathrm{Diag}(\xi_{11}, \ldots, \xi_{nn}) \in \mathcal{S}^n_+$, Definition 3.8 reduces to Definition 3.4 in the LCP context.

(ii) For $\mathbb{V} = \mathbb{R}^n$, $\mathcal{K} = \mathcal{L}^n_+$, we consider the canonical Jordan frame $\{e_1, e_2\}$(defined in Example 2.3(ii)). Then, for $\Xi = (\xi_{ij}) \in \mathcal{S}^2_+$, the Schur product of Ξ and $v \in \mathrm{SOL}(L, \mathcal{L}^n_+, 0)\backslash\{0\}$ reduces to

$$\Xi \bullet v = \xi_{11}(v_1 + v_2)e_1 + \xi_{22}(v_1 - v_2)e_2 + \xi_{12}(0, 0, v_3, \ldots, v_n),$$

where $v = (v_1 + v_2)e_1 + (v_1 - v_2)e_2 + (0, 0, v_3, \ldots, v_n)$ is its Peirce decomposition associated with $\{e_1, e_2\}$. Taking into account this, condition (ii) of Definition 3.8 is reduced to

$$\langle \Xi \bullet v, v\rangle = \frac{1}{2}(\xi_{11}(v_1 + v_2)^2 + \xi_{22}(v_1 - v_2)^2) + \xi_{12}\|(v_3, \ldots, v_n)\|^2 > 0.$$

(iii) For $\mathbb{V} = \mathcal{S}^n$ and $\mathcal{K} = \mathcal{S}^n_+$, we consider $\{E_1, \ldots, E_n\}$ the canonical Jordan frame (defined in Example 2.3(iii)). Then, for $\Xi = (\xi_{ij}) \in \mathcal{S}^n_+$, the Schur product of Ξ and $V \in \mathrm{SOL}(L, \mathcal{S}^n_+, 0)\backslash\{0\}$ coincide with the well-known Schur (or Hadamard) product of two symmetric matrices (see [27]). Thus, Definition 3.8 reduces to Definition of $\mathbf{F}_1$-transformation given in [15] in the SDLCP context: $L \in \mathcal{L}(\mathcal{S}^n)$ is an $\mathbf{F}_1$-transformation if for each $V \in \mathrm{SOL}(L, \mathcal{S}^n_+, 0)\backslash\{0\}$ there exists a matrix $\Lambda \in \mathcal{S}^n_+$ such that

$$(i)\ \Lambda \bullet V \in \mathcal{S}^n_+ \qquad (ii)\ \langle \Lambda \bullet V, V\rangle > 0 \qquad (iii)\ L^\top(\Lambda \bullet V) \in -\mathcal{S}^n_+.$$

In the following proposition we list various classes of linear transformations that are contained in the class $\mathbf{F_1}$.

Proposition 3.10. *$L \in \mathbf{F_1}$ if any of the following conditions is satisfied:*

- **(a):** *L is a star-transformation;*
- **(b):** *$L \in \mathbf{Z}$ and*
  - **(i):** *$-L$ is copositive or*
  - **(ii):** *L is normal;*

Proof. (a): Let $v \in \mathrm{SOL}(L, \mathcal{K}, 0)\backslash\{0\}$ with Peirce decomposition $v = \sum_{i\leq j} v_{ij}$. Since L is a star-transformation, we have $L^\top(v) \in -\mathcal{K}$. Then, conditions (i)–(iii) of Definition 3.8 can be easily checked provided that $\Xi = \mathbb{1}$, which denotes the $n \times n$ matrix whose entries are all equal to 1 (note that $\mathbb{1} \in \mathcal{S}^r_+$ and $\mathbb{1} \bullet v = v$). The result follows.

(b): Let $v \in \mathrm{SOL}(L, \mathcal{K}, 0)\backslash\{0\}$ with Peirce decomposition $v = \sum_{i\leq j} v_{ij}$, that is, $v, L(v) \in \mathcal{K}$ and $\langle L(v), v\rangle = 0$. Since $L \in \mathbf{Z}$, we get $\langle L(v), L(v)\rangle \leq 0$, and consequently $L(v) = 0$. We proceed to prove both cases.
(i): If $-L$ is copositive, then $\langle L(tx + v), tx + v\rangle \leq 0$ for all $x \in \mathcal{K}$ and for all $t > 0$. From this, after dividing by t we get $t\langle L(x), x\rangle + \langle L(x), v\rangle \leq 0$ for all $t > 0$. Taking limit $t \searrow 0$

we obtain $\langle x, L^\top(v)\rangle \le 0$ for all $x \in \mathcal{K}$. From Proposition 2.8, Part (a), we conclude that $L^\top(v) \in -\mathcal{K}$, that is, L is a star-transformation. The desired result follows from (a).
(ii): Since L is normal and $L(v) = 0$, we obtain that

$$\|L^\top(v)\|^2 = \langle v, L(L^\top(v))\rangle = \langle v, L^\top(L(v))\rangle = 0.$$

That is, $L^\top(v) = 0$, which is in $-\mathcal{K}$. Thus, the desired result follows again from (a).

□

T-Transformation. In the following definition, we extend the notion of T-property for matrices given in [16] to our SCLCP context.

Definition 3.11. *We say that $L \in \mathcal{L}(\mathbb{V})$ has the* $\mathbf{T}$*-property if for any $\Lambda \in \mathrm{Aut}(\mathbb{V})$ being orthogonal and any index set $\alpha = \{1, \dots, l\}$ $(1 \le l \le r)$, the existence of a solution $x \in \mathbb{V}$ to the system*

(7)
$$\begin{aligned} P^{(\alpha)}(x) \in \mathrm{int}(\mathcal{K}^{(\alpha)}), \quad (I - P^{(\alpha)})(x) = 0, \quad & P^{(\alpha)}(\widetilde{L}(x)) \in -\mathcal{K}^{(\alpha)}, \quad P^{(\bar\alpha)}(\widetilde{L}(x)) \in \mathcal{K}^{(\bar\alpha)}, \\ & (I - P^{(\alpha)} - P^{(\bar\alpha)})(\widetilde{L}(x)) = 0, \end{aligned}$$

$\bar\alpha = \{1, \dots, r\}\backslash\alpha$, *implies that there is a nonzero $y \in \mathbb{V}$ satisfying*

(8)
$$\begin{aligned} P^{(\alpha)}(y) \in \mathcal{K}^{(\alpha)},\ (I - P^{(\alpha)})(y) = 0, \ \ & P^{(\alpha)}(\widetilde{L}^\top(y)) \in -\mathcal{K}^{(\alpha)},\ P^{(\bar\alpha)}(\widetilde{L}^\top(y)) \in -\mathcal{K}^{(\bar\alpha)}, \\ & (I - P^{(\alpha)} - P^{(\bar\alpha)})(\widetilde{L}^\top(y)) = 0,\ \langle P^{(\alpha)}(y), P^{(\alpha)}(\widetilde{L}(x))\rangle = 0. \end{aligned}$$

We illustrate this concept in some known examples of Euclidean Jordan algebras.

