An Exact Algorithm for Power Grid Interdiction Problem with Line Switching

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***Abstract*—Power grid vulnerability analysis is often performed through solving a bi-level optimization problem, which, if solved to optimality, yields the most destructive interdiction plan with the worst loss. As one of the most effective operations to mitigate deliberate outages or attacks, transmission line switching recently has been included and modeled by a binary variable in the lower level decision model. Because this bi-level problem is a challenging nonconvex discrete optimization problem, no exact algorithm has been developed, and only a few recent heuristic procedures are available. In this paper, we present a novel tri-level reformulation of this bi-level problem, as well as its single-level equivalent form, and describe a finitely-convergent cutting plane algorithm to derive an exact solution. Numerical results are provided and benchmarked against existing ones, which confirms the quality of solutions and the computational efficiency of our algorithm.**

***Index Terms*—power grid interdiction, power flow, transmission line switching, mixed integer bi-level programming, cutting plane algorithm**

NOMENCLATURE

Indices and Sets

i	Generator
j	Demand
l	Transmission line
n	Node
m	Alias of n
$\mathcal{A}$	Set of attack decisions
$\mathcal{C}$	Set of switching decisions
H	$\lvert\mathcal{C}\rvert - 1$ where $h \in \{0, ..., H\}$
$\mathcal{U}$	Subset of $\{0, ..., H\}$

Parameters

x_l	Reactance at line l
M_l	A big number for line l
F_l	Flow limit at line l
P_i^{min}	Minimal generation of generator i
P_i^{max}	Maximal generation of generator i
D_j^n	Demand at node n
K	Cardinality of attack
$\overline{\theta}$	Maximum difference of connected phase angles

Decision variables

d_j^n	Satisfied demand at node n
θ_n	Phase angle at node n
f_l^{mn}	Power flow on line l from node m to n
z_l	Line switching, 0 if l is switched off
p_i^n	Generation level of generator at node n
w_l	Attack, 0 if line l is removed by attacker

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I. INTRODUCTION

The power grid interdiction problem often is modeled as an attack-defend model (or sometime as a defend-attack-defend model, if some defending decisions can be made before attacks) or a leader-follower game [8, 25, 27, 9]. The attacker seeks to minimize the satisfied load demand by removing up to K transmission lines; after the attack, the system operators try to mitigate the loss by adjusting nodal generation, load shedding, and other dispatching parameters. Identifying such a group of lines whose removal will lead to the severe loss of demand is critical for system daily operations and long-term security. For example, limited protective or hardening resources can be allocated to those lines to enhance the reliability of the system or reduce the probability of successfulness of the attacks. Also, those critical lines are of interest in the transmission network capacity expansion problem to better balance risk and economic advantages. In fact, when $K = 1$ and 2, the problem is closely related to $N - 1$ and $N - 2$ security criteria adopted in the power industry.

Because of its significance to the power industry, emergency planning centers, and homeland security, as well as its complexity from the underlining network/flow characteristics, an extensive amount of research effort has been dedicated to the power grid interdicting problem since 2000. As a result, many algorithms have been developed to address variants with different considerations and scales. Salmeron et al. [25] formulate the problem of optimal interdiction of an electric power network as a bi-level programming problem and develop a heuristic algorithm to solve the problem. Later, they extend the model to multiple periods with the considerations of repair times of the power grid after attack and the demand variation over time [27]. They propose and implement a *global Benders decomposition* method that can solve large-scale problems with high-quality solutions in a reasonable time. In parallel, Motto et al. [22] convert the bi-level power interdicting problem into an equivalent single-level mixed integer programming (MIP) model through dualizing the lower-level linear programming (LP) problem, which reduces the formulation from bi-level to single-level, and linearizing resulting binary-binary or binary-continuous terms, which finally yields an MIP model. They also employ Karush-Kuhn-Tucker (KKT) optimality conditions of the lower-level LP problem to achieve a similar transformation [5, 3]. Using the duality of LP and linearization techniques, Janjarassuk and Linderoth [19] also convert a stochastic network interdiction problem into an equivalent MIP. Due to the stochastic structure, they are able to apply an L-shaped decomposition technique with a sampling-based approach to solve their problem. To reduce the computational complexity, Bier et al. [7] develop a greedy-based algorithm to derive interdiction strategies, and report a set of vulnerable transmission lines that are different from those in [25]. Different from the above classical mathematical programming based approach, graph theory-based methods, in conjunction with KKT conditions, are developed in [12, 11, 23] to solve power grid interdicting problem. As a result, instances of large-scale power grids can be solved approximately within a short time [24].

