

A von Neumann Alternating Method for Finding Common Solutions to Variational Inequalities

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*Dedicated to Professor V. Lakshmikantham
on the occasion of his retirement*

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Abstract

Modifying von Neumann's alternating projections algorithm, we obtain an alternating method for solving the recently introduced Common Solutions to Variational Inequalities Problem (CSVIP). For simplicity, we mainly confine our attention to the two-set CSVIP, which entails finding common solutions to two unrelated variational inequalities in Hilbert space.

Keywords: Alternating method, averaged operator, fixed point, Hilbert space, inverse strongly monotone operator, metric projection, nonexpansive operator, resolvent, variational inequality.

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1 Introduction

A new problem, called the *Common Solutions to Variational Inequalities Problem* (CSVIP) has recently been introduced in [8, Subsection 7.2] and further studied in [9]. In [8] it was considered as a special case of the *Split Variational Inequality Problem* (SVIP) introduced therein. The CSVIP consists of finding common solutions to unrelated variational inequalities. In the present paper we propose a new algorithm for solving the two-set CSVIP, which stems from the classical von Neumann alternating projections algorithm [22]. We also extend this algorithm to two methods for solving the general CSVIP, which concerns any finite number of sets.

We first recall the general form of the CSVIP (for single-valued operators).

Problem 1.1 *Let \mathcal{H} be a real Hilbert space. Let there be given, for each $i = 1, 2, \dots, N$, an operator $A_i : \mathcal{H} \rightarrow \mathcal{H}$ and a nonempty, closed and convex subset $K_i \subset \mathcal{H}$, with $\bigcap_{i=1}^N K_i \neq \emptyset$. The CSVIP (for single-valued operators) is to find a point $x^* \in \bigcap_{i=1}^N K_i$ such that, for each $i = 1, 2, \dots, N$,*

$$\langle A_i(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in K_i, i = 1, 2, \dots, N. \quad (1.1)$$

For simplicity, in this paper we mainly confine our attention to the case where $i = 2$. Denoting $A_1 = f$, $A_2 = g$ and the nonempty, closed and convex subsets K_1 and K_2 by C and Q , respectively, we get the following two-set CSVIP.

Problem 1.2 *Let \mathcal{H} be a real Hilbert space, and let C and Q be two non-empty closed and convex subsets of \mathcal{H} with $C \cap Q \neq \emptyset$. Given two operators f and g from \mathcal{H} into itself, the two-set CSVIP is to find a point $x^* \in C \cap Q$ such that*

$$\langle f(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in C \quad (1.2)$$

and

$$\langle g(x^*), y - x^* \rangle \geq 0 \text{ for all } y \in Q. \quad (1.3)$$

If we denote by $SOL(C, f)$ and $SOL(Q, g)$ the solution sets of (1.2) and (1.3), respectively, then Problem 1.2 is to find a point $x^* \in SOL(C, f) \cap SOL(Q, g)$.

Looking at (1.2) separately, we get the well-known *Variational Inequality Problem* (VIP), first introduced by Hartman and Stampacchia in 1966

(see [15]). The importance of VIPs in Nonlinear Analysis and Optimization Theory stems from the fact that several fundamental problems can be formulated as VIPs, e.g., constrained and unconstrained minimization, finding solutions to systems of equations, and saddle-point problems. See the book by Kinderlehrer and Stampacchia [17] for a wide range of other applications of VIPs. For an excellent treatise on variational inequality problems in finite-dimensional spaces, see the two-volume book by Facchinei and Pang [12]. The books by Konnov [18] and Patriksson [24] contain extensive studies of VIPs including applications, algorithms and numerical results.

Another motivation for defining and studying the CSVIP in [8, 9] originates in the simple observation that if we choose all $A_i = 0$, then the problem reduces to that of finding a point $x^* \in \bigcap_{i=1}^N K_i$ in the nonempty intersection of a finite family of closed and convex sets, which is the well-known *Convex Feasibility Problem* (CFP). If the sets K_i are the fixed point sets of a family of operators $T_i : \mathcal{H} \rightarrow \mathcal{H}$, then the CFP is the *Common Fixed Point Problem* (CFPP). These problems have been intensively studied over the past decades both theoretically (existence, uniqueness, and properties of solutions) and algorithmically (devising iterative procedures which generate sequences that converge, finitely or asymptotically, to a solution).