Example 3.3. (i) For $\mathbb{V} = \mathbb{R}^n$ and $\mathcal{K} = \mathbb{R}^n_+$, we consider the Jordan frame $\{e_1, \dots, e_n\}$ (defined in Example 2.3(i)) and set $\alpha = \{1, \dots, l\}$ with $1 \le l \le n$. Then, it is easily seen that $\mathbb{V}^{(\alpha)} = \{(x_\alpha, 0) \in \mathbb{V} : x_\alpha \in \mathbb{R}^l\}$ and $\mathbb{V}^{(\bar\alpha)} = \{(0, x_{\bar\alpha}) \in \mathbb{V} : x_{\bar\alpha} \in \mathbb{R}^{|\bar\alpha|}\}$. Hence, the projection $P^{(\alpha)}$ of $x \in \mathbb{V}$ on $\mathbb{V}^{(\alpha)}$ is given by $P^{(\alpha)}(x) = \sum_{i=1}^{l} x_i e_i$, where $x = \sum_{i=1}^{n} x_i e_i$ is its Peirce decomposition. On the other hand, taking $I \in \mathrm{Aut}(\mathbb{R}^n)$ (because $I(e) = e$) we get that $P^{(\bar\alpha)}(\widetilde{L}(x)) = \sum_{i=l+1}^{n} z_i e_i$, where $L(x) = \sum_{i=1}^{n} z_i e_i$ is its Peirce decomposition associated to $\{e_1, \dots, e_n\}$. Hence, taking into account the above and that $L(x) = Mx$ with $M \in \mathbb{R}^{n\times n}$, Definition 3.13 reduces to saying: A matrix $M \in \mathbb{R}^{n\times n}$ has T-property if only if for any nonempty set $\alpha = \{1, \dots, l\} \subseteq \{1, \dots, n\}$, the existence of a vector $x_\alpha \in \mathbb{R}^{|\alpha|}$ satisfying

(9)
$$x_\alpha > 0, \quad M_{\alpha\alpha} x_\alpha \le 0 \quad \text{and} \quad M_{\bar\alpha\alpha} x_\alpha \ge 0,$$

implies that there exists a nonzero vector $y_\alpha \in \mathbb{R}^{|\alpha|}_+$ such that

(10)
$$y_\alpha^\top M_{\alpha\cdot} \le 0 \quad \text{and} \quad y_\alpha^\top (M_{\alpha\alpha} x_\alpha) = 0.$$

The last definition by using subclass, coincides with that given by Aganagić and Cottle [16] in the LCP context.

(ii) For $\mathbb{V} = \mathbb{R}^n$, $\mathcal{K} = \mathcal{L}^n_+$, we consider the canonical Jordan frame $\{e_1, e_2\}$(defined in Example 2.3(ii)). Then, corresponding to $\alpha = \{1, 2\}$ we have $\mathbb{V}^{(\alpha)} = \mathbb{V}$ (because $e_1 + e_2 = e$) and, associated with $\alpha = \{1\}$ and $\bar\alpha = \{2\}$ we have $\mathbb{V}^{(\alpha)} = \mathbb{V}_{11}$ and $\mathbb{V}^{(\bar\alpha)} = \mathbb{V}_{22}$, respectively. Moreover, $\mathcal{K}^{(\alpha)} = \{\kappa e_1 : \kappa \in \mathbb{R}_+\}$. Hence, for $\alpha = \{1\}$, the projection $P^{(\alpha)}$ of $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ on $\mathbb{V}^{(\alpha)}$ is given by $P^{(\alpha)}(x) = (x_1 + x_2)e_1$, where $x = (x_1 + x_2)e_1 + (x_1 - x_2)e_2 + (0, 0, x_3, \dots, x_n)$ is its Peirce decomposition associated with $\{e_1, e_2\}$. On the other hand, since $\mathbb{V}^{(\alpha)}$ are one dimensional spaces, their orthogonal projection are easily computed at the elements of $L(\mathbb{V}_1)$, with $L \in \mathcal{L}(\mathbb{R}^n)$, as follows:

$$P^{(\alpha)}(L(\kappa e_1)) = \kappa \frac{\langle L(e_1), e_1\rangle}{\|e_1\|^2} e_1, \quad P^{(\bar{\alpha})}(L(\kappa e_1)) = \kappa \frac{\langle L(e_1), e_2\rangle}{\|e_2\|^2} e_2, \quad \kappa \in \mathbb{R}. \tag{11}$$

So, for $L(x) = Mx$ with $M \in \mathbb{R}^{n\times n}$ and for $x \in \mathbb{R}^n$ with Pierce decomposition $x = (x_1 + x_2)e_1 + (x_1 - x_2)e_2 + (0, 0, x_3, \dots, x_n)$, Definition 3.11, for $\alpha = \{1\}$, reduces to:
A matrix $M : \mathcal{L}^n \to \mathcal{L}^n$ is said to have the T-property if for any matrix $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}$ with $D \in \mathbb{R}^{n-1\times n-1}$ being an orthogonal matrix, the existence of a solution $x \in \mathbb{R}^n$ to the system

$$\begin{aligned} x_1 = x_2 > 0,\ x_3 = \dots = x_n = 0, \quad & \langle \Lambda^\top M\Lambda(e_1), e_1\rangle \le 0,\ p = 0, \\ & \langle \Lambda^\top M\Lambda(e_1), e_2\rangle \ge 0, \end{aligned} \tag{12}$$

where $\widetilde{L}(x) = \kappa_1 e_1 + \kappa_2 e_2 + (0, 0, p)$, implies that there is a nonzero $y \in \mathbb{R}^n$ satisfying

$$\begin{aligned} y_1 = y_2 \ge 0,\ y_3 = \dots = y_n = 0, \quad & y_1\langle \Lambda^\top M^\top\Lambda(e_1), e_1\rangle \le 0,\ p' = 0, \\ & y_1\langle \Lambda^\top M^\top\Lambda(e_1), e_2\rangle \le 0,\ y_1\langle \Lambda^\top M\Lambda(e_1), e_1\rangle = 0, \end{aligned} \tag{13}$$

where $\widetilde{L}^\top(y) = \kappa'_1 e_1 + \kappa'_2 e_2 + (0, 0, p')$. Here we have used the Pierce decomposition of y with respect to $\{e_1, e_2\}$.

(iii) For $\mathbb{V} = \mathcal{S}^n$ and $\mathcal{K} = \mathcal{S}^n_+$, we consider the Jordan frame $\{E_1, \dots, E_n\}$ (defined in Example 2.3(iii)) and set $\alpha = \{1, \dots, l\}$ with $1 \le l \le n$. Then, taking into account Examples 2.4, 2.5 and 3.1, Definition 3.11 reduces to saying:
A linear transformation $L : \mathcal{S}^n \to \mathcal{S}^n$ is said to have the **T**-property if for any orthogonal matrix $U \in \mathbb{R}^{n\times n}$ and for any index set $\alpha = \{1, \dots, k\}(1 \le k \le n)$, the existence of a solution $X \in \mathcal{S}^n$ to the system

$$X_{\alpha\alpha} \in \mathcal{S}^{|\alpha|}_{++},\quad X_{ij} = 0, \forall i, j \notin \alpha,\ [\widetilde{L}_U(X)]_{\alpha\alpha} \in -\mathcal{S}^{|\alpha|}_+,\ [\widetilde{L}_U(X)]_{\alpha\bar{\alpha}} = 0,\quad [\widetilde{L}_U(X)]_{\bar{\alpha}\bar{\alpha}} \in \mathcal{S}^{|\bar{\alpha}|}_+, \tag{14}$$

implies that there is a nonzero matrix $Y \in \mathcal{S}^n$ satisfying

$$\begin{aligned} Y_{\alpha\alpha} \in \mathcal{S}^{|\alpha|}_+,\quad Y_{ij} = 0, \forall i, j \notin \alpha,\quad & [\widetilde{L}^\top_U(Y)]_{\alpha\alpha} \in -\mathcal{S}^{|\alpha|}_+,\quad [\widetilde{L}^\top_U(Y)]_{\bar{\alpha}\bar{\alpha}} \in -\mathcal{S}^{|\bar{\alpha}|}_+ \\ & [\widetilde{L}^\top_U(Y)]_{\alpha\bar{\alpha}} = 0,\quad \langle Y_{\alpha\alpha}, [\widetilde{L}_U(X)]_{\alpha\alpha}\rangle = 0. \end{aligned} \tag{15}$$

As far as, we know this is the first time that **T**-transformation is extend to a nonpolyhedral cone $\mathcal{K}$.

The following result is a extension of [16, Proposition 2] to our SCLCP context.

Proposition 3.12. *If a linear transformation $L : \mathbb{V} \to \mathbb{V}$ is monotone, then it has the* **T***-property.*

Proof. Let Λ be an orthogonal automorphism of $\mathbb{V}$, $\alpha = \{1, \dots, l\}$ $(1 \le l \le n)$ be a nonempty index set, and $x \in \mathbb{V}$ a solution of the system (7). Clearly, $\widetilde{L}$ and $\widetilde{L}^\top$ are monotone. Then, from the inequality $0 \le \langle x, \widetilde{L}(x)\rangle = \langle P^{(\alpha)}(x), P^{(\alpha)}(\widetilde{L}(x))\rangle$ and the fact that $P^{(\alpha)}(x) \in \mathrm{int}(\mathcal{K}^{(\alpha)})$ and $P^{(\alpha)}(\widetilde{L}(x)) \in -\mathcal{K}^{(\alpha)}$, we deduce that $P^{(\alpha)}(\widetilde{L}(x)) = 0$. Hence, $\langle x, \widetilde{L}(x)\rangle = \langle P^{(\alpha)}(x), P^{(\alpha)}(\widetilde{L}(x))\rangle = 0$.