It is noted the lower-level problems (or defending problems) are assumed to be LP problems in all the aforementioned studies. This assumption is essential for them in that duality theory and KKT conditions of LP play the central role in formulation transformations or algorithm development. Recently, transmission line switching has been analytically studied in order to reduce dispatch cost in power system scheduling. It is observed that up to 25 percent dispatch savings can be achieved [13, 16, 17, 20]. Given the fact that the topology of the power grid will be changed if one or more transmission lines are attacked or disconnected, transmission line switchings can also be incorporated into system operator's post-disruption decision for a better mitigation effect. For example, in PJM, Special Protection Schemes have included transmission line switching as one operation during contingencies [18]. Nevertheless, line-switching decisions are made with pre-defined rules, which are not analytical and could be less effective. Recently, this idea, i.e., including line switching into the lower level decision problem, is quantitatively studied by Arroyo and Fernandez [4] and Delgadillo et al. [10]. Because it is necessary to represent line-switching decisions by binary variables, the lower-level problem becomes non-convex, and strong duality theory and KKT conditions are not valid anymore. As a result, the previously-developed techniques to convert two-level to single-level are not applicable. Given the difficulties to solve this problem exactly, two heuristic procedures, a genetical algorithm and a *multi-start Benders decomposition* method, are developed to identify high-quality solutions [4, 10]. A numerical study shows that the latter could identify optimal solutions for some instances [10]. Nevertheless, the power grid interdiction problem with line switching remains an open problem for researchers in the power systems as well as operations research communities.

On the other hand, some recent research on robust optimization could be used to solve this open problem. To provide a solution method to this challenging problem that is of critical interest to the power industry, governmental organizations, and security agencies, we developed an exact algorithm based on strategies used to solve 2-stage robust optimization problems [29, 30]. Specifically, unlike existing work that either tries to derive a single-level equivalent formulation or directly deals with its two-level structure, we reformulated the original problem into a tri-level structure, which separates binary line switching variables from other continuous decision variables. Then, a cutting plane method within a master-subproblem framework can be used to dynamically convert the tri-level formulation into a (growing) single-level formulation, which can be readily solved by off-the-shelf MIP solvers. Because the whole procedure involves (provides) an upper bound and a lower bound, the quality of the best feasible solution found so far can be estimated, and a user-defined tolerance can be supplied to achieve a computational tradeoff, which provides a flexible mechanism for system operators in practice.

Our major contributions are listed as follows:

(i) We provide an equivalent tri-level formulation of the bilevel power grid interdicting problem by separating transmission line switching from the defending decision set. This tri-level formulation provides a framework to eliminate the difficulties brought by binary line switching decisions.

(ii) We develop a finitely-convergent cutting plane algorithm to exactly solve the tri-level formulation. The whole procedure does not depend on externally-supplied parameters and is easy to implement.

(iii) A set of preliminary computational results is presented. Through benchmarked against existing ones in [10], our results show that the developed algorithm can derive optimal solutions with significantly-reduced computation time. So, it greatly improves our solution capability on this challenging problem.

The paper is organized as follows. In Section II, we present the bilevel formulation of power grid interdicting problem with transmission line switching. In Section III we give the equivalent tri-level formulation and the cutting plane algorithm with a master-subproblem framework. In Section IV we report the results of the computational study. Section V concludes the paper and discusses future research directions.

II. Bilevel Attack-Defend Model

In the following, we present a bi-level $\min$-$\max$ formulation for the power grid interdicting problem with transmission line switching. The higher-level decision is made by the attacker, which seeks to minimize the served load by removing transmission lines. Then, after some lines are disrupted by the attacker, system operator solves the lower decision problem to compute the optimal operations, including line switching and other dispatching operations, to maximize the load that can be served. Note that an essentially similar model is proposed in [10], where the bi-level formulation takes $\max$-$\min$ format, i.e., both attacker and system operator consider the unmet load demand. In remainder of this paper, a vector of variables is in the boldface of the corresponding variable, and $\hat{\cdot}$ denotes a fixed decision variable.

$$\min_{\mathbf{w}\in\mathcal{A}} \max \sum_j d_j^n \tag{1}$$

$$st.\ \theta_m - \theta_n - x_l f_l^{mn} + (1 - z_l w_l) M_l \geq 0, \forall l \tag{2}$$

$$\theta_m - \theta_n - x_l f_l^{mn} - (1 - z_l w_l) M_l \leq 0, \forall l \tag{3}$$

$$-F_l z_l w_l \leq f_l^{mn} \leq z_l w_l F_l, \forall l \quad (4)$$

$$p_i^n \leq P_i^{max}, \forall i \quad (5)$$

$$d_j^n \leq D_j^n, \forall j \quad (6)$$

$$\sum_l f_l^{\cdot n} + p_i^n = \sum_l f_l^{n\cdot} + d_j^n, \forall n \quad (7)$$

$$p_i^n \geq 0, d_j^n \geq 0, f_l^{mn}, \theta_n \text{ free}, z_l \in \{0,1\} \quad (8)$$

where $\mathcal{A} = \{w_l \in \{0,1\}, \sum_l (1-w_l) \leq K\}$.