Our alternating method for solving the two-set CSVIP is inspired by von Neumann's original alternating projections method. Von Neumann [22] presented a method for calculating the orthogonal projection onto the intersection of two closed subspaces in Hilbert space. Let \mathcal{H} be a real Hilbert space, and let A and B be closed subspaces. Choose $x \in \mathcal{H}$ and construct the sequences $\{a^k\}_{k=0}^\infty$ and $\{b^k\}_{k=0}^\infty$ by

$$\begin{cases} b^0 = x, \\ a^k = P_A(b^{k-1}) \text{ and } b^k = P_B(a^k), \quad k = 1, 2, \dots, \end{cases} \quad (1.4)$$

where P_A and P_B denote the orthogonal projection operators of \mathcal{H} onto A and B , respectively. Von Neumann showed [22, Lemma 22, page 475] that both sequences $\{a^k\}_{k=0}^\infty$ and $\{b^k\}_{k=0}^\infty$ converge strongly to $P_{A \cap B}(x)$. This algorithm is known as von Neumann's alternating projections method. Observe that not only the sequences converge strongly, but also that their common limit is the nearest point to x in $A \cap B$. For recent elementary geometric proofs of von Neumann's result, see [19, 20]. In 1965 Bregman [5] established the weak convergence of the sequence of alternating nearest point mappings between two closed and convex intersecting subsets of a Hilbert

space. See also [1, 6]. In 2005 Bauschke, Combettes and Reich [2] studied the alternating resolvents method for finding a common zero of two maximal monotone mappings (see also the recent paper of Boikanyo and Moroşanu [4]). We propose an alternating method which employs two averaged operators in the sense of [1]. In this connection, we note that not all averaged operators are resolvents (of monotone mappings).

Our paper is organized as follows. Section 2 contains some preliminaries. In Section 3 we present our alternating method for solving the two-set CSVIP and establish its convergence. In Section 4 we extend our algorithm to two methods for solving the general CSVIP.

2 Preliminaries

Let \mathcal{H} be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$, and let D be a nonempty, closed and convex subset of \mathcal{H} . We write $x^k \rightharpoonup x$ to indicate that the sequence $\{x^k\}_{k=0}^{\infty}$ converges weakly to x , and $x^k \rightarrow x$ to indicate that the sequence $\{x^k\}_{k=0}^{\infty}$ converges strongly to x .

We now recall some definitions and properties of several classes of operators.

Definition 2.1 *Let $h : \mathcal{H} \rightarrow \mathcal{H}$ be an operator and let $D \subset \mathcal{H}$.*

(i) *The operator h is called **Lipschitz continuous** on $D \subset H$ with constant $L > 0$ if*

$$\|h(x) - h(y)\| \leq L\|x - y\| \text{ for all } x, y \in D. \quad (2.1)$$

(ii) *The operator h is called **nonexpansive** on D if it is 1-Lipschitz continuous.*

(iii) *The operator h is called **inverse strongly monotone** with constant $\beta > 0$ (β -ism) on D if*

$$\langle h(x) - h(y), x - y \rangle \geq \beta \|h(x) - h(y)\|^2 \text{ for all } x, y \in D. \quad (2.2)$$

(iv) *The operator h is called **firmly nonexpansive** [14] on D if*

$$\langle h(x) - h(y), x - y \rangle \geq \|h(x) - h(y)\|^2 \text{ for all } x, y \in D,$$

in other words, h is 1-ism.

(v) The operator h is called **averaged** [1] if there exists a nonexpansive operator $N : \mathcal{H} \rightarrow \mathcal{H}$ and a number $c \in (0, 1)$ such that

$$h = (1 - c)I + cN. \quad (2.3)$$

In this case, we say that h is *c-av* [7].

(vi) We say that a nonexpansive operator h **satisfies Condition (W)** [11] if whenever $\{x^k - y^k\}_{k=1}^{\infty}$ is bounded and $\|x^k - y^k\| - \|h(x^k) - h(y^k)\| \rightarrow 0$, it follows that $(x^k - y^k) - (h(x^k) - h(y^k)) \rightarrow 0$.

(vii) The operator h is called **strongly nonexpansive** [6] if it is nonexpansive and whenever $\{x^k - y^k\}_{k=1}^{\infty}$ is bounded and $\|x^k - y^k\| - \|h(x^k) - h(y^k)\| \rightarrow 0$, it follows that $(x^k - y^k) - (h(x^k) - h(y^k)) \rightarrow 0$.

Definition 2.2 Let \mathbb{N} be the set of natural numbers, $\{h_1, h_2, \dots\}$ be a sequence of operators, and $r : \mathbb{N} \rightarrow \mathbb{N}$. An unrestricted (or random) product of these operators is the sequence $\{S_n\}_{n \in \mathbb{N}}$ defined by $S_n = h_{r(n)}h_{r(n-1)} \cdots h_{r(1)}$.

Note that inverse strong monotonicity is also known as the **Dunn property** [10, 27]. It is easy to see that a β -ism operator is Lipschitz continuous with constant $1/\beta$. Some of the relations between these classes of operators are given below. For more details, see Bruck and Reich [6], Baillon et al. [1], Goebel and Reich [14], and Byrne [7].

Remark 2.3 (i) An operator h is averaged if and only if its complement $G := I - h$ is ν -ism for some $\nu > 1/2$.

(ii) The operator h is firmly nonexpansive if and only if its complement $I - h$ is firmly nonexpansive.