We claim that

$$P^{(\alpha)}(\widetilde{L}^\top(x)) \in -\mathcal{K}^{(\alpha)},\ P^{(\bar{\alpha})}(\widetilde{L}^\top(x)) \in -\mathcal{K}^{(\bar{\alpha})},\ (I - P^{(\alpha)} - P^{(\bar{\alpha})})(\widetilde{L}^\top(x)) = 0.$$

Indeed, since $\langle x, (\widetilde{L} + \widetilde{L}^\top)(x)\rangle = 0$ and $\widetilde{L} + \widetilde{L}^\top$ is a self-adjoint monotone linear transformation, it clearly follows that $(\widetilde{L} + \widetilde{L}^\top)(x) = 0$. But $\widetilde{L}(x) \in \mathcal{K}$ (because $P^{(\alpha)}(\widetilde{L}(x)) = 0$ and (7) holds), then $\widetilde{L}^\top(x) \in -\mathcal{K}$. From this, it follows that $P^{(\alpha)}(\widetilde{L}^\top(x)) \in -\mathcal{K}^{(\alpha)}$ and $P^{(\bar{\alpha})}(\widetilde{L}^\top(x)) \in -\mathcal{K}^{(\bar{\alpha})}$. On the other hand, since $P^{(\alpha)}(x) \in \operatorname{int}(\mathcal{K}^{(\alpha)})$, $P^{(\alpha)}(\widetilde{L}^\top(x)) \in -\mathcal{K}^{(\alpha)}$ and

$$\langle P^{(\alpha)}(\widetilde{L}^\top(x)), P^{(\alpha)}(x)\rangle = \langle \widetilde{L}^\top(x), x\rangle = 0,$$

it follows that $P^{(\alpha)}(\widetilde{L}^\top(x)) = 0$. This together with condition $-\widetilde{L}^\top(x) \in \mathcal{K}$ and Proposition 2.8, Part (e) implies that $(I - P^{(\bar{\alpha})})(\widetilde{L}^\top(x)) = 0$. Thus, $y = x$ solve (8). □

$\mathbf{F}_2$-Transformation.

Definition 3.13. *A linear transformation $L : \mathbb{V} \to \mathbb{V}$ is said to have the $\mathbf{F}_2$-property if for any $\Lambda \in \operatorname{Aut}(\mathbb{V})$ orthogonal and any index set $\alpha = \{1, \ldots, l\}$ $(1 \le l \le r)$, the existence of a solution $x \in \mathbb{V}$ to the system*

(16)
$$P^{(\alpha)}(x) \in \operatorname{int}(\mathcal{K}^{(\alpha)}), \quad (I - P^{(\alpha)})(x) = 0, \quad (I - P^{(\bar{\alpha})})(\widetilde{L}(x)) = 0, \quad P^{(\bar{\alpha})}(\widetilde{L}(x)) \in \mathcal{K}^{(\bar{\alpha})},$$

$\bar{\alpha} = \{1, \ldots, r\}\backslash\alpha$, *implies that there is a nonzero $y \in \mathbb{V}$ satisfying*

(17)
$$P^{(\alpha)}(y) \in \mathcal{K}^{(\alpha)}, \quad (I - P^{(\alpha)})(y) = 0, \quad (I - P^{(\bar{\alpha})})(\widetilde{L}^\top(y)) = 0, \quad P^{(\bar{\alpha})}(\widetilde{L}^\top(y)) \in -\mathcal{K}^{(\bar{\alpha})}.$$

We illustrate this definition in the following Euclidean Jordan algebras.

Example 3.4. (i) For $\mathbb{V} = \mathbb{R}^n$ and $\mathcal{K} = \mathbb{R}^n_+$, we consider the Jordan frame $\{e_1, \ldots, e_n\}$ (defined in Example 2.3(i)) and set $\alpha = \{1, \ldots, l\}$ with $1 \le l \le n$. Taking into account Example 3.3, Part (ii) and that $L(x) = Mx$ with $M \in \mathbb{R}^{n\times n}$, Definition 3.13 reduces to saying:
A matrix $M \in \mathbb{R}^{n\times n}$ is an F_1-matrix if only if for any nonempty set $\alpha = \{1, \ldots, l\} \subseteq \{1, \ldots, n\}$, the existence of a vector $x_\alpha \in \mathbb{R}^{|\alpha|}$ satisfying

$$x_\alpha > 0, \quad M_{\alpha\alpha}x_\alpha = 0 \quad \text{and} \quad M_{\bar{\alpha}\alpha}x_\alpha \ge 0, \tag{18}$$

implies that there exists a nonzero vector $y_\alpha \in \mathbb{R}^{|\alpha|}_+$ such that

$$y_\alpha^\top M_{\alpha\alpha} = 0 \quad \text{and} \quad y_\alpha^\top M_{\alpha\bar{\alpha}} \le 0. \tag{19}$$

The last definition by using subclass, coincides with that given by Flores and López [14] in LCP context.

(ii) For $\mathbb{V} = \mathbb{R}^n$ and $\mathcal{K} = \mathcal{L}^n_+$, we consider the Jordan frame $\{e_1, e_2\}$ defined in Example 3.3(ii). Then, taking into account the ideas of Example 3.3(ii) and letting $L(x) = Mx$ with $M \in \mathbb{R}^{n\times n}$ for all $x \in \mathbb{R}^n$, Definition 3.13, for $\alpha = \{1\}$, reduces to saying:
A matrix $M \in \mathbb{R}^{n\times n}$ is said to have the $\mathbf{F}_2$-property if for any matrix $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}$ with $D \in \mathbb{R}^{n-1\times n-1}$ an orthogonal matrix, the existence of a solution $x \in \mathbb{R}^n$ to the system

$$x_1 = x_2 > 0,\ x_3 = \cdots = x_n = 0,\ \langle \Lambda^\top M\Lambda(e_1), e_1\rangle = 0,\ p = 0,\ \langle \Lambda^\top M\Lambda(e_1), e_2\rangle \ge 0, \tag{20}$$

where $\widetilde{L}(x) = \kappa_1 e_1 + \kappa_2 e_2 + (0,0,p)$, implies that there is a nonzero $y \in \mathbb{R}^n$ satisfying

(21)
$$y_1 = y_2 \geq 0,\ y_3 = \cdots = y_n = 0,\ y_1\langle \Lambda^\top M^\top \Lambda(e_1), e_1\rangle = 0,\ p' = 0,\ y_1\langle \Lambda^\top M^\top \Lambda(e_1), e_2\rangle \leq 0,$$

where $\widetilde{L}^\top(y) = \kappa'_1 e_1 + \kappa'_2 e_2 + (0,0,p')$.