As reflected in objective function (1), the attacker can remove no more than K transmission lines trying to minimize the served demand. After the attack, the system operator tries to maximize the served demand (or equivalently minimize the loss) by adjusting network topology through line switching, phase angels and generation levels. Constraints (2-3), similar to those proposed in [1, 26], are direct current (DC) power flow approximations that follow Kirchhoff's Laws with additional attack and switching decisions. The parameter M_l is a sufficiently large number to be specified later. If the line l is removed by the attacker, i.e., $w_l = 0$, or disconnected by the operator, i.e., $z_l = 0$, the two inequalities will be satisfied regardless of the difference between two phase angles θ_m and θ_n at bus towers connected by line l. Otherwise, i.e. $w_l = z_l = 1$, the two constraints will form the traditional power flow equation $\theta_m - \theta_n - x_l f_l^{mn} = 0$ for line l. In [10], one equation $z_l w_l(\theta_m - \theta_n - x_l f_l^{mn}) = 0$ is formulated to capture the power laws as well as attack/switching decisions. Clearly, constraints (2-3) can be treated as a result of linearizing the aforementioned constraint.

Constraint (4) forces the power flow on a transmission line to be zero when the line is attacked or disconnected; otherwise, the flow will be restricted within $[-F, F]$ [17, 10]. Based on the joint restriction of constraints (2-4), the parameter M_l can be specified [1, 26], and the maximal difference of two phase angles at buses m, n connected by a line l, $\overline{\theta} = \theta_m^{max} - \theta_{\underline{n}}^{min}$, can be implicitly incorporated. To be specific, assume $\overline{\theta}$ is explicitly given and a line l is available (that is, $\theta_m - \theta_n - x_l f_l^{mn} = 0$), then (i) if $F_l \leq \overline{\theta}/x_l$, $|\theta_n - \theta_m| \leq \overline{\theta}$ will always be reductant; (ii) if $F_l > \overline{\theta}/x_l$, we replace F_l by a new value of $\overline{\theta}/x_l$ and then the difference of angular separation, i.e., $\theta_m - \theta_n$, could be automatically restricted. Meanwhile, we can have a valid value $\overline{\theta} + x_l F_l$ for parameter M_l. Therefore, unlike the model in [10], we do not explicitly include phase angle limits.

Constraint (5) describes the aggregated generation limit at buses with generators. Similarly, constraint (6) guarantees that the satisfied demand does not exceed the nominal demand value at load buses. The nonnegativity constraints on generation and demand variables ensure that generation/demand will always be generation/demand. Finally, constraint (7) is the traditional node balance equation such that the inflow and outflow of any node are equal [10, 26, 17, 1, 10].

III. An Exact Solution Approach through A Tri-level Reformulation

In this section, we first present a tri-level formulation of the power grid interdiction problem and its single-level equivalent form. Then, we describe an exact algorithm using cutting plane strategy that dynamically builds and solves the single-level equivalent form.

A. A Tri-level Formulation and Its Single-Level Equivalent Form

Note that the objective function in the original bi-level formulation in (1-7) can be equivalently expressed as

$$\min_{\mathbf{w}\in\mathcal{A}} \max_{\mathbf{z},\mathbf{p},\mathbf{f},\theta,\mathbf{d}} \sum_j d_j^n = \min_{\mathbf{w}\in\mathcal{A}} \max_{z\in\mathcal{C}} \max_{\mathbf{p},\mathbf{f},\theta,\mathbf{d}} \sum_j d_j^n$$

where $\mathcal{C}$ is the binary set including all possible line switching decisions.

Hence, for any given attack $\hat{\mathbf{w}} \in \mathcal{A}$ and switching decision $\hat{\mathbf{z}}$, the remaining problem, which is actually a dispatching problem, is a pure LP problem. Furthermore, this LP problem is always feasible in that the solution with $\{\mathbf{p}, \mathbf{f}, \theta, \mathbf{d}\} = \mathbf{0}$ is feasible in all cases. Therefore, the strong duality holds, and the maximization dispatching problem can be equivalently replaced by its minimization dual problem. Next, we present the corresponding dual problem in (9-14), where λ^1 and λ^2, λ^3 and λ^4, λ^5 and λ^6, λ^7 are dual variables for constraints (2-7), respectively. Note that $n|i@n$ and $n|j@n$ denote the bus n with generator i or demand j, respectively. Also, $l|l = m \rightarrow \cdot$ and $l|l = \cdot \rightarrow m$ denote the transmission line with bus m as the start and end bus, respectively.