(iii) The operator h is firmly nonexpansive if and only if it is $1/2$ -averaged.

(iv) If h_1 and h_2 are c_1 -av and c_2 -av, respectively, then their composition $S = h_1h_2$ is $(c_1 + c_2 - c_1c_2)$ -av.

(v) Every averaged operator is strongly nonexpansive and therefore satisfies condition (W).

Let D be a closed and convex subset of \mathcal{H} . For every point $x \in \mathcal{H}$, there exists a unique nearest point in D , denoted by $P_D(x)$. This point satisfies

$$\|x - P_D(x)\| \leq \|x - y\| \text{ for all } y \in D. \quad (2.4)$$

The operator P_D is called the metric projection or the nearest point mapping of \mathcal{H} onto D . The metric projection P_D is characterized by the fact that $P_D(x) \in D$ and

$$\langle y - P_D(x), x - P_D(x) \rangle \leq 0 \text{ for all } x \in \mathcal{H}, y \in D. \quad (2.5)$$

It is also well known that the operator P_D is averaged (see, e.g., [14, page 17]).

Definition 2.4 A sequence $\{x^k\}_{k=0}^\infty \subset \mathcal{H}$ is called *Fejér-monotone* with respect to D if for every $u \in D$

$$\|x^{k+1} - u\| \leq \|x^k - u\| \text{ for all } k \geq 0. \quad (2.6)$$

Next we recall the definition of a maximal monotone mapping.

Definition 2.5 Let $M : \mathcal{H} \rightarrow 2^{\mathcal{H}}$ be a set-valued mapping defined on a real Hilbert space \mathcal{H} .

(i) The *resolvent* of M with parameter λ is the operator $J_\lambda^M := (I + \lambda M)^{-1}$, where I is the identity operator.

(ii) M is called a *maximal monotone mapping* if M is monotone, i.e.,

$$\langle u - v, x - y \rangle \geq 0, \text{ for all } u \in M(x) \text{ and } v \in M(y), \quad (2.7)$$

and the graph $G(M)$ of M ,

$$G(M) := \{(x, u) \in H \times H \mid u \in M(x)\}, \quad (2.8)$$

is not properly contained in the graph of any other monotone mapping.

Definition 2.6 Let C be a nonempty, closed and convex subset of \mathcal{H} . Denote by $N_C(v)$ the *normal cone* of C at $v \in C$, i.e.,

$$N_C(v) := \{z \in \mathcal{H} \mid \langle z, y - v \rangle \leq 0 \text{ for all } y \in C\}. \quad (2.9)$$

Consider now the variational inequality with respect to the set D and the operator h :

$$\langle h(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in D. \quad (2.10)$$

Define the mapping M as follows:

$$M(v) := \begin{cases} h(v) + N_D(v), & v \in D, \\ \emptyset, & \text{otherwise.} \end{cases} \quad (2.11)$$

Under a certain continuity assumption on h (which every ism operator satisfies), Rockafellar [28, Theorem 5, p. 85] showed that M is a maximal monotone mapping and $M^{-1}(0) = \text{SOL}(D, h)$.

Remark 2.7 *It is well known that for $\lambda > 0$,*

(i) M is monotone if and only if the resolvent J_λ^M of M is single-valued and firmly nonexpansive.

*(ii) M is maximal monotone if and only if J_λ^M is single-valued, firmly nonexpansive and its **domain** is all of \mathcal{H} , where*

$$\text{dom}(J_\lambda^M) := \{x \in \mathcal{H} \mid J_\lambda^M(x) \neq \emptyset\}. \quad (2.12)$$

(iii)

$$0 \in M(x^*) \Leftrightarrow x^* \in \text{Fix}(J_\lambda^M), \quad (2.13)$$

where $\text{Fix}(J_\lambda^M)$ denotes the fixed point set of J_λ^M .

It is known that the metric projection operator coincides with the resolvent of the normal cone.

Now we recall the following lemma [8].

Lemma 2.8 *Let \mathcal{H} be a real Hilbert space and let $D \subset \mathcal{H}$ be nonempty, closed and convex. Let $h : \mathcal{H} \rightarrow \mathcal{H}$ be an α -ism operator. If $D \cap \{x \in \mathcal{H} \mid h(x) = 0\} \neq \emptyset$, then $x^* \in \text{SOL}(D, h)$ if and only if $h(x^*) = 0$.*

Using the characterization of the metric projection (2.5), we get another connection between the solution set of a variational inequality problem and the fixed point set of a certain operator, namely, for any $\lambda > 0$,

$$\text{SOL}(D, h) = \text{Fix}(P_D(I - \lambda h)). \quad (2.14)$$

Indeed,

$$x^* \in \text{Fix}(P_D(I - \lambda h)) \Leftrightarrow P_D(x^* - \lambda h(x^*)) = x^* \quad (2.15)$$

and by (2.5), we have for all $x \in D$ and $\lambda > 0$,

$$\begin{aligned} 0 &\leq \langle x^* - \lambda h(x^*) - P_D(x^* - \lambda h(x^*)), P_D(x^* - \lambda h(x^*)) - x \rangle \\ &= \langle x^* - \lambda h(x^*) - x^*, x^* - x \rangle = \langle -\lambda h(x^*), x^* - x \rangle \\ &= \lambda \langle h(x^*), x - x^* \rangle. \end{aligned} \quad (2.16)$$

Next we present another useful property of the operator $P_D(I - \lambda h)$ (cf. [21]).

