Algorithms for Bilevel Pseudomonotone Variational Inequality Problems

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Abstract. We propose easily implementable algorithms for minimizing the norm with pseudomonotone variational inequality constraints. This bilevel problem arises in the Tikhonov regularization method for pseudomonone variational inequalities. Since the solution set of the lower variational inequality is not given explicitly, the available methods of mathematical programming and variational inequality can not be applied directly. With these algorithms we want to give an answer to a question posed in the ref [1].

Keywords. Bilevel variational inequality, pseudomonotonicity, projection method, Armjio linesearch, convergence.

1 Introduction

Variational innequality (shortlyVI) is a fundamental topic in applied mathematics. VIs are used for formulating and solving various problems arising in mathematical physis, economics, engineering and other fields. Theory, methods and applications of VI can be found in some comprehensive books and monographs (see e.g.[2, 3, 4, 5]). Mathematical programs with variational inequality constraints can be considered as one of further development directions of variational inequality [6]. Recently, these problems have been received much attention of researchers due to its vast applications.

In this paper, we are concerned with a special case of VIs with variational inequality constraints. Namely, we consider the bilevel variational inequality problem

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(BVI):

$$\min\{\|x - x^g\| : x \in S\} \tag{1.1}$$

where $x^g \in C$ and

$$S = \{u \in C : \langle F(u), y - u\rangle \geq 0, \forall y \in C\},$$

i.e.. S is the solution set of the variational inequality VI(C, F) defined as

$$\text{Find } x^* \in C \text{ such that } \langle F(x^*), y - x^*\rangle \geq 0, \forall y \in C. \tag{1.2}$$

In what follows we suppose that C is a nonempty closed convex subset in the Euclidean space $I\!R^n$ and $F : I\!R^n \to I\!R^n$ such that $C \subseteq \text{dom}F$. We call problem (1.1) the upper problem and (1.2) the lower one. Monotone bilevel variational inequalities and equilibrium problems were considered in some articles (see e.g., [7, 8, 9, 10, 11, 12, 13, 14, 15]). In this case the subproblems to be solved are monotone. However, if the lover problem is pseudomonotone the subproblems to be solved do not inherit any monotonicity property. It should be noticed that the solution set S of the lower problem (1.2) is convex whenever F is pseudomonotone on C. However, the main difficulty is that, even the constrained set S is convex, it is not given explicitly as in a standard mathematical programming problem, and therefore the available methods of convex optimization and variational inequality cannot be applied directly to problem (1.1).

In recent years, regularization techniques for nonmonotone variational inequlities have been considered in some papers [1, 16, 17, 18, 19, 20]. In the Tikhonov regularization method applying to the variational inequality VI(C, F), the least norm solution of VI(C, F) can be obtained as the limit, as $\epsilon \to 0$, of the solutions of variational inequality subproblems VI(F_ϵ) where $F_\epsilon := F + \epsilon I$ with $\epsilon > 0$ and I being the identity operator. So the problem of finding the limit in the Tikhonov method applying to pseudomonotone VIs leads to the problem of the form (1.1) with S being the solution set of the original problem. It is well-known that if F is monotone, then $F + \epsilon I$ is strongly monotone for every $\epsilon > 0$. However, when F is pseudomonotone, the operator $F + \epsilon I$ may not be pseudomonotone for any $\epsilon > 0$ (see the counterexample 2.1 in [1]). This counter example raises a question posed in [1] for the Tikhonov regularization method "Why one has to replace the original pseudomonotone VI by the sequence of auxiliary problems VI(C, F_ϵ), $\epsilon > 0$, none of which is pseudomonotone?".

In this paper, we propose algorithms for solving bilevel variational inequality problem (1.1) when the lower problem is pseudomonotone with respect to its solution set. The latter pseudomonotonicity is somewhat more general than the pseudomonotonicity. With these algorithms we want to give an answer to the above mentioned question posed in [1] by showing that with the help of the auxiliary problems VI(C, F_ϵ), that may be not pseudomonotone for any $\epsilon > 0$, the same limit point of every Tikhonov trajectory can be obtained by solving bilevel problem (1.1).

2 Preliminaries

As usual, by P_C we denote the projection operator onto the closed convex set C with the norm $\|.\|$, that is

$$P_C(x) \in C: \ \|x - P_C(x)\| \leq \|x - y\| \ \forall y \in C.$$

The following well known results on the projection operator onto a closed convex set will be used in the sequel.