$$\begin{aligned}\min \ & \sum_l \lambda_l^1(1-\hat{z}_l\hat{w}_l)M_l + \sum_l \lambda_l^2(1-\hat{z}_l\hat{w}_l)M_l \\ & + \sum_l \lambda_l^3 F_l \hat{z}_l \hat{w}_l + \sum_l \lambda_l^4 F_l \hat{z}_l \hat{w}_l \\ & + \sum_i \lambda_i^5 P_i^{max} + \sum_j \lambda_j^6 D_j^n \end{aligned} \quad (9)$$

$$st. \ \lambda_i^5 + \lambda_{n|i@n}^7 \geq 0, \forall i \quad (10)$$

$$\lambda_j^6 - \lambda_{n|j@n}^7 \geq 1, \forall j \quad (11)$$

$$x_l\lambda_l^1 - x_l\lambda_l^2 - \lambda_l^3 + \lambda_l^4 - \lambda_m^7 + \lambda_n^7 = 0, \forall l(m \rightarrow n) \quad (12)$$

$$\sum_{l|l=m\rightarrow\cdot} (\lambda_l^2 - \lambda_l^1) + \sum_{l|l=\cdot\rightarrow m} (\lambda_l^1 - \lambda_l^2) = 0, \forall m \quad (13)$$

$$\lambda^7 \text{ free }, \lambda^1, ..., \lambda^6 \geq 0 \quad (14)$$

Then, we can reformulate the bilevel model into an equivalent tri-level model; see Proposition 1 in the following.

Proposition 1. *The bilevel problem defined in (1-8) is equivalent to the following tri-level programming problem:*

$$\begin{aligned}\min_{\mathbf{w}\in\mathcal{A}} \max_{\mathbf{z}\in\mathcal{C}} \min_{\lambda^1,\ldots,\lambda^7} & \sum_l \lambda_l^1(1-z_l w_l)M_l + \sum_l \lambda_l^2(1-z_l w_l)M_l \\ & + \sum_l \lambda_l^3 F_l z_l w_l + \sum_l \lambda_l^4 F_l z_l w_l + \sum_i \lambda_i^5 P_i^{max} + \sum_j \lambda_j^6 D_j^n \end{aligned} \quad (15)$$

$st. \ (10-14)$

We can interpret this tri-level model as follows: the attacker makes attack decisions first, then the system operator responds with line switching decisions to determine system configuration, and the attacker again makes a set of recourse decisions represented by $(\lambda^1, \dots, \lambda^7)$ after line switching decisions are disclosed. It is worth pointing out that for the grid interdiction problem (in either bi-level or tri-level formulations), the virtual solution we need is an optimal attack plan w^*. Once it is determined, other decisions are determined through solving an MIP problem.

Although such a transformation from a bi-level model into a tri-level model is anti-intuitive, it provides a mechanism to isolate the line switching set from other decisions. In particular, based on a solution strategy presented to solve tri-level robust optimization problem [29, 30], a singe-level equivalent form actually can be developed by enumerating all possible the system operator's responses, which constitute a finite set, given the binary property of switching decision. The key step is that, for any possible operator's decision $\hat{\mathbf{z}}^{(h)}$ we create a new set of recourse decision variables $(\lambda^{1(h)}, \dots, \lambda^{7(h)})$. Next, we present this single-level form.

Proposition 2. *The tri-level model defined above is equivalent to the single-level form defined in (16-22).*

$$\min \quad \eta \tag{16}$$

$$\begin{aligned} st.\ \eta \geq & \sum_l \lambda_l^{1(h)}(1-\hat{z}_l^{(h)}w_l)M_l + \sum_l \lambda_l^{2(h)}(1-\hat{z}_l^{(h)}w_l)M_l \\ & + \sum_l \lambda_l^{3(h)} F_l \hat{z}_l^{(h)} w_l + \sum_l \lambda_l^{4(h)} F_l \hat{z}_l^{(h)} w_l \\ & + \sum_i \lambda_i^{5(h)} P_i^{max} + \sum_j \lambda_j^{6(h)} D_j^n, \forall \hat{\mathbf{z}}^{(h)} \in \mathcal{C} \end{aligned} \tag{17}$$

$$\lambda_i^{5(h)} + \lambda_{n|i@n}^{7(h)} \geq 0, \forall i, h \tag{18}$$

$$\lambda_j^{6(h)} - \lambda_{n|j@n}^{7(h)} \geq 1, \forall j, h \tag{19}$$

$$\begin{aligned} & x_l\lambda_l^{1(h)} - x_l\lambda_l^{2(h)} - \lambda_l^{3(h)} + \lambda_l^{4(h)} - \lambda_m^{7(h)} + \lambda_n^{7(h)} = 0, \\ & \forall l(m \to n), h \end{aligned} \tag{20}$$

$$\begin{aligned} & \sum_{l|l=m\to\cdot} (\lambda_l^{2(h)} - \lambda_l^{1(h)}) + \sum_{l|l=\cdot\to m} (\lambda_l^{1(h)} - \lambda_l^{2(h)}) = 0, \\ & \forall m, h \end{aligned} \tag{21}$$

$$\mathbf{w} \in \mathcal{A}, \eta, \lambda^{7(h)} \textit{ free }, \lambda^{1(h)}, ..., \lambda^{6(h)} \geq 0 \tag{22}$$

where $\{\hat{\mathbf{z}}^{(h)}\}_{h=0}^{H} = \mathcal{C}$.