Lemma 2.9 *Let \mathcal{H} be a real Hilbert space and let $D \subset \mathcal{H}$ be nonempty, closed and convex. Let $h : \mathcal{H} \rightarrow \mathcal{H}$ be a β -ism operator on \mathcal{H} . If $\lambda \in (0, 2\beta)$, then the operator $P_D(I - \lambda h)$ is averaged.*

Proof. We first prove that the operator $I - \lambda h$ is averaged. More precisely, we claim that if h is β -ism, then the operator $I - \lambda h$ is averaged for $\lambda \in (0, 2\beta)$. Indeed, take $c \in (0, 1)$ such that $c \geq \lambda/(2\beta)$ and set $N := I - \frac{\lambda}{c}h$. Then $I - \lambda h = (1 - c)I + cN$ and N is nonexpansive:

$$\begin{aligned}
& \|x - y\|^2 - \|N(x) - N(y)\|^2 \\
&= \|x - y\|^2 - \left(\|x - y\|^2 - 2\frac{\lambda}{c} \langle h(x) - h(y), x - y \rangle + \left(\frac{\lambda}{c}\right)^2 \|h(x) - h(y)\|^2 \right) \\
&= 2\frac{\lambda}{c} \langle h(x) - h(y), x - y \rangle - \left(\frac{\lambda}{c}\right)^2 \|h(x) - h(y)\|^2 \\
&\geq \frac{\lambda}{c} \left(2\beta \|h(x) - h(y)\|^2 - \frac{\lambda}{c} \|h(x) - h(y)\|^2 \right) \\
&= \frac{\lambda}{c} \left(2\beta - \frac{\lambda}{c} \right) \|h(x) - h(y)\|^2 \geq 0. \tag{2.17}
\end{aligned}$$

Now, since the metric projection P_D is averaged, so is the composition $P_D(I - \lambda h)$ (see Remark 2.3(iii)). ■

While the metric projection operator is the resolvent operator of the normal cone operator, the composed operator $P_D(I - \lambda h)$ need not be a resolvent of a monotone mapping.

Lemma 2.10 [6, 7] *If $U : \mathcal{H} \rightarrow \mathcal{H}$ and $V : \mathcal{H} \rightarrow \mathcal{H}$ are averaged operators and $\text{Fix}(U) \cap \text{Fix}(V) \neq \emptyset$, then $\text{Fix}(U) \cap \text{Fix}(V) = \text{Fix}(UV) = \text{Fix}(VU)$.*

The next lemma was proved by Takahashi and Toyoda [29, Lemma 3.2]. In this connection, see also [25, Proposition 2.1].

Lemma 2.11 *Let \mathcal{H} be a real Hilbert space and let $D \subset \mathcal{H}$ be nonempty, closed and convex. Let the sequence $\{x^k\}_{k=0}^{\infty} \subset \mathcal{H}$ be Fejér-monotone with respect to D . Then the sequence $\{P_D(x^k)\}_{k=0}^{\infty}$ converges strongly to some $z \in D$.*

Next we recall a theorem of Opial's [23], which is also known in the literature as the Krasnosel'skiĭ-Mann theorem.

Theorem 2.12 *Let \mathcal{H} be a real Hilbert space and let $D \subset \mathcal{H}$ be nonempty, closed and convex. Assume that $h : D \rightarrow D$ is an averaged operator with $\text{Fix}(h) \neq \emptyset$. Then, for an arbitrary $x^0 \in D$, the sequence $\{x^{k+1} = h(x^k)\}_{k=0}^{\infty}$ converges weakly to $z \in \text{Fix}(h)$.*

Remark 2.13 *The convergence obtained in Theorem 2.12 is not strong in general [13, 3].*

3 The Algorithm

In this section we introduce our modified von Neumann alternating method for solving the two-set CSVIP (1.2) and (1.3). Let $\Gamma := \Gamma(C, Q, f, g) := SOL(C, f) \cap SOL(Q, g)$.

The following conditions are needed for our convergence theorem.

Condition 3.1 *The operators $f : \mathcal{H} \rightarrow \mathcal{H}$ and $g : \mathcal{H} \rightarrow \mathcal{H}$ are α_1 -ism and α_2 -ism, respectively.*

Condition 3.2 $\lambda \in (0, 2\alpha)$, where $\alpha := \min\{\alpha_1, \alpha_2\}$.