(iii) For $\mathbb{V} = \mathcal{S}^n$ and $\mathcal{K} = \mathcal{S}^n_+$, we consider the Jordan frame $\{E_1, \ldots, E_n\}$ (defined in Example 2.3(iii)) and set $\alpha = \{1, \ldots, l\}$ with $1 \leq l \leq n$. Then, taking into account Examples 2.4, 2.5(iii) and 3.1, Definition 3.13 reduces to Definition of $\mathbf{F}_2$-transformation given in [15] in SDLCP context: A linear transformation $L : \mathcal{S}^n \to \mathcal{S}^n$ is said to have the $\mathbf{F}_2$-property if for any orthogonal matrix $U \in \mathbb{R}^{n\times n}$ and for any index set $\alpha = \{1, \ldots, k\}(1 \leq k \leq n)$, the existence of a solution $X \in \mathcal{S}^n$ to the system

(22)
$$X_{\alpha\alpha} \in \mathcal{S}^{|\alpha|}_{++},\quad X_{ij} = 0, \forall i,j \notin \alpha,\ [\widetilde{L}_U(X)]_{\alpha\alpha} = 0,\ [\widetilde{L}_U(X)]_{\alpha\bar{\alpha}} = 0,\quad [\widetilde{L}_U(X)]_{\bar{\alpha}\bar{\alpha}} \in \mathcal{S}^{|\bar{\alpha}|}_+,$$

implies that there is a nonzero matrix $Y \in \mathcal{S}^n$ satisfying

(23)
$$Y_{\alpha\alpha} \in \mathcal{S}^{|\alpha|}_+,\quad Y_{ij} = 0, \forall i,j \notin \alpha,\ [\widetilde{L}^\top_U(Y)]_{\alpha\alpha} = 0,\quad [\widetilde{L}^\top_U(Y)]_{\alpha\bar{\alpha}} = 0\quad [\widetilde{L}^\top_U(Y)]_{\bar{\alpha}\bar{\alpha}} \in -\mathcal{S}^{|\bar{\alpha}|}_+.$$

Proposition 3.14. *If L has the $\mathbf{F}_2$-property, then L is an $\mathbf{F}$-transformation.*

Proof. Let v be a nonzero solution of LCP$(L, \mathcal{K}, 0)$. Consider an orthogonal automorphism $\Lambda \in$Aut$(\mathbb{V})$ such that

$$\Lambda^\top(v) = \Lambda^{-1}(v) = \sum_{i=1}^{r} \lambda_i(v)e_i = \lambda_1(v)e_1 + \ldots + \lambda_l(v)e_l + 0e_{l+1} + \ldots + 0e_r,$$

where $\{e_1, \ldots, e_r\}$ is a Jordan frame of $\mathbb{V}$ and $\lambda_i(v) > 0$ for all $i = 1, \ldots, l$, with $l \in \{1, \ldots, r\}$. We proceed to show that $x = \Lambda^{-1}(v)$ is a solution of (16). It is immediate that $x = P^{(\alpha)}(x) \in \text{int}(\mathcal{K}^{(\alpha)})$, where $\alpha = \{1, \ldots, l\}$, and hence x satisfies first two conditions of (16). Also, $\widetilde{L}(x) = \Lambda^\top(L(v))$. So, since $L(v) \in \mathcal{K}$ it follows that $P^{(\bar{\alpha})}(\widetilde{L}(x)) \in \mathcal{K}^{(\bar{\alpha})}$ (cf. [12, Remark 4.1] and Proposition 2.8, Part (d)). Moreover, condition $\langle L(v), v\rangle = 0$ implies that $\langle x, \widetilde{L}(x)\rangle = 0$. Then, by using Proposition 2.8, Part (b) and (c), we get that $\widetilde{L}(x)$ has the following spectral decomposition

$$\widetilde{L}(x) = \sum_{i=1}^{r} \lambda_i(\widetilde{L}(x))e_i = 0e_1 + \ldots + 0e_l + \lambda_{l+1}(\widetilde{L}(x))e_{l+1} + \ldots + \lambda_r(\widetilde{L}(x))e_r,$$

where $\lambda_i(\widetilde{L}(x))e_i \geq 0$ for all $i = l+1, \ldots, r$. From this, it follows that $\widetilde{L}(x) \in \mathcal{K}^{(\bar{\alpha})}$ and hence $P^{(\bar{\alpha})}(\widetilde{L}(x)) = \widetilde{L}(x)$. Then, $\widetilde{L}(x)$ satisfies the last two conditions of (16). Therefore, there exists a nonzero solution y of (17).

We claim that $\tau_v = \Lambda(y)$ satisfies conditions (**i**)–(**iii**) in (5). Indeed, it is obvious that $\tau_v \in \mathcal{K}$, because $y = P^{(\alpha)}(y) \in \mathcal{K}^{(\alpha)}$ and $\Lambda(\mathcal{K}) = \mathcal{K}$. So, due to that $x \in$int$(\mathcal{K}^{(\alpha)})$, $y \in \mathcal{K}^{(\alpha)}$ with $y \neq 0$, from Proposition 2.8, Part (a), we get

$$\langle \tau_v, v\rangle \quad = \quad \langle y, x\rangle > 0.$$

Finally, since $L^\top(\tau_v) = (\Lambda^\top)^{-1}(\widetilde{L}^\top(y))$ and $\widetilde{L}^\top(y) \in -\mathcal{K}$ (consequence of (17)), it follows that $L^\top(\tau_v) \in -\mathcal{K}$. We have thus deduced that L is an $\mathbf{F}$-transformation. $\square$

In the following proposition we list various classes of linear transformations that are contained in the class $\mathbf{F_2}$.

Proposition 3.15. *$L \in \mathbf{F_2}$ if any of the following conditions is satisfied:*

- **(a):** *L is a star-transformation;*
- **(b):** *$L \in \mathbf{Z}$ and*
  - **(i):** *$-L$ is copositive or*
  - **(ii):** *L is normal;*
- **(c):** *L has $\mathbf{T}$-property.*

Proof. (a): Let Λ be an orthogonal automorphism of $\mathbb{V}$, $\alpha = \{1, \ldots, l\}$ $(1 \leq l \leq n)$ be a nonempty index set, and $x \in \mathbb{V}$ be a solution of the system (16). Let us define $v = \Lambda(x)$. Clearly, v, $L(v) = (\Lambda^\top)^{-1}(\widetilde{L}(x)) \in \mathcal{K}$ (because $x, \widetilde{L}(x) \in \mathcal{K}$ and Λ, $(\Lambda^\top)^{-1}$ preserve $\mathcal{K}$) and $\langle v, L(v)\rangle = \langle x, \widetilde{L}(x)\rangle = 0$. Hence, v is a nonzero solution of $\mathrm{SOL}(L, \mathcal{K}, 0)$. Since L is a star-transformation, we have that $L^\top(v) \in -\mathcal{K}$. Then, $\widetilde{L}^\top(x) = \Lambda^{-1}L^\top(v) \in -\mathcal{K}$ (because Λ^{-1} preserve $\mathcal{K}$). On the other hand, since $P^{(\alpha)}(x) \in \mathrm{int}(\mathcal{K}^{(\alpha)})$, $P^{(\alpha)}(\widetilde{L}^\top(x)) \in -\mathcal{K}^{(\alpha)}$ (cf. Proposition 2.8, Part (d)) and

$$\langle P^{(\alpha)}(\widetilde{L}^\top(x)), P^{(\alpha)}(x)\rangle = \langle \widetilde{L}^\top(x), x\rangle = \langle x, \widetilde{L}(x)\rangle = 0,$$

it follows that $P^{(\alpha)}(\widetilde{L}^\top(x)) = 0$. This together with condition $-\widetilde{L}^\top(x) \in \mathcal{K}$ and Proposition 2.8, Part (e) implies that $(I - P^{(\bar{\alpha})})(\widetilde{L}^\top(x)) = 0$. Therefore, $y = x$ solves (17). Thus, we conclude that L has the $\mathbf{F}_2$-property.

(b): Let Λ be an orthogonal automorphism of $\mathbb{V}$, $\alpha = \{1, \ldots, l\}$ $(1 \leq l \leq n)$ be a nonempty index set, and $x \in \mathbb{V}$ a solution of the system (16). As before, $v = \Lambda(x)$ is a nonzero element of $\mathrm{SOL}(L, \mathcal{K}, 0)$. Since $L \in \mathbf{Z}$, we obtain $\langle L(v), L(v)\rangle \leq 0$ and consequently $L(v) = 0$. Hence, $\widetilde{L}(x) = \Lambda^\top L(v) = 0$. On the other hand, as $\langle \widetilde{L}(z), z\rangle = \langle L(\Lambda(z)), \Lambda(z)\rangle$ and $-L$ is copositive, it follows that $-\widetilde{L}$ is copositive (because Λ preserve $\mathcal{K}$). Moreover, as L is normal and Λ is an orthogonal automorphism, it follows that $\widetilde{L}$ also is normal. Thus, the arguments given in order to prove that $L \in \mathbf{F}_1$, but applied to $\widetilde{L}$ instead of L, imply that L has the $\mathbf{F}_2$-property.
(c): If L has the $\mathbf{T}$-property, then obviously $L \in \mathbf{F}_2$, since (16) implies that $P^{(\alpha)}(\widetilde{L}(x)) = 0$ and this implies $P^{(\alpha)}(\widetilde{L}^\top(y)) = 0$. $\square$

3.2. **Examples of transformations.** In this section, we present some linear in $\mathcal{L}(\mathbb{V})$ that belong to subclasses $\mathbf{F_1}$, $\mathbf{F_2}$ and $\mathbf{T}$. These linear transformations are intensively studied in the LCP literature.