In the above single-level equivalent form, decision variables are $\mathbf{w}$ and $(\lambda^{1(h)}, \dots, \lambda^{7(h)})$ for all h. We note that constraint (17) involves terms that are products of one binary variable, w_l, and one continuous variable $\lambda^{k(h)}$ with $k = 1, \dots, 4$. So, it would be natural to linearize those terms using standard linearization techniques so that professional MIP solvers can be called to solve this problem. Assume that M' is a sufficiently large number that provides an upper bound to optimal values of $\lambda^{k(h)}$ in (9-14). It can be observed that a unit change of the right-hand side of any constraint in (2-4) will incur some unserved load or bring additional served load, both are bounded by $\sum_j D_j^n$, i.e., the total nodal demand. Hence, $M' = \sum_j D_j^n$ is a valid upper bound for those dual variables (recourse decision variables). With M', we can have the following equivalent linear constraints.

Proposition 3. *Constraints defined in (17) can be linearized as (23-36).*

$$\begin{aligned} \eta \geq & \sum_l (\lambda_l^{1(h)} - \hat{z}_l^{(h)}\hat{\alpha}_l^{(h)})M_l + \sum_l (\lambda_l^{2(h)} - \hat{z}_l^{(h)}\hat{\beta}_l^{(h)})M_l \\ & + \sum_l \pi_l^{(h)} F_l \hat{z}_l^{(h)} + \sum_l \mu_l^{(h)} F_l \hat{z}_l^{(h)} + \sum_i \lambda_i^{5(h)} P_i^{max} \\ & + \sum_j \lambda_j^{6(h)} D_j^n, \forall \hat{\mathbf{z}}^{(h)} \in \mathcal{C} \end{aligned} \tag{23}$$

$$\alpha_l^{(h)} \leq \lambda_l^{1(h)}, \forall h, l \tag{24}$$

$$\alpha_l^{(h)} \geq \lambda_l^{1(h)} - (1-w_l)M', \forall h, l \tag{25}$$

$$\alpha_l^{(h)} \leq w_l M', \forall h, l \tag{26}$$

$$\beta_l^{(h)} \leq \lambda_l^{2(h)}, \forall h, l \tag{27}$$

$$\beta_l^{(h)} \geq \lambda_l^{2(h)} - (1-w_l)M', \forall h, l \tag{28}$$

$$\beta_l^{(h)} \leq w_l M', \forall h, l \tag{29}$$

$$\pi_l^{(h)} \leq \lambda_l^{3(h)}, \forall h, l \tag{30}$$

$$\pi_l^{(h)} \geq \lambda_l^{3(h)} - (1-w_l)M', \forall h, l \tag{31}$$

$$\pi_l^{(h)} \leq w_l M', \forall h, l \tag{32}$$

$$\mu_l^{(h)} \leq \lambda_l^{4(h)}, \forall h, l \tag{33}$$

$$\mu_l^{(h)} \geq \lambda_l^{4(h)} - (1-w_l)M', \forall h, l \tag{34}$$

$$\mu_l^{(h)} \leq w_l M', \forall h, l \tag{35}$$

$$\pi_l^{(h)}, \mu_l^{(h)}, \alpha_l^{(h)}, \beta_l^{(h)} \geq 0 \tag{36}$$

As a result, the tri-level model, as well as the bi-level one, can be reformulated as a linear model.

Corollary 1. *The equivalent single-level linear MIP formulation of the tri/bi-level power grid interdiction problem is the one obtained by replacing* (17) *with (23-36) in the formulation of (16- 22).*

Ideally, the tri/bi-level power grid interdiction problem could be easily solved using this single-level equivalent linear form. We note that, however, this single-level equivalent form is of theoretical interest only. As a set of $(\lambda^{1(h)}, \dots, \lambda^{7(h)})$ and their corresponding constraints must be introduced for any possible $\hat{\mathbf{z}}^h$ in set $\mathcal{C}$, which is actually exponential with respect to the number of lines in the grid; it is not feasible to explicitly list this single-level form to solve the tri-level problem for any real instances. Instead of using the complete enumeration to obtain the equivalent form, we next describe a solution algorithm that makes use of a partial enumeration strategy and cutting plane method. Our computational study shows that this algorithm is very effective in solving power grid interdiction problem.