Condition 3.3 $\Gamma \neq \emptyset$.

Algorithm 3.4

Initialization: *Select an arbitrary starting point $x^0 \in \mathcal{H}$.*

Iterative step: *Given the current iterate x^k , compute*

$$y^k = (P_Q(I - \lambda g))(x^k) \quad \text{and} \quad x^{k+1} = (P_C(I - \lambda f))(y^k). \quad (3.1)$$

Note that (3.1) is actually an alternating method, that is,

$$\begin{aligned} x^{k+1} &= (P_C(I - \lambda f))(P_Q(I - \lambda g))(x^k) \\ &= P_C(P_Q(x^k - \lambda g(x^k)) - \lambda f(P_Q(x^k - \lambda g(x^k))))). \end{aligned} \quad (3.2)$$

An illustration of the iterative step of Algorithm 3.4 is presented in Figure 1.

Theorem 3.5 *Let \mathcal{H} be a real Hilbert space, and let C, Q be two nonempty closed and convex subsets of \mathcal{H} . Assume that Conditions 3.1-3.3 hold and set $\alpha := \min\{\alpha_1, \alpha_2\}$. Then any sequence $\{x^k\}_{k=0}^\infty$ generated by Algorithm 3.4 converges weakly to a point $x^* \in \Gamma$, and furthermore,*

$$x^* = \lim_{k \rightarrow \infty} P_\Gamma(x^k). \quad (3.3)$$

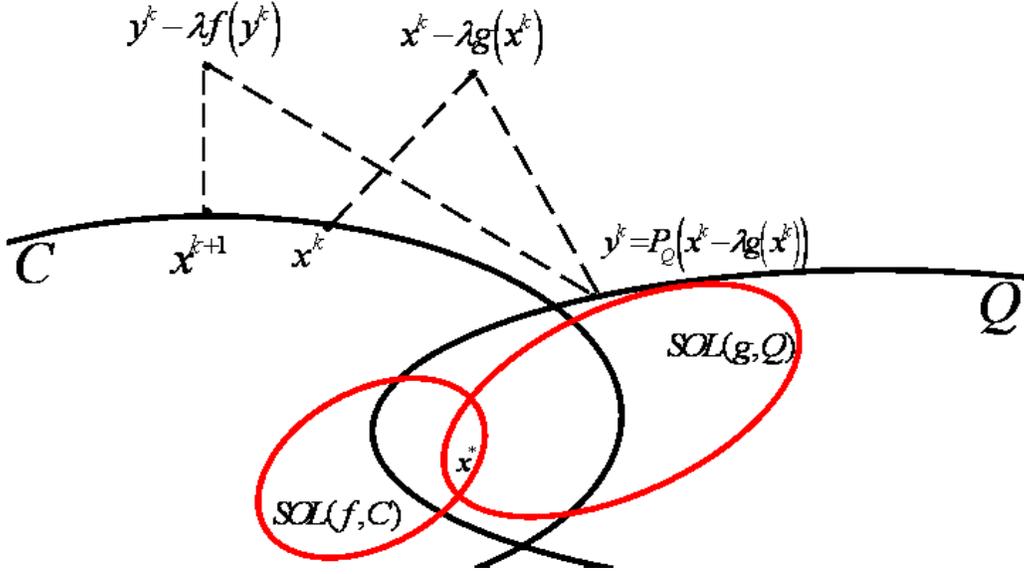


Figure 1: Illustration of the iterative step of Algorithm 3.4.

Proof. Let $\lambda \in (0, 2\alpha)$. By Lemma 2.9, the operators $P_C(I - \lambda f)$ and $P_Q(I - \lambda g)$ are averaged and so is their composition $(P_C(I - \lambda f))(P_Q(I - \lambda g))$ (Remark 2.3). Since $\Gamma \neq \emptyset$, Opial's theorem (Theorem 2.12) guarantees that any sequence $\{x^k\}_{k=0}^{\infty}$ generated by Algorithm 3.4 converges weakly to a point $x^* \in \text{Fix}((P_C(I - \lambda f))(P_Q(I - \lambda g)))$. Combining the assumption $\Gamma \neq \emptyset$ with Lemma 2.10, we obtain

$$\begin{aligned} \text{Fix}(P_C(I - \lambda f)) \cap \text{Fix}(P_Q(I - \lambda g)) &= \text{Fix}((P_C(I - \lambda f))(P_Q(I - \lambda g))) \\ &= \text{Fix}((P_Q(I - \lambda g))(P_C(I - \lambda f))), \end{aligned} \tag{3.4}$$

which means that $x^* \in \text{Fix}(P_C(I - \lambda f))$ and $x^* \in \text{Fix}(P_Q(I - \lambda g))$, and therefore by (2.14) $x^* \in \Gamma$. Finally, let $z \in \Gamma$, i.e., $z \in \text{SOL}(C, f) \cap \text{SOL}(Q, g)$. Then $P_Q(z - \lambda g(z)) = P_C(z - \lambda f(z)) = z$. Since the operators $P_C(I - \lambda f)$