Lemma 2.1 *([4]) Suppose that C is a nonempty closed convex set in $I\!R^n$. Then*
(i) $P_C(x)$ is singleton and well defined for every x;
(ii) $\pi = P_C(x)$ if and only if $\langle x - \pi, y - \pi\rangle \leq 0, \forall y \in C$;
(iii) $\|P_C(x) - P_C(y)\|^2 \leq \|x - y\|^2 - \|P_C(x) - x + y - P_C(x)\|^2, \ \forall x, y \in C$.

We recall some well known definitions on monotonicity (see e.g. [4, 5]).

Definition 2.1 *An operator $\phi : C \to I\!R^n$ with dom $\phi \subseteq C$ is said to be*
(a) strongly monotone on C with modulus γ, if

$$\langle \phi(x) - \phi(y), x - y\rangle \geq \gamma \|x - y\|^2, \ \forall x, y \in C;$$

(b) monotone on C if

$$\langle \phi(x) - \phi(y), x - y\rangle \geq 0, \ \forall x, y \in C;$$

(c) pseudomonotone on C if

$$\langle \phi(x), x - y\rangle \geq 0 \Longrightarrow \langle \phi(y), x - y\rangle \geq 0 \ \forall x, y \in C;$$

(d) ϕ is said to be pseudomonotone on C with respect to $x^* \in C$ if

$$\langle \phi(x^*), x - x^*\rangle \geq 0 \Longrightarrow \langle \phi(x), x - x^*\rangle \geq 0 \ \forall x \in C.$$

ϕ is pseudomonotone on C with respect to a set $A \subseteq C$ if it is pseudomonotne on C with respect to every point $x^* \in A$.

From the definitions it follows that $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \ \forall x^* \in C$.

In what follows we need the following blanket assumptions on F:

(A1) F is continuous on its domain;

(A2) F is pseudomonotone on C with respect to every solution of problem $VI(C, F)$;

Lemma 2.2 *Suppose that Assumtions (A1), (A2) are satisfied and that the variational inequality (1.2) admits a solution. Then the solution set of (1.2) is closed, convex.*

The proof of this lemma when F is pseudomonotone on C can be found in [4, 5, 21]. When F is pseudomonotone with respect to its solution set, the proof can be done by the same way.

Following the auxiliary problem principle [22, 23] let us define a bifunction $L : C \times C \to I\!R$ such that

(B1) $L(x, x) = 0,\ \exists \beta > 0 :\ L(x, y) \geq \frac{\beta}{2}\|x - y\|^2,\ \forall x, y \in C$;

(B2) L is continuous, $L(x, .)$ is differentiable, strongly convex on C for every $x \in C$ and $\nabla_2 L(x, x) = 0$ for every $x \in C$.

An example for such a bifunction is

$$L(x, y) := g(y) - g(x) - \langle \nabla g(x), y - x\rangle$$

with g being any differentiable, strongly convex function on C with modulus $\beta > 0$. particularly, $g(x) = \frac{1}{2}\|x\|^2$. The following lemma is well known from the auxiliary problem principle for VIs.

Lemma 2.3 *([23]) Suppose that F satisfies (A1), (A2) and L satisfies (B1), (B2). Then, for every $\rho > 0$, the following statements are equivalent:*

a) x^* is a solution to $VI(C, F)$;

b) $x^* \in C : \langle F(x^*), y - x^*\rangle + \frac{1}{\rho}L(x^*, y) \geq 0,\ \forall y \in C$;

c) $x^* = argmin\{\langle F(x^*), y - x^*\rangle + \frac{1}{\rho}L(x^*, y) : y \in C\}$;

d) $x^* \in C : \langle F(y), y - x^*\rangle \geq 0,\ \forall y \in C$.

3 The Algorithm and Its Convergence

In what follows we suppose that the solution set S of the lower variational inequality (1.2) is nonempty and that F is continuous, pseudomonotone on C with respect to S. In this case, S is closed and convex, The following algorithm can be considered as a combination of the extragradient method [4, 24, 25, 26] and the cutting techniques [27, 28] to the bilevel problem (1.1).

Algorithm 1. Choose $\rho > 0$ and $\eta \in (0, 1)$. Starting from $x^1 := x^g \in C$ (x^g plays the role of a guessed solution).