B. Algorithm Description

Based on observations made in [29, 30] on solving the tri-level 2-stage robust optimization problems, it is anticipated that only a very small part of $\mathcal{C}$ is critical in determining an optimal solution. So, one idea is to start with a single-level formulation with a small subset of $\mathcal{C}$ (i.e., their variables and constraints) and then gradually expand the formulation by including more significant components (i.e., their variables and constraints) of $\mathcal{C}$. Note from (16-22) that a single-level formulation with a subset of $\mathcal{C}$, which is named the *partial single-level formulation*, yields a lower bound to the actual optimal value. Also, any feasible solution to the bi-level formulation, which can be obtained for any attack plan, yields an upper bound, given that the ultimate objective function is minimization. Therefore, the expansion process can be terminated with an optimal solution whenever upper and lower bounds match. This idea is implemented within a master-subproblem framework using a cutting plane strategy as follows. The same strategy is used to solve 2-stage robust optimization problems, which have been proven very efficient comparing to other solution methods [29, 30].

To simplify our exposition, we use the compact forms of all problems in this subsection. Specifically, let $\mathbf{y}$ denote the set of decision variables, including line switching and other dispatching variables, made by system operator, then the lower level decision model given any attack $\hat{\mathbf{w}} \in \mathcal{A}$ will be

$$\max \mathbf{cy} \tag{37}$$
$$st.\ \mathbf{By} \leq \mathbf{g} - \mathbf{A}\hat{\mathbf{w}} \tag{38}$$
$$\text{variable restrictions on } \mathbf{y}, \tag{39}$$

where $\mathbf{c}, \mathbf{g}, \mathbf{B}, \mathbf{A}$ are appropriate vectors or matrices defined in (1-7). Similarly, for some known $\{\mathbf{z}^{(h)}\}_{h\in\mathcal{U}} \subseteq \mathcal{C}$, the partial single-level formulation is

$$\min \eta \tag{40}$$
$$st.\ \mathbf{Q}\eta + \mathbf{E}\lambda^{(h)} + \mathbf{Gw} \leq \mathbf{h}\hat{\mathbf{z}}^{(h)} + \mathbf{a}, \forall h \in \mathcal{U} \tag{41}$$
$$\mathbf{w} \in \mathcal{A} \tag{42}$$
$$\text{variable restrictions on } \mathbf{w}, \eta, \lambda^{(\mathbf{h})}, \tag{43}$$

where $\mathbf{h}, \mathbf{a}, \mathbf{Q}, \mathbf{E}, \mathbf{G}$ are appropriate vectors or matrices defined in (18-21, 23-36). Next we describe the algorithm in steps. Since this solution approach is based on a tri-level reformulation of the bi-level problem, we name it *Tri-level reformulation (TLR)* method for convenience.

Steps of TLR Algorithm

1) Set $LB = -\infty$, $UB = +\infty$, $h = 0$, $U = \emptyset$ and an optimality tolerance ϵ.
2) Solve the partial single-level formulation defined in (40-43) (as the *master problem*). Derive an optimal solution $(\mathbf{w}^{*_h}, \eta^{*_h}, \lambda^{*_h})$ and update $LB = \eta^{*_h}$.
3) Solve the lower level problem defined in (37-39) (as the *subproblem*) with $\hat{\mathbf{w}} = \mathbf{w}^{*_h}$ and update $UB = \min\{UB, \mathbf{cy}^{*_\mathbf{h}}\}$.
4) If $UB - LB \leq \epsilon$, return $\mathbf{w}^{*_h}$ as an optimal attack plan and terminate. Otherwise, update $U = U \cup \{h\}$ and $h = h + 1$, create new recourse decision variables $\lambda^{(h)}$ and add the corresponding constraints defined in (41) as cutting planes to the partial single-level problem. Go to step 2.

Based on the results presented in [29] and the fact that $\mathcal{C}$ is a finite binary set, it follows directly that TLR is finitely convergent to an optimal solution. To the best of our knowledge, it is the first exact algorithm developed to solve the power grid interdiction problem with line switching. In fact, its computational performance is also very promising, compared to the recent study on a multi-start Benders decomposition method [10].

TABLE I
CONFIGURATIONS OF COMPUTING FACILITIES

Platform	TLR	MSBD
CPU	1 processor at 3GHz	4 processors at 2.6GHz
RAM	3.25G	32G
CPLEX	12.2	11.0.1
Global Opt. Tolerance ϵ	1×10^{-4}	NA
ϵ for each problem	1×10^{-4}	1×10^{-2}

IV. Computational Study

This section presents a case study based on the IEEE 24-bus reliability test system [15]. The system has 24 buses, 11 of which are equipped with generators and 16 which are load buses. There are, in total, 38 transmission lines, and the cardinality of attacks is from 1 to 12. To benchmark against existing research in [10], we adopted their parameters, at our best efforts, to define the power grid. Specifically, parameters of power flow capacity F and nodal demand D were assumed to be those used in [10], which were converted into per unit (pu) quantities with a 100 MVA base. Also, the two lines connected with the same towers were treated as independent lines as in [10]. The reactance x of each transmission line and the maximal generation level p^{max} at each node are not listed in [10], so we adopted those parameters directly from [15]. All parameters are listed in the appendix. Given that the restrictions of angular differences are not presented in [10], we aimed to select those parameters close to practice. Observe that in previous research, different values were adopted for $\overline{\theta}$, such as 1.2 radian [17], 1 radian [1], etc. However, in practice "it is extremely rare to ever see such angular separation exceed 30 degrees" [21]. Also, the sufficient accuracy of DC simplification (2-3) of alternating current (AC) power flow relies on a small-valued θ as $sin(\theta) \cong \theta$ plays a central role in the approximation process [2, 14]. Hence, $\overline{\theta}$ is taken implicitly to be 0.5 in this paper.