and $P_Q(I - \lambda g)$ are averaged, they are also nonexpansive. Thus

$$\begin{aligned}
\|x^{k+1} - z\|^2 &= \|(P_C(I - \lambda f))(P_Q(I - \lambda g))(x^k) - z\|^2 \\
&= \|(P_C(I - \lambda f))(P_Q(I - \lambda g))(x^k) - P_C(I - \lambda f)(z)\|^2 \\
&\leq \|P_Q(x^k - \lambda g(x^k)) - z\|^2 \\
&= \|P_Q(x^k - \lambda g(x^k)) - P_Q(z - \lambda g(z))\|^2 \\
&\leq \|x^k - z\|^2.
\end{aligned} \tag{3.5}$$

So

$$\|x^{k+1} - z\|^2 \leq \|x^k - z\|^2, \tag{3.6}$$

which means that the sequence $\{x^k\}_{k=0}^\infty$ is Fejér-monotone with respect to Γ . Now, put

$$u^k = P_\Gamma(x^k). \tag{3.7}$$

Since the operators $P_C(I - \lambda f)$ and $P_Q(I - \lambda g)$ are nonexpansive, it follows from (2.14) that the sets $SOL(C, f)$ and $SOL(Q, g)$ are nonempty, closed and convex (see [14, Proposition 5.3, page 25]). In addition, since $\Gamma \neq \emptyset$, each u^k is well defined. So, applying (2.5) with $D = \Gamma$ and $x = x^k$, we get

$$\langle y - P_\Gamma(x^k), x^k - P_\Gamma(x^k) \rangle \leq 0 \text{ for all } k \geq 0 \text{ and } y \in \Gamma. \tag{3.8}$$

Taking, in particular, $y = x^* \in \Gamma$, we obtain

$$\langle x^* - u^k, x^k - u^k \rangle \leq 0. \tag{3.9}$$

By Lemma 2.11, $\{u^k\}_{k=0}^\infty$ converges strongly to some $u^* \in \Gamma$. Therefore

$$\langle x^* - u^*, x^* - u^* \rangle \leq 0 \tag{3.10}$$

and hence $u^* = x^*$, as asserted. ■

Remark 3.6 1. The sequence $\{y^k\}_{k=0}^\infty$ also converges weakly to $x^* \in \Gamma$.

2. Under the additional assumptions that C and Q are symmetric, and f and g are odd, that is, $f(-x) = -f(x)$ and $g(-x) = -g(x)$ for all $x \in \mathcal{H}$, we get from [1, Corollary 2.1] that any sequence $\{x^k\}_{k=0}^\infty$, generated by Algorithm 3.4, converges strongly to a point $x^* \in \Gamma$.

3. Strong convergence also occurs when either C or Q is compact.

4. According to [1, Corollary 2.2], if $\Gamma = \emptyset$, then $\lim_{k \rightarrow \infty} \|x^k\| = \infty$.

5. When C and Q are closed subspaces and $f = g = 0$ in the two-set CSVIP (1.2) and (1.3), we get von Neumann's original problem and then Algorithm 3.4 is the classical alternating projections method (1.4).

6. In [9, Algorithm 3.1] we presented an algorithm which can be applied to the solution of (1.2) and (1.3). The structure of this algorithm is quite different from that of Algorithm 3.4 in the sense that at each step there is the need to calculate the projection of the current iterate onto the intersection of three half-spaces. Although the latter calculation complicates the process, the sequence generated in this way converges strongly to a solution.

Following [26] and [11], we now present two more algorithms for solving the two-set CSVIP (1.2) and (1.3). Let \mathcal{H} be a real Hilbert space, and let C and Q be two nonempty, closed and convex subsets of \mathcal{H} .

Recall the following two lemmata [26, Lemmata 1.3 and 1.4].

Lemma 3.7 *A convex combination of strongly nonexpansive operators is also strongly nonexpansive.*

Lemma 3.8 *Let T be a convex combination of the strongly nonexpansive mappings $\{T_k \mid 1 \leq k \leq m\}$. If the set $\cap \{\text{Fix}(T_k) \mid 1 \leq k \leq m\}$ is not empty, then it equals $\text{Fix}(T)$.*

Now we can propose the following parallel algorithm.

Algorithm 3.9

Initialization: Select an arbitrary starting point $x^0 \in \mathcal{H}$, and let the numbers w_1 and w_2 be such that $w_1, w_2 \geq 0$ and $w_1 + w_2 = 1$.