Iteration $k (k = 1, 2, ...)$ Having x^k do the following steps:

Step 1. Solve the strongly convex program

$$\min\{\langle F(x^k), y - x^k\rangle + \frac{1}{\rho}L(x^k, y) : y \in C\} \qquad CP(x^k)$$

to obtain its unique solution y^k.

If $y^k = x^k$, take $u^k = y^k$ and go to Step 4. Otherwise, go to Step 2.

Step 2 (Armijo linesearch) Find m_k as the smallest nonnegative integer number m satisfying

$$z^{k,m} := (1-\eta^m)x^k + \eta^m y^k, \tag{3.1}$$

$$\langle F(z^{k,m}), y^k - z^{k,m}\rangle + \frac{1}{\rho}L(x^k, y^k) \leq 0. \tag{3.2}$$

Set $\eta_k = \eta^{m_k}$, $z^k := z^{k,m_k}$. If $0 = F(z^k)$, take $u^k = z^k$ and go to Step 4. Otherwise, go to Step 3.

Step 3. Compute

$$\sigma_k = \frac{-\eta_k\langle F(z^k), y^k - z^k\rangle}{(1-\eta_k)\|F(z^k)\|^2},\ u^k := P_C(x^k - \sigma_k F(z^k)). \tag{3.3}$$

Step 4. Having x^k and u^k construct two convex sets

$$C_k := \{y \in C : \|u^k - y\|^2 \leq \|x^k - y\|^2\};$$

$$D_k := \{y \in C : \langle x^g - x^k, y - x^k\rangle \leq 0\}.$$

Let $B_k := C_k \cap D_k$.

Step 5. Compute $x^{k+1} := P_{B_k}(x^g)$.

Increase k by one and go to iteration k.

Remark 3.1 *(i) If $m_k = 0$, then $x^k = y^k$, hence x^k solves $VI(C, F)$. Indeed, by the Armijo rule we have $z^k = y^k$, and therefore*

$$\frac{1}{\rho}L(x^k, y^k) = \langle F(z^k), y^k - z^k\rangle + \frac{1}{\rho}L(x^k, y^k) \leq 0,$$

which, together with nonnegativity of L, implies $L(x^k, y^k) = 0$. Since $L(x^k, y^k) \geq \frac{\beta}{2}\|x^k - y^k\|^2$, we have $x^k = y^k$.

(ii)

$$\sigma_k := \frac{-\eta_k\langle F(z^k), y^k - z^k\rangle}{(1-\eta_k)\|F(z^k)\|^2} > 0.$$

whenever $x^k \neq y^k$.

Before considering the convergence of our algorithm, let us emphasize that the main difference between our algorithm with the other available ones [8, 9, 11, 15, 28] for bilevel problems related to (1.1) is that the lower VI in Algorithm 1 is pseudomonotone with respect to its solution rather than monotone as in other algorithms. Moreover, the main subproblem in Step 1 in Algorithm 1 is a strongly convex program rather than a strongly momotone variational inequality as in the above mentioned algorithms. As we have mentioned, for pseudomonotonicity case, the latter subvariational inequality is no longer strongly monotone, even not pseudomonotone. In our recent paper [7], the lower variational inequality can be pseudomonotone. However, the algorithm proposed there consists of two loops, and for the convergence, we have to assume that the inner loop must terminate after a finite number of projections. For monotonicity case this assumption is satisfied, but the number of the projections, although is finite, cannot be estimated.

Lemma 3.1 *Under the assumptions of Lemma 2.3, it holds that*

$$\|u^k - x^*\|^2 \le \|x^k - x^*\|^2 - \sigma_k^2\|F(z^k)\|^2, \ \forall x^* \in S, \ \forall k. \tag{3.4}$$

Proof. The proof of this lemma can be done similarly as the proof of Lemma in [4] (see also [29]). So we give here only a sketch. For simplicity of notation, we write F^k for $F(z^k)$ and v^k for $x^k - \sigma_k F^k$. Since $u^k = P_C(v^k)$, by nonexpansiveness of the projection, we have

$$\begin{aligned} \|u^k - x^*\|^2 &= \|P_C(v^k) - P_C(x^*)\|^2 \le \|v^k - x^*\|^2 \\ &= \|x^k - x^* - \sigma_k F^k\|^2 \\ &= \|x^k - x^*\|^2 + \sigma_k^2\|F^k\|^2 - 2\sigma_k\langle F^k, x^k - x^*\rangle. \end{aligned} \tag{3.5}$$