The solution was implemented in C++ on a PC desktop, and the commercial solver IBM ILOG CPLEX was used as the MIP solver for all problems. In addition, we set bus 1 to be the reference bus, that is, $\theta_1 = 0$. The differences between computing facilities used in this study and in [10] are listed in Table I. In the remainder of the section, to avoid confusion, we use *MSBD* to denote the multi-start Benders decomposition method and the associated research presented in [10].

Our computational study included 12 instances, ranging from $K = 1$ to $K = 12$ for the attacker, basically replicating those made in [10]. To provide a comparable evaluation, we adopted a strategy by [10] in our TRL implementation, which pursued a high-quality initial attack by solving the bi-level programming problem without transmission switching. As reported in [10], such a strategy can often accelerate the computational speed. The comparison of the computational performance with respect to MSBD is shown in Table II together with the number of iterations in TLR. From the solution quality point of view, it is observed that TLR can

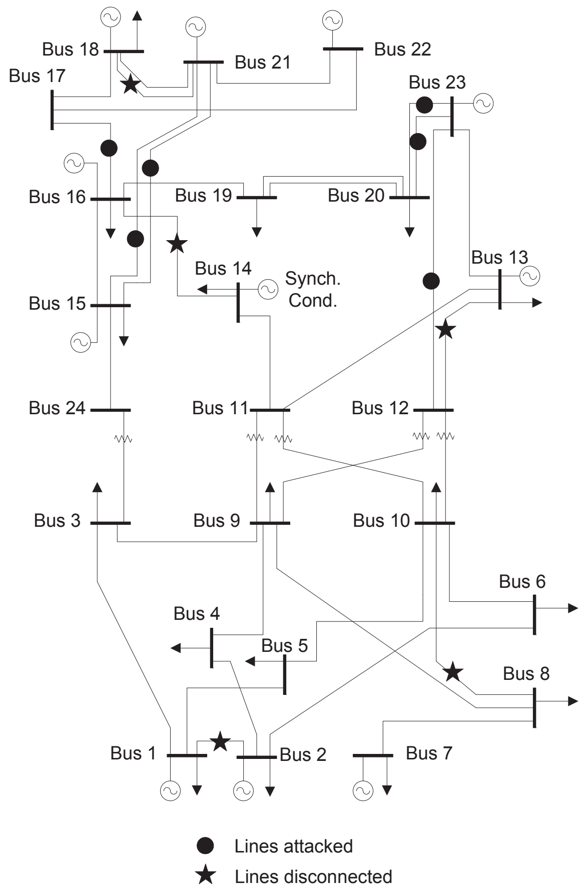

Fig. 1. Best attack plan for K=6 and the optimal switching

derive optimal solutions in all instances within a reasonable time, while MSBD cannot provide quality guaranteed solutions in most instances. From the computational speed point of view, TLR is comparable with MSBD for easy instances, while it performs an order of magnitude faster than MSBD for most challenging ones. Actually, for the most difficult instance where $K = 6$, as shown in Figure 1, the computational time could reduced to a few hundred (~ 600) seconds if the accelerating strategy [10] is not employed. Note that initial attack plans obtained by the accelerating strategy are reported to be optimal for instances $K = 7, 11, 12$ in [10]. In our study, we observe that initial attack plans are optimal for instances $K = 5, 7-12$. A possible explanation is that some parameters, which are not reported in [10], are different from those used in the numerical study by Delgadillo et al. [10]. It is also confirmed by results presented in Table III.

Table III presents the maximal load shedding in all instances. We mention that 168.5MW demand cannot be met even without any attacks, i.e., K=0, based on the parameters in our study. Optimal attack plans are shown in Table IV, from which we can identify critical transmission lines for different Ks in $N - K$ vulnerability consideration. For example, for $K \leq 2$ line 12-23 is critical; for $K \leq 5$, the important lines are 7-8, 12-23, 20-23A, and 20-23B, which actually is the best attack plan when K is four; similarly, for cases with $K \geq 6$,

TABLE II
ALGORITHM PERFORMANCE COMPARISON

K	Time (s) of MSBD [10]	Time (s) of TLR	Iterations of TLR
1	18.48	3.9	4
2	293.31*	11.6	5
3	2261.59*	12.7	4
4	2180.67*	67.6	6
5	1610.35*	6.1	1
6	1520.47*	1928.8	16
7	0.89	15.2	1
8	1312.42*	18.5	1
9	1155.64*	19.7	1
10	1029.01*	9.4	1
11	0.85	4.1	1
12	0.88	3.2	1

* indicating quality of the solution is unknown.