1. **Lyapunov transformation:** Let $a \in \mathbb{V}$ with $\mathbb{V}$ any Euclidean Jordan algebra. The Lyapunov transformation L_a defined in (2) is a Lyapunov-like transformation [13]. By Proposition 2.9, Part (d), Proposition 3.10, Part (a) and Proposition 3.15, Part (a) we have that $L_a \in \mathbf{F_1} \cap \mathbf{F_2}$. On the other hand, if $a \in \mathrm{int}(\mathcal{K})$, then L_a is strongly monotone (cf. Proposition 2.8, Part(a)); thus, L_a has the $\mathbf{T}$-property by Proposition 3.12.
2. **Quadratic representation:** Let $\mathbb{V}$ be a Euclidean Jordan algebra and $a \in \mathbb{V}$. If, in addition, V is simple and $\pm a \in \mathrm{int}(K)$, the transformation P_a is strongly monotone (see [13, Theorem 6.5]). By Proposition 2.9, Parts (a) and (c), Proposition 3.10, Part (a) and Proposition 3.15, Part (c) we have that $P_a \in \mathbf{F_1} \cap \mathbf{F_2}$. On the other hand, under the same assumptions, P_a also has the $\mathbf{T}$-property by Proposition 3.12.
3. **Stein transformation:** Let $a \in \mathbb{V}$ with $\mathbb{V}$ any Euclidean Jordan algebra. Consider the Stein transformation S_a defined by $S_a = I - P_a$. If $\lambda_i(\pm a) \subseteq (-1, 1)$, for all i,

then the transformation S_a is strongly monotone (see [28, Theorem 3.3]). Hence, by using Proposition 2.9, Parts (a) and (c), Proposition 3.10, Part (a) and Proposition 3.15, Part (c) we have that $S_a \in \mathbf{F_1} \cap \mathbf{F_2}$. On the other hand, under the same assumptions, S_a also has the **T**-property by Proposition 3.12.

(4) **The relaxation transformation:** Let $\{e_1, \ldots, e_r\}$ be a Jordan frame in $\mathbb{V}$ and $A \in \mathbb{R}^{r \times r}$. We define $R_A : \mathbb{V} \to \mathbb{V}$ as follows. For any $x \in \mathbb{V}$, write the Peirce decomposition $x = \sum_{i=1}^r x_i e_i + \sum_{i<j} x_{ij}$. Then

$$R_A(x) = \sum_{i=1}^{r} y_i e_i + \sum_{i<j} x_{ij},$$

where $[y_1, y_2, \ldots, y_r]^\top = A([x_1, x_2, \ldots, x_r]^\top)$. This is a generalization of a concept introduced in [29] for $\mathbb{V} = \mathcal{S}^n$. Let A be a P-matrix (i.e all its principal minors are positive). By [11, Proposition 5.1], the latter is equivalent to saying that R_A has the **P**-property, which in turn by Proposition 2.9, Parts (a) and (c), implies that R_A is star-transformation. Hence, by using Proposition 3.10, Part (a) and Proposition 3.15, Part (a) we have that $R_A \in \mathbf{F_1} \cap \mathbf{F_2}$. On the other hand, if A is a nonnegative diagonal matrix, then clearly R_A is monotone transformation. Hence, by using Proposition 3.12 we have that R_A has the **T**-property.

4. Existence results for symmetric cone LCP's

In this section, we present coercive and noncoercive existence results for symmetric cone SCLCP's. Our approach follows the same arguments of [30] for LCP's and of [6] for SDLCP's. For this, we recall that problem (1) is equivalent to the following variational inequality problem VIP$(L, \mathcal{K}, q)$: find an element $\bar{x}$ such that:

$$\bar{x} \in \mathcal{K} \text{ and } \langle L(\bar{x}) + q, x - \bar{x} \rangle \geq 0, \text{ for all } x \in \mathcal{K}. \tag{24}$$

We approximate this problem by the following sequence of variational inequality problems VIP(L, D_k, q): find an element x^k such that:

$$x^k \in D_k \text{ and } \langle L(x^k) + q, x - x^k \rangle \geq 0 \text{ for all } x \in D_k. \tag{25}$$

where $D_k := \{x \in \mathcal{K} : \langle d, x \rangle \leq \sigma_k\}$ with $d \in \text{int}(\mathcal{K})$ and $\sigma_k \to +\infty$. Since each set D_k is compact and convex, by Hartman-Stampacchia theorem we have that (25) has a nonempty solution set SOL(L, D_k, q). Moreover, it is clear that each solution x^k is a solution of (25) if and only if $X^k \in \Omega_k$ is an optimal solution of the linear program

$$\inf_x \Big[\langle L(x^k) + q, x \rangle : \ x \in \mathcal{K}, \ \langle d, x \rangle \leq \sigma_k \Big].$$

Applying optimality conditions, we obtain that x^k is a solution of (25) if and only if there exists $\theta_k \in \mathbb{R}$ such that (x^k, θ_k) is a solution of the following problem, called the augmented symmetric cone LCP: find $x^k \in \mathcal{K}$ and $\theta_k \geq 0$ such that

$$y^k := L(x^k) + q + \theta_k d \in \mathcal{K}, \ \langle d, x^k \rangle \leq \sigma_k, \langle y^k, x^k \rangle = 0 \text{ and } \theta_k(\sigma_k - \langle d, x^k \rangle) = 0. \tag{ASCLCP$_k$}$$

From this, we observe that

$$\langle d, x^k \rangle < \sigma_k \implies \theta_k = 0 \implies x^k \in \text{SOL}(L, \mathcal{K}, q). \tag{26}$$

Moreover, we have from (ASCLCP$_k$) that

$$\theta_k = -\Big\langle L(x^k)+q, \frac{x^k}{\sigma_k}\Big\rangle. \tag{27}$$

Implications (26) shows that only the case $\langle d, x^k\rangle = \sigma_k$, for all $k \in \mathbb{N}$, deserves further analysis. This analysis is carried out below by extending the arguments from [6, 30] to our symmetric cone framework via the spectral decomposition theorem. Since $\langle d, x^k\rangle = \sigma_k$ for all $k \in \mathbb{N}$, we are interested in to obtain asymptotic properties of the sequence $\{\frac{x^k}{\sigma_k}\}$.

As $y^k, \frac{x^k}{\sigma_k} \in \mathcal{K}$ and $\langle y^k, \frac{x^k}{\sigma_k}\rangle = 0$ for all $k \in \mathbb{N}$, by Proposition 2.8, Part (b) and (c), it follows that there exists a Jordan frame $\{e_1^k, \ldots, e_r^k\}$ such that

$$\frac{x^k}{\sigma_k} = \sum_{i=1}^r \lambda_i\left(\frac{x^k}{\sigma_k}\right) e_i^k, \quad y^k = \sum_{i=1}^r \lambda_i(y^k) e_i^k, \tag{28}$$

where $\lambda(\frac{x^k}{\sigma_k}) := (\lambda_1(\frac{x^k}{\sigma_k}), \ldots, \lambda_r(\frac{x^k}{\sigma_k}))$ and $\lambda(y^k) := (\lambda_1(y^k), \ldots, \lambda_r(y^k))$ denote the eigenvalues of $\frac{x^k}{\sigma_k}$ and y^k, respectively. Therefore, $\Big\langle d, \frac{x^k}{\sigma_k}\Big\rangle = \sum_{i=1}^r \lambda_i(\frac{x^k}{\sigma_k})\langle d, e_i^k\rangle = 1$ and since $\lambda_i(\frac{x^k}{\sigma_k})\langle d, e_i^k\rangle \geq 0$ for all $i \in \{1, \ldots, r\}$, we conclude that for all $k \in \mathbb{N}$ it holds that

$$\gamma^k := \Big(\lambda_1\left(\frac{x^k}{\sigma_k}\right)\langle d, e_1^k\rangle, \ldots, \lambda_r\left(\frac{x^k}{\sigma_k}\right)\langle d, e_r^k\rangle\Big) \in \Delta := \Big\{\gamma \in \mathbb{R}^r_+ :\ \sum_{i=1}^r \gamma_i = 1\Big\}.$$

As stated in [31, Theorem 18.2], the simplex Δ can be decomposed as the disjoint union of the relative interior of its extreme faces $\Delta_{J_i} = \mathrm{co}\{(\underbrace{0, \ldots, 0, 1}_{s}, 0, \ldots 0) :\ s \in J_i\}$, with J_i being a nonempty subindex set of $\{1, \ldots, r\}$ for each $i = 1, \ldots, 2^r - 1$; that is to say,

$$\Delta = \bigsqcup_{i=1}^{2^r-1} \mathrm{ri}\Big(\Delta_{J_i}\Big). \tag{29}$$

The next result describes the asymptotic behavior of the sequence $\{\frac{x^k}{\sigma_k}\}$.