Since $x^* \in S$, using d) of Lemma 2.3 we can write

$$\langle F^k, x^k - x^*\rangle = \langle F^k, x^k - z^k + z^k - x^*\rangle \ge \langle F^k, x^k - z^k\rangle. \tag{3.6}$$

Since $x^k - z^k = \frac{\eta_k}{1-\eta_k}(z^k - y^k)$,

$$\langle F^k, x^k - z^k\rangle = \frac{\eta_k}{1-\eta_k}\langle F^k, z^k - y^k\rangle = \sigma_k\|F^k\|^2. \tag{3.7}$$

The last equality comes from the definition of σ_k by (3.3) in the algorithm. Combining (3.5), (3.6) and (3.7) we obtain (3.4).

The following theorem shows validity and convergence of the algorithm.

Theorem 3.1 *Suppose that Assumptions (A1), (A2) and (B1), (B2) are satisfied and that $VI(C, F)$ admits a solution. Then the algorithm is well defined and both sequences $\{x^k\}$, $\{u^k\}$ converge to the unique solution of the original bilevel problem (1.1).*

Proof. First, we prove that the linesearch is well defined. Suppose by contradiction that for all nonnegative integer numbers m one has:

$$\langle F(z^{k,m}), y^k - z^{k,m}\rangle + \frac{1}{\rho}L(x^k, y^k) > 0. \tag{3.8}$$

Thus letting $m \to \infty$, by continuity of F, we have

$$\langle F(x^k), y^k - x^k\rangle + \frac{1}{\rho}L(x^k, y^k) \geq 0,$$

which, together with

$$\langle F(x^k), x^k - x^k\rangle + \frac{1}{\rho}L(x^k, x^k) = 0,$$

implies that x^k is the solution of the strongly convex program $CP(x^k)$. Thus $x^k = y^k$ which contradicts to the fact that the linesearch is performed only when $y^k \neq x^k$.

From Lemma 3.1, it follows that $\|u^k - x^*\| \leq \|x^k - x^*\|$ for every k and $x^* \in S$. Hence, by the definition of C_k, one has $S \subseteq C_k$ for every k. Moreover, $S \subseteq D_k$ for every k. In fact, since $x^1 = x^g$, $S \subseteq D_1 = I\!R^n$. By definition of x^{k+1}, it follows, by induction, that if $S \subseteq D_k$, then $S \subseteq D_{k+1}$. Consequently $S \subseteq C_k \cap D_k = B_k$ for every k.

By definition of D_k, we have $x^k = P_{D_k}(x^g)$, and $S \subset D_k$ implies $\|x^k - x^g\| \leq \|x^* - x^g\|$ for any $x^* \in S$ and for every k. Thus $\{x^k\}$ is bounded. Moreover, since $x^{k+1} \in B_k \subseteq D_k$,

$$\|x^k - x^g\| \leq \|x^{k+1} - x^g\| \ \forall k$$

Thus $\lim \|x^k - x^g\|$ exists and is finite.

Now we show that the sequence $\{x^k\}$ is asymtotically regular, i.e., $\|x^{k+1} - x^k\| \to 0$ as $k \to \infty$. Indeed, since $x^k \in D_k$ and $x^{k+1} \in D_k$, by convexity of D_k, one has $\frac{x^{k+1}+x^k}{2} \in D_k$. Then from $x^k \in P_{D_k}(x^g)$ by using the strong convexity of the function $\|x^g - .\|^2$ we can write

$$\begin{aligned}\|x^g - x^k\|^2 &\leq \|x^g - \frac{x^{k+1} + x^k}{2}\|^2 \\ &= \|\frac{x^g - x^{k+1}}{2} + \frac{x^g - x^k}{2}\|^2 \\ &= \frac{1}{2}\|x^g - x^{k+1}\|^2 + \frac{1}{2}\|x^g - x^k\|^2 - \frac{1}{4}\|x^{k+1} - x^k\|^2\end{aligned}$$

which implies that

$$\frac{1}{2}\|x^{k+1} - x^k\|^2 \leq \|x^g - x^{k+1}\|^2 - \|x^g - x^k\|^2.$$

Remember that $\lim \|x^k - x^g\|$ does exist, we obtain $\|x^{k+1} - x^k\| \to 0$.