TABLE III
LOAD SHEDDING (MW) CAUSED BY ATTACKS

K	MSBD [10]	TLR
1	131	398.5
2	279	486
3	429	657.5
4	538	745
5	688	825
6	775	884.5
7	855	972
8	905	1022
9	1002	1061
10	1017	1144
11	1131	1208
12	1194	1258

12-23, 15-21A, 15-21B, 16-17, 20-23A, and 20-23B are, in general, critical lines.

V. Conclusion and Discussion

In this paper, we proposed an equivalent tri-level formulation to the power grid interdicting problem. Then, we derived its single-level equivalent form and developed an exact solution approach using a cutting plane strategy within a master-subproblem framework. The formulations are nontrivial, and the algorithm is novel. In particular, a set of preliminary computational results show that it performs significantly better than existing work and is promising for dealing with real-size power grids.

TABLE IV
BEST ATTACK STRATEGY

k	Best attack plan
1	12-23
2	7-8, 12-23
3	12-23, 20-23A, 20-23B
4	7-8, 12-23, 20-23A, 20-23B
5	7-8, 12-23, 15-24, 20-23A, 20-23B
6	12-23, 15-21A, 15-21B, 16-17, 20-23A, 20-23B
7	7-8, 12-23, 15-21A, 15-21B, 16-17, 20-23A, 20-23B
8	7-8, 12-23, 13-23, 15-21A, 15-21B, 16-17, 20-23A, 20-23B
9	1-3, 1-5, 2-4, 2-6, 3-24, 7-8, 12-23, 20-23A, 20-23B
10	1-2, 2-4, 2-6, 7-8, 12-23, 15-21A, 15-21B, 16-17, 20-23A, 20-23B
11	1-3, 1-5, 2-4, 2-6, 7-8, 12-23, 15-21A, 15-21B, 16-17, 20-23A, 20-23B
12	1-3, 1-5, 2-4, 2-6, 7-8, 12-23, 13-23, 15-21A, 15-21B, 16-17, 20-23A, 20-23B

TABLE V
BRANCH DATA

Branch	From	To	x (pu)	Limit (pu)
1	1	2	0.014	0.875
2	1	3	0.211	0.875
3	1	5	0.085	0.875
4	2	4	0.127	0.875
5	2	6	0.192	0.875
6	3	9	0.119	0.875
7	3	24	0.084	0.8
8	4	9	0.104	1
9	5	10	0.088	1
10	6	10	0.061	0.875
11	7	8	0.061	0.875
12	8	9	0.165	0.5
13	8	10	0.165	0.875
14	9	11	0.084	0.5
15	9	12	0.084	2
16	10	11	0.084	0.5
17	10	12	0.084	2
18	11	13	0.048	2.5
19	11	14	0.042	0.5
20	12	13	0.048	2.5
21	12	23	0.097	2.5
22	13	23	0.087	0.5
23	14	16	0.059	0.5
24	15	16	0.017	0.5
25	15	21	0.049	2.5
26	15	21	0.049	2.5
27	15	24	0.052	0.8
28	16	17	0.026	2.5
29	16	19	0.023	0.5
30	17	18	0.014	2.5
31	17	22	0.105	2.5
32	18	21	0.027	2.5
33	18	21	0.027	2.5
34	19	20	0.04	2.5
35	19	20	0.04	2.5
36	20	23	0.022	2.5
37	20	23	0.022	2.5
38	21	22	0.068	2.5

In fact, the method can be readily modified to deal with other bi-level problems with binary or integer decision variables in the lower level model, which yields an impact on general methodology. Moreover, given that the bi-level interdiction problem can be effectively solved using our method, it is anticipated that the challenging extended tri-level problems, such as defend-attack-defend (DAD) problems arising from power, military logistics, or other infrastructure systems [9, 6, 28], can be solved with advanced algorithm development based on the current one. Developing such an algorithm and solving power grid applications are our on-going projects.

Appendix: Parameters

TABLE VI
GENERATOR DATA

Benerator	Bus	Capacity (pu)
1	1	1.72
2	2	1.72
3	7	2.4
4	13	2.85
5	14	0
6	15	2.15
7	16	1.55
8	18	4
9	21	4
10	22	3
11	23	6.6

TABLE VII
DEMAND DATA

Demand	Bus	Load (pu)
1	1	1.08
2	3	1
3	4	0.74
4	5	0.5
5	6	1.36
6	7	1.25
7	8	1.37
8	9	1.55
9	10	1.7
10	13	2.65
11	14	1
12	15	3.17
13	16	1
14	18	3.33
15	19	1.81
16	20	1.28

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