Lemma 4.1. *Let $\{x^k\}$ be a sequence of solutions to* (ASCLCP$_k$) *such that $\langle d, x^k\rangle = \sigma_k$ for all $k \in \mathbb{N}$ and $\frac{x^k}{\sigma_k} \to v$ for some $v \in \mathcal{K}$. Then*

(a): *$v \in \mathrm{SOL}(L, \mathcal{K}, \tau_v d)$ with $\tau_v = -\langle L(v), v\rangle \geq 0$.*

Moreover, there exist a nonempty subindex set $J_v \subseteq \{1, \ldots, r\}$, a Jordan frame $\{e_1, \ldots, e_r\}$ and a subsequence $\{k_m\}$ such that

(b): *$\{e_1^{k_m}, \ldots, e_r^{k_m}\} \to \{e_1, \ldots, e_r\}$ and $\lambda(\frac{x^{k_m}}{\sigma_{k_m}}) \to \lambda(v)$ as $m \to +\infty$; thus $\gamma^{k_m} \to \gamma := (\lambda_1(v)\langle d, e_1\rangle, \ldots, \lambda_r(v)\langle d, e_r\rangle) \in \Delta$.*

(c): *$\gamma^{k_m} \in \mathrm{ri}(\Delta_{J_v})$; i.e., $\mathrm{supp}\{\lambda(\frac{x^{k_m}}{\sigma_{k_m}})\} = J_v$ and $\lambda(y^{k_m})\Big|_{J_v} = 0$ for all $m \in \mathbb{N}$. As a consequence $\mathrm{supp}\{\lambda(v)\} \subseteq J_v$;*

Finally, for every $z \in \mathcal{K} \setminus \{0\}$ with $\mathrm{supp}\{\lambda(z)\} \subseteq J_v$ one has

(d): *$\langle y^{k_m}, z\rangle = 0$ for all $m \in \mathbb{N}$;*

(e): *$\Big\langle L(x^{k_m}) + q, \frac{z}{\langle d, z\rangle}\Big\rangle = \langle L(x^{k_m}) + q, v\rangle$ for all $m \in \mathbb{N}$;*

(f): *$\langle L(v), \frac{z}{\langle d, z\rangle}\rangle = \langle L(v), v\rangle$.*

Proof. (a): By dividing inequality (25) by σ_k^2, setting $x = 0$ and $x = \frac{\sigma_k}{\langle d,z\rangle} z$ for $z \in \mathcal{K} \setminus \{0\}$, and taking limit $k \to +\infty$ we get $\langle L(v), v\rangle \leq 0$ and $\langle L(v), \frac{z}{\langle d,z\rangle} - v\rangle \geq 0$ respectively. The results follows from this since $\langle d, v\rangle = 1$.

(b): Let us consider the case when $\mathbb{V}$ is not necessarily simple. Due to Theorem 2.4, it suffices to consider $\mathbb{V} = \mathbb{V}_1 \times \mathbb{V}_2 \times \cdots \times \mathbb{V}_{\bar{j}}$, where each $\mathbb{V}_j$ is a simple Jordan Algebra with the corresponding symmetric cone $\mathcal{K}_j$ and rank r_j. As before, the superscript (j) is used to denote the j-th block of a given vector in $\mathbb{V}$.

Remarks 2.2 and 2.5, applied to each $\mathbb{V}_j$, imply the existence of positive numbers θ_j, with $j = 1, \ldots, \bar{j}$, such that either $||(e_i^k)^{(j)}||^2 = 0$ or $||(e_i^k)^{(j)}||^2 = \theta_j$, for all $i \in \{1, \ldots, r\}$, $k \in \mathbb{N}$, where the latter holds for one and only one block j. Then $\|e_i^k\|^2 \leq \max\{\theta_j : j = 1, \ldots, \bar{j}\}$, for all $i \in \{1, \ldots, r\}$, $k \in \mathbb{N}$. Set $\bar{\theta} := \max\{\theta_j : j = 1, \ldots, \bar{j}\}$. Therefore, there exists a Jordan frame $\{e_1, \ldots, e_r\}$ and a subsequence $\{k_m\}$ such that $\{e_1^{k_m}, \ldots, e_r^{k_m}\}$ converges to $\{e_1, \ldots, e_r\}$. Moreover, Proposition 2.8, Part (h), yields [1]

$$\lambda_{\min}(d) \leq \frac{\sum_{j=1}^{\bar{j}} \theta_j \langle d^{(j)}, (e_i^k)^{(j)}\rangle}{\sum_{j=1}^{\bar{j}} \theta_j \|(e_i^k)^{(j)}\|^2} \leq \frac{\bar{\theta}\sum_{j=1}^{\bar{j}} \langle d^{(j)}, (e_i^k)^{(j)}\rangle}{\sum_{j=1}^{\bar{j}} \theta_j^2} = \frac{\bar{\theta}\langle d, e_i^k\rangle}{\sum_{j=1}^{\bar{j}} \theta_j^2},$$

for all $i \in \{1, \ldots, r\}$ and for all $k \in \mathbb{N}$. This together with the equality $\sum_{i=1}^r \lambda_i(\frac{x^k}{\sigma_k})\langle d, e_i^k\rangle = 1$ implies that eigenvalues $\lambda_i(\frac{x^k}{\sigma_k})$, with $i = 1, \ldots, r$, are bounded. Hence, passing to a subsequence if necessary, it follows that $\{\lambda(\frac{x^{k_m}}{\sigma_{k_m}})\}$ converges to $\lambda(v)$ as $m \to +\infty$. Consequently, $\gamma^{k_m} \to \gamma$ as $m \to +\infty$, and $\gamma \in \Delta$.

(c): Since $\gamma^k \in \Delta$ for all k, from decomposition (29) without loss of generality we may consider that there exists a nonempty subindex set $J_v \subseteq \{1, \ldots, r\}$ such that $\gamma^{k_m} \in \mathrm{ri}(\Delta_{J_v})$ for all $m \in \mathbb{N}$. Hence, we obtain that $\mathrm{supp}\{\lambda(\frac{x^{k_m}}{\sigma_{k_m}})\} = J_v$ for such m (see [25, Exercise 2.28(e)]). From this, we prove that $\mathrm{supp}\{\lambda(v)\} \subseteq J_v$. From the spectral decompositions (28) we get

$$\begin{aligned} 0 = \left\langle \frac{x^{k_m}}{\sigma_{k_m}}, y^{k_m} \right\rangle &= \sum_{i,j=1}^{r} \lambda_i\left(\frac{x^{k_m}}{\sigma_{k_m}}\right)\lambda_j(y^{k_m})\langle e_i^{k_m}, e_j^{k_m}\rangle \\ &= \sum_{i=1}^{r} \lambda_i\left(\frac{x^{k_m}}{\sigma_{k_m}}\right)\lambda_i(y^{k_m})\|e_i^{k_m}\|^2 \\ &= \sum_{i\in J_v} \lambda_i\left(\frac{x^{k_m}}{\sigma_{k_m}}\right)\lambda_i(y^{k_m})\|e_i^{k_m}\|^2. \end{aligned}$$

and thus $\lambda(y^{k_m})\Big|_{J_v} = 0$ since $\lambda_i(\frac{x^{k_m}}{\sigma_{k_m}}) > 0$ for $i \in J_v$.