On the other hand, since $x^{k+1} \in B_k \subseteq C_k$, by definition of C_k, $\|u^k - x^{k+1}\| \leq \|x^{k+1} - x^k\|$. Thus $\|u^k - x^k\| \leq 2\|x^{k+1} - x^k\|$, which together with $\|x^{k+1} - x^k\| \to 0$ implies $\|u^k - x^k\| \to 0$ as $k \to \infty$.

Next, we show that any cluster point of the sequence $\{x^k\}$ is a solution to variational inequality $VI(C,F)$. Indeed, let $\overline{x}$ be any cluster point of $\{x^k\}$. For simplicity of notation, without loss of generality we may suppose that $x^k \to \overline{x}$. We consider two distinct cases:

Case 1: The linesearch is performed only for finitely many k. In this case, by the algorithm, either $u^k = x^k$ or $u^k = z^k$ for infinitely many k. In the first case, x^k is a solution to $VI(C,F)$ while in the latter case, u^k is a solution to $VI(C,F)$ for infinitely many k. Hence, by $\|u^k - x^k\| \to 0$, we have that $\overline{x}$ is a solution to $VI(C,F)$.

Case 2: The linesearch is performed for infinitely many k. Then, by taking a subsequence, if necessary, we may assume that the linesearch is performed for every k.

We distinguish two possibilities:

(a) $\overline{\lim}_k \eta_k > 0$. From $x^k \to \overline{x}$ and $\|u^k - x^k\| \to 0$ follows $u^k \to \overline{x}$. Then applying (3.4) with some $x^* \in S$ we see that $\sigma_k \ \|F^k\|^2 \to 0$. Then by definition of σ_k, we have $-\frac{\eta_k}{1-\eta_k}\langle F^k, y^k - z^k\rangle \to 0$. Since $\overline{\lim}_k \eta_k > 0$, by taking again a subsequence if necessary, we may assume that $-\langle F^k, y^k - z^k\rangle \to 0$. On the other hand, using Assumption (B1) and the Armijo rule, we can wirte

$$0 \leq \frac{\beta}{\rho}\|x^k - y^k\|^2 \leq \frac{1}{\rho}L(x^k, y^k) \leq -\langle F^k, y^k - z^k\rangle \to 0.$$

Hence $\|x^k - y^k\| \to 0$. Then, since $x^k \to \overline{x}$, one has $y^k \to \overline{x}$. Since $y^k := y(x^k)$ is the unique solution of the problem

$$\min\left\{\langle F(x^k), y - x^k\rangle + \frac{1}{\rho}L(x^k, y) : \ y \in C\right\} \qquad CP(x^k)$$

whose objective function is lower semicontinuous, by the Berge Maximum Theorem ([30] Theorem 19), function $y(.)$ is closed at $\overline{x}$. Thus $\overline{x} = y(\overline{x})$, which means that $\overline{x}$ is the solution of $CP(\overline{x})$. and therfore, by Lemma 2.3, $\overline{x}$ solves $VI(C,F)$.

(b) $\lim_k \eta_k = 0$. Since $\{x^k\}$ is bounded and y^k is the unique solution of the problem

$$\min\left\{\langle F(x^k), y - x^k\rangle + \frac{1}{\rho}L(x^k, y) : \ y \in C\right\}, \qquad CP(x^k)$$

using again the Berge Maximum Theorem we may assume that $y^k := y(x^k) \to \overline{y}$ for some $\overline{y}$. Note that the function $y(.)$ is closed at $\overline{x}$, we have $\overline{y} = y(\overline{x})$, which means that $\overline{y}$ is the solution of $CP(\overline{x})$. Thus

$$\langle F(\overline{x}), \overline{y} - \overline{x}\rangle + \frac{1}{\rho}L(\overline{x}, \overline{y}) \leq \langle F(\overline{x}), y - \overline{x}\rangle + \frac{1}{\rho}L(\overline{x}, y) \ \forall y \in C. \qquad (3.9)$$

On the other hand, as m_k is the smallest natural number satisfying the Armijo linesearch rule, we have

$$\langle F(z^{k,m_k-1}), y^k - z^{k,m_k-1}\rangle + \frac{1}{\rho}L(x^k, y^k) > 0.$$

Note that $z^{k,m_k-1} \to \overline{x}$ as $k \to \infty$, from the last inequality, by continuity of F and L, we obtain in the limit that

$$\langle F(\overline{x}), \overline{y} - \overline{x}\rangle + \frac{1}{\rho}L(\overline{x}, \overline{y}) \geq 0. \tag{3.10}$$