Iterative step: Given the current iterate x^k , compute

$$x^{k+1} = w_1 P_C(x^k - \lambda f(x^k)) + w_2 P_Q(x^k - \lambda g(x^k)). \quad (3.11)$$

Theorem 3.10 *Let \mathcal{H} be a real Hilbert space, and let C and Q be two nonempty, closed and convex subsets of \mathcal{H} . Assume that Conditions 3.1-3.3 hold. Then any sequence $\{x^k\}_{k=0}^\infty$ generated by Algorithm 3.9 converges weakly to a point $x^* \in \Gamma$, and furthermore,*

$$x^* = \lim_{k \rightarrow \infty} P_\Gamma(x^k). \quad (3.12)$$

Proof. By Lemma 2.9, the operators $P_C(I - \lambda f)$ and $P_Q(I - \lambda g)$ are averaged, hence strongly nonexpansive (see Remark 2.3). According to Lemma 3.7, any convex combination of strongly nonexpansive mappings is also strongly nonexpansive. So the sequence $\{x^k\}_{k=0}^\infty$ generated by Algorithm 3.9 is, in fact, an iteration of a strongly nonexpansive operator and therefore the desired result is obtained by [6] and Lemma 3.8. ■

Remark 3.11 *The convergence obtained in Theorem 3.9 is not strong in general [3].*

Finally, we recall the following theorem [11, Theorem 1].

Theorem 3.12 *Let $T_1 : \mathcal{H} \rightarrow \mathcal{H}$ and $T_2 : \mathcal{H} \rightarrow \mathcal{H}$ be two nonexpansive operators which satisfy Condition (W), the fixed point sets of which have a nonempty intersection. Then any unrestricted product from T_1 and T_2 converges weakly to a common fixed point.*

Since every averaged operator is strongly nonexpansive and therefore satisfies Condition (W), we can apply the above theorem to obtain an algorithm for solving the two-set CSVIP by using any unrestricted product from $P_C(I - \lambda f)$ and $P_Q(I - \lambda g)$. Any such unrestricted product converges weakly to a point in Γ .

4 The general CSVIP

In this section we extend our algorithm to two methods for solving the general CSVIP with single-valued operators. Let \mathcal{H} be a real Hilbert space. Let there be given, for each $i = 1, 2, \dots, N$, an operator $f_i : \mathcal{H} \rightarrow \mathcal{H}$ and a nonempty, closed and convex subset $C_i \subset \mathcal{H}$, with $\bigcap_{i=1}^N C_i \neq \emptyset$. The CSVIP is to find a point $x^* \in \bigcap_{i=1}^N C_i$ such that, for each $i = 1, 2, \dots, N$,

$$\langle f_i(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in C_i, i = 1, 2, \dots, N. \quad (4.1)$$

Denote $\Psi := \bigcap_{i=1}^N SOL(C_i, f_i)$.

Algorithm 4.1

Initialization: Select an arbitrary starting point $x^0 \in \mathcal{H}$.

Iterative step: Given the current iterate x^k , compute the product

$$x^{k+1} = \prod_{i=1}^N (P_{C_i}(I - \lambda f_i))(x^k). \quad (4.2)$$

Theorem 4.2 Let \mathcal{H} be a real Hilbert space. For each $i = 1, 2, \dots, N$, let an operator $f_i : \mathcal{H} \rightarrow \mathcal{H}$ and a nonempty, closed and convex subset $C_i \subset \mathcal{H}$ be given. Assume that $\bigcap_{i=1}^N C_i \neq \emptyset$, $\Psi \neq \emptyset$ and that for $i = 1, 2, \dots, N$, f_i is α_i -ism. Set $\alpha := \min_i \{\alpha_i\}$ and take $\lambda \in (0, 2\alpha)$. Then any sequence $\{x^k\}_{k=0}^\infty$ generated by Algorithm 4.1 converges weakly to a point $x^* \in \Psi$, and furthermore,

$$x^* = \lim_{k \rightarrow \infty} P_\Psi(x^k). \quad (4.3)$$

Algorithm 4.3

Initialization: Select an arbitrary starting point $x^0 \in \mathcal{H}$ and a nonnegative finite sequence $\{w_i\}_{i=1}^N$ such that $\sum_{i=1}^N w_i = 1$.

Iterative step: Given the current iterate x^k , compute

$$x^{k+1} = \sum_{i=1}^N w_i (P_{C_i}(I - \lambda f_i))(x^k). \quad (4.4)$$

Theorem 4.4 Let \mathcal{H} be a real Hilbert space. For each $i = 1, 2, \dots, N$, let an operator $f_i : \mathcal{H} \rightarrow \mathcal{H}$ and a nonempty, closed and convex subset $C_i \subset \mathcal{H}$ be given. Assume that $\bigcap_{i=1}^N C_i \neq \emptyset$, $\Psi \neq \emptyset$, and that for $i = 1, 2, \dots, N$, f_i is α_i -ism. Set $\alpha := \min_i \{\alpha_i\}$ and take $\lambda \in (0, 2\alpha)$. Then any sequence $\{x^k\}_{k=0}^\infty$ generated by Algorithm 4.3 converges weakly to a point $x^* \in \Psi$, and furthermore,

$$x^* = \lim_{k \rightarrow \infty} P_\Psi(x^k). \quad (4.5)$$

The proofs of Theorem 4.2 and 4.4 are analogous to those of Theorems 3.5 and 3.10, respectively, and therefore are omitted.