(d): Let $z \in \mathcal{K}$ be such that $\mathrm{supp}\{\lambda(z)\} \subseteq J_v$. By applying part (g) of Proposition 2.8 and item (c) above, we obtain:

$$0 \leq \mathrm{tr}(y^{k_m} \circ z) \leq \langle \lambda(y^{k_m}), \lambda(z)\rangle = \langle \lambda(y^{k_m})\Big|_{J_v}, \lambda(z)\Big|_{J_v}\rangle = 0.$$

Therefore, $\mathrm{tr}(y^{k_m} \circ z) = 0$ for all $m \in \mathbb{N}$. The desired result follows from Parts (g) and (b) of Proposition 2.8.

[1]This upper bound can be slightly improved thanks to Remark 2.5. Indeed, one can obtain $\lambda_{\min}(d) \leq \frac{\langle d,e_i^k\rangle}{\theta_{j_{i,k}}} \leq \frac{\langle d,e_i^k\rangle}{\min_j \theta_j}$, where $j_{i,k}$ corresponds to the nonzero block of e_i^k.

(e): If $z \in \mathcal{K} \setminus \{0\}$ is such that $\operatorname{supp}\{\lambda(z)\} \subseteq J_v$, then equation (27) and item (d) yield

$$\Big\langle L(x^{k_m}) + q, \frac{x^{k_m}}{\sigma_{k_m}} \Big\rangle = \Big\langle L(x^{k_m}) + q, \frac{z}{\langle d, z\rangle} \Big\rangle.$$

Replacing z by v, we obtain item (e).

(f): After dividing the equality in item (e) by σ_{k_m} and taking the limit $m \to +\infty$ we obtain the desired result. □

The proof of Lemma 4.1, Part (a), shows us that sets $\mathrm{SOL}(L, \mathcal{K}, \tau d)$ for $\tau \geq 0$ play an important role in our analysis. Conditions imposed to these sets allow us to extend to SCLCP's the following classes of linear transformations that were introduced for LCP's by García (see [30] and the references therein) and that were extended to SDLCP's in [6].

Definition 4.2. *Let $L : \mathbb{V} \to \mathbb{V}$ be a linear transformation.*

- *L is a García's transformation if there exists a $d \in \operatorname{int}(\mathcal{K})$ such that $\mathrm{SOL}(L, \mathcal{K}, \tau d) = \{0\}$ for all $\tau > 0$. In this case we say that L is a $\mathbf{G}$-transformation with respect to d, or simply $L \in \mathbf{G}(d)$.*
- *L is a #-transformation if $[v \in \mathrm{SOL}(L, \mathcal{K}, 0) \implies (L + L^\top)(v) \in \mathcal{K}]$.*
- *L is a $\mathbf{G}^{\#}$-transformation if $L \in \mathbf{G}$ and it is a #-transformation. Similarly, a $\mathbf{G}(d)^{\#}$-transformation is defined for $d \in \operatorname{int}(\mathcal{K})$.*

Example 4.1. (1) *Monotone and copositive transformations are $\mathbf{G}$-transformations.*

(2) *Proceeding exactly as in* [6, Proposition 4.8] *one can prove that $L \in \#$ if any of the following conditions is satisfied: L is self-adjoint (that is, $L^\top = L$); L is skew-symmetric; $L \in \mathbf{R_0}$; L is copositive; $-L$ is a star-transformation; and L is a star-transformation and $-L^\top \in \mathbf{Z}$.*

The next result shows that the class $\mathbf{G}$ is invariant under automorphisms.

Lemma 4.3. *Let $L : \mathbb{V} \to \mathbb{V}$ be a linear transformation. For $\Gamma \in \operatorname{Aut}(\mathcal{K})$, define $\widehat{L} = \Gamma^\top L \Gamma$. Then, L is a $\mathbf{G}$-transformation with respect to d if and only if $\widehat{L}$ is a $\mathbf{G}$-transformation with respect to $\Gamma^\top(d)$.*

Proof. The result follows directly from $\Gamma^{-1}(\mathrm{SOL}(L, \mathcal{K}, d)) = \mathrm{SOL}(\widehat{L}, \mathcal{K}, \Gamma^\top(d))$ (see [12, Theorem 5.1]) and $\Gamma^\top \in \operatorname{Aut}(K)$. □

The next proposition provides two characterizations of the class of Garcia's linear transformations. This is an symmetric cone version of [30, Proposition 3.1] and [6, Proposition 4.6] proved for LCP's and SDLCP's ,respectively.

Proposition 4.4. *Let $d \in \operatorname{int}(\mathcal{K})$ and $L : \mathbb{V} \to \mathbb{V}$ be a linear transformation. Then, the following are equivalent:*

(*a*) $L \in \mathbf{G}(\mathbf{d})$;

(*b*) $\Big[v \in \mathcal{K},\ L(v) - \langle L(v), v\rangle d \in \mathcal{K},\ \langle d, v\rangle = 1\Big] \Longrightarrow \langle L(v), v\rangle \geq 0$;

(*c*) $\Big[v \in \mathcal{K},\ \langle d, v\rangle = 1,\ \langle L(v), v\rangle < 0\Big] \Longrightarrow L(v) - \langle L(v), v\rangle d \notin \mathcal{K}$.

Proof. $(a) \Rightarrow (b)$: We argue by contradiction. Suppose that the left-hand side of item (b) holds and $\langle L(v), v\rangle < 0$. It follows that $\langle L(v) - \langle L(v), v\rangle d, v\rangle = 0$ and hence $v \in \mathrm{SOL}(L, \mathcal{K}, \tau d)$ with $\tau = -\langle L(v), v\rangle > 0$. By linearity we have $v/\tau \in \mathrm{SOL}(L, \mathcal{K}, d)$, which implies that $v = 0$ by item (a), obtaining a contradiction with the fact that $\langle d, v\rangle = 1$.
$(b) \Rightarrow (c)$: We argue by contradiction. Suppose that the left-hand side of item (c) holds

and that $L(v) - \langle L(v), v\rangle d \in \mathcal{K}$. Then, by using item (b) we conclude that $\langle L(v), v\rangle \geq 0$, obtaining a contradiction.
$(c) \Rightarrow (a)$: We argue by contradiction. Suppose that for some $\tau > 0$ there exists a $v \neq 0$ such that $v/\tau \in \mathrm{SOL}(L, \mathcal{K}, d)$. By changing τ if necessary we may assume that $\langle d, v\rangle = 1$. By assumption we have that $v, L(v)+\tau d \in \mathcal{K}$ and $\langle v, L(v)+\tau d\rangle = 0$. From this, we deduce that $\langle L(v), v\rangle = -\tau < 0$. Then, by using item (c) we conclude that $L(v) - \langle L(v), v\rangle d \notin \mathcal{K}$, obtaining a contradiction with the fact that $L(v) + \tau d \in \mathcal{K}$. □

We now obtain a bound for the asymptotic cone of the solution set to symmetric cone LCP's for $\mathbf{G}^{\#}$-transformations.

Proposition 4.5. *If* $L \in \mathbf{G}^{\#}$, *then* $\mathrm{SOL}(L, \mathcal{K}, q)^{\infty} \subseteq \mathrm{SOL}(L, \mathcal{K}, 0) \cap \{-q\}^{+}$.