Substituting $y = \overline{x}$ into (3.9) we get

$$\langle F(\overline{x}), \overline{y} - \overline{x}\rangle + \frac{1}{\rho}L(\overline{x}, \overline{y}) \leq 0.$$

Thus

$$\langle F(\overline{x}), \overline{y} - \overline{x}\rangle + \frac{1}{\rho}L(\overline{x}, \overline{y}) = 0. \tag{3.11}$$

From (3.11) and

$$\langle F(\overline{x}), \overline{x} - \overline{x}\rangle + \frac{1}{\rho}L(\overline{x}, \overline{x}) = 0,$$

it follows that both $\overline{x}$ and $\overline{y}$ are solutions of the strongly convex program

$$\min\{\langle F(\overline{x}), y - \overline{x}\rangle + \frac{1}{\rho}L(\overline{x}, y) : y \in C\}.$$

Hence $\overline{x} = \overline{y}$ and therfore, by Lemma 2.3, $\overline{x}$ solves $VI(C, F)$. Moreover, from $\|u^k - x^k\| \to 0$, we can also conclude that every cluster point of $\{u^k\}$ is a solution to $VI(C, F)$.

Finally, we show that $\{x^k\}$ converges to the unique solution of the bilevel problem (1.1). To this end, let x^* be any cluster point of $\{x^k\}$. Then, there exists a subsequence $\{x^{k_j}\}$ such that $x^{k_j} \to x^*$ as $j \to \infty$. Since, by the algorithm, $x^{k_j} = P_{B_{k_j-1}}(x^g)$, we have

$$\langle x^{k_j} - x^g, y - x^{k_j}\rangle \geq 0,\ \forall y \in B_{k_j-1} \supseteq S,\ \forall j.$$

Letting $j \to +\infty$ we obtain

$$\langle x^* - x^g, y - x^*\rangle \geq 0,\ \forall y \in S,$$

which together with $x^* \in S$ implies that $x^* = P_S(x^g)$. Thus, the whole sequence must converge to the unique solution to the original bilevel problem (1.1). Then, from $\|x^k - u^k\| \to 0$, it follows that u^k converges to the solution of Problem (1.1) as well. □

Other Armijo rules can be chosen for the above algorithm. For example, one can use the linesearch in [27]. Then at each iteration k in Algorithm 1, the Step 2 and Step 3 are replaced by the Step 2a) and Step 3a) below.

Step 2a). Having $x^k \in C$ find m_k as the smallest nonnegative integer m such that

$$\langle F(x^k - \eta^m r(x^k), r(x^k) \geq \sigma \|r(x^k)\|^2 \tag{3.12}$$

where $\sigma \in (0,1)$, $r(x^k) = x^k - y^k$ with y^k being the unique solution of the problem $CP(x^k)$. Take $z^k := (1-\eta_k)x^k + \eta_k y^k$. If $F(z^k) = 0$, take $u^k := z^k$. Otherwise, go to Step 4 (in Algorithm 1).

Step 3a). Define

$$H_k := \{x \in C : \langle F(z^k), x - z^k\rangle \leq 0\}$$

and take $u^k := P_{H_k}(x^k)$.

It is easy to see that if F is pseudomonotone on C with respect to S, then $S \subseteq H_k$. By the same way as the linesearch rule (3.2) we can show that the linesearch (3.12) is well defined. Moreover as in the linesearch (3.2), for (3.12) the following lemma can be proved by the same idea as in the proof of Theorem 2.1 in [27].

Lemma 3.2 *Under the assumptions of Lemma 2.3, it holds that*

$$\|u^k - x^*\|^2 \leq \|x^k - x^*\|^2 - \frac{\eta_k \sigma}{\|F(z^k)\|^2}\|r(x^k)\|^4, \ \forall x^* \in S, \ \forall k. \tag{3.13}$$

Using this lemma we can prove the following convergence theorem by the same argument as in the proof of Theorem 3.2.

Theorem 3.2 *Suppose that Assumptions (A1), (A2) and (B1), (B2) are satisfied and that $VI(C,F)$ admits a solution. Then Algorithm 1 with Steps 2 and 3 are replaced by Steps 2a) and 3a) respectively is well defined and both sequences $\{x^k\}$, $\{u^k\}$ converges to the unique solution of bilevel problem (1.1).*

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