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References

- [1] J.-B. Baillon, R. E. Bruck and S. Reich, On the asymptotic behavior of nonexpansive mappings and semigroups in Banach spaces, *Houston Journal of Mathematics* **4** (1978), 1–9.
- [2] H. H. Bauschke, P. L. Combettes and S. Reich, The asymptotic behavior of the composition of two resolvents, *Nonlinear Analysis* **60** (2005), 283–301.
- [3] H. H. Bauschke, E. Matoušková, S. Reich, Projection and proximal point methods: convergence results and counterexamples, *Nonlinear Analysis* **56** (2004), 715–738.
- [4] O. A. Boikanyo and G. Moroşanu, On the method of alternating resolvents, *Nonlinear Analysis* **74** (2011), 5147–5160.
- [5] L. Bregman, The method of successive projection for finding a common point of convex sets, *Soviet Mathematics Doklady* **6** (1965), 688–692.
- [6] R. E. Bruck and S. Reich, Nonexpansive projections and resolvents of accretive operators in Banach spaces, *Houston Journal of Mathematics* **3** (1977), 459–470.
- [7] C. L. Byrne, A unified treatment of some iterative algorithms in signal processing and image reconstruction, *Inverse Problems* **20** (1999), 1295–1313.
- [8] Y. Censor, A. Gibali and S. Reich, Algorithms for the split variational inequality problem, *Numerical Algorithms* **59** (2012), 301–323.
- [9] Y. Censor, A. Gibali, S. Reich and S. Sabach, Common solutions to variational inequalities, *Set-Valued and Variational Analysis*, accepted for publication.
- [10] J. C. Dunn, Convexity, monotonicity, and gradient processes in Hilbert space, *Journal of Mathematical Analysis and Applications* **53** (1976), 145–158.
- [11] J. M. Dye and S. Reich, Unrestricted iterations of nonexpansive mappings in Hilbert space, *Nonlinear Analysis* **18** (1992), 199–207.

- [12] F. Facchinei and J. S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems, Volume I and Volume II*, Springer-Verlag, New York, NY, USA, 2003.
- [13] A. Genel and J. Lindenstrauss, An example concerning fixed points, *Israel Journal of Mathematics* **22** (1975), 81-85.
- [14] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Marcel Dekker, New York and Basel, 1984.
- [15] P. Hartman and G. Stampacchia, On some non-linear elliptic differential-functional equations, *Acta Mathematica* **115** (1966), 271–310.
- [16] S. Ishikawa, Fixed points and iteration of a nonexpansive mapping in a Banach Space, *Proceedings of the American Mathematical Society* **59** (1976), 65–71.
- [17] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York-London, 1980.
- [18] I. V. Konnov, *Combined Relaxation Methods for Variational Inequalities*, Springer-Verlag, Berlin, Heidelberg, 2001.
- [19] E. Kopecká and S. Reich, A note on the von Neumann alternating projections algorithm, *Journal of Nonlinear and Convex Analysis* **5** (2004), 379–386.
- [20] E. Kopecká and S. Reich, Another note on the von Neumann alternating projections algorithm, *Journal of Nonlinear and Convex Analysis* **11** (2010), 455–460.
- [21] E. Masad and S. Reich, A note on the multiple-set split convex feasibility problem in Hilbert space. *Journal of Nonlinear and Convex Analysis* **8** (2007), 367–371.
- [22] J. von Neumann, On rings of operators. Reduction theory, *Annals of Mathematics* **50** (1949), 401–485.
- [23] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, *Bulletin of the American Mathematical Society* **73** (1967), 591–597.

- [24] M. Patriksson, *Nonlinear Programming and Variational Inequality Problems, A Unified Approach*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1999.
- [25] S. Reich, Nonlinear evolution equations and nonlinear ergodic theorems, *Nonlinear Analysis* **1** (1977), 319–330.
- [26] S. Reich, A limit theorem for projections, *Linear and Multilinear Algebra* **13** (1983), 281–290.
- [27] A. Renaudi and G. Cohen, Conditioning and regularization of nonsymmetric operators, *Journal of Optimization Theory and Applications* **92** (1997), 127–148.
- [28] R. T. Rockafellar, On the maximality of sums of nonlinear monotone operators, *Transactions of the American Mathematical Society* **149** (1970), 75–88.
- [29] W. Takahashi and M. Toyoda, Weak convergence theorems for non-expansive mappings and monotone mappings, *Journal of Optimization Theory and Applications* **118** (2003), 417–428.