Proof. Let $d \in \mathrm{int}(\mathcal{K})$ be such that $L \in \mathbf{G}^{\#}(d)$ and $v \in \mathrm{SOL}(L, \mathcal{K}, q)^{\infty}$. If $v = 0$ then the inclusion is trivial. So, we consider that $v \neq 0$. Without loss of generality we assume that $\langle d, v\rangle = 1$. By definition there exists $\{x^k\}$ and $\{t_k\}$ such that $x^k \in \mathrm{SOL}(L, \mathcal{K}, q)$ for all $k \in \mathbb{N}$, $t_k \to +\infty$ and $\frac{x^k}{t_k} \to v$ as $k \to +\infty$. By defining $\sigma_k := \langle d, x^k\rangle$ for all $k \in \mathbb{N}$ it is easy to check that $\sigma_k \to +\infty$ and $\frac{x^k}{\sigma_k} \to v$. By Lemma 4.1, Part (a) we have $v \in \mathrm{SOL}(L, \mathcal{K}, \tau_v d) \setminus \{0\}$ with $\tau_v = -\langle L(v), v\rangle \geq 0$. If $\tau_v > 0$, then we get a contradiction to L being a $\mathbf{G}$-transformation. Therefore, $\tau_v = 0$ and we have $v \in \mathrm{SOL}(L, \mathcal{K}, 0)$. From this and Lemma 4.1, Part (f) for $z = \frac{x^{k_m}}{\sigma_{k_m}}$ we get $\langle L(v), \frac{x^{k_m}}{\sigma_{k_m}}\rangle = 0$. Then,

$$0 = \left\langle L(x^{k_m}) + q, \frac{x^{k_m}}{\sigma_{k_m}} \right\rangle = \langle L(x^{k_m}) + q, v\rangle = \langle x^{k_m}, (L + L^{\top})(v)\rangle + \langle q, v\rangle,$$

where we have used Lemma 4.1, Part (d) and the fact that each x^{k_m} is a solution to problem (1). As $L \in \#$, we have $(L+L^{\top})(v) \in \mathcal{K}$, which in turn by Proposition 2.8, Part (a) implies that $\langle x^{k_m}, (L + L^{\top})(v)\rangle \geq 0$. Hence, from the above equality we get $\langle q, v\rangle \leq 0$. □

We now obtain existence results that extend [32, Theorems 9 and 11] given for LCP's and [6, Theorem 5.1] given for SDLCP's.

Theorem 4.6. *Let* $q \in \mathbb{V}$ *and* $L \in \mathbf{G}^{\#}$.

- **(a):** *If* $q \in \mathrm{SOL}(L, \mathcal{K}, 0)^{+}$, *then* $\mathrm{SOL}(L, \mathcal{K}, q)$ *is nonempty (possibly unbounded);*
- **(b):** *If* $q \in \mathrm{int}[\mathrm{SOL}(L, \mathcal{K}, 0)^{+}]$, *then* $\mathrm{SOL}(L, \mathcal{K}, q)$ *is nonempty and compact.*

Proof. Let $d \in \mathrm{int}(\mathcal{K})$ be such that $L \in \mathbf{G}^{\#}(d)$.

(a): Let $\{(x^k, \theta_k)\}$ be a sequence of solutions to problems (ASCLCP_k). If there exists $k \in \mathbb{N}$ such that $\langle d, x^k\rangle < \sigma_k$, then by implication (26) we have that $x^k \in \mathrm{SOL}(L, \mathcal{K}, q)$ and we are done. On the contrary, if $\langle d, x^k\rangle = \sigma_k$ for all $k \in \mathbb{N}$, then up to subsequences $\frac{x^k}{\sigma_k} \to v$ for some v. By Lemma 4.1, Part (a) we have $v \in \mathrm{SOL}(L, \mathcal{K}, \tau_v d)$. Proceeding as in Proposition 4.5 we prove that $\tau_v = 0$; thus, $v \in \mathrm{SOL}(L, \mathcal{K}, 0)$. From this, by Lemma 4.1 there exists a nonempty subindex set $J_v \subseteq \{1, \ldots, r\}$ and a subsequence $\{\frac{x^{k_m}}{\sigma_{k_m}}\}$ such that $\mathrm{supp}\{\lambda(\frac{x^{k_m}}{\sigma_{k_m}})\} = J_v$ and $\langle L(v), x^{k_m}\rangle = 0$ for all $m \in \mathbb{N}$. By using this, equality (27), and Lemma 4.1, Part (e) we obtain

$$0 \leq \theta_{k_m} = -\left\langle L(x^{k_m}) + q, \frac{x^{k_m}}{\sigma_{k_m}} \right\rangle = -\langle L(x^{k_m}) + q, v\rangle = -\langle x^{k_m}, (L + L^{\top})(v)\rangle - \langle q, v\rangle.$$

As $v \in \mathrm{SOL}(L, \mathcal{K}, 0)$, by hypothesis we get $\langle q, v\rangle \geq 0$ and $(L+L^\top)(v) \in \mathcal{K}$; thus, $\langle x^{k_m}, (L+L^\top)(v)\rangle \geq 0$ by Proposition 2.8, Part (a). Consequently, $\theta_{k_m} = 0$ and by implication (26) we conclude that $x^{k_m} \in \mathrm{SOL}(L, \mathcal{K}, q)$ and we are done.

(b): From item (a) we conclude that $\mathrm{SOL}(L, \mathcal{K}, q) \neq \emptyset$. To prove that this set is bounded it is sufficient to show that $\mathrm{SOL}(L, \mathcal{K}, q)^\infty = \{0\}$. This follows from Proposition 4.5 since by hypothesis $\mathrm{SOL}(L, \mathcal{K}, 0) \cap \{-q\}^+ = \{0\}$. Indeed, if on the contrary we suppose that there exists $u \neq 0$ such that $u \in \mathrm{SOL}(L, \mathcal{K}, 0)$ and $\langle q, u\rangle \leq 0$, then as $q \in \mathrm{int}[\mathrm{SOL}(L, \mathcal{K}, 0)^+] = \mathrm{SOL}(L, \mathcal{K}, 0)^{s+}$ (see [25, Exercise 6.22]) we obtain $\langle q, u\rangle > 0$, a contradiction. □

The last theorem directly implies the following result.

Corollary 4.7. *If $L \in \mathbf{G}$, then $L \in \mathbf{R_0}$ if and only if $L \in \mathbf{Q_b}$.*

Remark 4.8. *The hypothesis on q of Theorem 4.6, Part* (a) *implies the following necessary condition:*

$$q \in \mathrm{SOL}(L, \mathcal{K}, 0)^+ \Longrightarrow \lambda_{\max}(q) \geq 0.$$

Indeed, if $q \in \mathrm{SOL}(L, \mathcal{K}, 0)^+$, then $\langle q, x\rangle \geq 0$ for all $x \in \mathrm{SOL}(L, \mathcal{K}, 0)$. In case when $\mathbb{V}$ is not simple, we denote by $\mathbb{V}_j$, with $j = 1, \ldots, \bar{j}$, the simple Jordan Algebra located in the j-th position, by $\mathcal{K}_j$ its corresponding symmetric cone, by r_j its rank, and by $x^{(j)}$ and $q^{(j)}$ the j-th block of x and q, respectively. So, Proposition 2.8, Part (f), applied to $\mathbb{V}_j$, implies that $\sum_{j=1}^{\bar{j}} \sum_{i=1}^{r_j} \lambda_i(q^{(j)})\lambda_i(x^{(j)}) \geq \langle q, x\rangle \geq 0$. But, since $\lambda_i(x^{(j)}) \geq 0$ for all $i = 1, \ldots, r_j$ and $j = 1, \ldots, \bar{j}$ (because $x^{(j)} \in \mathcal{K}_j$), it necessarily follows that $\lambda_{\max}(q) \geq 0$.

By taking into account Example 4.1 we now list some conditions under which the linear transformations defined in Section 3.2 are García's transformations.

Example 4.2. Let $\mathbb{V}$ be a Euclidean Jordan algebra and $a \in \mathbb{V}$.

(1) If $a \in$int$(\mathcal{K})$, then L_a is strongly monotone (cf. Proposition 2.8, Part(a)). Thus, $L_a \in \mathbf{G}(d) \cap \mathbf{R_0}$ for any $d \in \mathrm{int}(\mathcal{K})$.
(2) If $\mathbb{V}$ is simple and $\pm a \in$int$(\mathcal{K})$, the quadratic transformation P_a is strongly monotone by [13, Theorem 6.5]. Thus, $P_a \in \mathbf{G}(d) \cap \mathbf{R_0}$ for any $d \in \mathrm{int}(\mathcal{K})$.
(3) If $\lambda_i(\pm a) \subseteq (-1, 1)$, for all i, then the Stein transformation S_a is strongly monotone (see [28, Theorem 3.3]). Thus, $S_a \in \mathbf{G}(d) \cap \mathbf{R_0}$ for any $d \in \mathrm{int}(\mathcal{K})$.
(4) Let $A \in \mathbb{R}^{r\times r}$ be a nonnegative matrix, then the relaxation transformation R_A is copositive. Thus, $R_A \in \mathbf{G}(d)$ for any $d \in \mathrm{int}(\mathcal{K})$.

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